NUMERICAL CALCULATIONS ON THE NEW APPROACH TO THE CASCADE THEORY—I

By S. K. Srinivasan and N. R. Ranganathan

(*Department of Physics, University of Madras*)

Received December 3, 1956

(Communicated by Dr. Alladi Ramakrishnan, F.A.SC.)

ABSTRACT

In the new approach to the cascade theory suggested by Alladi Ramakrishnan and one of us (S. K. S.), we deal with the number of electrons produced between 0 and t in a soft cascade, each of these electrons having an energy greater than E at the point of its production as contrasted with the usual approach where we are interested in the number of electrons with energy greater than E at t. We present here numerical calculations of the mean numbers on the basis of the new approach.

In a previous paper (1956) in these proceedings (hereafter referred to as Paper I), Ramakrishnan and one of us (S. K. S.) suggested a new approach to cascade theory which yields equations having elegant asymptotic solutions. In the normal approach we are interested in $n(E; t)$ the number of particles above a certain energy E at a particular thickness t in a shower initiated by a particle of known energy or energy distribution. On the other hand, we now ask for $N(E; t)$ the number of particles produced between 0 and t each with energy greater than E at the point of its production. It is well known that $n(E; t)$ and $N(E; t)$ are stochastic variates and a comprehensive treatment should deal with their probability distribution functions. A more limited and tractable problem is to obtain the first and second moments of the distribution. In the case of electron-photon showers the first and second moments of $n(E; t)$ have been tabulated by other workers.* We now here present the numerical calculations relating to the first moment of $N(E; t)$.

In Paper I, Ramakrishnan and one of us (S. K. S.) have proved that the mean number of electrons produced between 0 and t with their primitive energies above E, in a shower initiated by a single electron of energy E_0 is given by

* For a comprehensive account of the treatment, see Ramakrishnan and Mathews (1954).

$$\epsilon\{N(E|E_0;t)\}$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{B_s C_s}{\mu_s-\lambda_s}\left(\frac{E_0}{E}\right)^{s-1}\cdot\frac{1}{s-1}\left\{\frac{1-e^{-\lambda_s t}}{\lambda_s}-\frac{1-e^{-\mu_s t}}{\mu_s}\right\}ds \quad (1)$$

where

$$B_s = 2\left\{\frac{1}{s}-\left(\frac{4}{3}+a\right)\frac{1}{(s+1)\ (s+2)}\right\}. \tag{2}$$

$$C_s = \frac{1}{s+1}+\left(\frac{4}{3}+a\right)\frac{1}{s(s-1)}\,. \tag{3}$$

$$\lambda_s = \tfrac{1}{2}(A_s+D)-\tfrac{1}{2}\{(A_s-D)^2+4\,B_sC_s\}^{\frac{1}{2}}. \tag{4}$$

$$\mu_s = \tfrac{1}{2}(A_s+D)+\tfrac{1}{2}\{(A_s-D)^2+4\,B_sC_s\}^{\frac{1}{2}}. \tag{5}$$

$$A_s = \left(\frac{4}{3}+a\right)\left\{\frac{d}{ds}\log|\bar{s}+\gamma-1+\frac{1}{s}\right\}+\tfrac{1}{2}-\frac{1}{s(s+1)}\,. \tag{6}$$

$$D = \frac{7}{9}-\frac{1}{6}a,\quad a = \cdot 0246. \tag{7}$$

For fairly large thicknesses (say $t \geqslant 4$), we can neglect the term containing $e^{-\mu_s t}$ and (1) can be written as

$$\epsilon\{N(E|E_0;t)\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{B_sC_s}{\mu_s\lambda_s}\cdot\frac{1}{s-1}\cdot\left(\frac{E_0}{E}\right)^{s-1}ds$$

$$-\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{B_s\,C_s}{\mu_s-\lambda_s}\cdot\frac{e^{-\lambda_s t}}{\lambda_s(s-1)}\left(\frac{E_0}{E}\right)^{s-1}ds$$

$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{\phi_1(S)}\,ds-\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{\phi_2(S;t)}\,ds \tag{8}$$

where

$$\phi_1(S) = \log\frac{B_sC_s}{\mu_s\lambda_s}-\log(s-1)+y(s-1);\ y = \log\frac{E_0}{E} \tag{9}$$

$$\phi_2(S;t) = \log\frac{B_sC_s}{(\mu_s-\lambda_s)\lambda_s}-\log(s-1)+y(s-1)-t\lambda_s \tag{10}$$

Noting that $\epsilon\{N(E/E_0; t)\}$ is a function only of E_0/E or $y = \log E_0/E$ we can write it as $\epsilon\{N(y; t)\}$. We next observe that as $t \rightarrow \infty$ the second intergral tends to 0 and hence we have

$$\epsilon\{N(y; \infty)\} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{\phi_1(s)}\, ds \tag{11}$$

As $s \rightarrow 2$, $\lambda_s \rightarrow 0$ and $B_s C_s/\mu_s$ remains finite and non zero at $s = 2$. Hence $\phi_1(s)$ and $\phi_2(s; t) \rightarrow +\infty$ as $s \rightarrow 2$. From (2) to (7), it follows that for very large real part of s,

$$A_s \simeq \left(\frac{4}{3} + \alpha\right)\left(\log s + \gamma - 1 + \frac{1}{2s}\right) + \tfrac{1}{2}$$

$$B_s \simeq \frac{2}{s}$$

$$C_s \simeq \frac{1}{s+1}$$

$$\lambda_s \simeq D - \frac{B_s C_s}{A_s - D} \simeq D - \frac{2}{\left(\frac{4}{3} + \alpha\right) s(s+1)\log s}$$

$$\mu_s \simeq A_s + \frac{B_s C_s}{A_s - D} \simeq A_s + \frac{2}{\left(\frac{4}{3} + \alpha\right) s(s+1)\log s} \tag{12}$$

