EXTENSION OF HUYGHEN'S CONSTRUCTION OF A WAVEFRONT TO A NONLINEAR WAVEFRONT AND A SHOCKFRONT

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ABSTRACT

Huyghen's method of wavefront construction is equivalent to integrating the ray equations. This article reviews a recent development which shows that the extension of the method to a nonlinear wavefront consists of integrating the ray equations along with a compatibility condition. The extension is also possible for the construction of weak shockfront.

INTRODUCTION

CHRISTIAN Huyghen (1629–1695), a Dutch mathematician and physicist, proposed that all points of a wavefront of light may be regarded as new sources of wavelets that expand in every direction at a rate depending on their velocities. Huyghen's proposal is a powerful method for studying various optical phenomena. The envelope of the wavefronts at a given time emitted by these sources constitutes the new wavefront at that time. This envelope or a new wavefront can also be obtained equivalently by drawing normals to the original wavefront, the length of the normals being equal to the distance travelled by the light and then considering the locus of the end points of the normals. The theory of complete integrals of the characteristic partial differential equation and the corresponding construction of envelopes show that the Huyghen's method of wavefront construction is true not only for light waves but also for waves governed by an arbitrary hyperbolic system of linear equations[1]. Huyghen's method of wavefront construction tells only about the location of the wavefront and not about the intensity of the wave on the wavefront. The latter forms an important problem whose approximate solution, in geometrical optics limit, will be discussed in the next section. For a hyperbolic system of quasilinear equations, we shall discuss a recent development in which the wavefront and the intensity on the wavefront are governed, in geometrical optics limit or short wave approximation, by quite a simple system of equations along, what we call, nonlinear rays[2,3]. The equations for the construction of the successive positions of the nonlinear wavefront are now coupled with the amplitude equation and the method can be regarded as an extension of the Huyghen's method. We shall also show that such an extension is possible for the construction of a shockfront (see ref. 4).

GEOMETRICAL OPTICS THEORY FOR THE (LINEAR) WAVE EQUATION

We shall first show that Huyghen's method of wavefront construction is embedded in a mathematical theory of finding approximate solution based on the fact that the discontinuities in a solution at the wavefront are represented by high frequency terms, which also form the dominant part of the solution. In this theory, known as 'geometrical optics', we assume that the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} - a_0^2 \nabla^2 u = 0, \ \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2},$$

$$a_0 = \text{constant} \qquad (1)$$

singular at a moving wavefront or a characteristic surface $\phi(x_\alpha, t) = 0$ and valid in the neighbourhood of the wavefront, is represented in the form

$$u(x_\alpha, t) = v_0(x_\alpha, t)\, H_0(\varphi) + v_1(x_\alpha, t)\, H_1(\varphi) + v_2(x_\alpha, t)\, H_2(\varphi) + \ldots, \qquad (2)$$

where $H_0(\varphi)$ is the Heaviside function: $H_0(\varphi) = 1$ for $\varphi \geq 1$, $= 0$ for $\varphi < 0$ and $H_n(\varphi)$ have weaker singularities for $n \geq 1$:

$$H_n(\phi) = \begin{cases} \frac{1}{n!}\varphi^n, & \geq 0 \\ 0, & < 0. \end{cases} \qquad (3)$$

The expression (2) implies that $v_0|_{\varphi=0}$ is the jump in the function u across $\varphi = 0$ and $v_1|_{\varphi=0}$ is the jump in the normal derivative of u across $\varphi = 0$.

It is not necessary to give explicit forms of the generalized functions whose coefficients are $v_0, v_1, \ldots$ but to write u as

$$u(x_\alpha, t) = \sum_{n=0}^{\infty} v_n(x_\alpha, t)\, g^{(n)}(\varphi) \qquad (4)$$

where the generalized functions $g^{(n)}(\varphi)$, with singularities at $\varphi = 0$, satisfy

$$\frac{d}{d\varphi} g^{(n)}(\varphi) = g^{(n-1)}(\varphi). \qquad (5)$$

Since the functions $g^{(n)}$ are regular at points where $\varphi \neq 0$, the coefficients $v_0, v_1, \ldots$ are uniquely determined only on the surface of discontinuity $\varphi = 0$ [this would not have been the case if we had taken v_n to be independent of t i.e. $v_n = v_n(x_\alpha)$].

