RESTRICTIONS OF REPRESENTATIONS OF CLASSICAL GROUPS: EXAMPLES

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Contents

1. Introduction

This paper is a sequel to [GGP], where we considered several restriction problems in the representation theory of classical groups over local and global fields. Assuming the Langlands-Vogan parameterization of irreducible representations, we formulated precise conjectures for the solution of these restriction problems. In the local case, our conjectural answer is given in terms of Langlands parameters and certain natural symplectic root numbers associated to them. In the global case, the conjectural answer is expressed in terms of the central critical value or derivative of a global L-function. For the precise statements of the restriction problems and our conjectures, we refer the reader to [GGP].

The conjectures for the case of special orthogonal groups were contained in the earlier papers [GP1] and [GP2] and were suggested by the results of Waldspurger [Wa1,2,3], Tunnell-Saito [Tu], [Sa], and Prasad [P1, 2, 3] in certain low rank cases. Since then, there have been further results in the orthogonal case, both locally and globally; see, for example [P4], [GR], [GJR2], and [PT]. Most notably, in a series of recent papers [Wa4-7] and [MW], Waldspurger and Moeglin-Waldspurger have established the local conjectures of [GP1, GP2], assuming certain expected properties of the characters of representations in tempered L-packets.

In this paper, we provide some evidence for the conjectures of [GGP] in the unitary case. More precisely, we shall consider the restriction problems in the following cases:

(i) the depth zero supercuspidal L-packets of DeBacker-Reeder [DR], which are associated to tame regular discrete L-parameters;

(ii) certain low rank cases, such as $\mathrm{U}(1) \times \mathrm{U}(1)$, $\mathrm{U}(1) \times \mathrm{U}(2)$, $\mathrm{U}(2) \times \mathrm{U}(2)$ and $\mathrm{U}(2) \times \mathrm{U}(3)$.

We conclude this introduction by summarizing some notations and conventions which are used throughout the paper. Let k be a local field, equipped with a non-trivial involution σ with fixed field k_0. We will always assume that the characteristic of k is not equal to 2. In Section 1, $k = \mathbb{C}$ and in Section 2, k is the unramified quadratic extension of k_0, but from Section 6 onwards, k is non-archimedean and there is no restriction on the ramification of k over k_0. We fix a non-zero element δ of k with trace 0 to k_0, so $k = k_0 + k_0 \cdot \delta$ and $\sigma(\delta) = -\delta$. In addition, ψ will denote a non-trivial additive character of k/k_0 whereas ψ_0 will denote a non-trivial character of k_0. We can pass from a character ψ_0 to a character ψ by defining $\psi(x) = \psi_0(\delta \cdot x)$ for all elements $x \in k$ of trace zero. In particular, this will be how ψ_0 and ψ are related in most parts of the paper. We will consider finite-dimensional hermitian or skew-hermitian spaces over k, typically denoted by V in the hermitian case and W in the skew-hermitian case. Given a hermitian space V, we may convert it to a skew-hermitian space by multiplying the hermitian form on V by the trace zero element δ; we denote the resulting skew-hermitian space by W_δ. Then one has an identification of the associated isometry groups: $\mathrm{U}(V) = \mathrm{U}(W_\delta)$.

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2. Discrete series parameters

We begin with the computation of the distinguished character in [GGP, Conjecture 17.3]

$$\chi = \chi_N \times \chi_M : A_M \times A_N \to < \pm 1 >,$$

which is defined using local root numbers, for some discrete series parameter for the group $G = \mathrm{U}(V) \times \mathrm{U}(V_0)$, where V_0 and V hermitian spaces over k, and $V_0 \subset V$ of odd codimension.

In general, these discrete series parameters have the form

$$\begin{aligned} M &= \bigoplus_i M_i \\ N &= \bigoplus_j N_j \end{aligned}$$

where the M_i are distinct conjugate-symplectic representations and the N_j are distinct conjugate-orthogonal representations of the Weil-Deligne group of k. The dimension of M is even and the dimension of N is odd. In this case, the centralizer $C_M \times C_N$ of the Langlands parameter is finite.

We will only consider the case where each $M_i = \mathbb{C}(\alpha_i)$ and each $N_j = \mathbb{C}(\beta_j)$ is one dimensional. Then α_i is a character of $k^\times/\mathbb{N}k^\times$ with $\alpha_i|_{k_0^\times} = \omega_{k/k_0}$, and β_j is a character of $k^\times/k_0^\times$. In this case, we have the component groups

$$\begin{aligned} A_M &= \bigoplus \mathbb{Z}/2\mathbb{Z} \cdot e_i \\ A_N &= \bigoplus \mathbb{Z}/2\mathbb{Z} \cdot f_j. \end{aligned}$$

These vector spaces have dimension equal to $\dim M$ and $\dim N$ over $\mathbb{Z}/2\mathbb{Z}$, which is as large as possible. We have

$$\begin{aligned} M^{e_i=-1} &= \mathbb{C}(\alpha_i) \\ N^{f_j=-1} &= \mathbb{C}(\beta_j). \end{aligned}$$

Fix a nontrivial additive character ψ of k which is trivial on k_0. By the definition of the character χ, we have the formulae

$$\begin{aligned} \chi(e_i) &= \epsilon(\mathbb{C}(\alpha_i) \otimes N, \psi) \\ \chi(f_j) &= \epsilon(M \otimes \mathbb{C}(\beta_j), \psi). \end{aligned}$$

Using the additivity of the local epsilon factors, this becomes

$$\begin{aligned} \chi(e_i) &= \prod_k \epsilon(\alpha_i\beta_k, \psi) \\ \chi(f_j) &= \prod_k \epsilon(\alpha_k\beta_j, \psi). \end{aligned}$$

Since the products $\alpha_i\beta_j$ are all conjugate-symplectic characters of $k^\times$, we need a formula to compute their root numbers. We will do this in two different cases - when $k/k_0 = \mathbb{C}/\mathbb{R}$, which we take up now, and then when k/k_0 is unramified which we do in the next section.

Proposition 2.1. *Assume that $k_0 = \mathbb{R}$ and choose an isomorphism $z : k \to \mathbb{C}$. Let*

$$\alpha = z^{-2a} \cdot (z\bar{z})^a = (\bar{z}/z)^a$$

be a conjugate-symplectic character of $k^\times$, where a is a half integer, and let

$$\psi = e^{2\pi i \mathrm{Tr}(iz)} = e^{2\pi(\bar{z}-z)}$$

Then

$$\epsilon(\alpha, \psi) = \begin{cases} +1 & \text{if } a > 0; \\ -1 & \text{if } a < 0. \end{cases}$$

Proof. Tate [T, (3.2.5)] gives the formula

$$\epsilon(\alpha, \psi_0) = i^{2a}$$

when $a > 0$ and $\psi_0(z) = e^{2\pi i \mathrm{Tr}(z)}$. Since $\psi(z) = \psi_0(iz)$, we find

$$\epsilon(\alpha, \psi) = i^{2a} \cdot \alpha(i) = +1$$

in this case. When $a < 0$ we must conjugate the isomorphism $z : k \to \mathbb{C}$ to use Tate's formula. This changes the character ψ, and hence the sign of ϵ. $\square$

Corollary 2.2. *Assume that $k_0 = \mathbb{R}$, choose an isomorphism $z : k \to \mathbb{C}$, and let $\psi = e^{2\pi(\bar{z}-z)}$. If M is the sum of the distinct symplectic characters $\alpha_i = (\bar{z}/z)^{a_i}$, where each a_i is a half integer, and N is the sum of the distinct orthogonal characters $\beta_j = (\bar{z}/z)^{b_j}$, where each b_j is an integer, then*

$$\begin{aligned} \chi(e_i) &= (-1)^{m_i} \\ \chi(f_j) &= (-1)^{n_j} \end{aligned}$$

where

$$\begin{aligned} m_i &= \#\{r : a_i + b_r < 0\} \\ n_j &= \#\{r : a_r + b_j < 0\}. \end{aligned}$$

Finally, we note that in the case when $k_0 = \mathbb{R}$, we may order the distinct characters α_i and β_j in the parameter φ so that

$$\begin{cases} a_1 > a_2 > a_3 \cdots \in \frac{1}{2}\mathbb{Z} - \mathbb{Z} \\ b_1 > b_2 > b_3 \cdots \in \mathbb{Z}. \end{cases}$$

Corollary 2.3. *For $i < j$, we have*

$$\begin{aligned} \chi(e_i)\chi(e_j) &= (-1)^{m_{ij}} \\ \chi(f_i)\chi(f_j) &= (-1)^{n_{ij}}. \end{aligned}$$

where

$$\begin{aligned} m_{ij} &= \#\{r : a_i + b_r > 0 > a_j + b_r\} \\ n_{ij} &= \#\{r : b_i + a_r > 0 > b_j + a_r\}. \end{aligned}$$

Since we know how to describe the representations in the L-packets of discrete series parameters when $k_0 = \mathbb{R}$ [GR], the calculation of $\chi(e_i)\chi(e_j)$ and $\chi(f_i)\chi(f_j)$ allows us to say something about the representation $\pi = \pi(\varphi, \chi) = \pi_1 \otimes \pi_2$ of $G(\mathbb{R})$ with $d(\pi) = 1$. The irreducible representations π_1 and π_2 are discrete series representations of even and odd dimensional unitary groups, with infinitesimal characters

$$\begin{aligned} a_1 > a_2 > a_3 > \cdots \\ b_1 > b_2 > b_3 > \cdots \end{aligned}$$

in $X^\star + \rho$ respectively. Moreover, in the chambers defined by their Harish-Chandra parameters, the simple root walls corresponding to

$$\begin{aligned} e_i - e_{i+1} \text{ is compact} &\iff \chi(e_i) \cdot \chi(e_{i+1}) = -1 \\ f_i - f_{i+1} \text{ is compact} &\iff \chi(f_i) \cdot \chi(f_{i+1}) = -1. \end{aligned}$$

More generally, for $i < j$, the positive root

$$\begin{aligned} e_i - e_j \text{ is compact} &\iff \chi(e_i) \cdot \chi(e_j) = (-1)^{i+j} \\ f_i - f_j \text{ is compact} &\iff \chi(f_i) \cdot \chi(f_j) = (-1)^{i+j}. \end{aligned}$$

This determines the signature of the unitary group $G(\mathbb{R})$, and in almost all cases the discrete series representation π.

We end this section with a remark about branching from $\mathrm{U}(n, 1)$ to $\mathrm{U}(n)$. According to a theorem of Harish-Chandra, an irreducible admissible $(\mathfrak{g}, K)$-module is determined by the action of $U(\mathfrak{g})^K$ on a given K-type which appears in the representation space. Here, $U(\mathfrak{g})$ denotes the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}$

of G and $U(\mathfrak{g})^K$ is the centralizer of K in $U(\mathfrak{g})$. Further, the action of $K \times U(\mathfrak{g})^K$ on the corresponding isotypical component is irreducible. By a theorem of Kostant, for

$$G = \mathrm{U}(n,1) \quad \text{and} \quad K = \mathrm{U}(n) \times \mathrm{U}(1),$$

$U(\mathfrak{g})^K$ is generated by the centers of the universal enveloping algebras of G and K, and thus is abelian. This proves that any irreducible representation of $\mathrm{U}(n)$ appears with multiplicity at most one in the sum of representations in a given L-packet of $\mathrm{U}(n,1)$ (since all the members of an L-packet have the same infinitesimal character).

3. Depth zero supercuspidals

In this section, we test the restriction conjecture for some tamely ramified discrete parameters φ of unitary groups. We begin by calculating the local root numbers, assuming that k_0 is non-archimedean with residue field $\mathbb{F}_q$ and k is the unramified quadratic extension of k_0.

Proposition 3.1. *Assume that k_0 is non-archimedean, and let k be the unramified quadratic field extension of k_0. Let ψ be an additive character of k which is trivial on both k_0 and the maximal ideal of the ring of integers A_k, but is nontrivial on A_k. Let α be a conjugate-symplectic character of $k^\times$ of conductor $f(\alpha)$. Then*

$$\epsilon(\alpha, \psi) = (-1)^{f(\alpha)+1}.$$

Proof. When k/k_0 is unramified, every conjugate-symplectic character α has the form

$$\alpha = \beta \cdot \mu,$$

where $\beta : k^\times/k_0^\times \to \mathbb{C}^\times$ is a conjugate-orthogonal character and μ is the unramified quadratic character of $k^\times$ (which is conjugate-symplectic). By [GGP, Section 5] and [FQ, Theorem 3], we have

$$\epsilon(\beta, \psi) = +1$$

for any character ψ of k which is trivial on k_0. Since μ is unramified, we have [T,(3.4.6)]

$$\epsilon(\alpha, \psi) = \epsilon(\beta, \psi) \cdot \mu(\pi^{f(\beta)+n(\psi)}).$$

Since $f(\beta) = f(\alpha)$ and $n(\psi) = -1$, this gives the formula in the proposition. □

Corollary 3.2. *Assume that k_0 is non-archimedean. Let k be the unramified quadratic field extension of k_0 and μ the quadratic unramified character of $k^\times$. Let ψ be an additive character of k which is trivial on both k_0 and the maximal ideal of the ring of integers A_k, but is nontrivial on A_k. Let*

$$M = \oplus_i \mathbb{C}(\alpha_i) \quad \textit{and} \quad N = \oplus_j \mathbb{C}(\beta_j)$$

where the α_i's are mutually distinct, tamely ramified, conjugate-symplectic characters, and the β_j's are mutually distinct, tamely ramified, conjugate-orthogonal characters. Order these characters so that

$$\alpha_1\beta_1 = \alpha_2\beta_2 = \cdots = \alpha_p\beta_p = \mu,$$

for $p \geq 0$ and $\alpha_i\beta_j \neq \mu$ for any pair $\{i, j\}$ with $i > p$ and $j > p$. Then

$$\chi(e_i) = \begin{cases} -1 & \text{when } i \leq p; \\ +1 & \text{when } i > p. \end{cases}$$

Similarly,

$$\chi(f_j) = \begin{cases} -1 & \text{when } j \leq p; \\ +1 & \text{when } j > p. \end{cases}$$

Finally, $\chi(-1, 1) = \chi(1, -1) = (-1)^p$.

Proof. Since our characters are all tamely ramified, we find

$$f(\alpha_i\beta_j) = 1,$$

unless $i = j \leq p$, in which case the product $\alpha_i\beta_i, i \leq p$, is equal to the unramified character μ and $f(\alpha_i\beta_i) = 0$. Taking the product of epsilon factors giving χ gives the desired result. □

We take the parameter

$$\begin{aligned} M &= \bigoplus \mathbb{C}(\alpha_i) \\ N &= \bigoplus \mathbb{C}(\beta_j) \end{aligned}$$

given by the sum of distinct conjugate-symplectic and distinct conjugate-orthogonal characters of $k^\times$. We assume that all of these characters are tamely ramified:

$$f(\alpha_i) = f(\beta_j) = 1.$$

The L-packet Π_φ of depth zero supercuspidal representations of the pure inner forms of $G = \mathrm{U}(V) \times \mathrm{U}(V_0)$ have been constructed by DeBacker and Reeder [DR]; we briefly summarize their results in this case. Let V be a hermitian space of dimension n over k. A parameter φ of the above type for the unitary group $\mathrm{U}(V) = \mathrm{U}_n$ gives, by restriction to the units of $k^\times$, a regular complex character ρ of the anisotropic torus $S = \mathrm{U}(1)^n$ (see [DR, Section 4.3]). The torus S comes equipped with $\nu_i : S \to \mathrm{U}(1)$ which are the projections onto the ith factor of $S = \mathrm{U}(1)^n$.

An embedding

$$\iota : S \to \mathrm{U}(V)$$

will be called *admissible* if V is the direct sum of orthogonal lines $L_i = kv_i$ on which S operates as

$$sv_i = \nu_i(s)v_i, \quad \text{for all } s \in S.$$

The $\mathrm{U}(V)$-conjugacy class of admissible embeddings ι depends only on the signs

$$\epsilon_i = (-1)^{\mathrm{ord}\langle v_i, v_i\rangle},$$

which must satisfy the one relation

$$\prod_i \epsilon_i = (-1)^{\mathrm{ord}(\mathrm{disc} V)}.$$

Since the two hermitian spaces V and V' of dimension n have distinct hermitian discriminants, all the values for ϵ_i are possible, and hence there are exactly 2^n conjugacy classes of admissible embeddings ι of S into $\mathrm{U}(V)$ and $\mathrm{U}(V')$. These conjugacy classes correspond bijectively to the characters $\chi = \chi_\iota$ of the group A_φ, where $\chi(e_i) = \epsilon_i$.

For each embedding $\iota : S \to \mathrm{U}(V)$, there is a unique maximal compact subgroup $K_\iota \subset \mathrm{U}(V)$ which contains the image. This is the subgroup stabilizing the lattice,

$$L_\iota = \bigoplus A_k v_i,$$

where we normalize the basis vectors of our S-stable lines to satisfy $0 \leq \mathrm{ord}\langle v_i, v_i\rangle \leq 1$. The compact-open subgroup K_ι is hyperspecial if and only if all of the inner products $\langle v_i, v_i\rangle$ have valuations of the same parity.

If we define

$$L_\iota^\vee = \{x \in V | \langle x, \ell\rangle \in A_k, \ \text{for all } \ell \in L_\iota\},$$

then

$$\varpi L_\iota^\vee \subset L_\iota \subset L_\iota^\vee.$$

The hermitian form on V restricted to L_ι gives rise to a non-degenenerate hermitian form on $L_\iota/(\varpi L_\iota^\vee)$ with values in A_k/ϖ, and the multiple of the hermitian form on V by ϖ gives rise to a non-degenerate hermitian form on $L_\iota^\vee/L_\iota$ with values in A_k/ϖ. Thus there is a natural map from K_ι to

$$\bar{K}_\iota(\mathbb{F}_q) = \mathrm{U}_r(\mathbb{F}_q) \times \mathrm{U}_{n-r}(\mathbb{F}_q),$$

where r is the number of v_i with $(-1)^{\mathrm{ord}\langle v_i, v_i\rangle} = -1$.

The torus $S(\mathbb{F}_q)$ embeds in $\bar{K}_\iota(\mathbb{F}_q)$, and the regular tame character ρ of $S(\mathbb{F}_q)$ allows us to construct an irreducible, supercuspidal representation $R_\iota(S, \rho)$ of the finite group $\bar{K}_\iota(\mathbb{F}_q)$, using the method of Deligne and Lusztig. We view this as a representation of the compact group K_ι, and define the representation

$$\pi_\chi = \pi_\iota, \quad \text{of } \mathrm{U}(V)$$

as the compact induction of $R_\iota(S, \rho)$. These are the 2^n depth zero supercuspidal representation in the L-packet Π_φ.

The Vogan bijection between the set Π_φ and the group of homomorphisms from A_φ to $\langle \pm 1\rangle$ is normalized as follows. Assume that the hermitian space V is split and even dimensional. Let L be an A-lattice in V with an orthogonal basis whose inner products are units in A. Let N_L be the unipotent radical of an Iwahori subgroup of the hyperspecial maximal subgroup $K = \mathrm{Aut}(L)$ in $\mathrm{U}(V)$. The construction of [GGP, §12] over the ring A gives a surjective homomorphism

$$f + f_0 : N_L \to A^{\frac{n}{2}-1} + A^-$$

where A^- is the eigenspace where $\sigma = -1$ on A, which consists of the elements of trace 0 to A_0.

By [DR2], the character $\chi = 1$ of A_φ corresponds to the unique representation π_1 in the L-packet of φ which is induced from a generic, cuspidal representation of the reductive quotient $\mathrm{U}_n(\mathbb{F}_q)$ of $K = \mathrm{U}(L)$. All of the generic characters of the unipotent radical $N(\mathbb{F}_q)$ of a Borel subgroup of $\mathrm{U}_n(\mathbb{F}_q)$ are conjugate, and we construct one of them in the following manner.

Let ψ be an additive character of k which is trivial on k_0 and the maximal ideal $\mathfrak{P}$ of A, but is nontrivial on A. Since A is unramified over A_0, we have

$$A_0 + A^- = 2 \cdot A + A^-.$$

Hence, for elements z in A^-, the character

$$z \mapsto \psi(z/2)$$

is nontrivial on $A^-/\mathfrak{P}^-$. Then the composition

$$n \mapsto \psi(\Sigma f(n)) \cdot \psi(f_0(n)/2)$$

defines a character of N_L which is the inflation of a generic character of $N(\mathbb{F}_q)$ under the natural homomorphism $N_L \to N(\mathbb{F}_q)$. Hence the representation π_1 corresponding to the trivial character of A_φ is generic for the character obtained by scaling the additive character ψ used in the computation of the root number in Proposition 3.1 and Corollary 3.2 by the factor 1/2, or equivalently by the factor 2. This is the normalization predicted in [GGP,Conjecture 17.3].

