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TECHNICAL NOTE

Mass transfer effects on the generalised vortex flow over a stationary surface with or without magnetic field

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Abstract. The effect of injection and suction on the generalised vortex flow of a steady laminar incompressible fluid over a stationary infinite disc with or without magnetic field under boundary-layer approximations has been studied. The coupled nonlinear ordinary differential equations governing the self-similar flow have been numerically solved using the finite-difference scheme. The results indicate that the injection produces a deeper inflow layer and de-stabilises the motion while suction or magnetic field suppresses the inflow layer and produces stability. The effect of decreasing n, the parameter characterising the nature of vortex flow, is similar to that of increasing the injection rate.

Keywords. Generalised vortex flow; injection; suction; boundary layer; magnetic field; infinite disc.

1. Introduction

The study of vortex flows above stationary surfaces has drawn considerable attention recently due to its applications in natural phenomena such as hurricanes, tornadoes, dust devils and spouts as also in industry such as in vortex chambers. The analogous hydromagnetic vortex flow with magnetic field finds application in cosmical and geophysical fluid dynamics such as in the evolution of rotating magnetic star, geomagnetic field of the earth etc. In addition, it can be applied in the industry such as in magnetohydrodynamic generators, gaseous-core nuclear reactors etc. Much of the earlier work (along with its applications) on vortex flows has been reviewed by Rott & Lewellen (1966), Morton (1966) and Lewellen (1971).

Solutions for the steady laminar incompressible boundary layer without mass transfer under vortex flow over a stationary disc have been obtained by several investigators. A generalised vortex is a circulatory motion in which the external tangential velocity V is a function of r, the distance from the axis of rotation. Simple examples are $V \propto r$ corresponding to a rigid rotation which has been studied by Bödewadt (1940), Browning (given in Schlichting 1968) and Millisaps & Nydahl (1973) and $V \propto r^{-1}$ corresponding to a potential vortex which has been studied by Burggraf *et al* (1971) and Serrin (1972). More general forms of the type $V \propto r^n$ where n is a constant are common; for example, in the atmosphere and in vortex chambers. Such forms are approximate fits of the observed velocity distributions. It has been found by Riehl (1954) that values of n between $-0{\cdot}6$ and $-0{\cdot}4$ are appropriate to hurricanes. Mack (1962), Kuo (1971) and Belcher *et al* (1972) have studied the above problem when $V \propto r^n$ where n varies from -1 to 1. It may be remarked that a single value of n cannot describe the velocity distribution within an entire vortex, but the various parts of the vortex may be represented by different values of n. For example, very close to the axes, the fluid is in solid-body rotation which corresponds to $n=1$ while outside the radius of the maximum wind, the velocity profile is usually close to that of a potential vortex which corresponds to $n=-1$. The transition region midway may be considered to be represented by the power-law distribution where n continuously decreases from 1 to -1. Therefore, it becomes necessary to study the flow for all values of n between -1 to 1. King & Lewellen (1964) studied the analogous hydromagnetic generalised vortex flow ($V \propto r^n$) for electrically conducting fluid with magnetic field on a stationary infinite disc in the absence of the mass transfer. Subsequently, Nath & Venkatachala (1977) extended the foregoing analysis to include the effect of suction when the fluid is in rigid-body rotation ($n = 1$). Moore (1956) observed that in the absence of magnetic field and suction, the similarity solution does not exist for the potential vortex ($n=-1$) or for $n=-0{\cdot}5$, but King & Lewellen (1964) and Stewartson & Troesch (1977) found that the solution does exist if the magnetic field exceeds a certain critical value. Nanbu (1971) observed that in the absence of the magnetic field, solution also exists provided the suction parameter is greater than a certain critical value.

The aim of the present analysis is to study the effect of mass transfer (both injection and suction) on generalised vortex flow ($V \propto r^n$, $-1 \leqslant n \leqslant 1$) over a stationary infinite porous flat plate with or without magnetic field. The coupled nonlinear ordinary differential equations governing the self-similar flow have been numerically solved using an implicit finite-difference scheme (Keller 1968). The results have been compared with those of Bödewadt (1940), Browning (see Schlichting 1968), Millisaps & Nydahl (1973), Kuo (1971), King & Lewellen (1964), Nath & Venkatachala (1977), and Nanbu (1971). It may be remarked that our results are more general than those available in the literature so far and they are only particular cases of our results.

2. Governing equations

We consider an infinite, insulated disc in a steady laminar incompressible electrically conducting fluid which at a large distance from the body is rotating with velocity $V \propto r^n$. We apply a magnetic field B perpendicular to its plane and assume that the magnetic Reynolds number is small. Hence, the induced magnetic field can be neglected as compared to the applied magnetic field. It is also assumed that the fluid has constant physical properties and that the flow is axi-symmetric. Under the foregoing conditions, the boundary-layer equations under similarity assumptions taking into account the effect of mass transfer can be expressed in dimensionless form as (King & Lewellen 1964)

$$H''' - [(n+3)/4]\, HH'' + (n/2)\, H'^2 + 2\,(1 - G^2) - SH' = 0, \qquad (1)$$

$$G'' - [n+3)/4]\, HG' + [(n+1)/2]\, H'\, G - S\,(G-1) = 0. \qquad (2)$$

The boundary conditions are

$$H(0) = H_w,\ H'(0) = G(0) = H'(\infty) = 0,\ G(\infty) = 1, \tag{3}$$

where $u = r^{-1} (\partial \psi / \partial z),\ w = -r^{-1} (\partial \psi / \partial r),$

