Proc. Indian Acad. Sci. (Math. Sci.), Vol. 100, No. 1, April 1990, pp. 87–92.

On shock dynamics

PHOOLAN PRASAD
Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012, India

MS received 30 May 1988; revised 11 September 1989

Abstract. This is in continuation of our paper 'On the propagation of a multi-dimensional shock of arbitrary strength' published earlier in this journal (Srinivasan and Prasad [9]). We had shown in our paper that Whitham's shock dynamics, based on intuitive arguments, cannot be relied on for flows other than those involving weak shocks and that too with uniform flow behind the shock. Whitham [12] refers to this as misinterpretation of his approximation and claims that his theory is not only correct but also provides a natural closure of the open system of the equations of Maslov [3]. The main aim of this note is to refute Whitham's claim with the help of an example and a numerical integration of a problem in gasdynamics.

Keywords. Shock propagation; non-linear waves; compressible flow.

1. Compatibility conditions on a shock manifold

Significance of a characteristic manifold of a hyperbolic system of first order partial differential equations in $m+1$ independent variables is well known and lies in the fact that one can derive from the equations a compatibility condition which contains only m independent interior derivatives in the manifold. One of the interior derivatives can be chosen in the direction of the bicharacteristic curve (or along a ray). It is also true that no surface other than the characteristic surface has the property that one can derive from the equations a compatibility condition on that surface containing only the interior derivatives. Therefore, if we take a shock manifold in $m+1$ dimensional space of all independent variables, the differential equations can be used to derive on it a relation which contains m interior derivatives in the shock manifold (one of which may be chosen to be the direction given by the shock ray) and one exterior derivative. In our paper [9] we derived one such compatibility condition for gasdynamics equations with special aim that all the derivatives appearing in it should be that of the excess of the density behind the shock (extension of ρ_1) over that ahead of the shock (ρ_0) divided by ρ_0 *i.e.* $(\rho_1-\rho_0)/\rho_0=\mu$, say, a measure of the shock strength. For two space dimensions, we have shown that

$$\frac{\partial\mu}{\partial s_1}=B_1(\mu)\frac{\partial\Theta}{\partial\eta}+B_2(\mu)\frac{\partial\mu}{\partial N},\tag{1}$$

and

$$\frac{\partial\Theta}{\partial s_1}=-\frac{\gamma+1}{2(1+\mu)\{2+\mu(1-\gamma)\}}\frac{\partial\mu}{\partial\eta},\tag{2}$$

where $(\cos\Theta, \sin\Theta)$ is the unit normal to shock surface at time t, $\partial/\partial s_1$ is the spatial rate of change in (x_1, x_2, t)-space along a shock ray, $\partial/\partial\eta$ and $\partial/\partial N$ are the tangential and normal derivatives for the shock surface at a fixed time, and B_1 and B_2 are functions of μ and the ratio γ of the specific heats (Srinivasan and Prasad [10]). The derivatives $\partial/\partial s_1$ and $\partial/\partial\eta$ are interior derivatives in the shock manifold in (x_1, x_2, t)-space and $\partial/\partial N$ is an exterior derivative.

Equation (1) is just one of the infinite sequence of compatibility conditions, which can be derived on the shock manifold by Grinfeld and Maslov's methods ([2], [3]). It throws a deep insight into a new structure of the gasdynamic equations: the rate of change of the shock amplitude depends not only on the distribution of the shock strength on the shock surface but also on the gradient of the flow in the direction of the normal to the shock. This is an exact result. At any time, this gradient can be made to have an arbitrary value by suitably prescribing the initial data influencing the flow behind the shock and hence the last term in (1) represents the effect of the waves which catch up with the shock from behind.

Whitham's theory [11] 'shock dynamics' is based on a number of assumptions (such as rays are particle paths, see page 277 [11]), the two main assumptions are

Assumption 1. Evolution of the shock strength on the shock front depends only on the initial position of the shock front and the distribution of the physical variables (and not on their derivatives) on the shock front.

Assumption 2. The rate of change of the shock strength along a ray is given by the characteristic rule.

