THE BIVARIATE GAMMA DISTRIBUTION AND THE RANDOM WALK PROBLEM *

BY R. SRINIVASAN

(*Centre of Advanced Study in Physics, University of Madras, Madras-25, India*)

Received April 2, 1965

(Communicated by Dr. G. N. Ramachandran, F.A.SC.)

1. INTRODUCTION

DURING the course of an investigation involving the application of statistical methods to some problems in X-ray crystallography some results, which appeared to be of basic interest in pure statistics, came to be noticed by the author. One of these, for example, is the possibility of setting up joint probability distributions of two correlated gamma variables. Towards the end of this work, however, it was brought to the notice of the author† that such a possibility has, in fact, been considered earlier by Kibble (1941) and extension of his results to the case of more than two variables has also been considered (Krishnamurthy and Parthasarathy, 1951).

Their approach is strictly on a mathematical basis while in the present paper the results (though in a simplified form) will be arrived at purely from a physical approach. This arises from a very interesting and intimate relation that exists between the random walk problem and these distributions.

2. THE BIVARIATE GAMMA DISTRIBUTION WITH PARAMETER $\frac{1}{2}$

The gamma distribution of parameter l denoted symbolically by $\gamma(l)$ for brevity is characterised by the probability density function [*e.g., see* Weatherburn (1961), p. 150].

$$\phi(x) = \frac{e^{-x} x^{l-1}}{\Gamma(l)}, \quad 0 \leq x \leq \infty \tag{1}$$

where $\Gamma(l)$ is the well-known gamma function

$$\Gamma(l) = \int_0^\infty e^{-t} t^{l-1} dt. \tag{2}$$

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* Contribution No. 172 from the Centre of Advanced Study in Physics, University of Madras.

† The author wishes to thank Dr. G. Siromani for bringing the above work to his notice.

The problem that we are interested in is to set up joint distributions of two or more such variables. This has, in fact, been done by Kibble (1941). He has shown that a two-variate distribution function in which each of the variates has the density function given by (1) may be represented by

$$\phi(x_1)\,\phi(x_2)$$

$$= \Big[1 - \frac{\rho^2}{l} L_1(x_1, l)\, L_1(x_2, l) + \frac{\rho^4}{2!\, l(l+1)}$$

$$\times L_2(x_1, l)\, L_2(x_2, l) + \ldots\Big] \tag{3}$$

where $L_r(x, l), l > 0$ is generalised Laguerre polynomials of degree r satisfying

$$L_r(x, l) \equiv r!\, L_r^{l-1}(x) = \frac{\left[-\frac{d}{dx}\right]^r x^r \phi(x)}{\phi(x)}. \tag{4}$$

In the case when the two-variates have similar distribution it has been shown that the distribution functions can be expressed in terms of modified Bessel functions. The treatment has been extended by Krishnamurthy and Parthasarathy (1951) to the case of n variates both for the case when each variate has a gamma distribution with the same parameter, as well as with different parameters.

We shall be concerned in this paper only with the two variate gamma distribution with the same parameter l. Let us first consider the case with $l = \frac{1}{2}$.

The relation between gamma variate and the normal distribution is well known. Thus if y has a normal distribution with mean a and standard deviation σ, given by

$$P(y)\, dy = \frac{1}{\sqrt{2\pi\sigma^2}} \exp. - \frac{(y-a)^2}{2\sigma^2}\, dy \tag{5}$$

then $u = (y-a)^2/2\sigma^2$ has a gamma distribution with parameter $\frac{1}{2}$. In order to set up joint distribution of correlated gamma variates with parameter $\frac{1}{2}$ it is natural to start with the joint distribution of two correlated normal variates given by

$$\phi(y_1, y_2)\, dy_1\, dy_2$$

$$= \frac{1}{2\pi(\sqrt{1-\rho^2})\sigma_1\sigma_2} \exp. -\left(\frac{1}{2(1-\rho^2)}\right) \times \left[\frac{y_1^2}{\sigma_1^2} + \frac{2\rho y_1 y_2}{\sigma_1\sigma_2} + \frac{y_2^2}{\sigma_2^2}\right] dy_1\, dy_2 \tag{6}$$

where σ_1 and σ_2 are the standard deviations of the two variates and ρ is the coefficient of linear correlation between the two variates. We have assumed the mean values of the two variates to be zero. The ranges of the variates are from $-\infty$ to $+\infty$.

