AN APPLICATION OF THE BI-PARTITIONAL FUNCTION Hg (P, Q) IN THE ENUMERATION OF DIFFERENT SAMPLES FROM FINITE POPULATION.

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THE bi-partitional function Hg (P, Q) is defined as

$$h_{p_1} h_{p_2} \cdots h_{p_\rho} = \Sigma \, \text{Hg} \, (P, Q) \, g \, (q_1 \, q_2 \cdots q_\sigma) \tag{1}$$

where P and Q stand for the partitions $(p_1 \, p_2 \cdots p_\rho)$ and $(q_1 \, q_2 \cdots q_\sigma)$ respectively of the same partible number $\omega \; (= \Sigma p_i = \Sigma q_i)$; $g \, (q_1 \, q_2 \cdots q_\sigma)$ is a monomial symmetric function of the quantities $x_1 \, x_2 \cdots$ given by

$$g \, (q_1 \, q_2 \cdots q_\sigma) = \Sigma \, (x_1^{q_1} \; x_2^{q_2} \cdots x_\sigma^{q_\sigma}), \tag{2}$$

the h-function of the pth degree called the homogeneous product sum is given by

$$h_p = \Sigma g \, (p) \tag{3}$$

the summation being taken over all the monomial symmetric functions represented by the partitions of the number p ; and the summation Σ in equation (1) is taken over all partitions Q of the weight ω. The properties of this bi-partitional function, its connection with distributions *in plano* and with the combinatorial problem of which it affords solution and the arithmetical method of evaluating it for large partitions are described in detail in a recent paper by the author in the *Phil. Trans.*[1] The practical method of its evaluation put forth in the above paper is primarily due to R. A. Fisher. It is proposed in this note to exhibit its application in the enumeration of different samples of given size drawn from a finite population.

We shall do well to recall the arithmetical method of evaluating Hg (P, Q). We need consider the special case when either P or Q contains two parts only. Let $P = (p_1, p_2 \cdots p_\rho)$ and $Q = (q, \omega - q)$. For evaluating Hg (P, Q), all the different partitions of q or $(\omega - q)$ whichever is less, having parts less than or equal to ρ are listed and the numbers of ways in which the parts of each of the different partitions of q or $(\omega - q)$ are obtained from the parts of P are recorded. The sum for all partitions represents Hg (P, Q). Thus to evaluate Hg $(42^2, 53)$: The different partitions of 3 having parts less than or equal to $\rho - 3$, are recorded. These are (3), (21) and (1^3). The

[1] P. V. Sukhatme, *Phil. Trans. Roy. Soc. London*, A, No. 780, Vol. **237**, 375-409.

part 3 of the partition (3) may be obtained from the parts of (42^2) in one way only; the parts 2, and 1, may be obtained in six ways from the parts of (42^2) and finally the parts of (1^3) may be obtained in one way only from the parts of (42^2). The process is sy tematical'y carried out as follows:

	(3)	(21)	(1^3)	
(42^2)	1	6	1	8

giving Hg (P, Q) = 8.

A partition $(p_1\, p_2 \cdots p_p)$ may be interpreted to mean that of the $\omega = \Sigma p_i$ quantities comprising a finite population p_1 quantities have a value 1 each; p_2 have a value 2 each and so on. Conversely a finite population of size ω can always be specified by a partition of the same weight ω.

The total number of ways in which samples of size q can be drawn from a population of size ω is $\frac{\omega!}{q!\,(\omega - q)!}$ and is obviously equal to the number of ways of drawing samples of size $(\omega - q)$ from a population of the same size. All the $\frac{\omega!}{q!\,(\omega - q)!}$ samples will not however be different in composition giving different moment statistics. The number of different samples of size q which can be drawn from the population $(p_1\, p_2 \cdots p_p)$, will necessarily be represented by such partit'ons of the number q, which have the number of parts less than or equal to p and whose parts are chosen in all different ways from the parts of the partition $(p_1, p_2 \cdots p_p)$. The arithmetical process described above clearly shows that the number of different samples of size q drawn from the population $(p_1\, p_2 \cdots p_p)$ of size ω is given by Hg (P, Q); where P = $(p_1\, p_2 \cdots p_p)$ and Q = $(q, \omega - q)$ or *vice versa* since Hg (P, Q) = Hg (Q, P).