$$\eta = (z/r_0)\,(v_0\, r_0/\nu)^{1/2}\,(r/r_0)^{(n-1)/2}, \tag{4a}$$

$$\psi = v_0\, r_0^2\,(\nu/v_0\, r_0)^{1/2}\,(r/r_0)^{(n+3)/2}\, F(\eta),$$

$$F = -H/2,\ u/v = F'(\eta) = -2^{-1} H'(\eta),$$

$$v/V = G(\eta),\ V = v_0\,(r/r_0)^n,$$

$$S = (\sigma B_0^2\, r_0) \,/\, (\rho\, v_0),\ B = B_0\,(r/r_0)^{(n-1)/2}, \tag{4b}$$

$$w_0/v_0 = c\,(r/r_0)^{(n-1)/2},\ \mathrm{Re} = v_0\, r_0/\nu,\ A = 4\,c\,\mathrm{Re}^{1/2}\,(n+3)^{-1}, \tag{4c}$$

$$H_w = -4\,(w_0/v_0)\,\mathrm{Re}^{1/2}\,(r/r_0)^{-(n-1)/2}\,(n+3)^{-1} = -A. \tag{4d}$$

Here r, θ and z are the radial, tangential and axial directions, respectively; u, v and w are the radial, tangential and axial velocity components, respectively; ψ and H are the dimensional and dimensionless stream functions, respectively; H' (or F'), G and H are the dimensionless radial, tangential and axial velocity components, respectively; v_0 and B_0 are the reference velocity and magnetic field; S is the magnetic parameter; σ is the electrical conductivity; w_0 is the dimensional surface mass transfer parameter; Re is the Reynolds number; ρ is the density; c, A and n are constants; and prime denotes derivatives with respect to η. We note that both w_0 and c are negative for suction and positive for injection and if w_0 varies according to equation (4c), then the dimensionless mass transfer parameter H_w is a constant (*i.e.* $H_w = -A$). The parameter $A \gtrless 0$ according to whether there is a suction or injection.

It may be remarked that equations (1) and (2) are the same as those of King & Lewellen (1964), if we put $H = -2F$ and $G = \Gamma$. They also reduce to those of Nath & Venkatachala (1977) if we put $n = 1$.

The skin-friction coefficients in the radial and tangential directions can be expressed as

$$C_f = 2\,\tau_r/\rho\, V^2 = (\mathrm{Re})^{1/2}\,(r/r_0)^{-(n+1)/2}\, H_w'',$$

$$\bar{C}_f = 2\,\tau_\theta/\rho\, V^2 = (\mathrm{Re})^{1/2}\,(r/r_0)^{-(n+1)/2}\, G_w', \tag{5a}$$

where $\tau_r = \mu\,(\partial u/\partial z)_w,\ \tau_\theta = \mu\,(\partial v/\partial z)_w,\ \mathrm{Re} = v_0\, r_0/\nu.$ (5b)

Here τ_r, C_f and H_w'' are, respectively, the shear stress, skin-friction coefficient and skin-friction parameter in the radial direction and τ_θ, $\bar{C}_f$ and G_w' are the corresponding expressions in the tangential direction.

3. Results and discussion

Equations (1) and (2) under conditions (3) have been solved numerically using a finite-difference scheme. The resulting nonlinear algebraic equations have been solved by Newton's iterative method. Since the method is described in complete detail in Keller (1968), its description is omitted here. The step size $\Delta\eta$ has been taken as 0·05 and η_∞ (value of η at the edge of the boundary layer) between 5 and 30 depending on the values of S, A and n. Further reduction in $\Delta\eta$ and change in η_∞ do not affect the results up to the 4th decimal place.

In order to assess the accuracy of the method for the problem under consideration, we have compared our velocity profiles (H', G, H) for $n = 1$, $A = S = 0$ with those of Bödewadt (1940), Browning (see Schlichting 1968), Millisaps & Nydahl (1973) and Kuo (1971), and for $n=-1$, $A=-1{\cdot}74$, $S=0$ with those of Nanbu (1971). Since the difference between the results obtained by Bödewadt, Millisaps & Nydahl and Kuo is very small, these results are not distinguishable in the scale used here. Hence, the comparison is shown only with those of Browning, Kuo and Nanbu (see figures 1 and 2). The skin-friction results (H''_w, G'_w) for $A = 0$, $-1 \leqslant n \leqslant 1$, $S \geqslant 0$ have been compared with those of King & Lewellen (1964) for $A=-2$, $n=-1$, $S = 0$ with those of Nanbu (1971), and for $A \leqslant 0$, $n = 1$, $S \geqslant 0$ with those of Nath & Venkatachala (1977) and they are given in tables 1 and 2. Both the velocity profiles (H', G, H) and skin-friction parameters (H''_w, G'_w) are found to be in good agreement.

The effect of mass transfer parameter A on the radial, tangential and axial velocity profiles (H', G, H) is shown in figures 1a to 1c, the effect of magnetic parameter S in figures 3a to 3c, and the effect of n characterising the nature of vortex flow in figures 4 and 5. Figures 1, 3 and 4 indicate that the velocity profiles (H', G, H) for $S = 0$, $A \geqslant 0$ approach their asymptotic values in an oscillatory manner. This oscillation is caused by surplus convection of angular