If we now consider the variates x_1 and x_2 given by the transformation $x_1 = y_1^2/\sigma_1^2$, $x_2 = y_2^2/\sigma_2^2$ the joint distribution of x_1 and x_2 is obtained easily from (6) to be

$$\phi(x_1, x_2) = \frac{1}{\pi\sqrt{1-\rho^2}} (x_1 x_2)^{-\frac{1}{2}} \exp. -\left(\frac{x_1 + x_2}{1-\rho^2}\right) \times \cosh\left(\frac{2\rho}{1-\rho^2}\sqrt{x_1 x_2}\right). \tag{7}$$

We could expect this to represent the joint distribution of two correlated gamma variates with parameter $\frac{1}{2}$. In fact, it does, as may be verified easily. Thus, if we put $\rho = 0$ in the above (which corresponds to x_1 and x_2 being independent) the right-hand side becomes the product of two $\gamma(\frac{1}{2})$ variates.

The question now arises whether one could set up similar joint distribution for other values of the parameter l. The necessary clue to this is obtained by first examining the problem of random walk in one dimension, to show its relation to the joint distribution of $\gamma(\frac{1}{2})$ variates.

3. THE RANDOM WALK IN ONE DIMENSION AND THE $\gamma(\frac{1}{2}, \frac{1}{2})$ DISTRIBUTION†

Consider the problem of random walk restricted to one dimension. The probability that the resultant r_N after N steps of equal length l, each taken either in the forward or backward direction with equal probability follows (for large value of N) the Gaussian law

$$P(r_N)\, dr_N = \frac{1}{\sqrt{2\pi\sigma_N^2}} \exp. -\left(\frac{r_N^2}{2\sigma_N^2}\right) dr_N \tag{8}$$

† The symbol $\gamma(l, m)$ is used adjectively to denote joint distribution of two gamma variates of parameters l and m.

where $\sigma_N^2 = Nl^2$. Let us consider now the distribution of r_N given that after making an initial P steps the resultant was r_P, where the total number of steps $N = P + Q$. One could see easily that so long as the number of additional steps (Q) is large, the required (conditional) distribution is given by

$$P(r_N; r_P) = \frac{1}{\sqrt{2\pi\sigma_Q^2}}\left[\exp. -\frac{(r_N - r_P)^2}{2\sigma_Q^2} + \exp. -\frac{(r_N + r_P)^2}{2\sigma_Q^2}\right] \quad (9\ a)$$

$$= \sqrt{\frac{2}{\pi\sigma_Q^2}} \exp. -\left(\frac{r_N^2 + r_P^2}{2\sigma_Q^2}\right)\cosh\left(\frac{r_N r_P}{\sigma_Q^2}\right) \quad (9\ b)$$

where σ_Q^2 stands for Ql^2.

The joint distribution of r_N and r_P is then obviously given by

$$\phi(r_N, r_P)\, dr_N dr_P$$

$$= P(r_N; r_P)\, P(r_P)\, dr_P$$

$$= \frac{2}{\pi\sigma_P^2\sigma_Q^2} \exp. -\frac{(r_N^2\sigma_P^2 + \sigma_N^2 r_P^2)}{2\sigma_P^2\sigma_Q^2}\cosh\left(\frac{r_N r_P}{\sigma_Q^2}\right) dr_N dr_P \quad (10)$$

where we have used the asymptotic distribution for r_P, namely

$$P(r_P)\, dr_P = \sqrt{\frac{2}{\pi\sigma_P^2}} \exp. -\left(\frac{r_P^2}{2\sigma_P^2}\right) dr_P \quad (11)$$

which is valid so long as the initial number of steps P is large. Making now the transforamtion