Substituting (4) in (1), using (5) and then equating the coefficients of functions $g^{(-2)}$, $g^{(-1)}, g^{(0)}, \ldots$ we get a sequence of conditions on the surface $\varphi = 0$, the first two being

$$\varphi_t^2 - a_0^2 \varphi_{x_\alpha} \varphi_{x_\alpha} = 0, \qquad (6)$$

$$2(\varphi_t v_{0t} - a_0^2 \varphi_{x_\alpha} v_{0x_\alpha}) + (\varphi_{tt} - a_0^2 \varphi_{x_\alpha x_\alpha}) v_0 = 0 \qquad (7)$$

where a repeated suffix indicates sum over the range 1, 2 and 3.

Equation (6) is the characteristic partial differential equation of the wave equation. Its characteristic curves give the bicharacteristic curves in (x_α, t) space and give the rays in (x_α)-space. The ray starting from a point $(x_{\alpha 0}, t_0)$ of the wavefront at time t_0 and having unit normal (n_α) at that point is a straight line given by

$$x_\alpha - x_{\alpha 0} = n_\alpha a_0 (t - t_0), \quad \alpha = 1, 2, 3. \qquad (8)$$

Equation (8) shows that the points (x_α) on the wavefront at time t are obtained by moving in the normal direction (n_α) by a distance equal to $a_0(t - t_0)$. This is equivalent to the Huyghen's construction.

We can easily show[5] that the term

$$\left(\frac{1}{a_0^2} \varphi_t v_{0t} - \varphi_{x_\alpha} v_{0x_\alpha}\right) \Big/ |\text{grad}\, \varphi|$$

in (7) represents the spatial rate of change dv_0/dl of the intensity v_0 along a ray. Since $n_\alpha = \varphi_{x_\alpha}/|\text{grad}\,\varphi|$ and $a_0 = -\varphi_t/|\text{grad}\,\varphi|$, we can show that

$$\varphi_{tt} - a_0^2 \varphi_{x_\alpha x_\alpha} = a_0^2 |\text{grad}\,\varphi| \frac{\partial n_\alpha}{\partial x_\alpha}$$

Let us define a function A along a ray as the limit (as the maximum diameter of the ray tube tends to zero) of the ratio of the cross-sectional area at any location of the ray tube to the area at a standard reference section. It follows from differential geometry that the mean curvature

$$-\frac{1}{2}\frac{\partial n_\alpha}{\partial x_\alpha} = -\frac{1}{2}\frac{1}{A}\frac{dA}{dl}.$$

Equaion (7) now takes the form

$$\frac{1}{v_0}\frac{dv_0}{dl} = -\frac{1}{2A}\frac{dA}{dl}, \qquad (9)$$

which leads to the important result regarding the law of variation of the amplitude of the wave

$$v_0(l) = \left(\frac{A(0)}{A(l)}\right)^{1/2} v(0). \qquad (10)$$

This equation leads to unbounded value of v_0 at the focus or caustic where $A(l) \to 0$. Hence, the expansion procedure (2) or (4) ceases to be valid near these singularities. The correct behaviour near such singularities was first investigated by Airy and has been given using boundary layer theory by Buchal and Keller[6].

EXTENSION OF HUYGHEN'S METHOD TO CONSTRUCTION OF A NONLINEAR WAVEFRONT

Huyghen's method of construction of a linear wavefront follows from the characteristic ordinary differential equations (or the ray equations) of the characteristic partial differential equation of the governing linear equations. The characteristic partial differential equation of a linear system is independent of the amplitude of the wave and hence the method of wavefront construction does not involve the amplitude. This is not so for the gas-dynamics waves, the characteristic partial differential equation for which, for a polytropic gas without dissipation and heat conduction, is

$$\varphi_t + u_\alpha \varphi_{x_\alpha} + a(\varphi_{x_\alpha}\varphi_{x_\alpha})^{1/2} = 0, \qquad (11)$$

where (u_α) is the velocity of the fluid and a is the local velocity of sound, given in terms of gas pressure p and mass density ρ as $a = (\gamma p/\rho)^{1/2}$. Equation (11) corresponds to the forward facing waves.

The characteristic ordinary differential equations of (11) can be written in the form

$$\mathrm{d}x_\alpha/\mathrm{d}t = u_\alpha + n_\alpha a \qquad (12)$$

and

$$\mathrm{d}n_\alpha/\mathrm{d}t = (n_\alpha n_\gamma - \delta_{\alpha\gamma})\left(n_\beta \frac{\partial u_\beta}{\partial x_\gamma} - \frac{\partial a}{\partial x_\gamma}\right), \qquad (13)$$

where unit normal n_α to wave front is given by $n_\alpha = \varphi_{x_\alpha}/|\mathrm{grad}\,\varphi|$. Without loss of generality (but for simplicity of the results) we shall consider hereafter the case of two space dimensions. Then the range of Greek subscripts α, β or γ will be 1 and 2 instead of 1, 2 and 3.