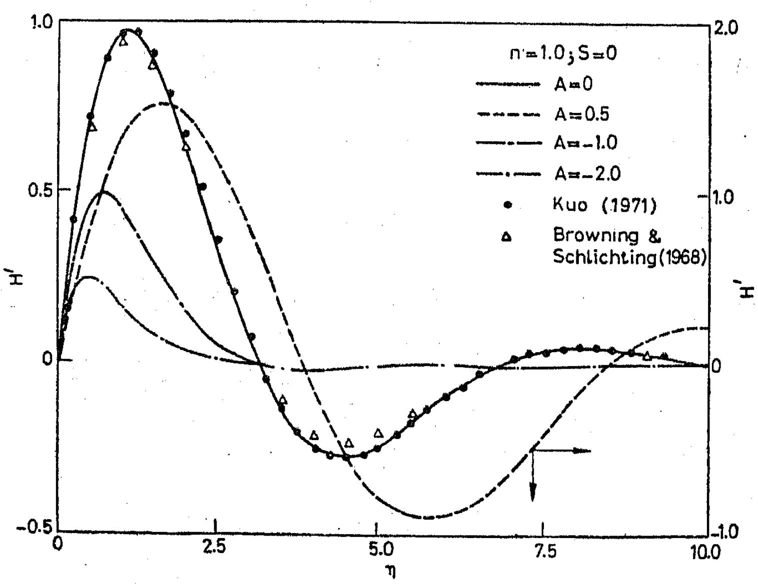

Figure 1a. Effect of mass transfer on radial velocity profiles.

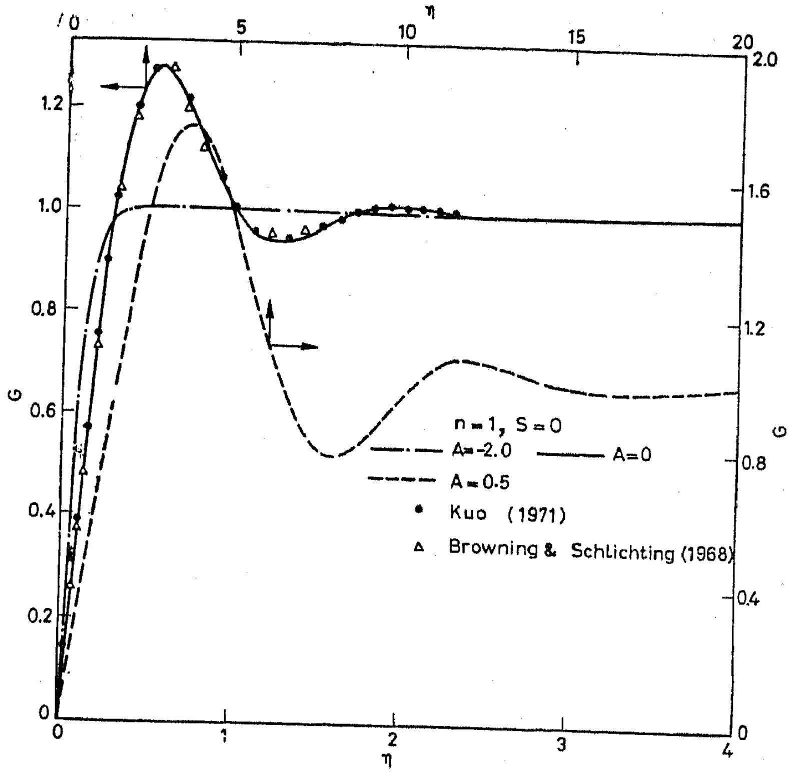

Figure 1b. Effect of mass transfer on tangential velocity profiles.

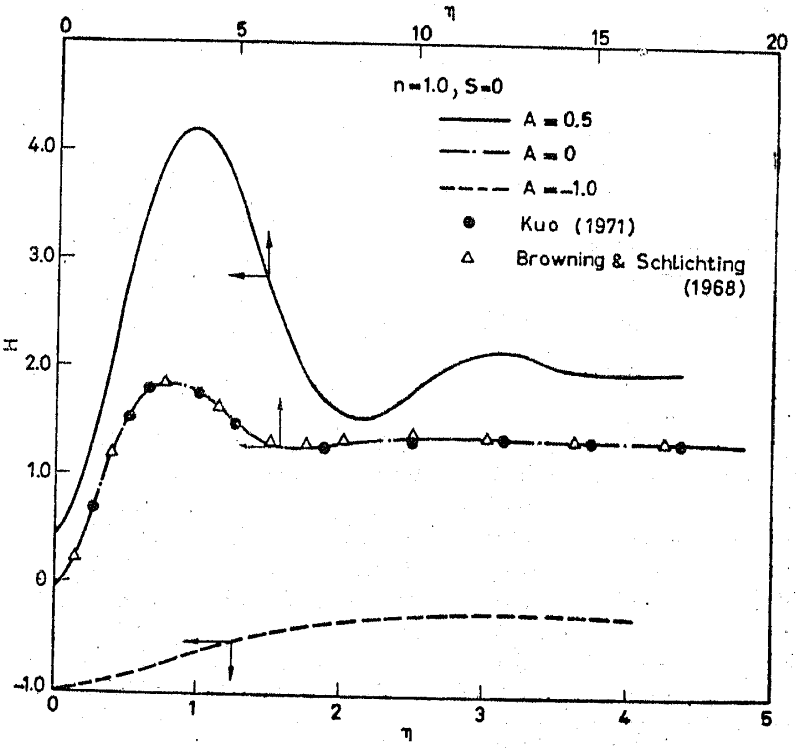

Figure 1c. Effect of mass transfer on axial velocity profiles.

Table 1. Radial and tangential skin-friction parameters at the wall.