$$x_1 = \frac{r_N^2}{2\sigma_N^2}, \quad x_2 = \frac{r_P^2}{2\sigma_P^2}$$

we get

$$\phi(x_1, x_2)\, dx_1\, dx_2$$

$$= \frac{1}{\pi\sigma_2}(x_1x_2)^{-\frac{1}{2}} \exp. -\left(\frac{x_1 + x_2}{\sigma_2^2}\right)\cosh\left(\frac{2\sigma_1}{\sigma_2^2}\sqrt{x_1x_2}\right) dx_1 dx_2 \quad (12)$$

where σ_1^2 and σ_2^2 stand for the ratios σ_P^2/σ_N^2 and σ_Q^2/σ_N^2 respectively so that $\sigma_1^2 + \sigma_2^2 = 1$. A comparison of (10) and (7) shows that the two expressions are identical excepting that ρ in (7) is replaced by σ_1 in (12). (Note $\sigma_2^2 = 1 - \sigma_1^2$.)

The above result thus shows us the relation between $\gamma\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution and the random walk in one dimension. The pair of variates r_N and

r_P which are correlated in the sense of partial random walk as discussed earlier leads us to the joint distribution of two γ ($\frac{1}{2}$) variates. This immediately suggests the possibility of setting up similar distributions for other values of l. In fact, one could anticipate that, in general, the bivariate gamma distribution γ (l, l) should be related to the random walk in $2l$ dimensions, the pair of variables to be considered being, as before, r_N and r_P which correspond to the distance from the origin at the end of N steps given that earlier, at the end of P steps it was r_P.

This conjecture in fact turns out to be true, as we shall show below. Let us first consider the two-dimensional case and later the general n-dimensional case.

Two-Dimensional Case

We have, in an obvious notation,

$$P(x, y)\, dxdy = \frac{1}{2\pi\sigma_x\sigma_y} \exp. - \left(\frac{x^2}{2\sigma_x^2} + \frac{y^2}{2\sigma_y^2}\right) dxdy \tag{13}$$

where x and y are the components of the resultant vector r_N from the origin. Assuming $\sigma_x^2 = \sigma_y^2 = \sigma^2/2$ (say), and since $x^2 + y^2 = r_N^2$, changing to polar co-ordinates

$$P\ (r_N, \alpha)\, dr_N d\alpha = \frac{r_N}{\pi\sigma^2} \exp. - \frac{r_N^2}{\sigma^2} \cdot dr_N d\alpha. \tag{14}$$

As in the previous case, we now ask for the conditional probability that after a total of N steps r_N lies between r_N and $r_N + dr_N$ given that initially after the first P steps it was r_P. This is obviously given by

$$P\ (r_N, \alpha\, ;\, r_P) = \frac{r_N}{\pi\sigma_Q^2} \exp. - \frac{(r_N - r_P)^2}{\sigma_Q^2}\, dr_N d\alpha \tag{15}$$

so that

$$P\ (r_N; r_P) = \int_0^{2\pi} \frac{r_N}{\pi\sigma_Q^2} \exp. - \frac{(r_N^2 + r_P^2 - 2r_N r_P \cos\alpha)}{\sigma_Q^2}\, d\alpha$$

$$= \frac{2r_N}{\sigma_Q^2} \exp. - \left(\frac{r_N^2 + r_P^2}{\sigma_Q^2}\right) I_0\left(\frac{2r_N r_P}{\sigma_Q^2}\right) \tag{16}$$

where $I_0(x)$ is the bessel function with imaginary argument, The joint probability distribution of r_N and r_P is thus given by

$$\phi(r_N, r_P) = P(r_N; r_P)\, P(r_P) \tag{17}$$

where $P(r_N; r_P)$ is given by (16) above and $P(r_P)$ is readily shown to be

$$P(r_P)\, dr_P = \frac{2r_P}{\sigma_P^2} \exp. - \frac{r_P^2}{\sigma_P^2}\, dr_P. \tag{18}$$

Thus

$$\phi(r_N, r_P) = \frac{4r_N r_P}{\sigma_P^2 \sigma_Q^2} \exp. - \left(\frac{r_N^2 + r_P^2}{\sigma_Q^2}\right) I_0\left(\frac{2r_N r_P}{\sigma_Q^2}\right). \tag{19}$$