The nonlinear ray equations (12) and (13) contain u_α and a, which depend on the amplitude of the wave. In order to find successive positions of a nonlinear wavefront, we must find rules of the variation of u_α and a along a nonlinear ray. This is not trivial since the compatibility condition on the characteristic surface of the gas-dynamic equations leads only to one relation along the nonlinear rays (12) and (13):

$$\rho a\left(n_1 \frac{\mathrm{d}u_1}{\mathrm{d}t} + n_2 \frac{\mathrm{d}u_2}{\mathrm{d}t}\right) + \frac{\mathrm{d}p}{\mathrm{d}t} + \rho a^2\left(n_1 \frac{\partial u_2}{\partial \lambda} - n_2 \frac{\partial u_1}{\partial \lambda}\right) = 0 \qquad (14)$$

where

$$\frac{\partial}{\partial \lambda} = n_1 \frac{\partial}{\partial x_2} - n_2 \frac{\partial}{\partial x_1} \qquad (15)$$

represents the spatial rate of change along the nonlinear wavefront at any time. However, under the assumption of short wave or high frequency wave (equivalent to the geometrical optics approximation of the linear theory) we can use the compatibility condition along the other two families of characteristic surfaces to show that each of u_1, u_2, p and ρ can be expressed in terms of a single variable, say w. This result has recently been obtained by Srinivasan (1987)[10], in the general case of an arbitrary amplitude. We shall, instead assume the amplitude to be small and discuss the nonlinear wave front in the case of weakly nonlinear waves[7].

The short wave assumption implies that the dimension of the flow region in a direction perpendicular to the wave front is small compared to the lateral extension of the wavefront and its radius of curvature. Consider a two-dimensional motion of an ideal gas under short wave assumption, where the nonlinear solution is the result of waves moving into a medium at rest i.e. with $(u_1, u_2, p, \rho) = (0, 0, p_0, \rho_0)$. Then the above mentioned relation between u_1, u_2, p, ρ and w, for small w, is

$$u_1 = n_1 w,\quad u_2 = n_2 w,\quad p - p_0 = \rho_0 a_0 w,\quad \rho - \rho_0 = (\rho_0/a_0) w. \qquad (16)$$

Let θ be the angle which the normal to the wavefront makes with the x_1-axis i.e. $n_1 = \cos\theta$, $n_2 = \sin\theta$. Now, equations (12), (13) and (14) lead to the following set of ray equations and the compatibility condition

$$\mathrm{d}x_1/\mathrm{d}t = \left(a_0 + \frac{\gamma+1}{2} w\right) \cos\theta, \quad (17)$$

$$\mathrm{d}x_2/\mathrm{d}t = \left(a_0 + \frac{\gamma+1}{2} w\right) \sin\theta, \quad (18)$$

$$\mathrm{d}\theta/\mathrm{d}t = -\frac{\gamma+1}{2}\frac{\partial w}{\partial \lambda}, \quad (19)$$

$$\mathrm{d}w/\mathrm{d}t = -\frac{1}{2} a_0 w \frac{\partial \theta}{\partial \lambda}. \quad (20)$$

These equations form a coupled system of ray equations for the determination of the successive positions of a weakly nonlinear wave front, the inclination to the x_1-axis and the wavefront intensity w. In the linear theory, the right hand side (19) is replaced by by zero showing that the rays are straight lines, w drops out from the equations (17) and (18) leading to (8) and the bicharacteristic equations (17)–(19) decouple from the compatibility condition (20). Equations (17)–(20) can be treated as a mathematical form of an extension of Huyghen's construction of a weakly nonlinear wave front. Apart from stretching of ray rays, the main new thing which appears in this extension is the turning of the wave front due to nonuniform distribution of the intensity on the wave front. Equation (20) is actually the same as that which leads to the amplitude-ray tube area relation in the linear theory but here the rays themselves are shifted due to (19).

Equations (17)–(20) can be used to derive many properties of the solution which are qualitatively very different from those obtained by the linear theory. One such result is that even in the first approximation of the short wave assumption (nonlinear geometrical optics), no infinite amplitude appears in the solution. For a discussion of these and many more new results, reference may be made to Ravindran and Prasad[3] and Ramanathan[2].