Assumption 1, which has never been stated explicitly by Whitham, was not realised and pointed out earlier to our work [4], [8]. This assumption is mathematically equivalent to having a compatibility condition on the shock manifold with only first order interior derivatives appearing in it. This assumption leads to Whitham's basic equations (equations (8.59) and (8.60) of [11]) and amounts to having only terms containing the derivatives $\partial/\partial s_1$ and $\partial/\partial\eta$ in (1). Assumption 2, giving a quantitative relation between shock strength and the ray tube area, is independent of assumption 1. The exact compatibility condition (1) shows that assumption 1 is not correct. To deduce his equation (8.59) in [11] from (1), Whitham assumes without any justification

$$\frac{\partial\mu}{\partial N} = a(\mu)\frac{\partial\mu}{\partial s_1}. \tag{3}$$

Since

$$\frac{\partial}{\partial s_1} = \frac{1}{C}\left(\frac{\partial}{\partial t} + C\cos\Theta\frac{\partial}{\partial x_1} + C\sin\Theta\frac{\partial}{\partial x_2}\right),$$

$\partial/\partial N = \cos\Theta\,\partial/\partial x_1 + \sin\Theta\,\partial/\partial x_2$ and μ has three independent arguments t, s and N (or t, x_1, x_2) (independent because at any time t the function μ *i.e.* the state behind the shock can be arbitrarily prescribed), there is no justification for the assumption (3). Assumption 2 is also not correct and it is well known that it gives reasonable results only for a few problems (Hayes [1]). In fact, Prasad *et al* [5] have recently worked out an example where the assumption can give a result with error as much as 800% or more. It is surprising that Whitham criticises our work, since he himself failed to give a mathematical justification for the assumption 2 as evident from his remark (page 272, [11]): 'When the quick derivation of the characteristic rule occurred to me, I hoped also that a full analysis of the approximation could be based directly on the original fluid dynamic equations. So far this has not been completed!' Of course, this cannot be completed because the characteristic method is not a correct approximation for a majority of problems as shown by the example in the next section.

2. An example

Example. Consider an one-dimensional single conservation law

$$u_t + (\tfrac{1}{2}u^2)_x = 0 \tag{4}$$

for which the shock velocity S is given in terms of the state u_l on the left and u_r on the right by

$$S = \tfrac{1}{2}(u_l + u_r). \tag{5}$$

We assume that the restriction of the solution to a region immediately on the left of the shock is smooth, then u satisfies $u_t + uu_x = 0$ or $u_t + \frac{1}{2}(u + u_r)u_x + \frac{1}{2}(u - u_r)u_x = 0$. Taking the limit as we approach the shock from the region on the left, we deduce the compatibility condition along the shock path in (x, t)-plane as

$$\frac{\mathrm{d}u_l}{\mathrm{d}t} + \tfrac{1}{2}(u_l - u_r)(u_x)_l = 0 \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{1}{2}(u_l + u_r). \tag{6}$$

Unlike the compatibility condition $\mathrm{d}u/\mathrm{d}t = 0$ on the characteristic curve along $\mathrm{d}x/\mathrm{d}t = u$, the condition (6) contains an exterior derivative with respect to the shock curve in addition to an interior derivative $\mathrm{d}/\mathrm{d}t = \partial/\partial t + S(\partial/\partial x)$. The exterior derivative term is $\frac{1}{2}(u_l - u_r)(u_x)_l$ which corresponds to the term $B_2(\mu)(\partial/\partial N)$ in (1).

While criticising our work, Whitham makes another (wrong) assumption, namely by the relation (3), which for this example, becomes

$$(u_x)_l = a(u_l)\left(\frac{\partial u_l}{\partial t} + S\frac{\partial u_l}{\partial x}\right).$$

Using it in (5), we get

$$\{1 + \tfrac{1}{2}a(u_l)(u_l - u_r)\}\frac{\mathrm{d}u_l}{\mathrm{d}t} = 0 \quad \text{along} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \tfrac{1}{2}(u_l + u_r),$$

a compatibility condition containing only the interior derivatives. This implies (unless $1 + \frac{1}{2}a(u_l)(u_l - u_r) = 0$) that u_l is constant along the shock path irrespective of the state behind the shock. Thus, this assumption (3) of Whitham leads to an absurd result. This is also shown by considering the following initial value problem (Ramanathan [7]) for the conservation law (4):

$$u(x, 0) = \begin{cases} 0, & x < -\eta \\ \dfrac{x + \eta}{\eta + 1}, & -\eta \leqslant x < 1 \\ 0, & x \geqslant 1 \end{cases} \tag{7}$$

with $\eta > -1$. Note that the initial data contains a single shock at $x = 1$. The shock position $X(t)$ at any time t is given by

$$X(t) = -\eta + (1+\eta)^{1/2}(1+\eta+t)^{1/2}. \tag{8}$$

As $\eta \to \infty$, $u(x,0)$ gives a uniform state behind the shock and in this case the shock moves with a constant velocity $\frac{1}{2}$. For all other values of η, $-1 < \eta < \infty$, the shock velocity and shock strength are respectively

$$S \equiv \frac{dX}{dt} = \frac{1}{2}\left(\frac{1+\eta}{1+\eta+t}\right)^{1/2} < \frac{1}{2} \quad \text{and} \quad u_1 = \left(\frac{\eta+1}{1+\eta+t}\right)^{1/2} \tag{9}$$

which can differ by any amount (in per cent error) from the respective values $1 + \frac{1}{2}t$ and $\perp$ given by Whitham's Characteristic rule.