S	A	$n = 1{\cdot}0$ H''_w	$n = 1{\cdot}0$ G'_w	$n = 0{\cdot}5$ H''_w	$n = 0{\cdot}5$ G'_w	$n = 0$ H''_w	$n = 0$ G'
0	0·5	1·9081	0·4950	2·0751	0·3612	2·4216	0·2061
0	0	1·8839 (1·884)[a] (1·8839)[b]	0·7727 (0·773)[a] (0·7729)[b]	2·0436 (2·044)[a]	0·6729 (0·673)[a]	2·4026 (2·402)[a]	0·4434 (0·443)[a]
0	−1·0	1·6707 (1·6700)[b]	1·3885 (1·3863)[b]	1·7832	1·1679	1·8122	1·0315
0	−2·0	1·2945 (1·2924)[b]	2·1568 (2·1550)[b]	1·5837	1·8482	1·6621	1·8571
1·0	0·5	1·1667	0·8683	1·2359	0·8308	1·3109	0·7983
1·0	0	1·2207 (1·220)[a] (1·2198)[b]	1·1119 (1·112)[a] (1·1119)[b]	1·3004 (1·262)[a]	1·0374 (1·043)[a]	1·4346	0·9315
1·0	−1·0	1·1653 (1·1634)[b]	1·7242 (1·7237)[b]	1·2659	1·5701	1·4735	1·3609
1·0	−2·0	0·9875 (0·9870)[b]	2·4803 (2·4794)[b]	1·0803	2·2312	1·2263	1·9569
2·0	0·5	0·8827	1·2239	0·9037	1·2228	0·9273	1·2185
2·0	0	0·9183 (0·918)[a] (0·9183)[b]	1·4606 (1·460)[a] (1·4602)[b]	0·9387	1·4280	0·9627 (0·928)[a]	1·3910 (1·431)[a]
2·0	−1·0	0·9058 (0·9062)[b]	2·0485 (2·0481)[b]	0·9354	1·9340	0·9708	1·8144
2·0	−2·0	0·8143 (0·8152)[b]	2·7690 (2·7682)[b]	0·8561	2·5521	0·9065	2·3320
5·0	0·5	0·5791	2·0094	0·5833	2·0294	0·5874	2·0496
5·0	0	0·5937 (0·594)[a]	2·2485 (2·248)[a]	0·5960	2·2396	0·5982	2·2307
5·0	−1·0	0·5985	2·8049	0·6023	2·7201	0·6055	2·6369
5·0	−2·0	0·5738	3·4624	0·5847	3·2806	0·5941	3·1038

[a]Results tabulated by King & Lewellen (1964).
[b]Results tabulated by Nath & Venkatachala (1977).

Table 2. Radial and tangential skin-friction parameters at the wall.

S	A	$n=-0{\cdot}5$ H''_w	$n=-0{\cdot}5$ G'_w	$n=-1{\cdot}0$ H''_w	$n=-1{\cdot}0$ G'_w
0	−2·0	1·6383	1·8764	1·6154 (1·6152)[a]	1·9372 (1·937)[a]
1·0	0·5	1·3583	0·7270	1·3987	0·6812
1·0	0	1·5061	0·8239	1·3165 (1·310)[b]	0·9551 (0·959)[b]
1·0	−1·0	1·8219	1·0599	2·5121	0·9971
1·0	−2·0	2·1470	1·3586	2·9722	1·2114
2·0	0·5	0·9543	1·2103	0·9845	1·1975
2·0	0	0·9910	1·3482	1·0234	1·3010
2·0	−1·0	1·0135	1·6875	1·0733	1·5494
2·0	−2·0	0·9650	2·1061	1·0532	1·8662
5·0	0·5	0·5917	2·0698	0·5959	2·0899
5·0	0	0·6006	2·2220	0·6030	2·2028
5·0	−1·0	0·6003	2·5558	0·6107	2·4753
5·0	−2·0	0·6023	2·9329	0·6092	2·7519

[a]Results tabulated by Nanbu (1971)
[b]Results tabulated by King & Lewellen (1964)

momentum present in the boundary layer (King & Lewellen 1964). It is found that both the amplitude and wavelength of oscillation increase as n decreases or injection ($A > 0$) increases, whereas the effect of suction ($A < 0$) and the magnetic field ($S > 0$) is just the reverse. It is evident from figures 1, 3, 4 and 5 that the maximum value of the radial and axial velocities (H', H) and the edge of the boundary layer increase as injection increases or n decreases. The effect of suction or magnetic field on them is just the opposite. There is a velocity overshoot in the tangential velocity G and it increases as injection increases or n decreases, but the magnitude of

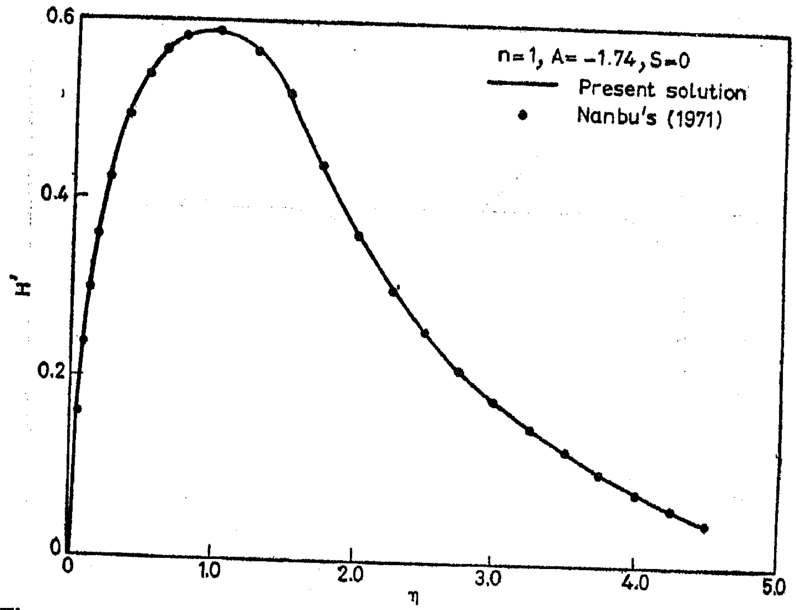

Figure 2. Comparison of radial velocity profiles with those of Nanbu (1971).