Making the transformations

$$x_1 = \frac{r_N^2}{\sigma_N^2}, \quad x_2 = \frac{r_P^2}{\sigma_P^2}$$

and with σ_1^2 and σ_2^2 defined as before, we get

$$\phi(x_1, x_2) = \frac{4x_1 x_2}{\sigma_2^2} \exp. - \frac{(x_1 + x_2)}{\sigma_2^2}\, I_0\left(\frac{2\sigma_1 \sqrt{x_1 x_2}}{\sigma_2^2}\right). \tag{20}$$

This is seen to reduce to the product of the distribution of two $\gamma(l)$ variates when we put $\sigma_1^2 = 0$.

The nature of the two functions $\cosh x$ in (12) and $I_0(x)$ in (20) suggests the underlying similarity. In fact if we make use of the well-known relation [*e.g.*, *see* McLachlan (1955), p. 202].

$$I_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cosh z \tag{21}$$

we can write (12) as

$$\phi(x_1, x_2) = \frac{\sigma_1^{\frac{1}{2}}}{\sqrt{\pi}} \frac{(x_1 x_2)^{-\frac{1}{4}}}{\sigma_2^2}\, I_{-\frac{1}{2}}\left(\frac{2\sigma_1 \sqrt{x_1 x_2}}{\sigma_2^2}\right) \exp. - \left(\frac{x_1 + x_2}{\sigma_2^2}\right) \tag{22}$$

whose similarity to (20) is brought out even more clearly. For the one-dimensional case we get an $I_{-\frac{1}{2}}(x)$ function while for the two-dimensional case it is an $I_0(x)$ function. We can therefore expect that for the n-dimensional we should get $I_{(n/2)-1}(x)$.

We shall now pass on to consider the general n-dimensional case.

If $x_1, x_2 \ldots x_n$ are the n-components of the n-dimensional vector r_N the joint normal distribution can be written as

$$\phi(x_1, x_2, \ldots x_n)\, dx_1 dx_2 \ldots dx_n$$

$$= \frac{1}{\left(\frac{2\pi}{n}\right)^{n/2} \sigma^n} \exp. - \frac{n}{2}\left[\frac{x_1^2 + x_2^2 + \ldots x_n^2}{\sigma^2}\right] dx_1 \cdots dx_n. \quad (23)$$

Transforming to spherical polar co-ordinates

$$\phi(r_N, \theta, \phi_1 \ldots \phi_{n-3})\, d\tau = \frac{n^{n/2}}{(2\pi)^{n/2}\sigma^n} \exp. - \left(\frac{n r_N^2}{2\sigma^2}\right) d\tau \quad (24)$$

where $d\tau$ stands for the n-dimensional volume element given by

$$r_N^{(n-1)} \sin^{(n-2)}\theta \sin^{(n-3)}\phi_1 \ldots \sin\phi_{n-3}\, dr d\theta \ldots d\phi_{n-3}. \quad (25)$$

We now require the conditional probability $P(r_N; r_P)$.

This is given by

$$P(r_N; r_P) = \frac{n^{n/2}}{(2\pi)^{n/2}\sigma_Q^n} \int\int \cdots \int \exp. - \frac{n}{2} \frac{(r_N - r_P)^2}{\sigma_Q^2} d\tau. \quad (26)$$

For convenience we may choose r_P as the polar axis so that $r_N - r_P$ becomes a function of r_N, r_P and θ only and independent of all the other angle variables. Thus expanding $(r_N - r_P)^2$, (26) can be written as

$$P(r_N; r_P) = \frac{n^{n/2}}{(2\pi)^{n/2}\sigma_Q^n} \int_0^\pi \exp. - \left(\frac{n r_N r_P \cos\theta}{\sigma_Q^2}\right) r_N^{(n-1)} \sin^{(n-2)}\theta d\theta$$

$$\times \int\int \cdots \int \sin^{(n-3)}\phi_1 \ldots \sin\phi_{n-3}\, d\phi_1 \ldots d\phi_{n-3}. \quad (27)$$