3. An example from gasdynamics

We have also used (1) and (2) to develop an algorithm to compute successive positions of a curved shock, when the shock is weak (Ramanathan [7], Ravindran and Prasad [8]). We present here the results of a computation using this algorithm and Whitham's theory from the Ph.D. thesis of Ramanathan for waves produced by a cuncave piston. The figure gives successive positions of a shock front according to

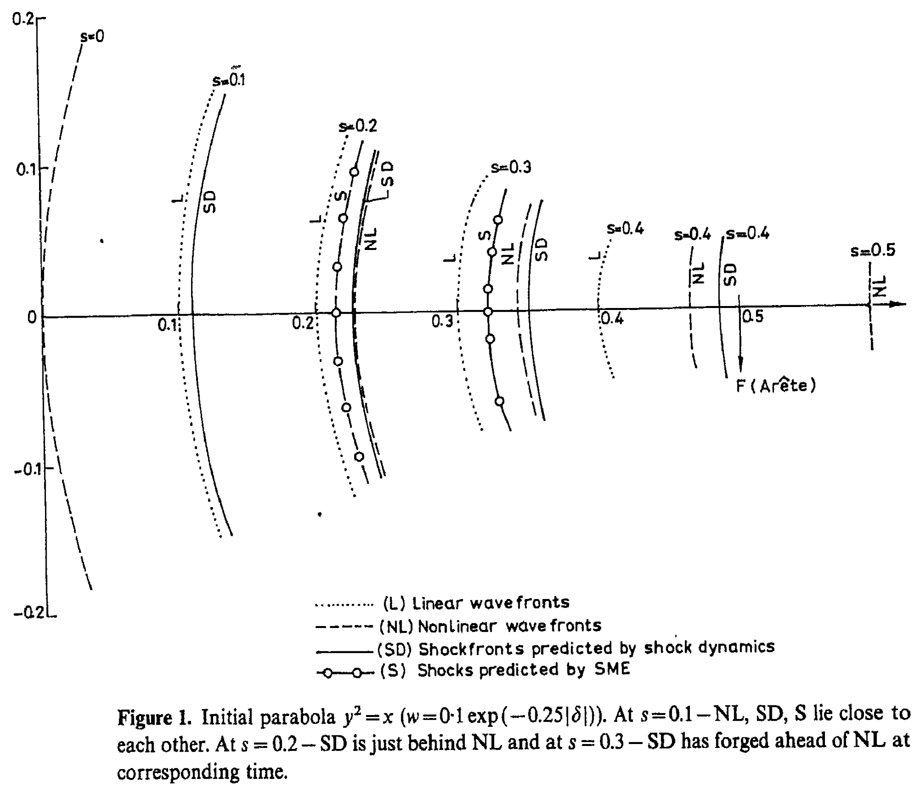

Figure 1. Initial parabola $y^2 = x$ ($w = 0{\cdot}1 \exp(-0.25|\delta|)$). At $s = 0.1$ – NL, SD, S lie close to each other. At $s = 0.2$ – SD is just behind NL and at $s = 0.3$ – SD has forged ahead of NL at corresponding time.

Whitham's shock dynamics (referred to SD in the figure), the shock fronts (S) as predicted by our theory, the linear wavefront (L) and the nonlinear wavefront (LS) (we have also developed a theory 'Kinematics of a nonlinear wavefront' to compute weakly nonlinear wavefronts). In this particular case, the shock predicted by Whitham's theory moves quite ahead of ours. At $t' = 0.3$ and $t' = 0.4$ it has moved even ahead of the nonlinear wavefront (NL), a result which is certainly not correct, since it contradicts a well known result 'the shock velocity of a weak shock is the mean of the characteristic velocities ahead and behind the shock'.