Now consider the parameter of $G = \mathrm{U}(V) \times \mathrm{U}(V_0) = \mathrm{U}_n \times \mathrm{U}_m$ which is given by

$$\begin{aligned} M &= \bigoplus \mathbb{C}(\alpha_i) \\ N &= \bigoplus \mathbb{C}(\beta_j). \end{aligned}$$

From the calculation of the character $\chi = \chi_N \times \chi_M$ of A_φ in the previous section, we conclude that the irreducible representation π_χ of $G = \mathrm{U}(V) \times \mathrm{U}(V_0)$ is compactly induced from a maximal compact subgroup with reduction isomorphic to

$$(\mathrm{U}_d(\mathbb{F}_q) \times \mathrm{U}_{n-d}(\mathbb{F}_q)) \times (\mathrm{U}_d(\mathbb{F}_q) \times \mathrm{U}_{m-d}(\mathbb{F}_q)).$$

Here $d \geq 0$ is the number of pairs (α_i, β_i) with $\alpha_i\beta_i = \mu$. The finite dimensional representation that we are inducing has the form

$$(R \otimes R(\alpha)) \otimes (R^\vee \otimes R(\beta))$$

where R is the Deligne-Lusztig representation of $\mathrm{U}_d(\mathbb{F}_q)$ associated to the character $(\alpha_1, \alpha_2, ..., \alpha_d)$ of the maximal torus $\mathrm{U}_1(\mathbb{F}_q)^d$ and $R^\vee$ is its dual representation, associated to the character $(\beta_1, \beta_2, ..., \beta_d) = (\alpha_1^{-1}, \alpha_2^{-1}, ..., \alpha_d^{-1})$. (We have abused notation here to denote α_i, β_j now to be characters of $\mathrm{U}(1)(\mathbb{F}_q)$ obtained from the corresponding characters of the local field in a natural way.) The remaining representations $R(\alpha)$ of $\mathrm{U}_{n-d}(\mathbb{F}_q)$ and $R(\beta)$ of $\mathrm{U}_{m-d}(\mathbb{F}_q)$ are associated to characters whose components α_i and β_j satisfy $\alpha_i\beta_j \neq \mu$ for all i, j.

As support for [GGP, Conjecture 17.3], we will prove:

Theorem 3.3. *Let π_χ be the depth zero supercuspidal representation of $G = \mathrm{U}(V) \times \mathrm{U}(V_0)$ defined above, which corresponds to the distinguished character in [GGP, conjecture 17.3]. Then π_χ possesses a Bessel model, in the sense that*

$$\dim \mathrm{Hom}_H(\pi_\chi, \nu) = 1$$

where (H, ν) is as defined in [GGP, §12].

To prove the existence of a (unique) Bessel model for π_χ, it is sufficient to establish the existence of a Bessel model for the representation

$$R(\alpha) \otimes R(\beta) \quad \text{of } \mathrm{U}_{n-d} \times \mathrm{U}_{m-d},$$

as there is clearly a unique $\mathrm{U}_d \times \mathrm{U}_d$ invariant linear form on $(R \otimes R^\vee)$. We will do this in the following two sections, after first studying the situation for general linear groups.

4. Branching laws for $\mathrm{GL}_n(\mathbb{F}_q)$

In this section, we calculate the restriction of a representation of $\mathrm{GL}_n(\mathbb{F}_q)$ to $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ where $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ sits inside $\mathrm{GL}_n(\mathbb{F}_q)$ in the natural way as

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

These branching laws are surely known in the literature, such as in the work of Thoma [Th]; however, we have preferred to give a different independent treatment.

We begin by recalling the notion of twisted Jacquet functor. Let $P = M \cdot N$ be any group such that N is a normal subgroup of P and let φ be a character of N whose stabilizer in M is denoted by M_φ. The data (N, φ) defines the twisted Jacquet functor from the category of smooth representations of P to the category of smooth representations of M_φ. It associates to a representation V of P the largest quotient

$V_{N,\varphi}$ of V on which N operates via the character φ; clearly $V_{N,\varphi}$ is a representation space for M_φ. The twisted Jacquet functor is exact.

Now let E_{n-1} be the mirabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of matrices whose last row is equal to $(0,0,\cdots,0,1)$ and let N_n be the group of upper triangular unipotent matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. We fix a nontrivial character ψ_0 of $\mathbb{F}_q$ and let ψ_n be the character of N_n, given by

$$\psi_n(u) = \psi_0(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}).$$

For a representation π of $\mathrm{GL}_n(\mathbb{F}_q)$, let

$$\pi^i = \text{ the } i\text{-th derivative of } \pi,$$

which is a representation of $\mathrm{GL}_{n-i}(\mathbb{F}_q)$. To recall the definition of π^i, if

$$R_{n-i} = \mathrm{GL}_{n-i}(\mathbb{F}_q) \cdot V_i$$

is the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of matrices

$$\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$$

with $g \in \mathrm{GL}_{n-i}(\mathbb{F}_q)$, $v \in M_{(n-i)\times i}$, $z \in N_i$, and if the character ψ_i of N_i is extended to V_i by extending it trivially across $M_{(n-i)\times i}$, then we have

$$\pi^i = \pi_{V_i,\psi_i}.$$

If π is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$, then $\pi^i = \pi$ for $i = 0$, and $\pi^n = 1$, the trivial representation of the trivial group $\mathrm{GL}_0(\mathbb{F}_q) = \{e\}$. All the other derivatives of π are 0.

The following proposition is from Bernstein-Zelevinsky [BZ, Lemma 4.5], where it was established for non-archimedean local fields, but their proof works for finite fields as well. It is known as the Leibnitz rule for derivatives.

Proposition 4.1. *For π_1 a representation of $\mathrm{GL}_{n_1}(\mathbb{F}_q)$ and π_2 of $\mathrm{GL}_{n_2}(\mathbb{F}_q)$, we let $\pi_1 \times \pi_2$ denote the representation of $GL_{n_1+n_2}(\mathbb{F}_q)$ induced from the corresponding representation of the parabolic subgroup with Levi subgroup $GL_{n_1}(\mathbb{F}_q) \times GL_{n_2}(\mathbb{F}_q)$. Then there is a composition series of the k-th derivative $(\pi_1 \times \pi_2)^k$ whose successive quotients are $\pi_1^i \times \pi_2^{k-i}$ for $i = 0, \cdots, k$.*

Here is a generality from Bernstein and Zelevinsky [BZ, §3.5].

Proposition 4.2. *Any representation Σ of E_{n-1} has a natural filtration of $E = E_{n-1}$ modules*

$$0 = \Sigma_0 \subset \Sigma_1 \subset \Sigma_2 \subset \cdots \subset \Sigma_n = \Sigma$$

such that

$$\Sigma_{i+1}/\Sigma_i = \mathrm{ind}_{R_i}^{E}(\Sigma^{n-i} \otimes \psi_{n-i}) \quad \textit{for } i = 0, \cdots, n-1,$$

where $R_i = \mathrm{GL}_i(\mathbb{F}_q) \cdot V_{n-i}$ is the subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ consisting of

$$\begin{pmatrix} g & v \\ 0 & z \end{pmatrix}$$

with $g \in \mathrm{GL}_i(\mathbb{F}_q)$, $v \in M_{i\times(n-i)}$, $z \in N_{n-i}$, and the character ψ_{n-i} on N_{n-i} is extended to V_{n-i} by extending it trivially across $M_{i\times(n-i)}$.

As a consequence of the above two propositions, we have the following corollary.

Corollary 4.3. *Let $n = n_1 + \cdots + n_r$ be a sum of positive integers, and let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{F}_q)$ for $i = 1, \cdots, r$. Let*

$$\Pi = \pi_1 \times \cdots \times \pi_r$$

be the corresponding parabolically induced representation of $\mathrm{GL}_n(\mathbb{F}_q)$. Then the restriction of the representation $\pi_1 \times \cdots \times \pi_r$ of $\mathrm{GL}_n(\mathbb{F}_q)$ to $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ is a sum of the following representations:

$$\pi_{i_1} \times \pi_{i_2} \times \cdots \times \pi_{i_s} \times \Sigma[n-1-(n_{i_1} + \cdots + n_{i_s})]$$

where $1 \le i_1 < i_2 < \cdots < i_s \le r$ (the empty sequence is allowed) with $n_{i_1} + \cdots + n_{i_s} < n$, and

$$\Sigma[m] = \mathrm{ind}_{N_m}^{\mathrm{GL}_m(\mathbb{F}_q)} \psi_m$$

denotes the Gelfand-Graev representation of $\mathrm{GL}_m(\mathbb{F}_q)$, with $\Sigma[1]$ equal to the regular representation of $\mathbb{F}_q^\times$ and $\Sigma[0]$ denoting the trivial representation of the trivial group.

Proof. By Proposition 4.2, the restriction of Π to E_{n-1} is the sum of

$$\Pi_{i+1}/\Pi_i = \mathrm{ind}_{R_i}^{E_{n-1}} (\Pi^{n-i} \otimes \psi_{n-i}).$$

Since $\mathrm{GL}_{n-1}(\mathbb{F}_q) \cdot R_i = E_{n-1}$ for any i, it follows that

$$(\Pi_{i+1}/\Pi_i)|_{\mathrm{GL}_{n-1}(\mathbb{F}_q)} = \Pi^{n-i} \times \Sigma[n-1-i],$$

where $\Sigma[n-1-i]$ is the Gelfand-Graev module of $\mathrm{GL}_{n-1-i}(\mathbb{F}_q)$. It only remains to calculate the derivatives Π^{n-i} of Π, but this follows from Proposition 4.1. $\square$

As a simple consequence of this corollary, we have the following.

Theorem 4.4. *Let $n = n_1 + \cdots + n_r$ be a sum of positive integers, and let π_i be an irreducible cuspidal representation of $\mathrm{GL}_{n_i}(\mathbb{F}_q)$, for $i = 1, \cdots, r$. Let $n-1 = m_1 + \cdots + m_s$ be a sum of positive integers, and let μ_i be an irreducible cuspidal representation of $\mathrm{GL}_{m_i}(\mathbb{F}_q)$. Assume that the representations $\mu_1, \cdots, \mu_s$ are pairwise distinct, so that the corresponding parabolically induced representation $\mu_1 \times \cdots \times \mu_s$ of $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ is irreducible. Then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{F}_q)}(\pi_1 \times \cdots \times \pi_r, \mu_1 \times \cdots \times \mu_s)$$

is equal to

$$\prod_{i=1}^{s}(1+d_i) \geq 1,$$

where d_i *is the multiplicity with which* μ_i *appears in the set* $\{\pi_1 \cdots, \pi_r\}$*. In particular, if the* π_i*'s are mutually distinct as well, then*

$$\dim \mathrm{Hom}_{\mathrm{GL}_{n-1}(\mathbb{F}_q)}(\pi_1 \times \cdots \times \pi_r, \mu_1 \times \cdots \times \mu_s) = 2^d$$

where d *is the cardinality of the set*

$$\{\pi_1, \cdots \pi_r\} \cap \{\mu_1, ..., \mu_s\}.$$

Corollary 4.5. *The restriction of the representation* $\pi_1 \times \cdots \times \pi_r$ *of* $\mathrm{GL}_n(\mathbb{F}_q)$ *to* $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ *contains the representation* $\mu_1 \times \cdots \times \mu_s$ *of* $\mathrm{GL}_{n-1}(\mathbb{F}_q)$ *(with* μ_i*'s cuspidal and mutually distinct) with multiplicity one if and only if the sets* $\{\pi_1, \cdots, \pi_r\}$ *and* $\{\mu_1, \cdots, \mu_s\}$ *have no common elements; in other cases, the multiplicity is a +ve even integer.*

5. Branching laws for $\mathrm{U}_n(\mathbb{F}_q)$

In this section, we use the method of base change, also called Shintani descent, to deduce some conclusions about branching laws for the restriction of a representation of $\mathrm{U}_n(\mathbb{F}_q)$ to $\mathrm{U}_{n-1}(\mathbb{F}_q)$ from the corresponding results for general linear groups obtained in the previous section. The result is then applied to give a proof of Theorem 3.3.

We make crucial use of the multiplicity 1 theorem for restriction of representations of unitary groups over p-adic fields, which was recently proved by Aizenbud, Gourevitch, Rallis and Schiffmann in [AGRS]. A simple consequence of their result is:

Proposition 5.1. *Let* π_1 *be an irreducible cuspidal representation of* $\mathrm{U}_{n-1}(\mathbb{F}_q)$ *and let*

$$\pi_2 = I_P(\sigma)$$

be a (possibly reducible) principal series representation of $\mathrm{U}_n(\mathbb{F}_q)$*, where* P *is a parabolic subgroup of* $\mathrm{U}_n(\mathbb{F}_q)$ *and* σ *is an irreducible cuspidal representation of a Levi factor of* P*. We allow the possibility that* $P = \mathrm{U}_n$*, in which case* $\pi_2 = \sigma$ *is cuspidal. Then*

$$\dim \mathrm{Hom}_{\mathrm{U}_{n-1}(\mathbb{F}_q)}(\pi_2, \pi_1) \leq 1.$$

Proof. Let k_0 be a local field with $\mathbb{F}_q$ as its residue field and let k be its unramified quadratic extension. Then one can find quasi-split unitary groups $\mathrm{U}(V_0)$ and $\mathrm{U}(V)$ with $V_0 \subset V$, such that $\mathrm{U}(V_0) \times \mathrm{U}(V)$ over k_0 contains a hyperspecial maximal compact subgroup $K_0 \times K$ with reductive quotient $\mathrm{U}_{n-1}(\mathbb{F}_q) \times \mathrm{U}_n(\mathbb{F}_q)$. Moreover, one may find a maximal parabolic subgroup $\tilde{P}$ of $\mathrm{U}(V)$, such that $\tilde{P} \cap K$ maps to the parabolic P in the reductive quotient $\mathrm{U}_n(\mathbb{F}_q)$.

We commit the usual abuse of notation in denoting by π_1 the representation of K_0 obtained from the representation π_1 of $\mathrm{U}_{n-1}(\mathbb{F}_q)$ through the natural map from K_0 to $\mathrm{U}_{n-1}(\mathbb{F}_q)$.

Let $\tilde{\pi}_1$ be the depth zero supercuspidal representation of $\mathrm{U}(V_0)$ which is obtained from π_1 by compact induction, so that

$$\tilde{\pi}_1 = \mathrm{ind}_{K_0}^{\mathrm{U}(V_0)} \pi_1.$$

Similarly, let $\tilde{\sigma}$ be a depth zero supercuspidal representation of the Levi factor of $\tilde{P}$ which contains σ as a type. Since the center of a Levi subgroup is non-compact, there are many choices for $\tilde{\sigma}$, and we may consider the principal series representation $I_{\tilde{P}}(\tilde{\sigma})$ of $\mathrm{U}(V)$ which is irreducible for a generic choice of $\tilde{\sigma}$; this is possible by a result of Waldspurger, cf. [Sau]. Moreover, if K_1 denotes the kernel of the natural projection map

$$K \twoheadrightarrow \mathrm{U}_n(\mathbb{F}_q),$$

then one has

$$I_{\tilde{P}}(\tilde{\sigma})^{K_1} = I_P(\sigma).$$

Now by Frobenius reciprocity, we have

$$\begin{aligned}\dim \mathrm{Hom}_{\mathrm{U}(V_0)}(I_{\tilde{P}}(\tilde{\sigma}), \tilde{\pi}_1) &= \dim \mathrm{Hom}_{K_0}(I_{\tilde{P}}(\tilde{\sigma}), \pi_1) \\ &= \dim \mathrm{Hom}_{\mathrm{U}_{n-1}(\mathbb{F}_q)}(I_{\tilde{P}}(\tilde{\sigma})^{K_{0,1}}, \pi_1)\end{aligned}$$

where $K_{0,1}$ is the kernel of the projection map $K_0 \twoheadrightarrow \mathrm{U}_{n-1}(\mathbb{F}_q)$. Since $K_{0,1} \subset K_1$, we have

$$I_{\tilde{P}}(\tilde{\sigma})^{K_{0,1}} \supset I_{\tilde{P}}(\tilde{\sigma})^{K_1} = I_P(\sigma) = \pi_2.$$

Thus we conclude that

$$\dim \mathrm{Hom}_{\mathrm{U}(V_0)}(I_{\tilde{P}}(\tilde{\sigma}), \tilde{\pi}_1) \geq \mathrm{Hom}_{\mathrm{U}_{n-1}(\mathbb{F}_q)}(\pi_2, \pi_1).$$

By [AGRS], the LHS is bounded above by 1 for a generic choice of $\tilde{\sigma}$ (so that $I_{\tilde{P}}(\tilde{\sigma})$ is irreducible), and hence so is the RHS. This proves the proposition. $\square$

Remark 5.2. We note that the above multiplicity one result for unitary groups over finite fields, proved via known multiplicity one result for p-adic fields, is weaker in some aspect, and stronger in some other aspect, than the corresponding result for p-adic fields. It is weaker since it assumes that the representation π_1 of $\mathrm{U}_{n-1}(\mathbb{F}_q)$ is cuspidal; it is stronger than the p-adic result in that it does not assume that the representation π_2 of $\mathrm{U}_n(\mathbb{F}_q)$ is irreducible, but only assumes that it is obtained from parabolic induction of an irreducible representation. Presumably such a stronger result should also be true in the p-adic context, and is in fact true if the cuspidal representation π_1 of U_{n-1} is compactly induced, which conjecturally is always the case (for cuspidal representations).

A corollary of the above proposition is the uniqueness of Bessel models for cuspidal representations of unitary groups over finite fields.

Proposition 5.3. *Let π_1 be an irreducible cuspidal representation of $\mathrm{U}_n(\mathbb{F}_q)$, and let π_2 be an irreducible cuspidal representation of $\mathrm{U}_m(\mathbb{F}_q)$ with $n > m$ but $m \not\equiv n \mod 2$.*
(i) Let P be a maximal parabolic subgroup of $\mathrm{U}_{n+1}(\mathbb{F}_q)$ with Levi factor $\mathrm{GL}_r(\mathbb{F}_{q^2}) \times \mathrm{U}_m(\mathbb{F}_q)$ (so that $m + 2r = n + 1$) and let τ be a cuspidal representation of $\mathrm{GL}_r(\mathbb{F}_{q^2})$. Consider the principal series representation $I_P(\tau \boxtimes \pi_2)$ of $\mathrm{U}_{n+1}(\mathbb{F}_q)$. Then, with the data (H, ν) defined as in [GGP, §12], we have

$$\mathrm{Hom}_{H(\mathbb{F}_q)}(\pi_1 \boxtimes \pi_2, \nu) \cong \mathrm{Hom}_{\mathrm{U}_n(\mathbb{F}_q)}(I_P(\tau \boxtimes \pi_2), \pi_1^\vee)$$

(ii) We have:

$$\dim \mathrm{Hom}_{H(\mathbb{F}_q)}(\pi_1 \boxtimes \pi_2, \nu) \le 1.$$

Proof. (i) This is the finite field analog of [GGP, Theorem 15.1], with the same proof.
(ii) If $n = m + 1$, (ii) is a special case of Proposition 5.1. In the general case when $n > m+1$, we choose τ in the context of (i) so that the induced representation $I_P(\tau \boxtimes \pi_2)$ is irreducible. Then (ii) follows immediately from (i) and Proposition 5.1. ☐

The above propositions allow us to study the restriction problem from $\mathrm{U}_n(\mathbb{F}_q)$ to $\mathrm{U}_{n-1}(\mathbb{F}_q)$ using Shintani descent. We begin by giving a brief review of this notion.