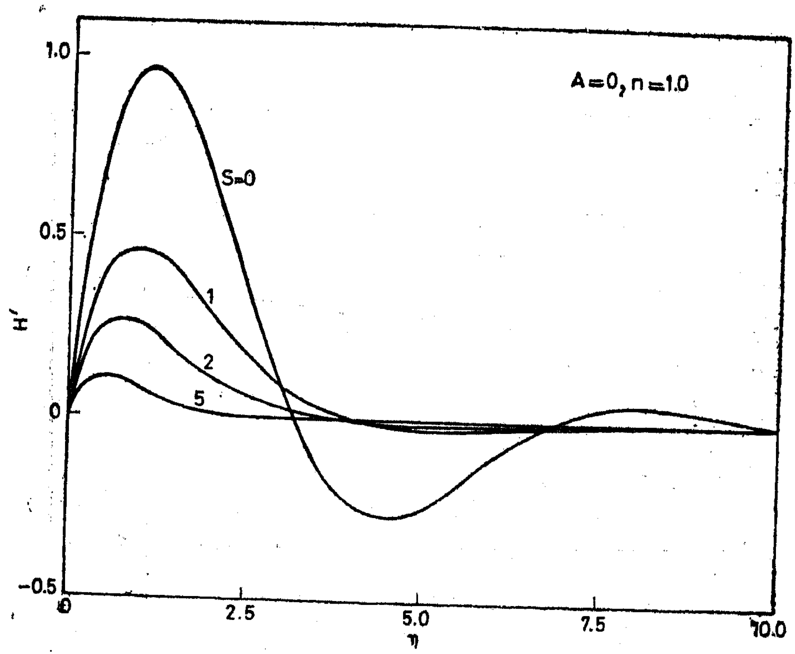

Figure 3a. Effect of magnetic field on radial velocity profiles.

velocity overshoot is reduced by suction or magnetic field (see figures 1b, 3b and 4b). Thus suction or injection produces strong effects on the radial and axial velocity profiles (H',H). Their respective maximum values are changed considerably over their no-mass transfer counterparts. The tangential velocity component G is comparatively less affected by injection or suction. Finally, it may be concluded that the injection produces a deeper inflow layer and de-stabilises the motion while suction or magnetic field depresses the inflow layer and produces stability. Similar effects have been observed by Nguyen *et al* (1975).

The radial and tangential skin-friction parameters H''_w and G'_w for various values of

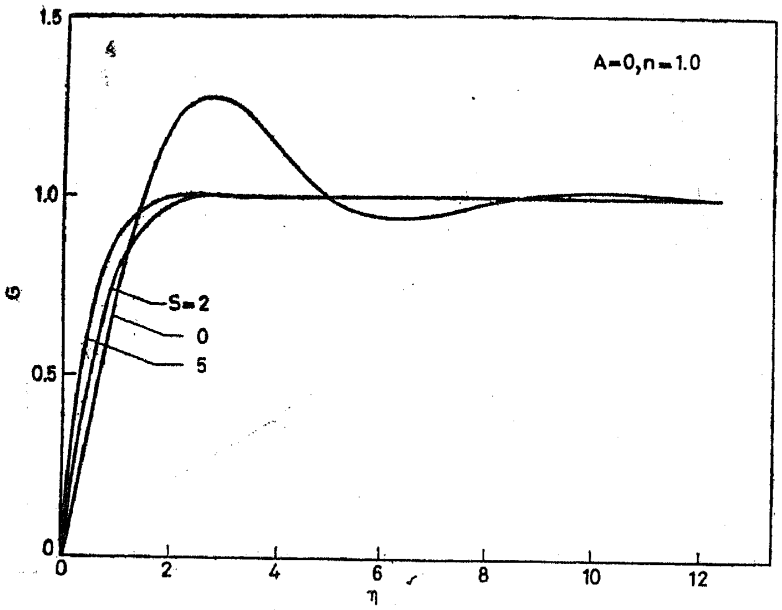

Figure 3b. Effect of magnetic field on tangential velocity profiles.

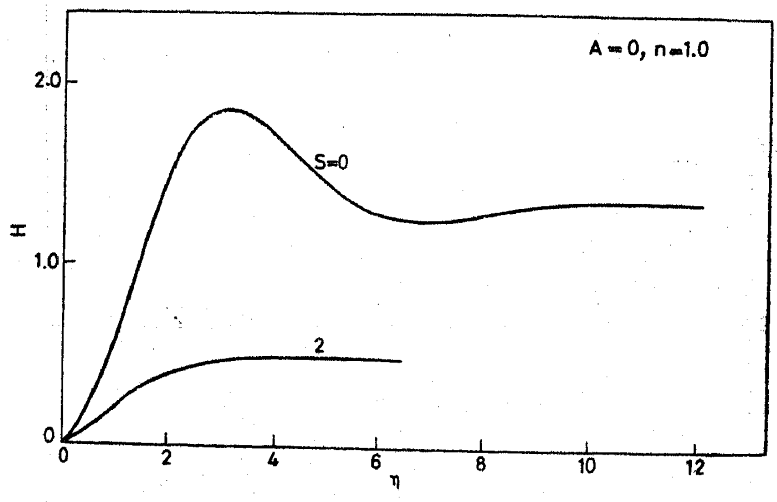

FIG.3c

Figure 3c. Effect of magnetic field on axial velocity profiles.

