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# NON-VANISHING OF THE FIRST COHOMOLOGY

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RÉSUMÉ. — On démontre que, pour les réseaux $\Gamma$ du type fini dans les groupes semi simples sur les corps locaux de caractéristique positive, $H^1(\Gamma, \mathrm{Ad})$ ne s'annule pas; ceci est bien différent de ce que passe dans le cas de caractéristique zéro.

ABSTRACT. — It is shown here that, for any finitely generated lattice $\Gamma$ in certain semi simple groups over local fields of positive characteristics, $H^1(\Gamma, \mathrm{Ad})$ is non-vanishing; this is in sharp contrast with the situation in characteristic zero.

Let $K$ be a local field (i. e. a non-discrete locally compact field), and let $\mathbf{G}$ be a connected semi simple algebraic group defined over $K$. Let $G = \mathbf{G}(K)$, and let $r = K-\mathrm{rank}\ \mathbf{G}$. The topology on $K$ induces a locally compact Hausdorff topology on $G$; in the sequel, we assume $G$ endowed with this topology. $G$ is then a $K$-analytic group. Let $\Gamma$ be a lattice in $G$ i.e., a discrete subgroup of $G$ such that $G/\Gamma$ carries a finite $G$-invariant Borel measure. We assume that $\Gamma$ is *irreducible*, i.e. no subgroup of $\Gamma$ of finite index is a direct product of two infinite normal subgroups.

In case $K = \mathbf{R}$ and $G$ is not locally isomorphic to either $SL(2, \mathbf{R})$ or $SL(2, \mathbf{C})$, it is known that $H^1(\Gamma, \mathrm{Ad}) = 0$; where, as usual, Ad denotes the adjoint representation of $G$ on its Lie algebra (*see* WEIL [9], [10] for uniform lattices; for non-uniform lattices in groups of $\mathbf{R}$-rank $> 1$, this vanishing theorem follows from the results of RAGHUNATHAN [8], combined with the results of MARGULIS [4] on arithmeticity; for non-uniform lattices in groups of $\mathbf{R}$-rank 1, it is contained in GARLAND-RAGHUNATHAN [2]).

It is also known, in view of a recent result of MARGULIS ([5], theorem 8), that in case $K$ is non-archimedean but of characteristic zero, $H^1(\Gamma, \mathrm{Ad}) = 0$ when $r > 1$.

The object of this note is to show that *when $K$ is of positive characteristic, then it is not in general true that $H^1(\Gamma, \mathrm{Ad}) = 0$.*

We shall in fact prove the following theorem.

THEOREM. — *Let $F$ be a finite field, and let $K$ be the local field $F((t))$. Let $\mathbf{G}$ be a connected semi simple algebraic group, with trivial center, defined over $F$. Let $G = \mathbf{G}(K)$, let $\Gamma$ be a finitely generated lattice in $G$. Then $H^1(\Gamma, \mathrm{Ad}) \neq 0$.*

*Remark.* — If $\mathbf{G}$ has no $K$-rank 1 factors, then according to a well-known theorem of D.A. KAZHDAN (*see* [1]), every lattice in $G$ is finitely generated.

For the proof of the theorem, we need to recall a result of WEIL [10].

We introduce some notation and a definition.

Let $\Lambda$ be a finitely generated abstract group. We shall let $\mathscr{A}(\Lambda, G)$ denote the space of all homomorphisms of $\Lambda$ in $G$ with the topology of pointwise convergence. There is a natural action of $G$ on $\mathscr{A}(\Lambda, G)$ induced by the inner automorphism.

Now assume that $\Lambda$ is a finitely generated subgroup of $G$, and let $\iota : \Lambda \to G$ be the natural inclusion. Then $\Lambda$ is said to be *locally* (or *infinitisimally*) *rigid* if the orbit of $\iota$ under $G$ is open in $\mathscr{A}(\Lambda, G)$. According to a result of WEIL [10], vanishing of $H^1(\Lambda, \mathrm{Ad})$ implies local rigidity of $\Lambda$.

*Proof of the theorem.* — In view of the above result of WEIL, to prove that $H^1(\Gamma, \mathrm{Ad}) \neq 0$, it suffices to show that $\Gamma$ is not infinitisimally rigid.

For $i > 1$, $t \mapsto t + t^i$ extends uniquely to give a continuous automorphism $a_i$ of $F((t))/F$. It is evident that, for any fixed $x \in F((t))$, the sequence $\{ a_i(x) \}$ converge to $x$.

Now since $\mathbf{G}$ is defined over $F$, $a_i$ induces a continuous automorphism $\alpha_i$ of $G$. Therefore, for all $i$, $\alpha_i . \iota$ is an embedding of $\Gamma$ in $G$; where $\iota : \Gamma \to G$ is the natural inclusion of $\Gamma$ in $G$. It is also obvious that the sequence $\{ \alpha_i . \iota \}$ converges to $\iota$ in $\mathscr{A}(\Gamma, G)$. We shall show that none of the $\alpha_i . \iota$ lie in the $G$-orbit of $\iota$. This will prove that $\Gamma$ is not locally rigid and hence $H^1(\Gamma, \mathrm{Ad}) \neq 0$.

If possible, assume that, for some $i$, $\alpha_i . \iota = \mathrm{Int}\, g_i . \iota$. Then $(\mathrm{Int}\, g_i^{-1} . \alpha_i) . \iota = \iota$, and the main theorem of PRASAD [6] implies that $\mathrm{Int}\, g_i^{-1} . \alpha_i$ is the identity automorphism of $G$. Hence, $\alpha_i = \mathrm{Int}\, g_i$.

We now fix a 1-dimensional torus $\mathbf{T}$ ($\subset \mathbf{G}$) which is defined and split over the finite field $F$ (existence of such a torus follows from Lang's theorem [3]). Let $T = \mathbf{T}(K)$. Then since $\mathbf{T}$ is defined over $F$, $\alpha_i(T) = T$. Moreover, for any rational character $\chi$ on $\mathbf{T}$ and all $t \in T$,

$$\chi(\alpha_i(t)) = a_i(\chi(t)).$$

Since $\alpha_i = \operatorname{Int} g_i$ and $\alpha_i(T) = T$, it follows that $g_i$ normalizes $T$ and hence also $\mathbf{T}$. Therefore, for any rational character $\chi$ on $\mathbf{T}$ :

$$\chi(\alpha_i(t)) = \chi(g_i t g_i^{-1}) = \chi^d(t),$$

where $d = +1$ or $-1$. Hence,

$$(\star) \qquad a_i(\chi(t)) = \chi^d(t), \qquad \text{where} \quad d = +1 \quad \text{or} \quad -1.$$

Now take $\chi$ to be one of the generators of the group of rational characters on $\mathbf{T}$. Then it follows from $(\star)$ that, for all $k \in K$, either

$$a_i(k) = k \qquad \text{or} \quad a_i(k) = k^{-1}.$$

But it is obvious from the definition of $a_i$, that this is not the case. Hence, none of the $\alpha_i . \iota$ lie in the $G$-orbit of $\iota$. This proves that $H^1(\Gamma, \mathrm{Ad}) \neq 0$.

*Remark.* — As the above proof shows, $\Gamma$ is not locally rigid. However, in case $K$-rank $\mathbf{G} > 1$ and $\Gamma$ is an irreducible uniform lattice, it is *strongly rigid* (*see* PRASAD [7], § 8).

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