Let G be a connected reductive algebraic group over $\mathbb{F}_q$ and let $m \ge 1$ be a fixed integer. The group $G(\mathbb{F}_{q^m})$ comes equipped with its Frobenius automorphism F, whose set of fixed points is $G(\mathbb{F}_q)$. There is a natural map, called the norm mapping,

$$\{F\text{-conjugacy classes in } G(\mathbb{F}_{q^m})\} \longrightarrow \{\text{conjugacy classes in } G(\mathbb{F}_q)\}$$

which is a bijection. The norm mapping thus induces an isomorphism of vector spaces

$$\{\text{class functions on } G(\mathbb{F}_q)\} \longrightarrow \{F\text{-class functions on } G(\mathbb{F}_{q^m})\},$$

which is called the base change map, and whose inverse is called Shintani descent. Furthermore, the base change map is an isometry:

$$\frac{\langle \chi_1, \chi_2 \rangle_{G(\mathbb{F}_q)}}{\# G(\mathbb{F}_q)} = \frac{\langle \chi_1', \chi_2' \rangle_{G(\mathbb{F}_{q^m})}}{\# G(\mathbb{F}_{q^m})},$$

where χ_1 and χ_2 are class functions on $G(\mathbb{F}_q)$ which are Shintani descents of the F-class functions χ_1' and χ_2' on $G(\mathbb{F}_{q^m})$. Here we have used the standard notation

$$\langle f_1, f_2 \rangle_G = \sum_G f_1(g) f_2(g^{-1}).$$

According to Deligne-Lusztig, given a maximal torus T of G defined over $\mathbb{F}_q$, and a character

$$\theta : T(\mathbb{F}_q) \to \mathbb{C}^\times,$$

there is a (virtual) representation of $G(\mathbb{F}_q)$ denoted by $R(T,\theta)$, which is called a Deligne-Lusztig representation. Now given a character θ as above, one has the character

$$\theta' : T' = T(\mathbb{F}_{q^m}) \to \mathbb{C}^\times$$

obtained by composing θ with the norm mapping: $T(\mathbb{F}_{q^m}) \to T(\mathbb{F}_q)$. Thus one may consider the Deligne-Lusztig representation $R(T',\theta')$. The following lemma is [DL, 5.16]:

Lemma 5.4. *Suppose that G has connected center. Then if $R(T,\theta)$ is irreducible, so is $R(T',\theta')$.*

Henceforth, we assume that G has connected center and that $R(T,\theta)$ is irreducible. The irreducible representation $R(T',\theta')$ is invariant by F and thus can be extended (in m ways) to the semi-direct product $G(\mathbb{F}_{q^m}) \rtimes \langle F\rangle$. For any such extension, the restriction of its character to the coset $G(\mathbb{F}_{q^m}) \cdot F$ is a F-class function, and one may consider its Shintani descent. The following is a basic fact in the theory of Shintani descent:

Proposition 5.5. *There is an extension of the irreducible representation $R(T',\theta')$ of $G(\mathbb{F}_{q^m})$ to $G(\mathbb{F}_{q^m}) \rtimes \langle F\rangle$ whose associated Shintani descent is the representation $R(T,\theta)$ of $G(\mathbb{F}_q)$.*

Now we can begin our study of the restriction problem for unitary groups over finite fields. Let

$$\begin{aligned} \pi_1 &= R(T_1,\theta_1) \\ \pi_2 &= R(T_2,\theta_2) \end{aligned}$$

be irreducible Deligne-Lusztig representations of $\mathrm{U}_n(\mathbb{F}_q)$ and $\mathrm{U}_{n-1}(\mathbb{F}_q)$ respectively, and let χ_i be the character of π_i. We shall consider the quadratic base change of π_i. By Proposition 5.5, there are extensions of the irreducible representations

$$\begin{aligned} \pi_1' &= R(T_1',\theta_1') \quad \text{of} \quad \mathrm{GL}_n(\mathbb{F}_{q^2}), \\ \pi_2' &= R(T_2',\theta_2') \quad \text{of} \quad \mathrm{GL}_{n-1}(\mathbb{F}_{q^2}), \end{aligned}$$

to $\mathrm{GL}_n(\mathbb{F}_{q^2}) \rtimes \mathbb{Z}/2$ and $\mathrm{GL}_{n-1}(\mathbb{F}_{q^2}) \rtimes \mathbb{Z}/2$ respectively, whose associated Shintani descents are χ_1 and χ_2 respectively. Fixing such an extension in each case, we denote the corresponding character of this distinguished extension to $\mathrm{GL}_n(\mathbb{F}_{q^2}) \rtimes \mathbb{Z}/2$ and $\mathrm{GL}_{n-1}(\mathbb{F}_{q^2}) \rtimes \mathbb{Z}/2$ respectively, by χ_1' and χ_2'.

From

$$\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})\rtimes\langle F\rangle} = \langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} + \langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})\cdot F}.$$

we find,

$$\begin{aligned}\frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})\rtimes\langle F\rangle}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} &= \frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} + \frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})\cdot F}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})},\\ &= \frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} + \frac{\langle \chi_1, \chi_2\rangle_{\mathrm{U}_{n-1}(\mathbb{F}_{q})}}{\#\mathrm{U}_{n-1}(\mathbb{F}_{q})}.\end{aligned}$$

Equivalently,

$$2\left[\frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})\rtimes\langle F\rangle}}{2\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}\right] = \frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} + \frac{\langle \chi_1, \chi_2\rangle_{\mathrm{U}_{n-1}(\mathbb{F}_{q})}}{\#\mathrm{U}_{n-1}(\mathbb{F}_{q})}.$$

Now we observe that:

(i) the left hand side of this last equality is an *even* integer;

(ii) the quantity $\frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}$ was computed in Theorem 4.4, under the assumption that $R(T_2', \theta_2')$ is an irreducible representation;

(iii) the quantity $\frac{\langle \chi_1, \chi_2\rangle_{\mathrm{U}_{n-1}(\mathbb{F}_{q})}}{\#\mathrm{U}_{n-1}(\mathbb{F}_{q})}$ is equal to 0 or 1 in certain cases, by Proposition 5.1.

Together, these observations allow one to compute $\frac{\langle \chi_1, \chi_2\rangle_{\mathrm{U}_{n-1}(\mathbb{F}_{q})}}{\#\mathrm{U}_{n-1}(\mathbb{F}_{q})}$ in certain situations.

Namely, let us assume that π_1 and π_2 are irreducible Deligne-Lusztig representations, and suppose further that π_2 is cuspidal. Then the quadratic base change π_1' and π_2' of π_1 and π_2 are irreducible full principal series representations of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ and $\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})$. Thus, Theorem 4.4 implies that

$$\frac{\langle \chi_1', \chi_2'\rangle_{\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})}}{\#\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})} = \begin{cases} 1, \text{ if the cuspidal supports of } \pi_1' \text{ and } \pi_2' \text{ are disjoint,}\\ \text{an even integer, otherwise.}\end{cases}$$

On the other hand, by Proposition 5.1, $\frac{\langle \chi_1, \chi_2\rangle_{\mathrm{U}_{n-1}(\mathbb{F}_{q})}}{\#\mathrm{U}_{n-1}(\mathbb{F}_{q})}$ is either 0 or 1. Therefore we get the following theorem as our only option.

Theorem 5.6. *Let π_1 and π_2 be irreducible Deligne-Lusztig representations of $\mathrm{U}_n(\mathbb{F}_q)$ and $\mathrm{U}_{n-1}(\mathbb{F}_q)$ respectively, and suppose that π_2 is cuspidal. Then*

$$\dim \mathrm{Hom}_{\mathrm{U}_{n-1}(\mathbb{F}_q)}(\pi_1, \pi_2) \neq 0$$

if and only if the cuspidal supports of the base change representations π_1' and π_2' of $\mathrm{GL}_n(\mathbb{F}_{q^2})$ and $\mathrm{GL}_{n-1}(\mathbb{F}_{q^2})$ respectively are disjoint, in which case the Hom space has dimension 1.

In particular, this theorem completes the proof of Theorem 3.3. Indeed, in the setting of Theorem 3.3, we need to show that the distinguished representation $\pi_\chi = \pi_1 \times \pi_2$ of

$\mathrm{U}(V) \times \mathrm{U}(V_0)$ satisfies

$$\mathrm{Hom}_H(\pi_\chi, \nu) \neq 0.$$

By the argument in the proof of Proposition 5.1, it is sufficient to show that the representation

$$R(\alpha) \otimes R(\beta) \quad \text{of } \mathrm{U}_{n-d}(\mathbb{F}_q) \times \mathrm{U}_{m-d}(\mathbb{F}_q)$$

satisfies

$$\mathrm{Hom}_{H(\mathbb{F}_q)}(R(\alpha) \otimes R(\beta), \nu) \neq 0.$$

The desired nonvanishing then follows from Proposition 5.3(i) and the above theorem, using the fact that the quadratic base change of $R(\alpha)$ and $R(\beta)$ have disjoint cuspidal support.

6. Langlands-Vogan packets for small unitary groups

The rest of this paper is devoted to verifying [GGP, Conjecture 17.3] or its variant [GGP, Conjecture 20.1] in various low rank examples in the unitary and symplectic cases. In this section, we explicate the Langlands-Vogan parameterization of irreducible representations of $\mathrm{U}(V)$ where V is a hermitian (or skew-hermitian) space over k of dimension ≤ 3.

When $\dim_k V = 1$, the group $\mathrm{U}(V)$ is naturally isomorphic to the subgroup k^1 of norm one elements in $k^\times$, via its scalar action on V. The map

$$x \mapsto x/x^\sigma$$

gives an isomorphism of $k^\times/k_0^\times$ with $\mathrm{U}(V)$. The only other pure inner form of $\mathrm{U}(V)$ is the group $\mathrm{U}(V')$ where V' is obtained from V by scaling the hermitian form on V by an element in $k_0^\times \smallsetminus \mathbb{N}k^\times$.

In this case, an L-parameter for $\mathrm{U}(V)$ is a 1-dimensional conjugate-orthogonal representation M of $WD(k)$, which corresponds via local class field theory to a character of $k^\times/k_0^\times$, and hence to characters χ_M of $\mathrm{U}(V)$ and χ'_M of $\mathrm{U}(V')$. The Vogan packet associated to M is then

$$\Pi_M = \{\chi_M, \chi'_M\}.$$

The component group A_M is $\mathbb{Z}/2\mathbb{Z}$ and the trivial character of A_M corresponds to the character χ_M of $\mathrm{U}(V)$.

Now consider the case when $\dim V = 2$. We take V to be the split hermitian space, and denote the other rank 2 hermitian space (which is anisotropic) by V'. In this case, the groups $\mathrm{U}(V)$ and $\mathrm{U}(V')$ are closely related to the group GL_2 and its inner form $D^\times$, where D is the unique quaternion division algebra over k_0.

More precisely, given a quaternion algebra B over k_0 (possibly split), we fix an embedding

$$k \hookrightarrow B$$

of algebras over k_0 and regard B as a 2-dimensional vector space over k via left multiplication. All such embeddings of k into B are conjugate under $\mathrm{Aut}_{k_0}(B)$ by the Skolem-Noether theorem. There is an element $b \in B$ (of trace zero) which normalizes k and whose conjugation action on k is the involution σ; moreover, all other such elements are of the form $\lambda \cdot b$ for $\lambda \in k$. We thus have a decomposition

$$B = k \cdot 1 + k \cdot b.$$

Define a nondegenerate hermitian form on B by

$$\langle x, y\rangle = \text{projection of } x \cdot \overline{y} \text{ onto } k \cdot 1,$$

where $y \mapsto \overline{y}$ is the canonical involution on B; let V_B be the associated hermitian space. If B is split, then V_B is the split hermitian space V, whereas if B is the quaternion division algebra D over k_0, then V_B is the anisotropic hermitian space V'.

The associated unitary similitude group is given by

$$\mathrm{GU}(V_B) \cong (B^\times \times k^\times)/\Delta k_0^\times$$

with an element $(b,t) \in B^\times \times k^\times$ acting on B by

$$(b,t)(x) = txb^{-1}.$$

The similitude character is given by

$$(b,t) \mapsto \mathbb{N}t \cdot \mathbb{N}b^{-1},$$

so that

$$\mathrm{U}(V_B) = \{(b,t) \in \mathrm{GU}(V_B) : \mathbb{N}b = \mathbb{N}t\}.$$

Observe that $\mathrm{U}(V_B)$ is a subgroup of

$$\mathrm{GU}^+(V_B) = ((B^\times)^+ \times k^\times)/\Delta k_0^\times,$$

where

$$(B^\times)^+ = \{b \in B^\times : \mathbb{N}b \in \mathbb{N}k^\times\}.$$

Moreover, it is easy to see that

$$\mathrm{GU}^+(V_B) = \mathrm{U}(V_B) \cdot Z_{\mathrm{GU}(V_B)},$$

where

$$Z_{\mathrm{GU}(V_B)} = (k_0^\times \times k^\times)/\Delta k_0^\times \cong k^\times$$

is the center of $\mathrm{GU}(V_B)$.

For later purposes, we describe here a nondegenerate rank 1 hermitian subspace of V_B. Consider the nondegenerate subspace

$$L_B = k \cdot b \hookrightarrow B$$

and observe that its orthogonal complement $L_B^\perp = k \cdot 1$ is isomorphic to $\langle 1\rangle$. The pointwise stabilizer of $L_B^\perp$ in $\mathrm{U}(B)$ is the diagonal subgroup

$$\mathrm{U}(L_B) \cong k^\times/k_0^\times \xrightarrow{\Delta} (B^\times \times k^\times)/\Delta k_0^\times.$$

We now come to the representation theory of $\mathrm{U}(V_B)$. Observing that the L-packets of $\mathrm{GU}(V_B)$ are all singletons, we take an L-packet of $\mathrm{U}(V_B)$ to be the set of irreducible constituents of the restriction of an irreducible representation of $\mathrm{GU}(V_B)$ to $\mathrm{U}(V_B)$. Since

$$\mathrm{GU}^+(V_B) = \mathrm{U}(V_B) \cdot Z_{\mathrm{GU}(V_B)},$$

when considering the restriction of an irreducible representation of $\mathrm{GU}(V_B)$ to $\mathrm{U}(V_B)$, we may as well consider the restriction problem to $\mathrm{GU}^+(V_B)$ in place of $\mathrm{U}(V_B)$.

Note that if $\tau \boxtimes \chi$ is an irreducible representation of

$$\mathrm{GU}(V_B) = (B^\times \times k^\times)/\Delta k_0^\times,$$

then its restriction to $\mathrm{GU}^+(V_B)$ is equal to

$$\tau|_{(B^\times)^+} \boxtimes \chi,$$

and it is known that $\tau|_{(B^\times)^+}$ is either irreducible or is the sum of two inequivalent irreducible summands. Moreover, the latter case holds if and only if $\tau \otimes \omega_{k/k_0} \cong \tau$, in which case we say that τ is dihedral with respect to k/k_0. Then the L-packet of $\mathrm{U}(V_B)$ associated to τ is the set

$$\Pi_{B,\tau,\chi} = \{(\tau^\alpha \boxtimes \chi)|_{\mathrm{U}(V_B)} : \tau^\alpha \text{ is an irreducible summand of } \tau|_{(B^\times)^+}\},$$

which has cardinality 1 or 2. Observe that if μ is any character of $k_0^\times$, then

$$\Pi_{B,\tau\otimes(\mu^{-1}\circ\det),\chi\cdot(\mu\circ\mathbb{N})} = \Pi_{B,\tau,\chi}.$$

If N is the L-parameter of τ, we also write $\Pi_{B,N,\chi}$ for $\Pi_{B,\tau,\chi}$.

To attach L-parameters to these packets, recall that an L-parameter in this case is a two dimensional conjugate-symplectic representation M of $WD(k)$. Now we note:

Proposition 6.1. *(i) Let $\tau\boxtimes\chi$ be an irreducible representation of $\mathrm{GU}(V) = (\mathrm{GL}_2(k_0)\times k^\times)/\Delta k_0^\times$, so that $\omega_\tau \cdot \chi|_{k_0^\times} = 1$. If N is the L-parameter of τ, then the representation*

$$M = N|_{WD(k)} \otimes \chi$$

of $WD(k)$ is conjugate-symplectic.

(ii) Conversely, any 2-dimensional conjugate-symplectic representation M of $WD(k)$ arises in this way from an irreducible representation $\tau \boxtimes \chi$ of $\mathrm{GU}(V)$, which is well-defined up to twisting by $(\mu^{-1} \circ \det) \boxtimes \mu \circ \mathbb{N}$ for some character μ of $k_0^\times$.

Proof. By [GGP, Thm. 8.1], we know that giving a parameter for the unitary group $\mathrm{U}(V)$ is equivalent to giving a 2-dimensional conjugate-symplectic representation M of $WD(k)$. Thus, it suffices to compare the standard description of the L-group of $\mathrm{U}(V)$, or rather $\mathrm{GU}(V)$, with that which arises from the identification

$$\mathrm{GU}(V) \cong (\mathrm{GL}_2(k_0) \times k^\times)/\Delta k_0^\times.$$

The L-group of $\mathrm{GU}(V)$ is

$$^L\mathrm{GU}(V) = [\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times] \rtimes \mathbb{Z}/2\mathbb{Z}$$

in which the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times$ is via the automorphism

$$(g, \alpha) \to (w_0\,{}^t g^{-1} w_0^{-1}, \alpha \det g) = ((\det g)^{-1} \cdot g, \alpha \cdot \det g),$$

with

$$w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

On the other hand, the L-group of $H = (\mathrm{GL}_2(k_0) \times k^\times)/\Delta k_0^\times$ is

$$^L H = [\mathrm{GL}_2(\mathbb{C}) \times \{(\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \mathbb{Z}/2\mathbb{Z}\}]^1,$$

where the $\mathbb{Z}/2\mathbb{Z}$ action on $\mathbb{C}^\times \times \mathbb{C}^\times$ in $^L H$ is via permuting the two factors, and $[\ ]^1$ in $^L H$ refers to the set of elements $(g; \alpha, \beta)$ with $\alpha \cdot \beta \cdot \det g = 1$. The isomorphism $H \cong \mathrm{GU}(V)$ induces a natural isomorphism

$$^L\mathrm{GU}(V) = [\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times] \rtimes \mathbb{Z}/2\mathbb{Z} \quad \longrightarrow \quad {}^L H = [\mathrm{GL}_2(\mathbb{C}) \times \{(\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \mathbb{Z}/2\mathbb{Z}\}]^1$$

given by

$$(g, \alpha) \mapsto (g\alpha; \alpha^{-1} \cdot \det g^{-1}, \alpha^{-1})$$

To complete the proof of the proposition, given a representation $\tau \boxtimes \chi$ of $H = (\mathrm{GL}_2(k_0) \times k^\times)/\Delta k_0^\times$, we get a Langlands parameter with values in $^L H$ which, by the isomorphism above, gives a parameter with values in $^L\mathrm{GU}(V)$. Composing this with the natural projection map

$$^L\mathrm{GU}(V) = [\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C}^\times] \rtimes \mathbb{Z}/2\mathbb{Z} \to \mathrm{GL}_2(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z} = {}^L\mathrm{U}(V)$$

whose kernel is $\mathbb{C}^\times$, we get a parameter in $^L\mathrm{U}(V)$, and therefore a conjugate-symplectic representation of $WD(k)$. This representation of $WD(k)$ is none other than $N|_{WD(k)} \otimes \chi$ where N is the L-parameter of τ. This proves (i).

Conversely, given a parameter for $\mathrm{U}(V)$, we lift it to $^L\mathrm{GU}(V)$ using a well-known theorem of Tate (on the vanishing of the 2nd cohomology group of $W(k)$ with values in $\mathbb{C}^\times$), and thus obtain a parameter for H. This proves (ii). $\square$

In view of the above proposition, we set the L-parameter associated to the packet $\Pi_{B,\tau,\chi}$ to be the conjugate-symplectic representation

$$M = N|_{WD(k)} \otimes \chi,$$

with N the L-parameter of τ. Given a conjugate-symplectic M, with associated pair (τ, χ) as in Proposition 6.1(ii), the associated Vogan packet is

$$\Pi_M = \dot{\bigcup}_B \Pi_{B,N,\chi},$$

where the union is taken over the two quaternion algebras over k_0.

Remark 6.2. It has been shown by Konno-Konno [KK] that the above construction of L-parameters agrees with the one supplied by the theory of twisted endoscopy (i.e. base change to GL(2) over k), which has been achieved by Rogawski [Ro] using the stable trace formula.

The following table lists the various possibilities of M, Π_M and the component group A_M, depending on the type of τ's.

τ	M	Π_M	A_M
non-dihedral principal series (with respect to k/k_0)	$P + {}^\sigma P^\vee$, $P \not\cong {}^\sigma P^\vee$	1 representation on $\mathrm{U}(V)$	trivial
non-dihedral discrete series (with respect to k/k_0)	irreducible conjugate-symplectic	1 representation on $\mathrm{U}(V)$ and 1 on $\mathrm{U}(V')$	$\mathbb{Z}/2\mathbb{Z}$
dihedral principal series (with respect to k/k_0)	$2 \cdot M'$, M' conjugate-symplectic	2 representations on $\mathrm{U}(V)$	$\mathbb{Z}/2\mathbb{Z}$
dihedral discrete series (with respect to k/k_0)	$M_1 + M_2$, $M_1 \not\cong M_2$ conjugate-symplectic	2 representations on $\mathrm{U}(V)$ and 2 on $\mathrm{U}(V')$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

If the conjugate-symplectic representation M is of the last two types in the above table, we shall call M dihedral with respect to k/k_0. If it is of the first two type, we shall call it non-dihedral with respect to k/k_0.

From the above table, we see that $\#\Pi_M = \#A_M = \#\mathrm{Irr}(A_M)$. To index the representations in Π_M by $\mathrm{Irr}(A_M)$, we need to fix a generic character of $\mathrm{U}(V)$ (where V is the split hermitian space). According to [GGP, Prop. 12.1(2)], a generic character of $\mathrm{U}(V)$ is specified by giving a nontrivial additive character $\psi : k/k_0 \to \mathbb{S}^1$. We briefly recall how this is done. Let $\{e, f\}$ be a basis of V such that

$$\langle e, e\rangle = 0 = \langle f, f\rangle, \quad \langle e, f\rangle = 1.$$

This is a unique such basis up to conjugation by $\mathrm{U}(V)$. Let N be the unipotent radical of the Borel subgroup of $\mathrm{U}(V)$ fixing the line spanned by e. Then there is a natural map $N \to k$ defined by

$$n \mapsto \langle nf - f, f\rangle,$$

which takes values in the subspace of trace zero elements in k. Composing this map with the non-trivial character $\psi : k/k_0 \to \mathbb{S}^1$, we get a unitary character $\theta : N \to \mathbb{C}^\times$ in general position, and the pair (N, θ) is unique up to conjugacy by $\mathrm{U}(V)$, for a fixed choice of ψ. If a representation of $\mathrm{U}(V)$ has nonzero Whittaker model with respect to (N, θ), we shall say that it is ψ-generic.