To take an example, suppose we have a population represented by the partition (1, 6, 8 12,). The interpretation attached to this partition is that of the 27 quantities comprising this population, 1 has a value one, 6 have values two each, 12 have values three each and the remaining 8 have values 4 each. The population will be represented by means of a frequency distribution as follows:

Value of the variate ..	1	2	3	4	
Frequency ..	1	6	12	8	= 27

The problem is to enumerate all the different samples of size, say 8, which can be drawn from this population. The necessary ca'cu'ations are shown below:

Partitions of 8 of parts ⩽ 4	No. of different samples	Specification 1	2	3	4	Frequency
(8)	2	·	·	8	·	495
		·	·	·	8	1
(71)	6	·	·	1	7	96
		·	1	·	7	48
		1	·	·	7	8
		·	·	7	1	6336
		·	1	7	·	4752
		1	·	7	·	792
(62)	6	·	2	·	6	420
		·	·	2	6	1848
		·	2	6	·	13860
		·	·	6	2	25872
		·	6	2	·	66
		·	6	·	2	28
(53)	6	·	5	3	·	1320
		·	5	·	3	336
		·	3	5	·	15840
		·	·	5	3	44352
		·	3	·	5	1120
		·	·	3	5	12320
(4^2)	3	·	4	4	·	7425
		·	·	4	4	34650
		·	4	·	4	1050
(61^2)	9	1	6	1	·	12
		1	6	·	1	8
		1	6	1	1	96
		1	1	6	·	5544
		1	·	6	1	7392
		·	1	6	1	44352
		1	1	·	6	168
		1	·	1	6	336
		·	1	1	6	2016
(521)	12	1	2	5	·	11880
		1	2	·	5	840
		1	·	2	5	3696
		1	5	2	·	396
		1	5	·	2	168
		1	·	5	2	22176
		·	1	5	2	133056
		·	1	2	5	22176
		·	2	5	1	95040

Partitions of 8 of parts ≤ 4	No. of different samples	Specification 1	2	3	4	Frequency
		·	2	1	5	10080
		·	5	1	2	2016
		·	5	2	1	3168
(431)	12	1	4	3	·	3300
		1	3	4	·	9900
		1	·	4	3	27720
		1	·	3	4	15400
		1	4	·	3	840
		1	3	·	4	1400
		·	1	4	3	166320
		·	1	3	4	8400
		·	4	3	1	26400
		·	4	1	3	10080
		·	3	1	4	16800
		·	3	4	1	79200
(42^2)	3	·	4	2	2	27720
		·	2	4	2	207900
		·	2	2	4	69300
(3^22)	3	·	3	3	2	123200
		·	3	2	3	73920
		·	2	3	3	184800
(51^3)	3	1	5	1	1	576
		1	1	5	1	38016
		1	1	1	5	4032
(421^2)	6	1	4	2	1	7920
		1	4	1	2	5040
		1	2	4	1	59400
		1	2	1	4	12600
		1	1	4	2	83160
		1	1	2	4	27720
(32^21)	3	1	3	2	2	36960
		1	2	3	2	92400
		1	2	2	3	55440
(3^21^2)	3	1	3	3	1	35200
		1	3	1	3	13440
		1	1	3	3	73920
TOTAL ..	77					$\left(\frac{27\,!}{8\,!\ 19\,!}\right)$

It will be seen that the number of different samples is 77 which is equal to Hg {(1, 6, 12, 8) ; (8, 19)} = Hg {(8, 19,) ; (1, 6, 12, 8)}.