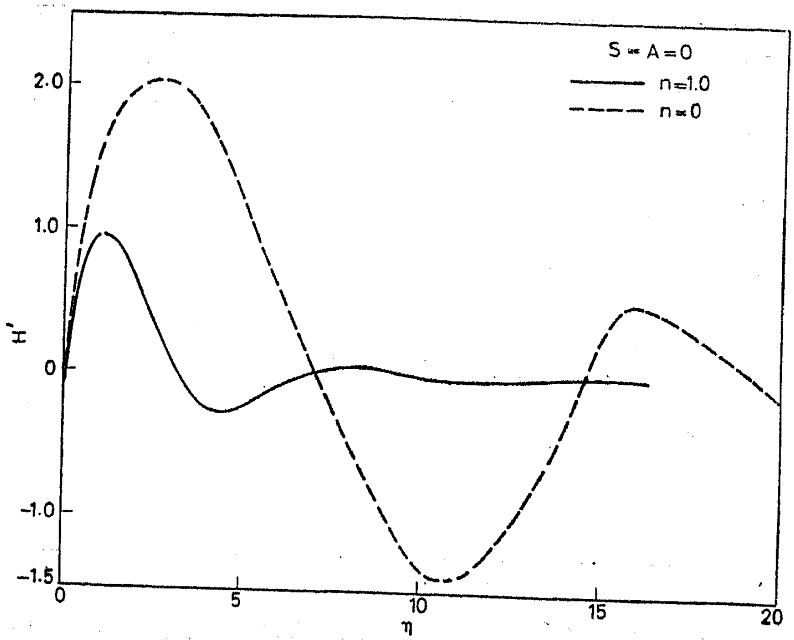

Figure 4a. Effect of the parameter n on radial velocity profiles ($S = 0$).

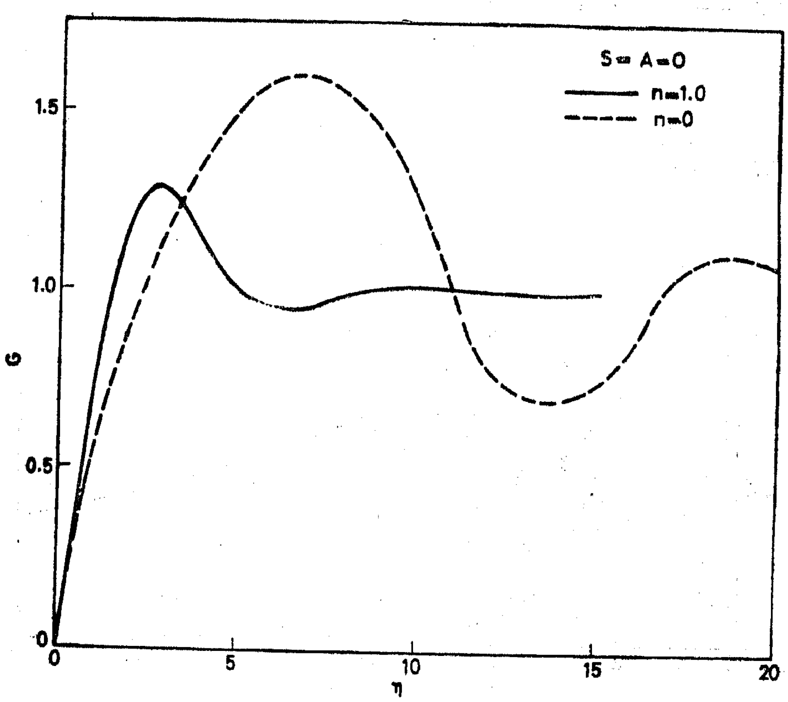

Figure 4b. Effect of the parameter n on tangential velocity profiles ($S = 0$).

S, n and A are given in tables 1 and 2. It is evident from the tables that for $S = 0$ and $A \geqslant 0$, the parameter n has strong influence on H''_w and G'_w, but in the presence of a large magnetic field its effect is rather small. For a given n and A, H''_w decreases, but G'_w increases as S increases. Also for a given S, H''_w decreases, but G'_w increases as n increases whatever may be the values of A (see figure 6). The effect of suction is to increase G'_w and the effect of injection is just the reverse and

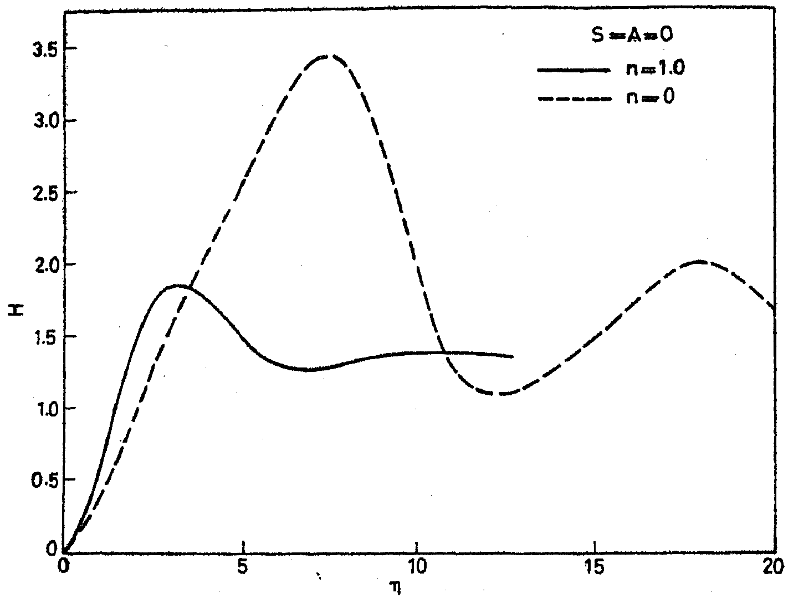

Figure 4c. Effect of the parameter n on axial velocity profiles ($S = 0$).