Having fixed $\psi : k/k_0 \to \mathbb{S}^1$, we then decree that

(i) the trivial character of A_M corresponds to the ψ-generic element in Π_M;

(ii) a character of A_M corresponds to a representation of $\mathrm{U}(V)$ if and only if it is trivial on the image of the central element $-1 \in {}^L\mathrm{U}(V)$.

From the above table, we see that these requirements completely determine the bijection

$$J(\psi) : \Pi_M \leftrightarrow \mathrm{Irr}(A_M),$$

except in the last case, where τ is a dihedral (with respect to k/k_0) discrete series representation of $\mathrm{U}(V)$ which is a compact unitary group, using the two characters of A_M which are nontrivial on the central -1. However, in §8, we shall resolve this issue when we describe an alternative construction of these Vogan packets using theta correspondence.

Finally, we consider the case when $\dim V = 3$. In this case, the only other pure inner form of $\mathrm{U}(V)$ is the group $\mathrm{U}(V')$ where V' is the hermitian space obtained from V via scaling by an element of $k_0^\times \smallsetminus \mathbb{N}k^\times$. In this case, the Vogan packets have been defined by Rogawski [Ro] via base change to $\mathrm{GL}(3)$ over k using the stable trace formula.

The L-parameters are conjugate-orthogonal representations M of $WD(k)$ of dimension 3. When M is irreducible, the associated Vogan packet is said to be stable; it consists of a representation of $\mathrm{U}(V)$ and the same representation regarded as a representation of $\mathrm{U}(V')$. The component group A_M is $\mathbb{Z}/2\mathbb{Z}$ and we decree that the trivial character correspond to a representation of $\mathrm{U}(V)$. On the other hand, when M is reducible, the associated Vogan packet is said to be endoscopic. In §8, we shall describe a construction of the endoscopic packets, and the labelling of their representations by $\mathrm{Irr}(A_M)$, via the approach of theta correspondence.

7. Theta correspondence

The goal of this section is to review the necessary background and framework for the theta correspondence for unitary groups. This is necessary for the construction of endoscopic Vogan packets of $\mathrm{U}(2)$ and $\mathrm{U}(3)$ which will be given in the following section.

Let V be a hermitian space and W a skew-hermitian space over k. To consider the theta correspondence for the dual pair $\mathrm{U}(V) \times \mathrm{U}(W)$, one requires certain additional data:

(i) an additive character $\psi_0 : k_0 \to \mathbb{S}^1$;

(ii) a character $\mu : k^\times \to \mathbb{C}^\times$ such that $\mu|_{k_0^\times} = \omega_{k/k_0}$;

(iii) a trace zero element $\delta \in k^\times$.

To elaborate, the tensor product $\mathrm{Res}_{k/k_0}(V \otimes_k W)$ has a natural symplectic form defined by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle = \mathrm{Tr}_{k/k_0}(\langle v_1, v_2 \rangle_V \cdot \langle w_1, w_2 \rangle_W).$$

Note that many authors (for example [HKS]) include a factor 1/2 on the right hand side, but we shall not follow this convention here. In any case, there is a natural map

$$i : \mathrm{U}(V) \times \mathrm{U}(W) \longrightarrow \mathrm{Sp}(V \otimes W/k_0).$$

One has the metaplectic $\mathbb{S}^1$-cover $\mathrm{Mp}(V \otimes W)$ of $\mathrm{Sp}(V \otimes W)$, and the character ψ_0 (together with the form $\langle -, - \rangle$ on $V \otimes W$) determines a Weil representation ω_{ψ_0} of $\mathrm{Mp}(V \otimes W)$. To obtain a representation of $\mathrm{U}(V) \times \mathrm{U}(W)$ from ω_{ψ_0}, however, one needs to specify a splitting of the map i to the metaplectic cover. This is quite subtle, but was completely understood by Gelbart-Rogawski [GRO], Kudla [K] and Harris-Kudla-Sweet [HKS]; it requires the additional data above.

More precisely, the data (V, ψ_0, μ) determines a splitting

$$i_{V,\mu,\psi_0} : \mathrm{U}(W) \hookrightarrow \mathrm{Mp}(V \otimes W),$$

whereas the data (W, ψ_0, μ, δ) determines a splitting

$$i_{W,\mu,\delta,\psi_0} : \mathrm{U}(V) \hookrightarrow \mathrm{Mp}(V \otimes W)$$

whose image commutes with that of i_{V,μ,ψ_0}. In [HKS], such splittings are constructed for any pair of characters (χ, χ') of $k^\times$ satisfying

$$\chi|_{k_0^\times} = \omega_{k/k_0}^{\dim V} \quad \text{and} \quad \chi'|_{k_0^\times} = \omega_{k/k_0}^{\dim W}.$$

In their terminology, our splittings are relative to the pair of characters

$$\chi = \mu^{\dim V} \quad \text{and} \quad \chi' = \mu^{\dim W}.$$

In particular, by [HKS, Corollary A.8], a property of this splitting is that the images of the centers of $\mathrm{U}(V)$ and $\mathrm{U}(W)$ are identified, so that the resulting theta correspondence preserves the central characters.

Using the above splittings, one obtains a Weil representation

$$\omega_{\psi_0,\mu} = \omega_{\psi_0} \circ (i_{W,\mu,\psi_0,\delta} \times i_{V,\mu,\psi_0})$$

of $\mathrm{U}(V) \times \mathrm{U}(W)$, where we have suppressed the data (V, W, δ) from the notation. The Weil representation $\omega_{\psi_0,\mu}$ depends only on the orbit of ψ_0 under $\mathbb{N}k^\times$. Thus, given an irreducible representation π of $\mathrm{U}(W)$, we have its big and small theta lift $\Theta_{\psi_0,\mu}(\pi)$ and $\theta_{\psi_0,\mu}(\pi)$ on $\mathrm{U}(V)$. By a result of Waldspurger, $\theta_{\psi_0,\mu}(\pi)$ is either zero or is irreducible when $p \neq 2$. For the groups of low rank discussed in this paper, one can check that this is true for all p.

It would appear that, by restricting (χ_1, χ_2) (as in [HKS]) to have the special form taken here, we are losing one degree of freedom. However, this lost degree of freedom can be regained by allowing twisting of the theta lifts by 1-dimensional characters of $\mathrm{U}(V)$, i.e. if we consider $\theta_{\psi_0,\mu}(\pi) \otimes (\chi \circ \det)$ as well.

It is also useful to consider the theta correspondence for similitude groups. Let

$$R \subset \mathrm{GU}(V) \times \mathrm{GU}(W)$$

be the subgroup consisting of elements (g, h) such that the product of the similitude factors, $\mathrm{sim}(g) \cdot \mathrm{sim}(h) = 1$. Then the Weil representation $\omega_{\psi_0,\mu}$ has a natural extension to R. Now observe that

$$R \subset \mathrm{GU}^+(V) \times \mathrm{GU}^+(W)$$

where $\mathrm{GU}^+(V)$ consists of those elements $g \in \mathrm{GU}(V)$ such that $\mathrm{sim}(g)$ lies in the image of the similitude map of $\mathrm{GU}(W)$, and analogously for $\mathrm{GU}^+(W)$. Then one may consider the induced representation

$$\Omega_{\psi_0,\mu} = \mathrm{ind}_R^{\mathrm{GU}^+(V)\times\mathrm{GU}^+(W)} \omega_{\psi_0,\mu}$$

of $\mathrm{GU}^+(V) \times \mathrm{GU}^+(W)$, which depends only on the orbit of ψ_0 under $\mathbb{N}k^\times$ (and is independent of ψ_0 in some cases). We can now consider the theta correspondence for $\mathrm{GU}^+(V) \times \mathrm{GU}^+(W)$ associated to $\Omega_{\psi_0,\mu}$. In particular, for a representation π of $\mathrm{GU}^+(W)$, we have its big and small theta lifts $\Theta_{\psi_0,\mu}(\pi)$ and $\theta_{\psi_0,\mu}(\pi)$ on $\mathrm{GU}^+(V)$.

In this paper, we will be considering the theta correspondence for $\mathrm{U}(V) \times \mathrm{U}(W)$ with $|\dim V - \dim W| \le 1$. In this case, there are some rather precise conjectures about the behavior of the theta correspondence in the literature (see for example [HKS, §7] and [P5]). We formulate these as the following working hypothesis.

Working hypothesis: Let V be a hermitian space and let W be a skew-hermitian space, and consider the theta correspondence for $\mathrm{U}(V) \times \mathrm{U}(W)$ relative to the data (ψ_0, μ). For an irreducible representation π of $\mathrm{U}(V)$, let $\theta_{\psi_0,\mu}(\pi)$ denote the (small) theta lift of π to $\mathrm{U}(W)$.

(a) If $\dim V = \dim W$, then the Langlands parameters of π and $\theta_{\psi_0,\mu}(\pi)$ are the same (if the latter is nonzero). For a given L-parameter M, the theta correspondence induces a permutation of the Vogan packet Π_M to itself. This bijection is given by translation by a character of the component group A_M, as given in [P5] in terms of the root numbers of conjugate-symplectic representations of the Weil-Deligne group.

(b) If $\dim V = \dim W - 1$, then the Langlands parameters M of π and N of $\theta_{\psi_0,\mu}(\pi)$ are related to each other by:

$$N = \mu^{-1} M + \mu^{\dim V}.$$

The theta correspondence relative to (ψ_0, μ) gives an injection

$$\theta_{\psi_0,\mu,V,W} : \Pi_{V,M} \hookrightarrow \Pi_{W,N}.$$

This injection can be naturally described in terms of the characters of the component groups of M and N as follows. Assume for simplicity that $\mu^{\dim V}$ does not occur in $\mu^{-1}M$, so that $A_N = \mathbb{Z}/2\mathbb{Z} \times A_M$. For an appropriately normalized Langlands-Vogan parameterization, the above injection is described by the natural map

$$\mathrm{Irr}(A_M) \longrightarrow \mathrm{Irr}(A_N) = \{\pm 1\} \times \mathrm{Irr}(A_M)$$

given by

$$\rho \mapsto (\epsilon, \rho)$$

where the sign ϵ is completely determined by ρ and the space W.

Moreover, as V and W vary over all hermitian and skew-hermitian spaces of the specified dimensions, one has

$$\Pi_N = \dot{\bigcup}_{V,W} \theta_{\psi_0,\mu,V,W}(\Pi_{V,M}),$$

where the union is disjoint and we ignore the theta lifts which are zero. The disjointness of the union means that if $V \neq V'$ and $W \neq W'$, then

$$\theta_{\psi_0,\mu,V,W}(\Pi_{V,M}) \cap \theta_{\psi_0,\mu,V',W}(\Pi_{V',M}) = \emptyset,$$

and,

$$\theta_{\psi_0,\mu,V,W}(\Pi_{V,M}) \cap \theta_{\psi_0,\mu,V,W'}(\Pi_{V,M}) = \emptyset.$$

While the second statement is part of definitions (since $\mathrm{U}(W)$ and $\mathrm{U}(W')$ are to be considered as different groups, even though they may be isomorphic), the first statement is in fact a consequence of the main result of [HKS] on theta dichotomy (as extended by [Go-Gr]).

In the following, we shall consider the low rank cases, with $\dim V \leq 2$ and $\dim W \leq 3$. In these cases, we shall use the above working hypothesis as a guide to label the representations in endoscopic L-packets of $\mathrm{U}(2)$ and $\mathrm{U}(3)$ which can be constructed using the theta correspondence. We note that these low rank cases are the only ones in which the Langlands-Vogan parameterization is fully understood for $\mathrm{U}(V)$ and $\mathrm{U}(W)$.

For example, statement (a) for $\dim V = 1$ is a result of Moen [Mo], Rogawski [Ro2] and Harris-Kudla-Sweet [HKS] (see Theorem 9.1 below), whereas the case when $\dim V = 2$ is verified in Theorem 11.2 below. On the other hand, statement (b) for $\dim V = 1$ is easy to check, and the case of $\dim V = 2$ is due to Gelbart-Rogawski-Soudry [GRS].

8. Endoscopic packets and theta correspondence

The goal of this section is to describe an alternative construction of the endoscopic packets of the unitary group $\mathrm{U}(V)$, via theta correspondence, when $\dim V = 2$ or 3. We shall rely heavily on the framework and notation of the previous two sections.

Our first case of interest is the theta correspondence for a skew-hermitian space W and a hermitian space V with

$$\dim W = 1 \quad \text{and} \quad \dim V = 2.$$

We shall use the associated theta correspondence to construct certain Vogan packets on $\mathrm{U}(V)$. Recall that in §6, we have given a construction of the rank 2 hermitian spaces V_B in terms of quaternion algebras B over k_0. Suppose that

$$M = M_1 + M_2$$

is a 2-dimensional conjugate-symplectic representation of $WD(k)$, with M_i conjugate-symplectic (but not necessarily distinct). As we explained in §6, such an M gives rise to a Vogan packet Π_M of $\mathrm{U}(V_B)$. If we fix an additive character

$$\psi : k/k_0 \longrightarrow \mathbb{S}^1$$

then there should be an associated bijection

$$J(\psi) : \Pi_M \longleftrightarrow \mathrm{Irr}(A_M).$$

It is the Vogan packet Π_M, together with the bijection $J(\psi)$, that we would like to construct using theta correspondence. In fact, since the Vogan packets on $\mathrm{U}(V_B)$ are defined by restriction from $\mathrm{GU}(V_B)$, it will be better to consider the theta correspondence for the similitude groups $\mathrm{GU}(W) \times \mathrm{GU}^+(V_B)$, with

$$\mathrm{GU}(W) \cong k^\times \quad \text{and} \quad \mathrm{GU}^+(V_B) = ((B^\times)^+ \times k^\times)/k_0^\times.$$

To set up the theta correspondence, we need to fix the data ψ_0, μ, and the trace zero element δ; these are as in the introduction.

Let W be a rank 1 skew-hermitian space with discriminant δ, and W' the other rank 1 skew-hermitian space. For any $a \in k_0^\times$, let W_a denote the rank 1 skew-hermitian space obtained from W by scaling by a. Finally, with $M = M_1 + M_2$ as above, we set

$$\mu = M_1,$$

and let χ be any character of $k^\times$ such that

$$\chi/\chi^\sigma = M_1 \cdot M_2.$$

This is possible since $M_1 \cdot M_2$ is a character of $k^\times/k_0^\times$. The choice of χ is not unique but any two choices differ by a character of $k^\times$ which is σ-invariant, or equivalently by one that factors through the norm map to $k_0^\times$. In any case, we have

$$M = M_1 + M_2 = \mu + \chi/\chi^\sigma \cdot \mu^{-1},$$

and the packet Π_M is obtained by the restriction of $\tau \boxtimes \chi$, where τ is the representation of $B^\times$ with L-parameter

$$N = \mathrm{Ind}_{WD(k)}^{WD(k_0)} \mu\chi^{-1}.$$

Now we may consider the theta correspondence associated to the Weil representation $\Omega_{\psi_0,\mu}$ of $\mathrm{GU}(W_a) \times \mathrm{GU}^+(V_B)$. Regarding χ as a character of $\mathrm{GU}(W_a)$, we have the theta lift

$$\Theta_{W_a,V_B,\psi_0,\mu}(\chi) = \theta_{W_a,V_B,\psi_0,\mu}(\chi)$$

on $\mathrm{GU}^+(V_B)$. With $B^\times = \mathrm{GL}_2(k_0)$, the character ψ determines a generic character of $\mathrm{GU}^+(V_B)$. We let τ^+ be the constituent of $\tau|_{\mathrm{GL}_2(k_0)^+}$ such that the representation $\tau^+ \boxtimes \chi$ of $\mathrm{GU}^+(V_B)$ is ψ-generic, and let τ^- denote the other constituent. We also let τ' be the Jacquet-Langlands lift of τ to $D^\times$, if it exists.

With these notations, we have the following proposition which follows by a computation of the Whittaker module of the Weil representation with respect to the maximal unipotent subgroup of $\mathrm{U}(1,1)$; this computation is standard and will therefore be not carried out here.

Proposition 8.1. *If B is split, so that $V_B = V$, then*

$$\begin{cases} \theta_{\psi_0,\mu,V,W}(\chi) = \tau^+ \boxtimes \chi, \\ \theta_{\psi_0,\mu,V,W'}(\chi) = \tau^- \boxtimes \chi. \end{cases}$$

If B is non-split, so that $V_B = V'$, then

$$\theta_{\psi_0,\mu,V',W}(\chi) + \theta_{\psi_0,\mu,V',W'}(\chi) = \tau' \boxtimes \chi,$$

where the RHS is interpreted as 0 if τ' does not exist. In particular, upon restriction to $\mathrm{U}(V)$ or $\mathrm{U}(V')$, the set

$$\{\theta_{\psi_0,\mu,V,W}(\chi), \theta_{\psi_0,\mu,V,W'}(\chi), \theta_{\psi_0,\mu,V',W}(\chi), \theta_{\psi_0,\mu,V',W'}(\chi)\}$$

is the Vogan packet Π_M associated to the L-parameter

$$M = M_1 + M_2 = \mu + \mu^{-1}\chi/\chi^\sigma.$$

Using the above construction of endoscopic packets of $\mathrm{U}(V)$, we can define the bijection

$$J(\psi) : \Pi_M \longleftrightarrow \mathrm{Irr}(A_M),$$

as follows. Consider the case when $M_1 \neq M_2$, so that $A_M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; this is the only case where the bijection $\Pi_M \leftrightarrow \mathrm{Irr}(A_M)$ has some ambiguity. We set

$$\begin{cases} \pi^{++} = \theta_{\psi_0,\mu,V,W}(\chi) \\ \pi^{--} = \theta_{\psi_0,\mu,V,W'}(\chi) \\ \pi^{+-} = \theta_{\psi_0,\mu,V',W'}(\chi) \\ \pi^{-+} = \theta_{\psi_0,\mu,V',W}(\chi). \end{cases}$$

In other words, the recipe for labelling is that

$$\pi^{\epsilon_1,\epsilon_2} = \theta_{\psi,\mu,V_B,W_a}(\chi)$$

where

$$\epsilon_1 \cdot \epsilon_2 = \epsilon(B) = \begin{cases} 1 \text{ if } B \text{ is split;} \\ -1, \text{ if } B \text{ is not split,} \end{cases}$$

and

$$\epsilon_2 = \omega_{k/k_0}(a).$$

Equivalently, if η is a a character of A_M, then

$$\pi_\eta = \theta_{\psi,\mu,V_B,W_a}(\chi)$$

if and only if

$$\begin{cases} \eta(a_1) = \epsilon(B) \cdot \omega_{k/k_0}(a), \\ \eta(a_2) = \omega_{k/k_0}(a). \end{cases}$$

We leave it to the reader to verify that under this system of bijections $J(\psi)$, the various desiderata of the Vogan parameterization listed in [GGP, §9 and §10] are satisfied. In particular, the trivial character of A_M corresponds to the unique ψ-generic representation of the packet, and if ψ' belongs to the other $\mathbb{N}k^\times$-orbit, then the unique ψ'-generic representation corresponds to the character

$$\eta_0(a_i) = (-1)^{\dim M_i}.$$

Indeed, when M is irreducible, η_0 is trivial, whereas when $M = M_1 + M_2$ is reducible, then η_0 is the character $(--)$ of $A_M = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

It will be useful to convert the above classification into the setting of rank 2 skew-hermitian spaces. Using the trace zero element δ, let $W_{B,\delta}$ be the skew-hermitian space obtained from V_B by scaling by δ; we shall frequently write W_B for $W_{B,\delta}$. Then we have

$$\mathrm{GU}(W_B) = \mathrm{GU}(V_B)$$

as subsets of $\mathrm{End}_k(B)$. Moreover, the notions of L-parameters and L-packets are the same for $\mathrm{U}(V_B)$ and $\mathrm{U}(W_B)$. The only difference lies in the data needed to specify a bijection of a Vogan packet with the set of characters of the component group. In the case of V_B, we used an additive character

$$\psi : k/k_0 \longrightarrow \mathbb{S}^1,$$

whereas for the case of W_B, one needs an additive character of k_0. However, it is easy to check that if a representation π of $\mathrm{U}(V_B)$ is generic with respect to ψ, then regarded as a representation of $\mathrm{U}(W_B)$, π is generic with respect to the character ψ_0 of k_0 which we have fixed, and the bijection

$$J(\psi) : \Pi_M \longleftrightarrow \mathrm{Irr}(A_M)$$

for $\mathrm{U}(V_B)$ is the bijection $J(\psi_0)$ for $\mathrm{U}(W_B)$. For a character η of A_M, we then have

$$\pi_\eta = \theta_{\psi_0,\mu,V_a,W_B}(\chi)$$

where μ and χ are obtained from M as before, V_a is the rank 1 hermitian space with discriminant a, and

$$\begin{cases} \eta(a_1) = \epsilon(B) \cdot \omega_{k/k_0}(a) \\ \eta(a_2) = \omega_{k/k_0}(a). \end{cases}$$

Let N be a conjugate-symplectic representation of $WD(k)$ of dimension 2 considered as an L-parameter for $\mathrm{U}(W_B)$. Let Π_N be the Vogan packet associated to N, together with the bijection

$$J(\psi_0) : \Pi_N \longleftrightarrow \mathrm{Irr}(A_N)$$

associated to the additive character ψ_0. Then for $\eta \in \mathrm{Irr}(A_N)$, we may consider the theta lift

$$\theta_{\psi_0,\mu,W_B,V_a}(\pi_\eta),$$

where $\pi_\eta \in \Pi_N$ is the representation of $\mathrm{U}(W_B)$ (this uniquely specifies B) indexed by η under $J(\psi_0)$. As the element a varies over the two representatives of $k_0^\times/\mathbb{N}k^\times$, and the character η varies over $\mathrm{Irr}(A_N)$, we obtain a collection of $2 \cdot \#\Pi_N$ representations (some of which might be zero). It was shown by Gelbart-Rogawski-Soudry [GRS] that the set of representations so obtained is the Vogan packet associated to the endoscopic parameter M given by:

$$M = \mu^2 + N \cdot \mu^{-1}.$$

The following lemma, which was shown in [GRS], addresses more precisely the issue of nonvanishing of these theta lifts.