EXTENSION OF HUYGHEN'S METHOD TO CONSTRUCTION OF A SHOCK FRONT

For this extension, firstly we need a first order partial differential equation, which we may call a shock manifold partial differential equation (SME) whose characteristic ordinary differential equations would give shock rays. Next we need a compatibility condition along a shock ray. A shock is a surface of discontinuity and hence the high frequency assumption or the short wave assumption is automatically satisfied. It was not until Maslov's work[8] in 1978 (English translation in 1980) and our paper[9] that the concept of SME was clear. Successive positions of a forward facing shock can indeed be determined by a SME

$$s_t + u_{\alpha 0} s_{x_\alpha} + A(s_{x_1}^2 + s_{x_2}^2 + s_{x_3}^2)^{1/2} = 0, \quad (21)$$

where $(u_{\alpha 0})$ is the fluid velocity on one side of the shock, say ahead of the shock on the right and A is the velocity of the shock relative to the gas ahead:

$$A^2 = -\frac{\rho_l}{\rho_l}\frac{p_l - p_r}{\rho_l - \rho_r}. \quad (22)$$

The subscripts l and r refer to states on the left and on the right sides of the shock. There is a difficulty in treating (21) as a partial differential equation, the functions $u_{\alpha 0}$, p_l, p_r, ρ_l and ρ_r are defined only on one side of the shock either in the left subdomain or right subdomain. This difficulty is overcome by continuing these functions on the otherside as infinitely differentiable functions, though such a continuation would not be unique. The second difficulty in accepting (11) as a SME is that there are more than one such equations which can qualify for SME. We showed[9], taking two different SMEs, that the shock rays given by them indeed coincide and hence both SMEs give the same successive positions of the shock.

The compatibility condition along a shock ray was first given by Maslov[8] for an isentropic flow of an ideal gas. For a shock of arbitrary strength, the assumption of constant entropy is not correct. Srinivasan and Prasad[4] derived the compatibility condition for a non-isentropic flow for a shock of arbitrary strength. In this article we use the compatibility condition for a weak shock propagating in two space dimensions into a gas at rest in uniform state [see Ravindran and Prasad (1985)]. Let Θ represent the inclination of the normal to the shock front to the x_1-axis and let w be an appropriate measure of the amplitude of the shock. Then a shock ray and the compatibility condition along it are given by

$$\frac{dx_1}{dt} = \left(a_0 + \frac{\gamma+1}{4} w\right) \cos\Theta \tag{23}$$

$$\frac{dx_2}{dt} = \left(a_0 + \frac{\gamma+1}{4} w\right) \sin\Theta \tag{24}$$

$$\frac{d\Theta}{dt} = -\frac{\gamma+1}{4}\frac{\partial w}{\partial \eta}, \tag{25}$$

and

$$\frac{dw}{dt} = -\frac{1}{2} a_0 w \frac{\partial \Theta}{\partial \eta} - \frac{\gamma+1}{4} w \frac{\partial w}{\partial N}, \tag{26}$$

where

$$\frac{\partial}{\partial \eta} = -\sin\Theta \frac{\partial}{\partial x_1} + \cos\Theta \frac{\partial}{\partial x_2} \quad \text{and}$$

$$\frac{\partial}{\partial N} = \cos\Theta \frac{\partial}{\partial x_1} + \sin\Theta \frac{\partial}{\partial x_2} \tag{27}$$

represent tangential and normal derivatives with respect to the shock surface at any fixed time. Comparing (17)–(19) with (23)–(25) we verify a general result[9] that the shock ray velocity components and the rate of rotation of a shock front is mean of those of the nonlinear wave fronts on the two sides of the shock but instantaneously coincident with the shock.

The right hand side of (17) and (20) contains only interior derivative $\partial/\partial\lambda$ in the nonlinear wavefront and hence given the initial position of the wavefront and the distribution of w on it, we can formulate an initial-value problem for (17)–(20) to solve for the position of a nonlinear wavefront at a later time and the distribution of the amplitude on it. Such a procedure is simply not possible for a shock-front due to the presence of the term $\partial w/\partial N$ which can be evaluated only if the solution is known at any time not only on the shock front but up to a short distance behind the shock front. Correctly speaking, the term $\partial w/\partial N$ in (26) represents the influence of the nonlinear waves which catch up with the shock from behind. This term is zero only when the flow behind the shock is a uniform state as in the case of one-dimensional piston problem moving with uniform speed. Fortunately, in the case of weak shocks, the term $\partial w/\partial N$ can be calculated with the help of the equations (17)–(20) [see Ramanathan[2], Ravindran and Prasad[3] and Prasad and Ravindran[5]]. Thus equations (23)–(26) really form a very good method for solving a large class of practical problems involving propagation of weak shocks.

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