This

$$\log \frac{B_s C_s}{\mu_s \lambda_s} - \log(s-1)$$

and

$$\log \frac{B_s\, C_s}{(\mu_s - \lambda_s)\, \lambda_s} - \log(s-1) - t\lambda_s$$

tend to $-\infty$ logarithmically. Since $y(s-1) \rightarrow +\infty$ much more rapidly, $\phi_1(s)$ and $\phi_2(s; t)$ tend to $+\infty$, as $s \rightarrow \infty$. Hence ϕ_1 and ϕ_2 must have a minimum as s increases along the real axis from 2 to ∞. Let s_1 and s_2 be the points at which

$$\left(\frac{d\phi_1}{ds}\right)_{s=s_1} = 0, \quad \left(\frac{\partial\phi_2}{\partial s}\right)_{s=s_2} = 0 \tag{13}$$

We can shift the line of integration in each of the integrals so that it passes through the saddle points. This can always be achieved since the only restriction on the line of integration is that it should be to the right of imaginary axis and also to the right of all singularities, *i.e.*, $\sigma > 2$. Having chosen the contour to pass through the saddle point, we have the well-known approximation of the saddle point method obtained by replacing each of the functions ϕ_1 and ϕ_2 by the first three terms of its Taylor expansion about the saddle point.

Thus

$$\phi_1(s) \simeq \phi_1(s_1) - \xi^2 \left(\frac{d^2\phi_1}{ds^2}\right)_{s=s_1} \tag{14}$$

$$\phi_2(s;t) \simeq \phi_2(s_1;t) - \eta^2 \left(\frac{\partial^2\phi_2}{\delta s^2}\right)_{s=s_2} \tag{15}$$

where

$$i\xi = s - s_1 \text{ in (14) and } i\eta = s - s_2 \text{ in (15).}$$

Substituting (14) and (15) in (8), we have

$$\epsilon\{\mathrm{N}(y;t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\phi_1(s_1) - \frac{1}{2}\xi^2\phi_1''(s_1)}\, d\xi - \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{\phi_2(s_2;t) - \frac{1}{2}\eta^2\phi_2''(s_2;t)}\, d\eta \tag{16}$$

(primes indicate differentiation with respect to s).
Hence

$$\epsilon\{\mathrm{N}(y;t)\} = \frac{e^{\phi_1(s_1)}}{\sqrt{2\pi\phi_1''(s_1)}} - \frac{e^{\phi_2(s_2;t)}}{\sqrt{2\pi\phi_2''(s_2;t)}} \tag{17}$$

The mean numbers $\epsilon\{\mathrm{N}(y;t)\}$ for various values of y and t have been calculated numerically using (17). In order to obtain the saddle points, it was found necessary to sub-tabulate at intervals of ·025, the basic functions A_s, B_s, C_s, λ_s and μ_s tabulated by Janossy and Messel (1951) at intervals of ·1. The mean number of electrons produced in infinite thickness and fairly large thickness are given in the following table. For convenience of comparison, we have also tabulated $\epsilon\{n(y;t)\}$, the mean number of electrons that exist at t with an energy above E.†

† These are taken from the table of Ramakrishnan and Mathews (1954).

$\epsilon\{N(y;t)\}$

$\epsilon\{n(y;t)\}$ is given in brackets.

y/t	4	6	8	16	∞
8	293 (94·2)	528 (149)	851 (146)	1372	1406
7	124 (48·8)	249 (61·3)	355 (49·6)	502	508
6	64·5 (23·4)	111 (22·9)	146 (15·1)	183	184·5
5	30·6 (10·2)	41·6 (7·55)	57·9 (4·01)	66·4	66·5
4	10·9 (3·85)	14·7 (2·11)	18·9 (1·897)	20·51	20·53
3	6·65 (1·21)	7·86 ·475)	8·56 (·160)	8·82	8·82

$\epsilon\{N(y;t)\}$ unlike $\epsilon\{n(y;t)\}$ has an interesting asymptotic property. While $\epsilon\{n(y;t)\} \rightarrow 0$ as $t \rightarrow \infty$, $\epsilon\{N(y;t)\}$ tends to a finite value for any finite y as $t \rightarrow \infty$. The smaller the y we observe from the table, the more quickly does $\epsilon\{N(y;t)\}$ approach the limit. For almost all the y given in the table the limit is nearly reached at $t = 16$.

For small thickness, we cannot neglect the term containing $e^{-\mu_s t}$ in (1). Then the saddle point formula as adapted here cannot be applied to this term. An alternative method of calculation of (1) for small thickness is being attempted and the mean numbers for small thickness will be reported in a subsequent contribution.

We are deeply indebted to Dr. Alladi Ramakrishnan for suggesting this problem.

References

Janossy and Messel, H. .. *Proc. Roy. Irish Acad.*, 1951, **54**, 217.

Ramakrishnan, Alladi and Mathews, P. M. *Prog. Theo. Physics*, 1954, **11**, 95.

——— and Srinivasan, S. K. .. *Proc. Ind. Acad. Sci. A*, 1956, **44**, 263.