Lemma 8.2. *Let $M = M_1 + M_2 = \mu^2 + N \cdot \mu^{-1}$ as above. If $M \ncong 3M_1$, assume without loss of generality that M_1 is distinct from any irreducible constituent of M_2.*

(i) If $M \ncong 3M_1$, then the representations $\theta_{\psi_0,\mu,V_a,W_B}(\pi_\eta)$ are always nonzero.

(ii) If $M = 3M_1$, then $N = 2 \cdot \mu^3$ and $A_N \cong \mathbb{Z}/2\mathbb{Z}$, so that we may regard $\eta = \pm 1$, depending on whether η is trivial or not. The representation $\theta_{\psi_0,\mu,V_a,W_B}(\pi_\eta)$ is nonzero if and only if

$$\omega_{k/k_0}(\mathrm{disc} V_a) = \eta.$$

In each case above, the non-zero representations are mutually distinct. Moreover, the representation $\theta_{\psi_0,\mu,V_a,W_B}(\pi_\eta)$ is generic if and only if π_η is generic with respect to $\psi_{0,\mathrm{disc}(V_a)}$.

We may now define a labeling of the elements in Π_M by the irreducible characters of A_M.

(i) If $M \ncong 3M_1$, and M_1 does not occur in M_2, then

$$A_M = A_{M_1} \times A_{M_2} = A_{M_1} \times A_N.$$

For a character $\chi = (\epsilon, \eta) \in \mathrm{Irr}(A_{M_1}) \times \mathrm{Irr}(A_N)$, we set

$$\pi^\chi = \pi^{\epsilon,\eta} = \theta_{\psi_0,\mu,W_B,V_a}(\pi_{\eta_V})$$

with

$$\epsilon \cdot \eta(-1) = \omega_{k/k_0}(a),$$

and

$$\eta_V = \begin{cases} \eta, \text{ if } \omega_{k/k_0}(\mathrm{disc}V) = 1; \\ \eta \cdot \eta_{N,0}, \text{ if } \omega_{k/k_0}(\mathrm{disc}V) = -1, \end{cases}$$

where $\eta_{N,0}$ is the character of A_N which indexes the ψ'-generic element of Π_N (where ψ' is a character of k/k_0 which is not in the $\mathbb{N}k^\times$-orbit of ψ). More simply, when $\mathrm{disc}(V) = 1$, we have

$$\chi(a_1) = \omega_{k/k_0}(a) \cdot \eta(-1) = \omega_{k/k_0}(a) \cdot \epsilon(B)$$

and

$$\chi|_{A_{M_2}} = \eta.$$

In particular, for a character χ of $A_M = A_{M_1} \times A_N$, π_χ is a representation of $\mathrm{U}(V)$ if and only if $\chi(-1,-1) = 1$.

(ii) If $M = 3M_1 = 3\mu^2$, then

$$A_M \cong A_N = \mathbb{Z}/2\mathbb{Z}.$$

For a character $\eta = \pm$ of A_M, we set

$$\pi^\eta = \theta_{\psi_0,\mu,V_a,W_B}(\pi_{\eta\cdot\omega_{k/k_0}(\mathrm{disc}V)})$$

with

$$\omega_{k/k_0}(a) = \eta.$$

By part (ii) of the above lemma, this condition ensures that the theta lift above is nonzero. In particular, the trivial character of A_M corresponds to a representation of $\mathrm{U}(V)$ whereas the nontrivial character corresponds to the same representation regarded on $\mathrm{U}(V')$.

Note that since $\dim V = 3$, there is only one orbit of generic characters for $\mathrm{U}(V)$, and hence the Vogan parameterization in this case is canonical. So it is instructive to observe that the above parameterization is independent of the choice of ψ_0 (or equivalently (ψ, δ)). We leave this to the reader, as well as the verification that the above definition satisfies the desiderata of the Vogan parameterization listed in [GGP, §9 and §10].

9. Skew-hermitian case: $\mathrm{U}(1) \times \mathrm{U}(1)$

Having explicated the Langlands-Vogan parameterization of the unitary groups $\mathrm{U}(V)$ with $\dim V \leq 3$, we are now in a position to verify instances of [GGP, Conjecture 17.3].

We begin with the case when $W_0 = W$ are skew-hermitian spaces with $\dim W_0 = \dim W = 1$. Let W' be the other skew-hermitian space of dimension 1. In this case the following result from [HKS, Corollary 8.5] is equivalent to our conjecture:

Theorem 9.1. *For each $a \in k_0^\times$, let W_a be the rank 1 skew-hermitian space with discriminant $a \cdot \delta$, and for each $b \in k_0^\times$, let V_b be the rank 1 hermitian space with discriminant b. Given a character η of $k^\times/k_0^\times$, which can be regarded as a character of $\mathrm{U}(W_a)$, we have*

$$\mathrm{Hom}_{\mathrm{U}(W_a)}(\eta, \omega_{W_a,V_b,\psi_0,\mu}) \neq 0 \Longleftrightarrow \epsilon(\eta \cdot \mu^{-1}, \psi_0(\mathrm{Tr}(\delta -))) = \omega_{k/k_0}(a \cdot b).$$

Remark 9.2. We note that our convention here differs from [HKS] in two aspects. Namely, we have adopted the convention that on $W_a \otimes V_b$, the symplectic form is $\mathrm{Tr}(\langle -, - \rangle_{W_a} \otimes \langle -, - \rangle_{V_b})$. In [HKS], the symplectic form is

$$\frac{1}{2} \cdot \mathrm{Tr}(\langle -, - \rangle_{W_a}^\sigma \otimes \langle -, - \rangle_{V_b}).$$

Besides the factor of $1/2$, the skew-hermitian form on W_a is conjugated by σ, which is necessitated by the convention adopted by [HKS] that skew-hermitian forms are linear in the second variable and hermitian forms are linear in the first variable. Conjugating the form on W_a by σ has the effect of replacing δ by $-\delta$ in [HKS, Corollary 8.5].

To apply the above theorem to [GGP, Conjecture 17.3], set $\eta = \alpha \cdot \beta$ for α, β characters of $\mathrm{U}(1)$, in the theorem, and note that the distinguished character χ_0 of $A_M \times A_N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given in [GGP, Conjecture 17.3] satisfies

$$\chi_0(-1, 1) = \chi_0(1, -1) = \epsilon(M \otimes N(\mu^{-1}), \psi_0(\mathrm{Tr}(\delta -))).$$

Thus, Theorem 9.1 implies that

$$\chi_0 \text{ is trivial} \Longleftrightarrow \mathrm{Hom}_{\mathrm{U}(W)}(\alpha \cdot \beta, \omega_{W,\psi_0,\mu}) \neq 0$$

and

$$\chi_0 \text{ is nontrivial} \Longleftrightarrow \mathrm{Hom}_{\mathrm{U}(W')}(\alpha' \cdot \beta', \omega_{W',\psi_0,\mu}) \neq 0.$$

This verifies [GGP, Conjecture 17.3] for this case.

10. Restriction from $\mathrm{U}(2)$ to $\mathrm{U}(1)$

In this section, we consider the restriction problem from $\mathrm{U}(2)$ to $\mathrm{U}(1)$. This problem has been studied by H. Saito [Sa2] and T. Konno [Ko], but we shall give an independent treatment here and relate the result to [GGP, Conjecture 17.3].

Recall that in §6, we have given a construction of rank 2 hermitian spaces V_B using quaternion algebras B over k_0, together with a non-degenerate rank 1 subspace:

$$L_B \hookrightarrow V_B,$$

such that

$$L_B^{\perp} = \langle 1 \rangle.$$

When B is split, this gives a pair of split hermitian spaces $L \subset V$, with

$$\mathrm{disc}(L) = -1.$$

On the other hand, if B is the quaternion division algebra D, one obtains a relevant pair $L' \subset V'$ with V' anisotropic. The groups

$$G = G(V) \times G(L) \quad \text{and} \quad G' = G(V') \times G(L')$$

are relevant pure inner forms of each other.

Suppose that M is a conjugate-symplectic 2-dimensional representation of $WD(k)$, with component group A_M, so that M determines a Vogan packet Π_M of $\mathrm{U}(V)$. In this section we will be interested in determining

$$\mathrm{Hom}_{\mathrm{U}(L_B)}(\pi_B \otimes \eta, \mathbb{C})$$

for $\pi_B \in \Pi_{M,B}$ and η the character of $\mathrm{U}(L_B)$ corresponding to N.

Since the embedding

$$\mathrm{U}(L_B) \hookrightarrow \mathrm{U}(V_B) \subset GU^+(V_B)$$

is given by the diagonal map

$$k^\times / k_0^\times \hookrightarrow (B^\times \times k^\times)/\Delta k_0^\times,$$

we see that

$$\bigoplus_B \bigoplus_{\pi_B \in \Pi_{M,B}} \mathrm{Hom}_{\mathrm{U}(L_B)}(\pi_B \otimes \eta, \mathbb{C}) = \mathrm{Hom}_{k^\times}(\tau, \chi^{-1}\eta^{-1}) + \mathrm{Hom}_{k^\times}(\tau', \chi^{-1}\eta^{-1}).$$

Now we note the following theorem of Waldspurger [Wa2], Tunnell [Tu] and Saito [Sa]:

Theorem 10.1. *Let τ be an irreducible admissible representation of $\mathrm{GL}_2(k_0)$ with L-parameter $N(\tau)$ and Jacquet-Langlands lift τ' on $D^\times$. For any character ν of $k^\times$, with $\nu|_{k_0^\times} = \omega_\tau$, we have*

$$\dim \mathrm{Hom}_{k^\times}(\tau, \nu) + \dim \mathrm{Hom}_{k^\times}(\tau', \nu) = 1.$$

Moreover,

$$\mathrm{Hom}_{k^\times}(\tau, \nu) \neq 0 \iff \epsilon(N(\tau)|_{WD(k)} \otimes \nu^{-1}, \psi) = 1,$$

where ψ is any non-trivial character of k/k_0.

Applying this theorem to the case at hand, with $\nu = \chi^{-1} \cdot \eta^{-1}$, we immediately deduce [GGP, Conjecture 17.1] (multiplicity one in L-packets). In fact, when τ is not dihedral with respect to k/k_0, this theorem also implies [GGP, Conjecture 17.3]. Indeed, in this case, $\tau \boxtimes \chi$ remains irreducible when restricted to $\mathrm{U}(V)$, so that

$$\Pi_M = \{\pi_M, \pi'_M\}.$$

Moreover, $A_M \cong A_N \cong \mathbb{Z}/2\mathbb{Z}$ and the distinguished character χ_0 of $A_M \times A_N$ satisfies

$$\chi_0(-1,1) = \chi_0(1,-1) = \epsilon(N(\tau)|_{WD(k)} \otimes \chi \cdot \eta, \psi).$$

Hence we deduce that

$$\chi_0 \text{ is trivial} \Longleftrightarrow \mathrm{Hom}_{\mathrm{U}(L)}(\pi_M \otimes \eta, \mathbb{C}) \neq 0$$

and

$$\chi_0 \text{ is nontrivial} \Longleftrightarrow \mathrm{Hom}_{\mathrm{U}(L')}(\pi'_M \otimes \eta, \mathbb{C}) \neq 0.$$

Suppose then that τ is dihedral with respect to k/k_0, so that

$$N(\tau)|_{WD(k)} = \alpha + \alpha^\sigma$$

for a character α of $k^\times$. In this case, τ is the sum of two distinct irreducible summands when restricted to $\mathrm{GL}_2(k_0)^+$ and the same holds for its Jacquet-Langlands lift τ' (if it exists). A refinement of Theorem 10.1 was obtained in the paper [P3] of the third author, as well as in [Sa2]. However, the results in the papers [P3] and [Sa2] fall slightly short of establishing [GGP, Conjecture 17.3]. The rest of this section completes the analysis of [P3] and [Sa2], thus proving [GGP, Conjecture 17.3].

When τ is dihedral with respect to k/k_0, we have

$$M = M_1 + M_2$$

with M_i conjugate-symplectic (not necessarily distinct). Using theta correspondence, we have described in §8 a construction of the packet Π_M as well as a bijection $J(\psi) : \Pi_M \leftrightarrow \mathrm{Irr}(A_M)$, depending on an additive character ψ of k/k_0. Thus, if $M_1 \neq M_2$, then each element $\pi^{\epsilon_1,\epsilon_2}$ of Π_M is specified by a pair of signs (ϵ_1, ϵ_2). Similarly, If $M_1 = M_2$, then Π_M contains two representations π^{++} and π^{--}. In either case, the representation π^{++} is the unique ψ-generic representation in Π_M.

Here is the main theorem of this section, which completes the verification of [GGP, Conjecture 17.3].

Theorem 10.2. *Suppose that $V_B = L_B \oplus L_1$ is a 2-dimensional hermitian space, where L_1 is a hermitian line with discriminant 1 and $\omega_{k/k_0}(-\mathrm{disc}(L_B)) = \epsilon(B)$. Suppose that $M = M_1 + M_2$ is an L-parameter of $\mathrm{U}(V_B)$ with M_i conjugate-symplectic, and let Π_M be its associated Vogan packet and A_M its component group. Let ψ be a non-trivial character of k/k_0, which induces a bijection $J(\psi) : \Pi_M \leftrightarrow Irr(A_M)$.*

Then for any character η of $\mathrm{U}(L_B)$,

$$\mathrm{Hom}_{\mathrm{U}(L_B)}(\pi^{\epsilon_1,\epsilon_2} \otimes \eta, \mathbb{C}) \neq 0$$

if and only if

$$\epsilon(M_1 \otimes \eta, \psi_2) = \epsilon_1 \quad \textit{and} \quad \epsilon(M_2 \otimes \eta, \psi_2) = \epsilon_2,$$

where $\psi_2(x) = \psi(2x)$.

Remark: Note that when $M_1 = M_2$, then there are no representations on the anisotropic $\mathrm{U}(V')$ to consider, and the two root numbers in question must have the same sign.

Proof. We assume that $M_1 \neq M_2$, since the case $M_1 \cong M_2$ is similar. Then we have

$$A_M = \mathbb{Z}/2\mathbb{Z}a_1 \times \mathbb{Z}/2\mathbb{Z}a_2.$$

Let us first recall the construction of the associated packet Π_M and the bijection $J(\psi)$: $\Pi_M \leftrightarrow \mathrm{Irr}(A_M)$ Setting

$$\mu = M_1 \quad \text{and} \quad \chi/\chi^\sigma = M_1 \cdot M_2,$$

the packet Π_M consists of the representations (with B, c varying):

$$\pi^{\epsilon_1,\epsilon_2} = \theta_{\psi_0,\mu,V_B,W_c}(\chi)$$

where B's are the two quaternion algebras over k_0 considered as hermitian spaces over k; W_c is the rank 1 skew-hermitian space of discriminant $c\delta$; and ψ is related to ψ_0 as everywhere else in the paper by the identity $\psi(x) = \psi_0(\delta x)$ for all trace zero elements x of k. Moreover, the bijection $J(\psi)$ is specified by:

$$(*) \qquad \epsilon_1 = \epsilon(B) \cdot \omega_{k/k_0}(c) \quad \text{and} \quad \epsilon_2 = \omega_{k/k_0}(c).$$

Now consider the seesaw diagram

$$\begin{tikzcd}
\mathrm{U}(L_B + L_1) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(W_c) \times \mathrm{U}(W_c) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(L_B) \times \mathrm{U}(L_1) & \Delta \mathrm{U}(W_c).
\end{tikzcd}$$

We start with the character χ on $\Delta\mathrm{U}(W_c)$ and the character η^{-1} on $\mathrm{U}(L_B)$, and consider the theta correspondence with respect to the additive character ψ_0. Then the seesaw identity gives

$$\mathrm{Hom}_{\mathrm{U}(L_B)}(\pi^{\epsilon_1,\epsilon_2}, \eta^{-1}) = \mathrm{Hom}_{\mathrm{U}(W_c)}(\theta_{\psi_0,\mu,W_c,L_B}(\eta^{-1}) \otimes \omega_{\psi_0,\mu,W_c}, \chi).$$

Hence,

$$\mathrm{Hom}_{\mathrm{U}(L_B)}(\pi^{\epsilon_1,\epsilon_2}, \eta^{-1}) \neq 0$$

if and only if the following two conditions hold:

(a) $$\theta_{\psi_0,\mu,W_c,L_B}(\eta^{-1}) \neq 0,$$

in which case, $\theta_{\psi_0,\mu,W_c,L_B}(\eta^{-1}) = \eta^{-1}$; and

(b) $$\mathrm{Hom}_{\mathrm{U}(W_c)}(\eta^{-1} \otimes \omega_{\psi_0,\mu,W_c}, \chi) \neq 0.$$

But both (a) and (b) are special cases of Theorem 9.1 [HKS, Corollary 8.5]. We deduce that (a) holds if and only if

$$\epsilon(\mu^{-1}\eta^{-1}, \psi_0(\mathrm{Tr}(\delta -)) = \omega_{k/k_0}(\mathrm{disc}(L_B)) \cdot \omega_{k/k_0}(c)$$

or equivalently

(c) $$\epsilon(M_1 \otimes \eta, \psi_2) = \omega_{k/k_0}(-\mathrm{disc} L_B) \cdot \omega_{k/k_0}(c) = \epsilon(B) \cdot \omega_{k/k_0}(c).$$

Similarly, (b) holds if and only if

$$\epsilon(\mu^{-1} \cdot \eta \cdot \chi/\chi^\sigma, \psi_0(\mathrm{Tr}(\delta -))) = \omega_{k/k_0}(c),$$

or equivalently

(d) $$\epsilon(M_2 \otimes \eta, \psi_2) = \omega_{k/k_0}(c).$$

In view of $(*)$, the theorem is proved. □

11. Theta correspondence for $\mathrm{U}(2) \times \mathrm{U}(2)$

Before moving on to the next case of [GGP, Conjecture 17.3], we need to establish some results about the theta correspondence for $\mathrm{U}(2) \times \mathrm{U}(2)$. More precisely, let V_B be the rank 2 hermitian space introduced in §6, and let $W_{B'}$ be the rank 2 skew-hermitian space obtained from $V_{B'}$ by scaling by the trace zero element $\delta \in k^\times$ fixed in the introduction. In this section, we will be interested in establishing the precise theta correspondence for the dual pair

$$\mathrm{U}(V_B) \times \mathrm{U}(W_{B'})$$

relative to the data (ψ_0, μ, δ).

The first result is the following proposition due to Harris [Ha, Lemma 4.3.3] and Konno-Konno [KK, Prop. 5.3 and Thm. 5.4].

Proposition 11.1. *Let M be a 2-dimensional conjugate-symplectic representation of $WD(k)$ which gives rise to a L-packet $\Pi_{M,B}$ for $\mathrm{U}(V_B)$ and $\Pi_{M,B'}$ for $\mathrm{U}(W_{B'})$.*

(i) For any $\pi \in \Pi_{M,B}$,

$$\theta_{\psi_0,V_B,W_{B'},\mu}(\pi) \neq 0 \Longleftrightarrow \epsilon(M \otimes \mu^{-2}, \psi) = \epsilon(B) \cdot \epsilon(B').$$

Note that the root number above is independent of the choice of the additive character ψ of k/k_0.

(ii) If the condition of (i) holds, then $\theta_{\psi_0,V_B,W_{B'},\mu}(\pi)$ belongs to $\Pi_{M,B'}$. In other words, the theta correspondence is the identity map on L-parameters.