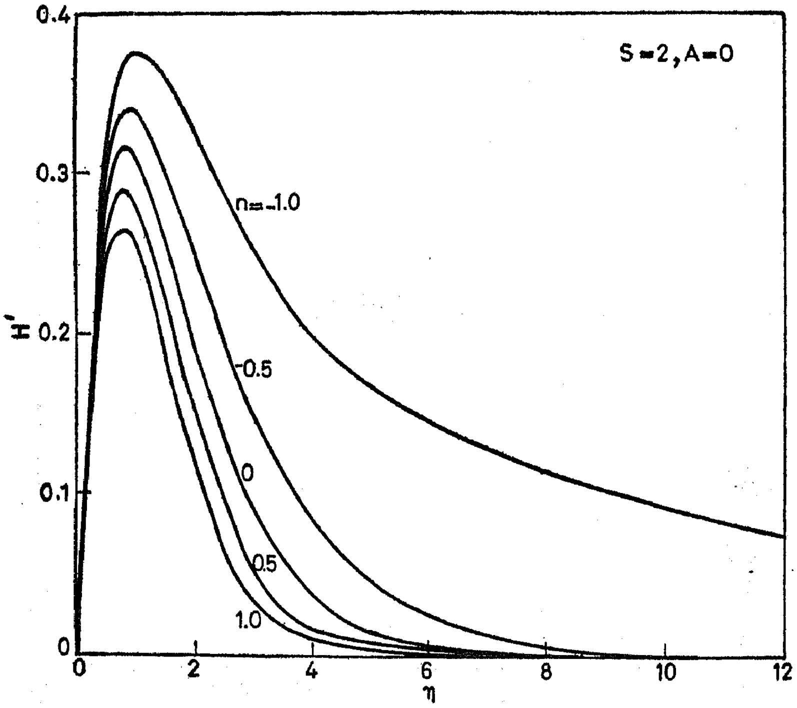

Figure 5a. Effect of the parameter n on radial velocity profiles ($S = 2$).

this is true for all values of S and n. However, H_w'', in general, decreases due to suction. For large S, the effect of mass transfer on H_w'' is small.

It may be remarked that for $S = 0$ (no magnetic field) and $A = 0$ (no mass transfer), the similarity solution does not exist for $n = -1$ or for $n = -0.5$ which has also been observed by Moore (1956). In fact, Belcher *et al* (1972) and Stewartson & Troesch (1977) have found that for S = A = 0 the similarity solution exists when $n > -0.1217$ ($-0.1217 < n \leqslant 1.0$). Our analysis also gives the same results. For $S = 0$ and $n = -1$, the similarity solution exists when $A = 1.74$ which coincides with that of Nanbu (1971). The similarity solution also exists for all n ($-1 \leqslant n \leqslant 1$) when $S \geqslant 0.1$ and $A = 0$ which is the same as that found by Stewartson & Troesch (1977).

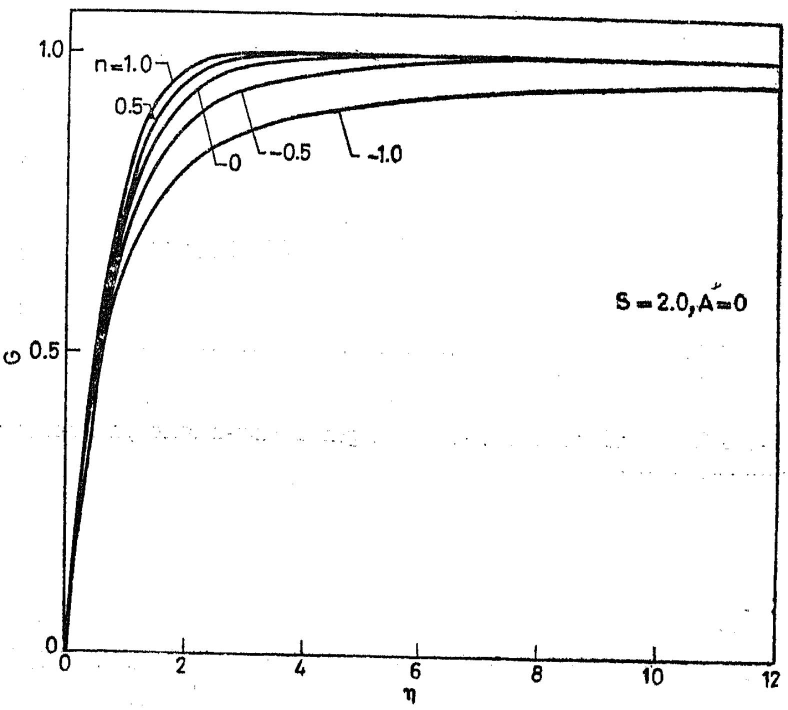

Figure 5b. Effect of the parameter n on tangential velocity profiles ($S = 2$).