4. Conclusion

Equation 1 is just one of the infinite sequence of compatibility conditions, which can be derived on the shock manifold. It throws a deep insight into a new structure of the gasdynamic equations, the rate of change of the shock amplitude depends not only on the distribution of the shock strength on the shock surface but also on the gradient of the flow in the direction of the normal to the shock. This is an exact result. The term $\partial\mu/\partial N$ is zero when the flow behind the shock is taken uniform in the normal direction at each instant. Except for this very special case, $\partial\mu/\partial N \neq 0$ and, in general, would contribute to the solution. As shown in the example, the value of $\partial\mu/\partial N$ will significantly depend on the initial data and hence this term must be treated as independent of the other two terms in (1). This being so, if we wish to compare our theory with Whitham's shock dynamics we must compare the coefficients B_1 and B_3 as done in our previous paper. Even in the case when the flow behind the shock is uniform i.e. $\partial\mu/\partial N = 0$, the shock dynamics of Whitham is not fully justified. Equations (1) and (2) reduce to equations similar to the shock dynamics equations but the coefficient B_1 differs significantly from B_3 in the shock dynamics for all values of μ other than those of weak shocks (see [10]). The column for $|((B_3 - B_1)/B_1)|$ in the table shows that the error is more than 50% for $M = 2.5$, about 100% for $M \simeq 3.5$ and increases to 255% for strong shocks. While criticising our work, Whitham completely ignores the significant difference present in the table for intermediate values of M.

We finally note that there is neither a mathematical proof for the derivation of Whitham's shock dynamics equations nor has any one shown that it can be derived as an approximate theory. There are only two justifications for its use: (i) it has given accurate results for certain problems (specially, some of the results of characteristic rule are amazingly good) and (ii) it is a simple and elegant method to solve a large variety of complex problems (Whitham [11]). However, there are situations (Hayes, [1]) of exact similarity solutions where the error by the characteristic rule (which forms the basis of shock dynamics) is as much as 15%. Our example in § 2 shows that when $t \gg (1 + \eta)$ the characteristic rule gives an error for the shock position which is even more than 100%. Our computation of a flow problem in § 3 shows that Whitham's theory gives a result which is absurd since the shock front moves ahead of the nonlinear wavefront. In mathematics, there is a very important difference between verification and proof. Verifying any number of times does not constitute a proof of a theory. A single large disagreement with the results of a theory, verified even hundred times for particular problems, immediately shows that the theory is not based on a sound foundation. In Whitham's shock dynamics and the characteristic rule, there is no well defined set of problems where they can give results in close agreement with the exact solution. Unless the set of problems, where shock dynamics can be safely used, is identified and unless an estimation of error is found, it is not correct to use Whitham's method. Not only that, it leads to a great misunderstanding about the correct form of equations on the shock manifold. Actually, Whitham shock dynamics for a weak shock is nothing but the kinematics of a nonlinear wavefront developed by us (Ravindran and Prasad, [8]) with some changes in the coefficients of the equations.

References

[1] Hayes W D, Self-similar strong shocks in an exponential medium, *J. Fluid Mech.* **32** (1968) 305–315

[2] Grinfel'd M A, Ray method for calculating the wavefront intensity in nonlinear elastic material, *PMM J. Appl. Math. Mech.* **42** (1978) 958–977

[3] Maslov V P, Propagation of shock waves in an isentropic non-viscous gas, *J. Sov. Math.* **13** (1980) 119–163

[4] Prasad P, Extension of Huyghen's construction of a wavefront to a nonlinear wavefront and a shockfront, *Curr. Sci.* **56** (1987) 50–54

[5] Prasad P, Ravindran R and Sau A, On the characteristic rule for shocks, *Appl. Math. Lett.*, (To appear)

[6] Prasad P and Srinivasan R, On methods of calculating successive positions of a shock front, *Acta Mech.* **74** (1988) 81–93

[7] Ramanathan T M, Huyghen's method of construction of weakly nonlinear wavefronts and shockfronts with application to hyperbolic caustic, Ph.D. Thesis, Indian Institute of Science, Bangalore, 1985

[8] Ravindran R and Prasad P, Kinematics of a shockfront and resolution of a hyperbolic caustic, in *Advances in nonlinear waves* (Ed) L Debnath, 1985, Pitman Research Notes in Mathematics, Vol II No. 111

[9] Srinivasan R and Prasad P, On the propagation of a multidimensional shock of arbitrary strength, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **94** (1985) 27–42

[10] Srinivasan R and Prasad P, Corrections to some expressions in "On the propagation of a multi-dimensional shock of arbitrary strength", *Proc. Indian Acad. Sci.* (*Math. Sci.*) **100** (1990) 93–94

[11] Whitham G B, *Linear and Nonlinear Waves*, (New York: John Wiley and Sons) 1974

[12] Whitham G B, On shock dynamics, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **96** (1987) 71–73