Thus, under the theta correspondence for (ψ_0,μ,δ), there is a unique B' such that the theta lift gives a bijection

$$\Pi_{M,B} \longleftrightarrow \Pi_{M,B'}.$$

If the parameter M is non-dihedral (with respect to k/k_0), then $\#\Pi_{M,B} = 0$ or 1. Hence the above proposition completely determines the theta lift of the representations in Π_M. When M is dihedral with respect to k/k_0, then $\#\Pi_{M,B} = 0$ or 2, and in the latter case, there are two possible bijections

$$\Pi_{M,B} \longleftrightarrow \Pi_{M,B'},$$

which the above proposition does not resolve. In [P5], the third author has formulated a precise conjecture addressing this issue. The following theorem confirms the conjecture in [P5] for this case:

Theorem 11.2. *Suppose that $M = M_1 + M_2$ is dihedral with respect to k/k_0. Fix the additive character ψ of k/k_0 which gives bijections*

$$\Pi_M \longleftrightarrow Irr(A_M),$$

and let ψ_0 be the additive character of k_0 such that ψ is related to ψ_0 as everywhere else in the paper by the identity $\psi(x) = \psi_0(\delta \cdot x)$ for all trace zero elements of k. Then the permutation of Π_M induced by the theta correspondence associated to (ψ_0,μ,δ) is given by multiplication by the character ρ_0 of A_M defined by

$$\rho_0(a_i) = \epsilon(M_i \otimes \mu^{-2}, \psi_2)$$

with

$$\psi_2(x) = \psi(2x) = \psi_0(\mathrm{Tr}(\delta x)).$$

Proof. Consider first the case where B' is split whereas B is arbitrary. In this case, the two elements in $\Pi_{M,B'}$ can be distinguished by the Whittaker models they support. Computing Whittaker models of the Weil representation $\omega_{\psi_0,V_B,W_{B'},\mu}$, one sees that for $\pi_\rho \in \Pi_{M,B}$, the representation $\theta_{\psi_0,V_B,W_{B'},\mu}(\pi_\rho)$ of $\mathrm{U}(W_{B'})$ is ψ_0-generic if and only if

$$\mathrm{Hom}_{\mathrm{U}(L_B)}(\pi_\rho^\vee, \mu^{-2}) \neq 0.$$

By the result of the previous section, this holds if and only if

$$\rho(a_1) = \epsilon(M_1 \otimes \mu^{-2}, \psi_2) \quad \text{and} \quad \rho(a_2) = \epsilon(M_2 \otimes \mu^{-2}, \psi_2),$$

as desired. This establishes the result when one of B or B' is split.

The only remaining case is where B and B' are both non-split, so that

$$\epsilon(M_1 \otimes \mu^{-2}, \psi_2) \cdot \epsilon(M_2 \otimes \mu^{-2}, \psi_2) = 1.$$

In this case, the desired result can be proved by a global method. We give a brief sketch of this.

Let π be a representation in $\Pi_{M,B}$, so that $\theta_{\psi_0,\mu}(\pi)$ also belongs to $\Pi_{M,B}$. We have:

Proposition 11.3. *Using the above notations, one can find:*

(1) *a totally real number field F of* odd *degree over $\mathbb{Q}$ and such that $F_{v_0} = k_0$ for some finite place v_0 of F;*

(2) *an additive character Ψ of $\mathbb{A}_F/F$ such that $\Psi_{v_0} = \psi_0$;*

(3) *a totally imaginary quadratic extension E of F such that $E_{v_0} \cong k$;*

(4) *a trace zero element $\Delta \in E$ such that $\Delta_v = \delta$ up to $\mathbb{N}k^\times$;*

(5) *an idele class character Σ of $\mathbb{A}_E{}^\times$ such that $\Sigma_{v_0} = \mu$ and $\Sigma|_{\mathbb{A}_F^\times} = \omega_{E/F}$;*

(6) *a quaternion algebra $\mathbb{B}$ over F ramified precisely at v_0 and all the infinite places, so that $\mathbb{B}_{v_0} = B$; this gives a hermitian space $V_{\mathbb{B}}$ over F which is isomorphic to V_B over F_{v_0};*

(7) *a cuspidal representation Π of $\mathrm{U}(V_{\mathbb{B}})$ such that*

(a) $\Pi_{v_0} = \pi$;

(b) Π belongs to a global endoscopic packet (i.e. the base change of Π to E is non-cuspidal);

(c) $L(BC_{E/F}(\Pi) \otimes \Sigma^{-2}, 1/2) \neq 0$

Proof. One can certainly find the number fields F and E satisfying (1) and (3) (see Lemma 15.3 below), after which one can find Ψ as in (2), Δ as in (4), Σ as in (5) and $\mathbb{B}$ as in (6). With these objects fixed, we need to find a cuspidal representation Σ as in (7). Clearly, there is no difficulty in find Π satisfying (7a) and (7b). The main difficulty is to find Π which satisfies (7c) as well.

Recall that the representation π is a summand in the restriction of a representation $\tau \boxtimes \chi$ of $(B^\times \times k^\times)/\Delta k_0^\times$, so that $\omega_\tau \cdot \chi|_{k_0^\times} = 1$ and the L-parameter of π is the L-parameter of the representation $BC(\tau) \otimes \chi$ of $B \otimes_{k_0} k \cong \mathrm{GL}_2(k)$. The fact that π is dihedral means that τ is dihedral, so that $BC(\tau) = \alpha \oplus \alpha^\sigma$ for some character α of $k^\times$, so that

$$M = M_1 + M_2 = \alpha\chi + \alpha^\sigma\chi.$$

Before commencing the construction of Π, we recall that we are assuming that

$$\epsilon(BC(\tau) \otimes \chi\mu^{-2}) = \epsilon(M_1 \otimes \mu^{-2}, \psi_2) \cdot \epsilon(M_2 \otimes \mu^{-2}, \psi_2) = 1.$$

By Tunnell-Saito [Tu, Sa], this condition implies that

$$\mathrm{Hom}_{k^\times}(\tau, \chi^{-1} \cdot \mu^2) = 0,$$

and if $JL(\tau)$ is the Jacquet-Langlands lift of τ to $\mathrm{GL}_2(k_0)$, then

$$\mathrm{Hom}_{k^\times}(JL(\tau), \chi^{-1} \cdot \mu^2) \neq 0.$$

By globalizing the character α of $k^\times$, one can find a dihedral cuspidal representation $\mathfrak{T}$ of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\mathfrak{T}_{v_0} = JL(\tau)$. Then using [P6, Lemma 1], one can find a character $\mathfrak{C}$ of $\mathbb{A}_E^\times$ such that $\mathfrak{C}_{v_0} = \chi$ and such that $\mathfrak{T}$ is globally distinguished by $\mathfrak{C} \cdot \Sigma^2$; necessarily we have $\omega_{\mathfrak{T}} \cdot \mathfrak{C}|_{\mathbb{A}_F^\times} = 1$. Then by Waldspurger [Wa3], one concludes that

$$L(BC(\mathfrak{T}) \otimes \mathfrak{C}\Sigma^{-2}, 1/2) \neq 0.$$

Now let

$$\Pi = JL_{\mathbb{B}}(\mathfrak{T}) \boxtimes \mathfrak{C}^{-1} \quad \text{on } \mathrm{U}(V_{\mathbb{B}}),$$

so that

$$L(BC(\Pi) \otimes \Sigma^{-2}, 1/2) = L(BC(\mathfrak{T}) \otimes \mathfrak{C}\Sigma^{-2}, 1/2) \neq 0.$$

This completes the construction of Π. □

Using the Π constructed in the proposition, we have:

$$\epsilon(BC(\Pi) \otimes \Sigma^{-2}, 1/2) = 1.$$

In particular, the set

$$S = \{v : \epsilon(BC_{E_v/F_v}(\Pi_v) \otimes \Sigma_v^{-2}, \Psi_{v,2}) = -1\}$$

has even cardinality and does not contain the place v_0. Let $\mathbb{B}'$ be the quaternion algebra over F such that

$$\epsilon(\mathbb{B}'_v) \neq \epsilon(\mathbb{B}_v) \Longleftrightarrow v \in S.$$

In other words, $\mathbb{B}'$ is obtained from $\mathbb{B}$ by switching the local invariants of $\mathbb{B}$ at the set S. Since $v_0 \notin S$, we have

$$\mathbb{B}'_{v_0} \cong B.$$

Moreover, by Proposition 11.1, for each place v of F,

$$\Theta_{\Psi_v, \Sigma_v, V_{\mathbb{B}_v}, W_{\mathbb{B}'_v}}(\Pi_v) \neq 0.$$

By [Ha], the nonvanishing of the central L-value above implies that the global theta lift is nonvanishing as well:

$$\Theta_{\Psi, \Sigma, \Delta, V_{\mathbb{B}}, W_{\mathbb{B}'}}(\Pi) \neq 0.$$

Now the assertion of the theorem has been checked for all finite places of F outside v_0, since at least one of $\mathbb{B}_v$ or $\mathbb{B}'_v$ is split at any $v \neq v_0$. At the archimedean places, the groups $\mathrm{U}(V_{\mathbb{B}} \otimes F_v)$ are compact and the theta correspondence over $\mathbb{R}$ involving compact groups is completely known (c.f. [Pa] or [HLS] for example). Using this, one can verify the analog of the assertion of the theorem over $\mathbb{R}$ (cf. [P5]); we omit the details here.

Thus the assertion of the theorem is true for all places of F over v_0. If the result of the theorem is not true at the place v_0, we would have a cuspidal representation

$\Theta_{\Psi,\Sigma,V_{\mathbb{B}},W_{\mathbb{B}'}}(\Pi)$ of $\mathrm{U}(W_{\mathbb{B}'})$ which violates the Labesse-Langlands multiplicity formula for global endoscopic packets of U(2). This gives the desired contradiction.

For example, suppose that S is empty so that $\mathbb{B} = \mathbb{B}'$. Then if the result of the theorem holds at all $v \neq v_0$ but fails at v_0, the cuspidal representation $\Theta_{\Psi,\Sigma,V_{\mathbb{B}},W_{\mathbb{B}'}}(\Pi)$ of $\mathrm{U}(W_{\mathbb{B}'})$ would differ from the cuspidal representation Π at an odd number of places v. This is a contradiction. □

12. Trilinear forms for U(2)

In this section, we return to the skew-hermitian case of [GGP, Conjecture 17.3]. In particular, we consider the case when

$$W_0 = W \quad \text{with } \dim W_0 = \dim W = 2.$$

Thus, let $W_B = W_{B,\delta}$ be the rank 2 skew-hermitian case obtained from V_B by scaling by δ. Fix an additive character ψ_0 of k_0, and a character μ of $k^\times$ so that

$$\mu|_{k_0^\times} = \omega_{k/k_0}.$$

This determines the Weil representation $\omega_{\psi_0,\mu}$ for $\mathrm{U}(W_B)$. Given two conjugate-symplectic representations M and N of $WD(k)$ of dimension 2, with corresponding Vogan packet Π_M and Π_N of $\mathrm{U}(W_B)$, we are interested in computing

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}}, \mathbb{C})$$

as π_M and π_N vary over all representations in Π_M and Π_N.

Note that the representation $\omega_{\psi_0,\mu}$ is not an irreducible representation of $\mathrm{U}(W_B)$. However, we may decompose $\omega_{\psi_0,\mu}$ according to central characters

$$\omega_{\psi_0,\mu} = \bigoplus_{\chi} \omega_{\psi_0,\mu}[\chi]$$

as χ runs over characters of $Z_{\mathrm{U}(W_B)} \cong k^\times/k_0^\times$. In fact, this decomposition is simply the decomposition of the Weil representation for the dual pair $\mathrm{U}(V_1) \times \mathrm{U}(W_B)$ where V_1 is the one dimensional hermitian space of discriminant 1. Thus, each summand $\omega_{\psi_0,\mu}[\chi]$ is an irreducible representation of $\mathrm{U}(W_B)$. Moreover, it belongs to an endoscopic packet of $\mathrm{U}(W_B)$ constructed in Proposition 8.1.

Now, because of central character reasons, it is clear that

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}[\chi]}, \mathbb{C}) = 0$$

unless

$$\det M \cdot \det N = \chi.$$

For this χ, we have

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}[\chi]}, \mathbb{C}).$$

In particular, [GGP, Conjecture 17.3] amounts to a question about invariant trilinear forms on $\mathrm{U}(W_B)$.

Given that the group $\mathrm{U}(W_B)$ can be described in terms of $\mathrm{GL}_2(k_0)$ and its inner form, we shall see that this question can be related to a question about invariant trilinear forms for GL_2 which has been addressed in a series of papers by the third author [P1,2,6]; we recall his result here:

Theorem 12.1. *Let N_1, N_2 and N_3 be 2-dimensional representations of $WD(k_0)$, with associated representations $\pi_{i,B}$ of $B^\times$. Assume that* $\det N_1 \cdot \det N_2 \cdot \det N_3 = 1$*. Then*

$$\sum_B \dim \mathrm{Hom}_{B^\times}(\pi_{1,B} \otimes \pi_{2,B} \otimes \pi_{3,B}, \mathbb{C}) = 1.$$

Moreover,

$$\mathrm{Hom}_{B^\times}(\pi_{1,B} \otimes \pi_{2,B} \otimes \pi_{3,B}, \mathbb{C}) \neq 0 \Longleftrightarrow \epsilon(N_1 \otimes N_2 \otimes N_3) = \epsilon(B).$$

To apply this theorem to the case of $\mathrm{U}(W_B)$, we need to consider the group $(B^\times)^+$ and calculate

$$\dim \mathrm{Hom}_{(B^\times)^+}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}).$$

More generally, let G be a subgroup of $\mathrm{GL}_2(k_0)$ containing $\mathrm{SL}_2(k_0)$. The group G is uniquely determined by the subgroup

$$k_G^\times \subset k_0^\times$$

consisting of determinant of elements of G. Thus, for any quaternion algebra B, it makes sense to define a corresponding subgroup G_B inside $B^\times$ containing $\mathrm{SL}_1(B)$. Restricting representations of $B^\times$ to G_B, one gets a notion of L-packet of representations of G_B. It is known that representations of $\mathrm{GL}_2(k_0)$ restrict to G with multiplicity 1, but this need not be the case for representations of $B^\times$ if B is non-split. For a representation π_B of G_B, let $m(\pi_B)$ denote the multiplicity with which it appears in the restriction of an irreducible representation of $B^\times$.

Now we have:

Theorem 12.2. *For $i = 1$, 2 and 3, let N_i be a 2-dimensional representation of $WD(k_0)$ with associated representation $\tilde{\pi}_{B,i}$ of $B^\times$. Assume that* $\prod_i \det N_i = 1$*. Then*

$$\sum_B \dim \mathrm{Hom}_{G_B}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}) = \#(k_0^\times / k_0^{\times 2} k_G^\times).$$

In particular,

$$\sum_B \sum_{\pi_{B,1},\pi_{B,2},\pi_{B,3}} m(\pi_{B,1}) \cdot m(\pi_{B,2}) \cdot m(\pi_{B,3}) \cdot \dim \mathrm{Hom}_{G_B}(\pi_{B,1} \otimes \pi_{B,2} \otimes \pi_{B,3}, \mathbb{C})$$

is equal to

$$\#(k_0^\times / k_0^{\times 2} k_G^\times),$$

where the inner sum is taken over irreducible representations $\pi_{B,i}$ of G_B which are contained in the representations $\tilde{\pi}_{B,i}$ of $B^\times$.

Proof. Clearly,

$$\mathrm{Hom}_{G_B}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}) \cong \sum_{\chi: k_0^\times / k_G^\times \to \mathbb{Z}/2} \mathrm{Hom}_{B^\times}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}_\chi),$$

where the χ's range over characters of $B^\times$ trivial on G_B identified to characters of $k_0^\times / k_G^\times$ with values in $\mathbb{Z}/2$, and $\mathbb{C}_\chi$ denotes the 1-dimensional representation $\chi \circ \mathbb{N}_B$ of $B^\times$. By Theorem 12.1, we have

$$\sum_B \dim \mathrm{Hom}_{B^\times}(\tilde{\pi}_{B,1} \otimes \tilde{\pi}_{B,2} \otimes \tilde{\pi}_{B,3}, \mathbb{C}_\chi) = 1,$$

for all characters χ of order ≤ 2 (by absorbing χ in one of the $\tilde{\pi}_{B,i}$'s without affecting the central character). Adding up the contribution of the various χ's, we get the conclusion of the theorem. □

Specializing this theorem to the case $G_B = (B^\times)^+$ and noting that, in this case, $m(\pi_{B,i}) = 1$ for each B, we obtain:

Corollary 12.3. *In the context of Theorem 12.2, let $G = \mathrm{GL}_2(k_0)^+$. Then one has,*

$$\sum_B \sum_{\pi_{B,1}, \pi_{B,2}, \pi_{B,3}} \dim \mathrm{Hom}_{G_B}(\pi_{B,1} \otimes \pi_{B,2} \otimes \pi_{B,3}, \mathbb{C}) = 2,$$

where the inner sum is taken over irreducible representations $\pi_{B,i}$ of G_B which are contained in the representations $\tilde{\pi}_{B,i}$ of $B^\times$.

We can now apply the corollary to the group $\mathrm{GU}^+(W_B)$ or equivalently $\mathrm{U}(W_B)$.

Corollary 12.4. *Let M_i be conjugate-symplectic representations of $WD(k)$ with associated L-packet $\Pi_{M_i,B}$ of $\mathrm{U}(W_B)$. Assume that* $\det M_1 \cdot \det M_2 \cdot \det M_3 = 1$. *Then*

(i)

$$\sum_B \sum_{\pi_i \in \Pi_{M_i,B}} \dim \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) = 2.$$

(ii) If one of the M_i's, say M_1, is dihedral with respect to k/k_0, so that $\#\Pi_{M_1,B_0} = 2$ for B_0 split, then

$$\dim \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \leq 1$$

for each B. If the above Hom space is nonzero, then

$$\dim \mathrm{Hom}_{\mathrm{U}(W_{B'})}(\pi_1' \otimes \pi_2' \otimes \pi_3', \mathbb{C}) = 0$$

for $B' \neq B$.

Proof. The first assertion follows immediately from the previous corollary and the definition of L-packets for $\mathrm{U}(W)$ given in §6. To deduce the last assertion, note that if

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) \neq 0,$$

then we also have

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1^c \otimes \pi_2^c \otimes \pi_3^c, \mathbb{C}) \neq 0,$$

where π_i^c denotes the conjugate of π_i by an element $c \in \mathrm{GU}(W_B) \smallsetminus \mathrm{GU}^+(W_B)$. Since

$$\dim \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}) + \dim \mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_1^c \otimes \pi_2^c \otimes \pi_3^c, \mathbb{C}) \leq 2,$$

each of these dimensions must be equal to 1, and all other Hom spaces must be 0. □

Remark 12.5. Since $k_0^\times / k_0^{\times 2}$ is a 2-group whose cardinality can be made arbitrarily large by choosing k_0 appropriately, and since the L-packet of representations of $\mathrm{SL}_2(k)$ is bounded by 4 [LL], it follows from Theorem 12.2 that

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(k)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C})$$

can be made arbitrarily large.

Now we can return to [GGP, Conjecture 17.3], so that M and N are two 2-dimensional conjugate-symplectic representations of $WD(k)$ which determine Vogan packets Π_M and Π_N of $\mathrm{U}(W_B)$. For a fixed additive character ψ_0 of k_0, we have obtained a bijection

$$J(\psi_0) : \Pi_M \longleftrightarrow \mathrm{Irr}(A_M)$$

and similarly for Π_N. We are interested in computing

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_M \otimes \pi_N \otimes \overline{\omega_{\psi_0,\mu}})$$

for $\pi_M \in \Pi_M$ and $\pi_N \in \Pi_N$.

If M and N are non-dihedral (with respect to k/k_0), so that Π_M and Π_N both contain at most one representation of each $\mathrm{U}(W_B)$ (as B varies), then [GGP, Conjecture 17.3] is a consequence of Theorem 12.1. Indeed, we have

$$A_M \times A_N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

and the distinguished character χ_0 satisfies

$$\chi_0(-1, 1) = \chi_0(1, -1) = \epsilon(M \otimes N(\mu^{-1}), \psi)$$

for any character ψ of k/k_0. On the other hand, if Π_M is obtained by the restriction of the representation $\tau_M \boxtimes \chi_M$ of $\mathrm{GU}(W_B)$ and Π_N is obtained from $\tau_N \boxtimes \chi_N$, then the epsilon factor occurring in Theorem 12.1 is

$$\begin{aligned} \epsilon(\rho_{\tau_M} \otimes \rho_{\tau_N} \otimes \mathrm{Ind}(\mu^{-1}\chi_M\chi_N), \psi_0) &= \epsilon(\rho_{\tau_M}|_{WD(k)} \otimes \rho_{\tau_N}|_{WD(k)} \otimes \mu^{-1} \cdot \chi_M \cdot \chi_N, \psi_0(\mathrm{Tr})) \\ &= \epsilon(M \otimes N(\mu^{-1}), \psi_0(\mathrm{Tr})) \\ &= \epsilon(M \otimes N(\mu^{-1}), \psi). \end{aligned}$$

This verifies [GGP, Conjecture 17.3] in this case.