The second part of the integral over the angle variables $\phi_1 \ldots \phi_{n-3}$ can be shown to be

$$\frac{(2\pi)^{n/2-1/2}}{\Gamma\left(\frac{n}{2} - \frac{1}{2}\right)}$$

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while, the first can be shown [*e.g.*, using Eqn. (9), p. 79, of Watson (1944)] to be

$$I_{(n/2)-1}\left(\frac{nr_Nr_P}{\sigma_Q^2}\right)\frac{\Gamma\left(\frac{n}{2}-\frac{1}{2}\right)}{\left(\frac{nr_Nr_P}{2\sigma_Q^2}\right)^{n/2-1/2}}\Gamma\left(\frac{1}{2}\right). \tag{29}$$

Thus we have finally

$$P(r_N;\, r_P) = \frac{n}{\sigma_Q^2}\frac{r_N^{n/2}}{r_P^{(n/2)-1}}\exp. -\frac{n}{2}\left(\frac{r_N^2+r_P^2}{\sigma_Q^2}\right)I_{(n/2)-1}\left(\frac{nr_Nr_P}{\sigma_Q^2}\right). \tag{30}$$

Now the function $P(r_P)$ itself is given by [*e.g.*, *see* Watson (1944) p. 421].

$$\frac{1}{2^{(n/2)-1}}\frac{n^{n/2}r_P^{n-1}}{\sigma_P^n\,\Gamma\left(\frac{n}{2}\right)}\exp. -\frac{n}{2}\frac{r_P^2}{\sigma_P^2}. \tag{31}$$

Thus the joint distribution $\phi(r_N, r_P)$ is obtained by taking the product of (30) and (31). We obtain after some simplification.

$$\phi(r_N, r_P) = \frac{n^{(n/2)+1}(r_Nr_P)^{n/2}}{\Gamma\left(\frac{n}{2}\right)2^{(n/2)-1}\sigma_Q^2\sigma_P^n}\,I_{(n/2)-1}\left(\frac{nr_Nr_P}{\sigma_Q^2}\right)$$

$$\times\exp. -\frac{n}{2}\left[\frac{\sigma_P^2r_N^2+\sigma_N^2r_P^2}{\sigma_P^2\sigma_Q^2}\right]. \tag{32}$$

We now make the transformation

$$x_1 = \frac{nr_N^2}{2\sigma_N^2},\quad x_2 = \frac{nr_P^2}{2\sigma_P^2}$$

and using the usual symbols

$$\sigma_1^2 = \frac{\sigma_P^2}{\sigma_N^2},\quad \sigma_2^2 = \frac{\sigma_Q^2}{\sigma_N^2},$$

we get

$$\phi(x_1, x_2) = \frac{2^{(n/2)-1}}{\sigma_2^n}\frac{(x_1x_2)^{(n/2)-1}}{\Gamma\left(\frac{n}{2}\right)}\exp. -\left(\frac{x_1+x_2}{\sigma_2^2}\right)$$

$$\times\,\Phi_{(n/2)-1}\left(\frac{2\sigma_1}{\sigma_2^2}\sqrt{x_1x_2}\right) \tag{34}$$

where $\Phi_\nu(x)$ stands for $I_\nu(x)/x^\nu$.

It is readily verified that substitution of $n = 1$, and 2, leads respectively to expression (22) and (20) for the one and two-dimensional cases.

SUMMARY

The relation between the random walk problem in one dimension and the joint distribution of two correlated gamma variates of parameter $\frac{1}{2}$ is firstly established. It is shown that the two variates which represent respectively the squares of distances from the origin (properly normalised) after P steps and $P + Q (= N)$ steps in one dimension are a pair of correlated gamma variates with parameter $\frac{1}{2}$. This is used as a heuristic principle to set up similar bivariate gamma distribution with parameters 1 by examining the corresponding problem of random walk in two dimensions. Finally, the results are extended to the case of general n-dimensional random walk where it is shown to lead to the bivariate gamma distribution with parameter $n/2$.

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706-65. Printed at The Bangalore Press, Mysore Road, Bangalore City, by K. V. Varadarajan, Manager, and Published by B. S. Venkatachar, Editor, "Proceedings of the Indian Academy of Sciences," Bangalore