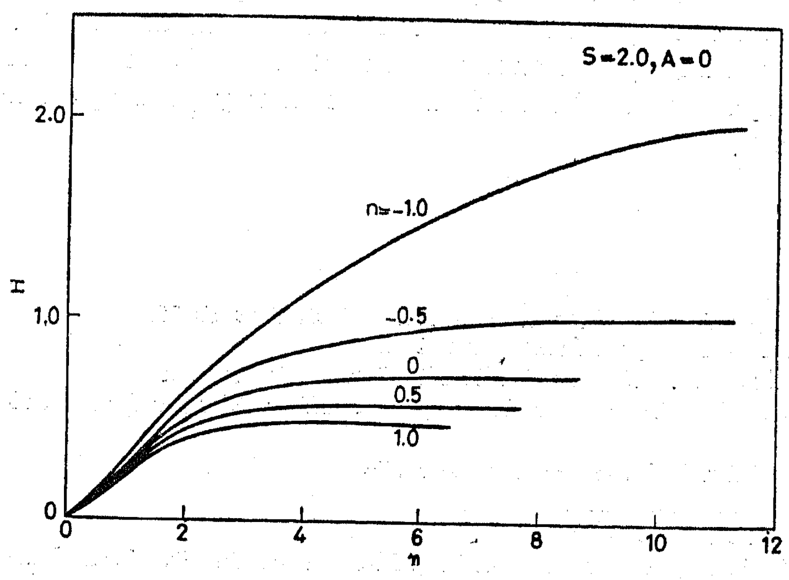

Figure 5c. Effect of the parameter n on axial velocity profiles ($S = 2$).

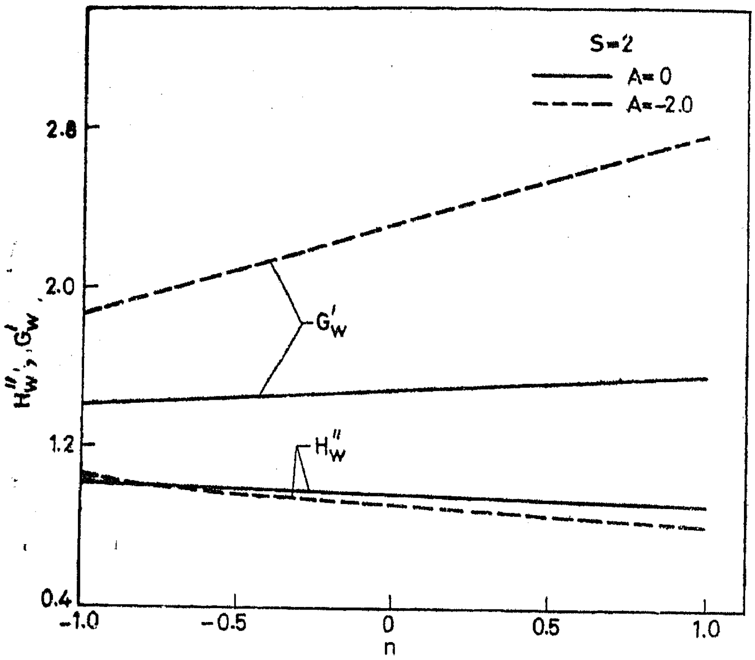

Figure 6. Variation of radial and tangential skin-friction parameters on the surface with respect to *n*.

4. Conclusions

The velocity profiles in the absence of the magnetic field approach their asymptotic values in an oscillatory manner and the amplitude and wavelength of oscillation increase as the injection increases or the parameter n, which characterises the nature of vortex flow, decreases from 1 to -1, whereas the effect of suction and magnetic field is just the reverse. There is a velocity overshoot in the tangential velocity and it increases as injection increases, but the magnitude of the velocity overshoot is reduced by suction or magnetic field. The injection or suction produces strong effect on the radial and axial velocity profiles as compared to that on the tangential profiles. The nature of the vortex flow has a strong effect on the skin friction when the magnetic field is not large. The results are found to be in very good agreement with the available results.

References

Belcher R J, Burggraf O R & Stewartson K 1972 *J. Fluid Phys.* **52** 753
Bödewadt U T 1940 *Z. Angew. Math. Mech.* **20** 241
Burggraf O R, Stewartson K & Belcher R 1971 *Phys. Fluids* **14** 1821
Keller H B 1968 *Numerical methods for two-point boundary value problems* (London: Blaisdell) p. 162
King W S & Lewellen W S 1964 *Phys. Fluids* **7** 1674
Kuo H L 1971 *J. Atmos. Sci.* **28** 20
Lewellen W S 1971 NASA CR-1772
Mack L 1962 Tech. Rep. No. 32-224, Jet Propulsion Laboratory, Pasadena, Calif., USA
Mallisaps K & Nydahl J E 1973 *Z. Angew. Math. Mech.* **53** 241

Moore F K 1956 *Advances in applied mechanics* eds L Dryden, T von Karman and G Kuerti (New York: Academic Press) Vol. 4, p. 159
Morton B R 1966 *Progress in aeronautical science* ed D Kucheman (Oxford: Pergamon Press) Vol. 7, p. 145
Nanbu K 1971 *AIAA J.* 9 1642
Nath G & Venkatachala B J 1977 *Proc. Indian Acad. Sci.* **A85** 332
Nguyen N D, Ribault J P & Florent P 1975 *J. Fluid Mech.* **68** 369
Riehl H 1954 *Tropical meteorology* (New York: McGraw Hill)
Rott N & Lewellen W S 1966 *Progress in aeronautical science* ed. D. Kucheman (Oxford: Pergamon Press) Vol. 7, p. 111
Schlichting H 1968 *Boundary-layer theory* (New York: McGraw Hill) 6th ed. p. 213
Serrin J 1972 *Philos. Trans. R. Soc. London* **A270** 325
Stewartson K & Troesch B A 1977 *Z. Angew. Math. Phys.* **28** 951