When at least one of M or N is dihedral with respect to k/k_0, we may appeal to the theta correspondence. Since the case when exactly one of them is dihedral with respect to k/k_0 is similar and easier, we shall give the details only when both M and N are dihedral with respect to k/k_0 Thus, let

$$M = M_1 + M_2 \quad \text{and} \quad N = N_1 + N_2,$$

with M_i and N_i conjugate-symplectic (not necessarily distinct), and write their component groups as

$$A_M = \mathbb{Z}/2\mathbb{Z}e_1 \times \mathbb{Z}/2\mathbb{Z}e_2 \quad \text{and} \quad A_N = \mathbb{Z}/2\mathbb{Z}f_1 \times \mathbb{Z}/2\mathbb{Z}f_2.$$

In this case, the packet Π_M can be obtained by theta correspondence from $\mathrm{U}(1)$. Set

$$\nu = M_1$$

and

$$M_1 \cdot M_2 = \eta/\eta^\sigma,$$

for some character η of $k^\times$. If L_a denote the rank 1 hermitian space with discriminant a, then

$$\Pi_M = \{\theta_{\psi_0,\nu,W_B,L_a}(\eta|_{\mathrm{U}(L_a)}) : \quad a \in k_0^\times/\mathbb{N}k^\times,\, \epsilon(B) = \pm 1\}.$$

Relative to the additive character ψ of k/k_0, we have the labelling

$$\pi_{\rho_M} = \theta_{\psi_0,\nu,W_B,L_a}(\eta|_{\mathrm{U}(L_a)})$$

if and only if

$$\rho_M(e_1) = \epsilon(B) \cdot \omega_{k/k_0}(a) \quad \text{and} \quad \rho_M(e_2) = \omega_{k/k_0}(a).$$

Similarly, a representation in Π_N has the form π_{ρ_N}, so that

$$\rho_N(f_1) \cdot \rho_N(f_2) = \epsilon(B).$$

Now consider the seesaw diagram

$$\begin{tikzcd}
\mathrm{U}(L_a + L_{-1}) \arrow[d, dash] \arrow[dr, dash] & \mathrm{U}(W_B) \times \mathrm{U}(W_B) \arrow[d, dash] \arrow[dl, dash] \\
\mathrm{U}(L_a) \times \mathrm{U}(L_{-1}) & \Delta\mathrm{U}(W_B)
\end{tikzcd}$$

and note that the rank 2 hermitian space $L_a + L_{-1}$ is isomorphic to $V_{B'}$ with $\epsilon(B') = \omega_{k/k_0}(a)$. We start with the representation $\eta|_{\mathrm{U}(L_a)}$ of $\mathrm{U}(L_a)$, so that the representation we obtain on $\mathrm{U}(W_B)$ is precisely

$$\pi_{\rho_M} = \theta_{\psi_0,\nu,W_B,L_a}(\eta|_{\mathrm{U}(L_a)}).$$

On the other side of the seesaw, we start with the representation $\mu \cdot \nu \cdot \pi_{\rho_N}^\vee$ of $\mathrm{U}(W_B)$. Note that taking contragredient has the following effect on the Vogan parameterization: for any character ρ_N of A_N, the representation $\pi_{\rho_N}^\vee$ has Vogan parameter

$$(N^\vee, \rho_N \cdot \beta_0)$$

where β_0 is the character of $A_{N^\vee} = A_N$ given by

$$\beta_0(b_i) = \omega_{k/k_0}(-1).$$

Now the seesaw identity gives:

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_{\rho_M} \otimes \omega_{\psi_0^{-1},\nu,W_B}, \mu \cdot \nu \cdot \pi_{\rho_N}^\vee) = \mathrm{Hom}_{\mathrm{U}(L_a)}(\Theta_{\psi_0,\nu^2,W_B,L_a+L_{-1}}(\mu\nu\pi_{\rho_N}^\vee), \eta|_{\mathrm{U}(L_a)}).$$

Since

$$\omega_{\psi_0^{-1},\nu,W_B} = \omega_{\psi_0^{-1},\mu^{-1},W_B} \otimes \mu\nu = \overline{\omega_{\psi_0,\mu}} \otimes \mu\nu,$$

we see that the LHS of this identity is equal to the desired space

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N} \otimes \overline{\omega_{\psi_0,\mu}}, \mathbb{C}).$$

On the other hand, the RHS is nonzero if and only if conditions (a) and (b) below are satisfied:

(a) $\Theta_{\psi_0,\nu^2,W_B,L_a+L_{-1}}(\mu \cdot \nu \cdot (\pi_{\rho_N})^\vee) \neq 0$. According to Theorem 11.2, this holds if and only if

$$\epsilon(N^\vee \otimes \mu\nu\nu^{-2}, \psi) = \epsilon(B) \cdot \omega_{k/k_0}(a),$$

or equivalently

$$\epsilon(N \otimes M_1(\mu^{-1}), \psi) = \epsilon(B) \cdot \omega_{k/k_0}(a) = \rho_M(e_1).$$

If this is satisfied, then by Theorem 11.2, the theta lift is equal to the representation

$$\pi_{\rho_{N^\vee}} \cdot \mu\nu$$

of $\mathrm{U}(L_a + L_{-1})$, with

$$\rho_{N^\vee}(f_i) = \rho_N(f_i) \cdot \omega_{k/k_0}(-1) \cdot \epsilon(N_i \otimes M_1(\mu^{-1}), \psi_{-2}).$$

(b) $\mathrm{Hom}_{\mathrm{U}(L_a)}(\pi_{\rho_{N^\vee}}\mu\nu, \eta/\eta^\sigma) \neq 0$. This is a branching problem for $\mathrm{U}(2)\times\mathrm{U}(1)$ which we have resolved in §10. Using the results there, we see that the desired nonvanishing holds if and only if

$$\rho_{N^\vee}(f_i) = \rho_N(f_i) \cdot \omega_{k/k_0}(-1) \cdot \epsilon(N_i \otimes M_1(\mu^{-1}), \psi_{-2}) = \epsilon(N_i \otimes M_2(\mu^{-1}), \psi_2)$$

or equivalently

$$\rho_N(f_i) = \epsilon(N_i \otimes M(\mu^{-1}), \psi_2) = \epsilon(N_i \otimes M(\mu^{-1}), \psi).$$

Finally, since

$$\rho_M(-1) = \rho_N(-1) = \epsilon(B),$$

we conclude that

$$\rho_M(e_2) = \epsilon(N \otimes M_2(\mu^{-1}), \psi).$$

Thus we conclude that

$$\mathrm{Hom}_{\mathrm{U}(W_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N} \otimes \overline{\omega_{\psi_0,\mu}}, \mathbb{C}) \neq 0$$

if and only if $\rho_M \times \rho_N$ is the distinguished character χ_0 of [GGP, Conjecture 17.3].

13. Restriction from U(3) to U(2): endoscopic case

In this section, we consider the restriction problem for $\mathrm{U}(3) \times \mathrm{U}(2)$. Using theta correspondence, we establish [GGP, Conjecture 17.3] for endoscopic packets of U(3). In the following section, we shall consider the stable packets of U(3).

We fix a pair

$$V_0 \subset V$$

of split hermitian spaces of dimensions 2 and 3 respectively with V/V_0 of discriminant 1. Let $V_0' \subset V'$ be the other pair of hermitian spaces of dimensions 2 and 3, such that $V/V_0 \cong V'/V_0'$.

More concretely, for each quaternion algebra B over k_0, we have a rank 2 hermitian space V_B. Then the rank 3 hermitian space

$$V_{B,b} = V_B + L_b$$

has discriminant satisfying

$$\omega_{k/k_0}(\mathrm{disc}(V_{B,b})) = \epsilon(B) \cdot \omega_{k/k_0}(b).$$

If we take $b = 1$, then as B varies, the pair

$$V_B \subset V_{B,1}$$

gives the pairs $V_0 \subset V$ and $V_0' \subset V'$.

Suppose first that N is a 2-dimensional conjugate-symplectic representation of $WD(k)$ with associated Vogan packet Π_N of $\mathrm{U}(V_B)$. If $N = \oplus_i N_i$, then we write

$$A_N = \prod_i A_{N_i} = \prod_i \mathbb{Z}/2\mathbb{Z} f_i.$$

For the fixed additive character ψ of k/k_0, we translate ψ by $-2 \cdot \mathrm{disc}(V) = -2$ and use the resulting character ψ_{-2} to fix the Vogan parameterization

$$J(\psi_{-2}) : \Pi_N \longleftrightarrow \mathrm{Irr}(A_N).$$

Now consider a 3-dimensional conjugate-orthogonal representation

$$M = M_1 + M_2$$

with $\dim M_i = i$ and such that each M_i is conjugate-orthogonal. Unless, $M \cong 3M_1$, we may further assume that M_1 does not occur in M_2. We shall assume that this is the case (i.e., $M \ncong 3M_1$), since the other case is similarly handled. Then

$$A_M = A_{M_1} \times A_{M_2}$$

and we write:

$$A_{M_1} = \mathbb{Z}/2\mathbb{Z}e \quad \text{and} \quad A_{M_2} = \prod_i \mathbb{Z}/2\mathbb{Z}e_i$$

if $M_2 = \oplus_i M_{2,i}$.

Moreover, we shall assume that the conjugate-orthogonal character M_1 has a conjugate-symplectic square root. This can be achieved by twisting M, and since this twist can be absorbed into N for the purpose of the restriction problem, there is no loss of generality in making this assumption on M_1. Under this assumption on M_1, we have described in §8 a construction of the Vogan packet Π_M as well as a bijection

$$\Pi_M \longleftrightarrow \mathrm{Irr}(A_M)$$

which is canonical in this case (i.e. independent of the additive character). To recall the construction briefly, we set

$$M_1 = \mu^2$$

for some conjugate-symplectic character μ and set

$$N' = M_2 \cdot \mu,$$

so that N' is conjugate-symplectic and $A_{N'} = A_{M_2}$. Then, for quaternion algebras B and B' over k_0, one considers the theta correspondence for

$$\mathrm{U}(W_{B'}) \times \mathrm{U}(V_{B,1})$$

relative to the data $(\psi_{0,-2}, \mu, \delta)$, where ψ_0 is our fixed additive character of k_0. The packet Π_M is then the theta lift of the packet $\Pi_{N'}$ of $\mathrm{U}(W_{B'})$. For the labelling of the representations in Π_M by $\mathrm{Irr}(A_M)$, we refer the reader to the end of §8.

Now we would like to determine

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\pi_M \otimes \pi_N, \mathbb{C}),$$

for $\pi_M \in \Pi_M$ and $\pi_N \in \Pi_N$. We examine this restriction problem using the seesaw diagram

$$\begin{tikzcd}
\mathrm{U}(V_B + L_1) \arrow[d, no head] \arrow[dr, no head] & \mathrm{U}(W_{B'}) \times \mathrm{U}(W_{B'}) \arrow[d, no head] \arrow[dl, no head] \\
\mathrm{U}(V_B) \times \mathrm{U}(L_1) & \mathrm{U}(W_{B'}).
\end{tikzcd}$$

On $\mathrm{U}(W_{B'})$, we start with a representation $\pi^{\eta}_{N'} \in \Pi_{N'}$ indexed by a character η of $A_{N'}$, so that

$$\eta(-1) = \epsilon(B').$$

On $\mathrm{U}(V_B)$, we start with a representation $(\pi_{\rho_N})^{\vee}$ associated to a character ρ_N of A_N, so that

$$\rho_N(-1) = \epsilon(B).$$

Then we have the seesaw identity:

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\Theta_{\psi_{0,-2},\mu}(\pi^{\eta}_{N'}) \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(W_{B'})}(\Theta_{\psi_{0,-2},\mu^2,V_B,W_{B'}}(\pi^{\vee}_{\rho_N}) \otimes \omega_{\psi_{0,-2},\mu,L_1,W_{B'}}, \pi^{\eta}_{N'}).$$

Further, for the representations we have at hand, one can easily check that the two big theta lifts in the see-saw identity are equal to their respective small theta lifts.

Now note that

$$\pi_{\rho_M} = \theta_{\psi_{0,-2},\mu}(\pi^{\eta}_{N'})$$

with

$$\rho_M|_{A_{N'}} = \eta \quad \text{and} \quad \rho_M(e) = \epsilon(B') \cdot \eta(-1) = \epsilon(B) \cdot \epsilon(B').$$

Moreover, $(\pi_{\rho_N})^{\vee}$ has Vogan parameter (relative to $J(\psi_{0,-2})$)

$$(N^{\vee}, \rho_{N^{\vee}}) = (N^{\vee}, \rho_N \cdot \beta_0)$$

with

$$\beta_0(f_i) = \omega_{k/k_0}(-1).$$

Then the seesaw identity reads:

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(W_{B'})}(\theta_{\psi_{0,-2},\mu^2,V_B,W_{B'}}(\pi_{\rho_{N^{\vee}}}) \otimes \omega_{\psi_{0,-2},\mu,W_{B'}}, \pi^{\eta}_{N'}).$$

The RHS is nonzero if and only if (i) and (ii) below hold.

(i) $\theta_{\psi_{0,-2},\mu^2,V_B,W_{B'}}(\pi_{\rho_{N^{\vee}}}) \neq 0$. By proposition 11.1, this holds if and only if

$$\epsilon(N^{\vee}\mu^{-2}, \psi_{-2}) = \epsilon(B) \cdot \epsilon(B') = \rho_M(e),$$

or equivalently

$$\epsilon(N \otimes M_1, \psi) = \rho_M(e).$$

Moreover, by Theorem 11.2, when this holds, we have

$$\theta_{\psi_0,\mu^2,V_B,W_{B'}}((\pi_{\rho_{N^{\vee}}}) = \pi_{\rho_{N^{\vee}} \cdot \rho_0}$$

where ρ_0 is the character of $A_{N^\vee} = A_N$ given by

$$\rho_0(f_i) = \epsilon(N_i^\vee \mu^{-2}, \psi_{-1}) = \epsilon(N_i \otimes M_1, \psi).$$

(ii) $\mathrm{Hom}_{\mathrm{U}(W_{B'})}(\pi_{\rho_{N^\vee}\cdot\rho_0} \otimes \omega_{\psi_{0,-2},\mu,W_{B'}}, \pi^{\eta}_{N'}) \neq 0$. This question was addressed in the previous section, and we deduce that the desired nonvanishing holds if and only if the character

$$(\rho_N \cdot \rho_0,\ \eta) \in \mathrm{Irr}(A_N) \times \mathrm{Irr}(A_{N'})$$

is the distinguished character χ_0 in [GGP, Conjecture 17.3] for the skew-hermitian case for $(W_{B'}, \mu)$. More precisely, the desired nonvanishing holds if and only if

$$\rho_N(f_i) \cdot \epsilon(N_i \otimes M_1, \psi) = \epsilon(N_i^\vee \otimes (N')^\vee(\mu), \psi_{-1}) = \epsilon(N_i \otimes M_2, \psi),$$

so that

$$\rho_N(f_i) = \epsilon(N_i \otimes M, \psi),$$

and

$$\eta(e_i) = \epsilon((N_i')^\vee \otimes N^\vee(\mu), \psi_{-1}) = \epsilon(M_{2,i} \otimes N, \psi).$$

This shows that

$$\mathrm{Hom}_{\mathrm{U}(V_B)}(\pi_{\rho_M} \otimes \pi_{\rho_N}, \mathbb{C}) \neq 0$$

if and only if the character $\rho_M \times \rho_N$ is the distinguished character χ_0 of [GGP, Conjecture 17.3], computed using the additive character ψ of k/k_0.

14. Restriction from U(3) to U(2): stable case

We now consider the restriction problem for stable Vogan packets of U(3). We preserve the notation of the previous sections. In particular, we have the pairs of spaces $V_0 \subset V$ and $V_0' \subset V'$, with $\dim V = \dim V' = 3$, $\dim V_0 = \dim V_0' = 2$, with V_0 the split hermitian space, and $\mathrm{disc}(V/V_0) = \mathrm{disc}(V'/V_0') = 1$. We will use the additive character ψ_{-2} to normalize the Vogan parameterization for $\mathrm{U}(V_0)$.

Let M be an irreducible 3-dimensional conjugate-orthogonal representation of $WD(k)$, so that its associated Vogan packet has the form

$$\Pi_M = \{\pi_M, \pi_M'\},$$

where π_M is a representation of $\mathrm{U}(V)$ and π_M' is the same representation considered on $\mathrm{U}(V')$. If M is an irreducible representation of the Weil group $W(k)$, then the representation π_M is supercuspidal. Otherwise,

$$M = \mu \boxtimes St_3$$

where μ is a conjugate-orthogonal character of $W(k)$ and St_3 denotes the irreducible 3-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. In this case, the representation π_M is a twisted Steinberg representation

$$\pi_M = St \otimes (\mu \circ \det).$$

On the other hand, let N be an arbitrary 2-dimensional conjugate-symplectic representation of $WD(k)$ with associated Vogan packet Π_N of $\mathrm{U}(V_0)$. We would like to determine

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M \otimes \pi_N, \mathbb{C})$$

for $\pi_M \in \Pi_M$ and $\pi_N \in \Pi_N$. We shall reduce this question to the case when Π_M and Π_N are both supercuspidal packets, by first treating the other cases directly. The supercuspidal case will then be handled by a global method in the next two sections.

We first consider the case when $M = \mu \boxtimes St_3$. Since we can absorb the twist by μ into the parameter N, we may assume without loss of generality that $\mu = 1$. In this case, $\pi_M = St_{\mathrm{U}(V)}$ is a quotient of a (un-normalized) principal series representation:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ind}_{B_V}^{\mathrm{U}(V)}(1) \longrightarrow St_{\mathrm{U}(V)} \longrightarrow 0,$$

where B_V denotes a Borel subgroup in $\mathrm{U}(V)$. We now have the following proposition.

Proposition 14.1. *(i) If N is not the parameter of the Steinberg representation of $\mathrm{U}(V_0)$, we have*

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(St_{\mathrm{U}(V)} \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(V_0)}(\left[\mathrm{Ind}_{B_V}^{\mathrm{U}(V)}(1)\right] \otimes \pi_{\rho_N}, \mathbb{C}) = \mathrm{Hom}_{\mathrm{U}(L)}(\pi_{\rho_N}, \mathbb{C}).$$

In particular, $\mathrm{Hom}_{\mathrm{U}(V_0)}(St_{\mathrm{U}(V)} \otimes \pi_{\rho_N}, \mathbb{C}) \neq 0$ *if and only if*

$$\rho_N(f_i) = \epsilon(N_i, \psi) = \epsilon(N_i \otimes M, \psi).$$

(ii) If N is the parameter of the Steinberg representation of $\mathrm{U}(V_0)$, so that $\Pi_N = \{St_{\mathrm{U}(V_0)}, 1_{\mathrm{U}(V_0')}\}$, we have

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(St_{\mathrm{U}(V)} \otimes St_{\mathrm{U}(V_0)}, \mathbb{C}) \neq 0.$$

On the other hand,

$$\mathrm{Hom}_{\mathrm{U}(V_0')}(St_{\mathrm{U}(V')}, \mathbb{C}) = 0.$$

Proof. (i) Part (i) is proved by a standard application of Mackey theory, which reduces the restriction problem for $\mathrm{U}(V) \times \mathrm{U}(V_0)$ to one for $\mathrm{U}(V_0) \times \mathrm{U}(L)$. Indeed, it is a special case of [GGP, Theorem 15.1], and so we omit its proof here.

(ii) The case of $\mathrm{U}(V_0')$ is obvious by Mackey theory, as in (i). The statement for $\mathrm{U}(V_0)$ is a special case of the following general lemma. □

Lemma 14.2. *Let $G = \mathrm{U}(V)$, and $H = \mathrm{U}(V_0)$ for V_0 a codimension one subspace of V such that a maximal isotropic subspace of V_0 continues to be maximal isotropic in V. Then the Steinberg representation St_G of G contains the Steinberg representation St_H of H as a quotient.*

Proof. Let $L_1 \subset L_2 \subset \cdots \subset L_d$ be a maximal isotropic flag in V_0 with $\dim L_r = r$ for all $1 \le r \le d$. By the hypothesis of the lemma, this is also a maximal isotropic flag in V. Let B_H and B_G be the stabilizer of this flag in H and G respectively. These are Borel subgroups in H and G respectively, and it is known that any parabolic in H (resp. G) containing B_H (resp. B_G) is obtained as the stabilizer of a partial flag $L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_j}$. It follows that intersection with H gives a bijection between parabolics in G containing B_G and parabolics in H containing B_H.

Now note that

$$St_G = \mathrm{Ind}_{B_G}^G(1) / \sum_{P \supset B_G} \mathrm{Ind}_P^G(1),$$

where P run over all parabolics containing but not equal to B_G, and induction refers to un-normalized induction. It follows that the restriction map from functions on $B_G \backslash G$ to $B_H \backslash H$ gives a surjection from the Steinberg representation of G to the Steinberg representation of H. □

Remark 14.3. The previous lemma and the proof works exactly the same way for orthogonal groups too, except for the pair (V, V_0) for which the even dimensional quadratic space is split. The reason being that for even dimensional split quadratic space V_0, with a maximal isotropic flag $L_1 \subset L_2 \subset \cdots \subset L_d$, the parabolics which contain the stabilizer of this flag (which is a Borel subgroup in $\mathrm{SO}(V_0)$) are not parametrized by the stabilizer of a partial flag $L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_j}$. This description is valid for all other quadratic spaces (except direct sum of hyperbolic planes), cf. [MVW, Chapter 1.III.2].

The proposition verifies [GGP, Conjecture 17.3] when $M = \mu \otimes St_3$ and N is arbitrary. We may thus restrict attention to the case when M is an irreducible representation of $W(k)$, so that Π_M is a stable supercuspidal packet consisting of the supercuspidal representation $\pi_M = \pi'_M$ on $\mathrm{U}(V) = \mathrm{U}(V')$.

We first consider the case when

$$N = P + (P^\sigma)^\vee \quad \text{or} \quad \mu \otimes St_2,$$

where P and $(P^\sigma)^\vee$ are not necessarily distinct. In such cases, the associated representations of $\mathrm{U}(V_0)$ are contained in principal series representations of $\mathrm{U}(V_0)$ induced from a Borel subgroup B_0. Thus, we need to compute:

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M, \mathrm{Ind}_{B_0}^{\mathrm{U}(V_0)}(\chi))$$

for a supercuspidal representation π_M of $\mathrm{U}(V)$. By Frobenius reciprocity, we see that this is equal to

$$\mathrm{Hom}_T((\pi_M)_{U_0}, \chi)$$

where U_0 is the unipotent radical of the Borel subgroup B_0 of $\mathrm{U}(V_0)$. We note that $U_0 = U_V^1$ with U_V^1 the center of the unipotent radical U_V of a Borel subgroup B_V of $\mathrm{U}(V)$.

Before proceeding further, let us note the following lemma.

Lemma 14.4. *Let π be an irreducible generic supercuspidal representation of* $\mathrm{U}(V)$ *(*$\dim V = 3$*) with central character ω. Let B_V be a Borel subgroup of* $\mathrm{U}(V)$*, and U_V the unipotent radical of B_V with center $U_V^1 = [U_V, U_V]$. Let $\psi : U_V \to \mathbb{C}^\times$ be a nondegenerate character of U_V. Then there is an isomorphism*

$$\pi_{U_V^1} \cong \mathrm{ind}_{Z_V \cdot U_V}^{B_V}(\omega \boxtimes \psi),$$

of B_V-modules, where Z_V denotes the center of $\mathrm{U}(V)$*.*

Proof. Let $\ell : \pi \to \mathbb{C}$ be a Whittaker functional for the character $\psi : U_V \to \mathbb{C}^\times$. Since ψ restricted to $U_V^1 = [U_V, U_V]$ is trivial, Frobenius reciprocity gives a homomorphism

$$\phi_\ell : \pi_{U_V^1} \to \mathrm{Ind}_{Z_V \cdot U_V}^{B_V}(\omega \boxtimes \psi),$$

of B_V-modules.

Since π is supercuspidal, by the standard argument of Kirillov theory, the image of ϕ_ℓ lands inside the compactly induced representation which is easily seen to be irreducible, hence ϕ_ℓ is a surjective homomorphism onto the compactly supported induced representation.

Since B_V operates transitively on the set of nontrivial characters of U_V, uniqueness of Whittaker models implies that the map ϕ_ℓ must be injective. □

It follows from the lemma above that $(\pi_M)_{U_V^1}$ is isomorphic to the regular representation of $T \cong k^\times$ on $\mathcal{S}(k^\times)$ where T is the quotient of a maximal torus in B_V by the center of $\mathrm{U}(V)$. Thus we have

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M, \mathrm{Ind}_{B_0}^{\mathrm{U}(V_0)}(\chi)) = \mathrm{Hom}_{k^\times}(\mathcal{S}(k^\times), \chi) = \mathbb{C}.$$

In particular, this verifies [GGP, Conjecture 17.3] when

$$N = P + (P^\sigma)^\vee, \quad \text{with } P \neq (P^\sigma)^\vee,$$

as the principal series representation on $\mathrm{U}(V_0)$ is irreducible. If $P \cong (P^\sigma)^\vee$, then the parameter N is dihedral and the corresponding principal series is the sum of two irreducible summands. In this case, we have not determined which of these summands contributes to the 1-dimensional Hom space above. The issue of which representation supports the Hom will be settled by Theorem 16.1 below.

Finally, when $N = \mu \otimes St_2$, we may assume without loss of generality that $\mu = 1$ (by absorbing μ into M). Then

$$\Pi_N = \{St_{\mathrm{U}(V_0)}, 1_{\mathrm{U}(V_0')}\},$$

and

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathrm{Ind}_{B_0}^{\mathrm{U}(V_0)} 1 \longrightarrow St_{\mathrm{U}(V_0)} \longrightarrow 0.$$

The above computation shows that

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M, \mathrm{Ind}_{B_0}^{\mathrm{U}(V_0)} 1) = \mathbb{C}.$$

On the other hand, by [GRS], we have

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M, \mathbb{C}) = 0$$

and

$$\mathrm{Hom}_{\mathrm{U}(V_0')}(\pi_M', \mathbb{C}) = 0.$$

Indeed, if these Hom spaces were not zero, π_M and π_M' would be obtainable as a theta lifting from some U(2), contradicting the fact that M is a stable parameter of U(3). Thus, we conclude that

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M, St_{\mathrm{U}(V_0)}) \neq 0,$$

which is what [GGP, Conjecture 17.3] predicts.

15. A global argument

The methods of theta correspondence pursued in the previous sections are inadequate to handle those representations of U(3) whose Langlands parameters M are irreducible, since such representations do not figure in the theta correspondence with a smaller unitary group. For such representations, however a global argument can be provided. The global argument rests on our ability to globalize the local situation such that the following hold:

(i) the global cuspidal representation Π has nonzero global period;

(ii) the analogous branching laws are known for all local components of Π other than that at the place of interest;

(iii) the nonvanishing of the global period implies the non-vanishing of a certain central critical L-value, as suggested by our global conjectures in [GGP].

We shall be able to achieve (i) and (ii) using a result of the third author and Schulze-Pillot [PS] (and also [P6]), and the requirement (iii) is a theorem due to Ginzburg, Jiang, and Rallis [GJR3, Theorem 4.6] in certain cases.

The main result of this section is the following theorem.

Theorem 15.1. *Let V_0 be a 2 dimensional hermitian subspace of a hermitian space V of dimension 3 over k. Suppose that π_M (resp. π_N) is an irreducible representation of $\mathrm{U}(V)$ (resp. $\mathrm{U}(V_0)$) with Langlands parameter M (resp. N). Then $\epsilon(M \otimes N, \psi)$ is independent of the additive character ψ of k/k_0 and so may be denoted as $\epsilon(M \otimes N)$. Suppose that*

$$\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_M \otimes \pi_N, \mathbb{C}) \neq 0.$$

Then

$$\epsilon(M \otimes N) = \begin{cases} 1 & \text{if } \mathrm{U}(V) \times \mathrm{U}(V_0) \text{ is quasi} - \text{split} \\ -1 & \text{otherwise} \end{cases}$$

Remark 15.2. Let St_n denote the unique irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n, considered as an irreducible representation of W'_k. From the formulae about epsilon factors, cf. [T], it follows that $\epsilon(St_n) = \pm 1$ for all integers n, and $\epsilon(St_n) = -1$ if and only if n is even. Therefore by the Clebsch-Gordan theorem about tensor product of representations of $\mathrm{SL}_2(\mathbb{C})$, $\epsilon(St_{n+1} \otimes St_n) = (-1)^n$, hence $\epsilon(St_{n+1} \otimes St_n) = 1$ if and only if n is even. Therefore theorem 15.1 (stated and proved here only for $n = 2$) is in accordance with Lemma 14.2 about Steinberg representation of $\mathrm{U}(n)$ whose parameter is St_n for general n.

The method that we follow to prove this theorem is pretty general, but it is based on a global theorem of Ginzburg, Jiang, and Rallis [GJR3, theorem 4.6] which assumes that automorphic forms on unitary groups $\mathrm{U}(n)$ have base change to $\mathrm{GL}(n)$. This is known at the moment only for generic automorphic representations on quasi-split unitary groups. However, by Rogawski [Ro], base change is known for any unitary group in 3 variables, which is why we have restricted ourselves to $\mathrm{U}(3)$ in the above theorem. Nonetheless, we have formulated some of the preliminary results below in greater generality.

We begin with the following globalization result about local fields, which will be applied to globalize hermitian spaces over local fields keeping unitary groups at infinity compact.

Lemma 15.3. *Let k be a quadratic extension of a non-archimedean local field k_0. Then there exists a totally real number field F with k_0 as its completion, and a quadratic totally imaginary extension E of F with corresponding completion k; further, we can assume that the degree of F over $\mathbb{Q}$ is any integer $d \geq$ the degree of k over the corresponding $\mathbb{Q}_p$.*

Proof. This follows from combining the weak approximation theorem (for the additive group) with the Krasner's lemma. $\square$

For the globalization of hermitian forms over a local field, we will need the well-known classification of a hermitian form over a number field, according to which a hermitian form over a number field is determined by

(1) the discriminant, and
(2) the signatures at the infinite places.

Moreover, given any discriminant, and signatures at infinite places (except for obvious compatibility between discriminant and signatures), there is such a global hermitian form with the given local constraints.

We also note the following exact sequence from classfield theory,

$$0 \to F^\times/\mathbb{N}E^\times \to \mathbb{A}_F^\times/\mathbb{N}\mathbb{A}_E^\times \to \mathrm{Gal}(E/F) \to 0,$$

from which it follows that one can construct an element in $F^\times$ which is trivial in $F_v^\times/\mathbb{N}E_v^\times$ at all the finite places except k_0, and which at the infinite places has the desired signs, except that the product of the signs is 1 or -1, depending on whether the element in $k_0^\times/\mathbb{N}k^\times$ is trivial or nontrivial.

Before proceeding further, let's recall that a hermitian space of dimension n is said to be quasi-split if it contains a maximal isotropic subspace of dimension d where d is the integral part of $n/2$. It is known that an even dimensional hermitian space over a non-archimedean local field is quasi-split if and only if its discriminant is $(-1)^d$ where $d = n/2$, and any odd dimensional hermitian space over a non-archimedean local field is quasi-split. (A hermitian space is quasi-split if and only if the corresponding unitary group is quasi-split in the sense of algebraic groups.)

From the classification theorem of hermitian forms over a number field recalled above, the following lemma follows easily; we omit the proof.

Lemma 15.4. *Let V be a hermitian space over k of dimension $n = 2d$. Let F be a totally real number field with completion k_0 at a place v_0 of F, and let E be a quadratic totally imaginary extension of F with corresponding completion k. Then there is a hermitian space $\mathbb{V}$ over E satisfying*

(a) $\mathbb{V} \otimes_F k_0 = V$;

(b) $\mathrm{U}(\mathbb{V} \otimes_F F_v)$ *is quasi-split for all finite places* $v \neq v_0$;

(c) *at all the infinite places v of F, $\mathbb{V} \otimes_F F_v$ has signature* $(n, 0)$

if and only if we are in one of the following situations:

(1) The integer d is odd, the hermitian space V is quasi-split, and the degree of F over $\mathbb{Q}$ is even.

(2) The integer d is odd, the hermitian space V is not quasi-split, and the degree of F over $\mathbb{Q}$ is odd.

(3) The integer d is even, the hermitian space V is quasi-split.

Corollary 15.5. *Let $V_0 \subset V$ be hermitian spaces over k, with $\dim_k(V/V_0) = 1$. Let F be a totally real number field with completion k_0 at a place v_0 of F, and let E be a totally imaginary quadratic extension of F with corresponding completion k. Then there are hermitian spaces over E*

$$\mathbb{V}_0 \subset \mathbb{V}$$

satisfying

(a) $\mathbb{V}_0 \otimes_E k = V_0$ *and* $\mathbb{V} \otimes_E k = V$, *so that* $\dim_E(\mathbb{V}/\mathbb{V}_0) = 1$

(b) *the corresponding unitary groups* $\mathrm{U}(\mathbb{V}_0)$ *and* $\mathrm{U}(\mathbb{V})$ *are quasi-split at all the finite places of F different from v_0;*

(c) *for all infinite places v of F,* $\mathrm{U}(\mathbb{V} \otimes F_v)$ *is the compact group* $\mathrm{U}(n+1, 0)$,

if and only if the even dimensional hermitian space in the pair $(\mathbb{V}, \mathbb{V}_0)$ *satisfies one of the three options of the previous lemma.*

Proof. The necessity of the condition is obvious. For the other direction, observe that since an odd dimensional hermitian space is automatically quasi-split at any finite place, we first construct the even dimensional hermitian space in the pair $(\mathbb{V}, \mathbb{V}_0)$, and construct the odd dimensional one by adding or subtracting a one dimensional hermitian space from the even dimensional one, keeping track only of the place corresponding to k_0, and the places at infinity. □

Proof of Theorem 15.1: By the results of the previous two sections, we already know the desired result if M is reducible or is the parameter of a twisted Steinberg representation. So we assume that M is an irreducible representation of $W(k)$, so that π_M is a supercuspidal representation of $\mathrm{U}(V)$. Similarly the theorem is already known if π_N is a principal series representation, or a twisted Steinberg representation of $\mathrm{U}(V_0)$. So we will assume in the rest of the proof that both π_M and π_N are supercuspidal representations.

We globalize the local spaces $V_0 \subset V$ to $\mathbb{V}_0 \subset \mathbb{V}$ as in the above corollary, so that $\mathrm{U}(\mathbb{V})$ is compact at infinity. It is then easy to see that we can globalize the representation π_M of $\mathrm{U}(V)$ to a cuspidal automorphic representation Π_1 of $\mathrm{U}(\mathbb{V})(\mathbb{A})$ in such a way that it is unramified at all the finite places of F except k_0. It is important to note that all local components of Π_1 belong to generic L-packets. Indeed, by the results of Rogawski [Ro], one knows that the base change $BC(\Pi_1)$ of Π_1 to $\mathrm{GL}_3(E)$ is cuspidal, since the base change of π_M to $\mathrm{GL}_3(k)$ is supercuspidal. Thus, all the local components of $BC(\Pi_1)$ are generic, so that the L-parameters of all local components of Π_1 are generic.

By [P6, Lemma 1], we can globalize π_N to an automorphic representation Π_0 such that the global period integral

$$\int_{\mathrm{U}(\mathbb{V}_0)\backslash \mathrm{U}(\mathbb{V}_0)(\mathbb{A})} f_0 f_1 \neq 0,$$

for some f_0 in Π_0, and f_1 in Π_1. By the theorems due to Ginzburg, Jiang, and Rallis [GJR3, Theorem 4.6], the non-vanishing of the global period integral implies the non-vanishing of a central critical L-value:

$$L(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E) \neq 0,$$

where Π_0^E and Π_1^E denote base change of Π_0 and Π_1 to E. This implies that the global root number,

$$\epsilon(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E) = 1.$$

Let

$$\Pi_0 = \otimes_w \Pi_{0,w}, \text{ and } \Pi_1 = \otimes_w \Pi_{1,w},$$

with $\Pi_{0,v} = \pi_N$, and $\Pi_{1,v} = \pi_M$. From the nonvanishing of the period integral, it follows that

$$\mathrm{Hom}_{\mathrm{U}(\mathbb{V}_{0,w})}(\Pi_{0,w} \otimes \Pi_{1,w}, \mathbb{C}) \neq 0$$

for all places w of F. Since, by construction, the representations $\Pi_{1,w}$ are unramified and generic for all finite places $w \neq v$, we know the validity of Theorem 15.1 for such representations. Thus we have:

$$\epsilon_w(\frac{1}{2}, \Pi_{0,w}^E \otimes \Pi_{1,w}^E) = 1$$

for all finite places $w \neq v$. Since the global epsilon factor is a product of local epsilon factors, we have

$$\epsilon(\frac{1}{2}, M \otimes N) \cdot \epsilon_\infty(\frac{1}{2}, \Pi_{0,\infty}^E \otimes \Pi_{1,\infty}^E) = 1.$$

Thus, to complete the proof of Theorem 15.1, we need to address the branching problem at the infinite places. In particular, we shall show:

Proposition 15.6. *Let V_0 be a codimension 1 hermitian subspace of a positive definite hermitian space V of dimension $n+1$ over $\mathbb{C}$. Suppose that π_1 (resp. π_0) is a finite dimensional irreducible representation of* $\mathrm{U}(V)$ *(resp.* $\mathrm{U}(V_0)$*). Let the Langlands parameter of π_1 (resp. π_0) be σ_1 (resp. σ_0). Suppose that* $\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_1 \otimes \pi_0, \mathbb{C}) \neq 0$. *Then*

$$\epsilon(\sigma_1 \otimes \sigma_0) = (-1)^{\frac{n(n+1)}{2}} \quad = \quad \begin{cases} 1 & if\ n \equiv 0, 3 \mod 4 \\ -1 & if\ n \equiv 1, 2 \mod 4. \end{cases}$$

This proposition completes the proof of Theorem 15.1, since one knows by Lemma 15.4 that there are an even number of places at infinity if $\mathrm{U}(\mathbb{V}_0)$ is quasi-split, and an odd number of places at infinity when $\mathrm{U}(\mathbb{V}_0)$ is not quasi-split since $\dim V_0 = 2$ (or any odd multiple of 2).

The rest of the section is devoted to the proof of the proposition. In fact, it is a simple consequence of the well-known branching law, recalled below in Lemma 15.8, from the compact group $\mathrm{U}(n+1)$ to $\mathrm{U}(n)$, combined with the value of the epsilon factor given by the following Lemma 15.7, which has been demonstrated in Proposition 2.1.

Lemma 15.7. *Let ψ be the additive character on $\mathbb{C}$ given by $\psi(z) = e^{-2\pi i y}$ where $z = x + iy$. For n a half-integer but not an integer, i.e., $n \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, let χ_n denote the character $\chi_n(z) = (\bar{z}/z)^n = e^{-2ni\theta}$ for $z = re^{i\theta} \in \mathbb{C}^\times$. Then for $n \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$,*

$$\epsilon(\chi_n, \psi) \quad = \quad \begin{cases} 1 & if\ n > 0 \\ -1 & if\ n < 0. \end{cases}$$

Lemma 15.8. *Let π_0 (resp. π_1) be a finite dimensional irreducible representation of the compact group* $\mathrm{U}(n)$ *(resp.* $\mathrm{U}(n+1)$*) with L-parameter restricted to $\mathbb{C}^\times$ given by an n-tuple of half-integers $\sigma_0 = \{-\lambda_n < -\lambda_{n-1} < \cdots < -\lambda_1\}$ (resp. $\sigma_1 = \{\mu_1 < \mu_2 < \cdots < \mu_{n+1}\}$ an $(n+1)$-tuple of half-integers), where all the λ_i's are half-integers but not integers if n is even, and are integers if n is odd, and the μ_i's are all integers if n is even, and half-integers but not integers if n is odd, i.e.,*

$$\begin{aligned}\sigma_0 &= \chi_{-\lambda_n} + \cdots + \chi_{-\lambda_1}, \text{ and}\\ \sigma_1 &= \chi_{\mu_1} + \cdots + \chi_{\mu_{n+1}}.\end{aligned}$$

Then

$$\mathrm{Hom}_{\mathrm{U}(n)}(\pi_1 \otimes \pi_0, \mathbb{C}) \neq 0$$

if and only if

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \cdots < \lambda_n < \mu_{n+1}.$$

Corollary 15.9. *With notation as in Lemma 15.8, and assuming that $\pi_0^\vee$ appears in the restriction of π_1, one has*

$$\epsilon(\chi_{\mu_k} \otimes \sigma_0) = (-1)^{n-k+1}, \ for\ all\ k,$$

and therefore,

$$\begin{aligned}\epsilon(\sigma_1 \otimes \sigma_0) &= \prod_{k=1}^{n+1} (-1)^{n-k+1}\\ &= (-1)^{\frac{n(n+1)}{2}}.\end{aligned}$$

Remark 15.10. It should be mentioned that the global method followed in the proof of Theorem 15.1 proves that if there is an invariant linear form, then the epsilon factor has the expected value predicted in [GGP]. The natural variant for unitary groups of the theorem of Waldspurger in [Wa4] will prove that such an invariant form always exists on a relevant pair of unitary groups. This will then strengthen Theorem 15.1 to an if and only if statement.

16. A finer global argument

In the previous section, we used a global argument to prove Theorem 15.1, which says that a nonzero invariant form for a Vogan packet $\Pi_M \times \Pi_N$ is supported on the quasi-split group $\mathrm{U}(V) \times \mathrm{U}(V_0)$ if and only if $\epsilon(M \otimes N) = 1$. One can refine this argument to compute other epsilon factors which arise in [GGP, Conjecture 17.3] when N is reducible. We give a sketch of this refined argument in this section.

Suppose that $N = N_1 + N_2$, where N_i is conjugate symplectic of dimension 1, with associated component group A_N. In §8, we have defined a bijection

$$J(\psi) : \Pi_N \leftrightarrow \mathrm{Irr}(A_N)$$

which depends on the fixed additive character $\psi : k/k_0 \to \mathbb{C}^\times$. When $N_1 \neq N_2$, $A_N = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and thus a representation $\pi_0 \in \Pi_N$ is labelled by a pair of signs $(\eta_1(\pi_0, \psi), \eta_2(\pi_0, \psi))$. When $N_1 = N_2$, (which corresponds to a reducible unitary principal series), $A_N = \mathbb{Z}/2$, and we have the label $\eta_1(\pi_0, \psi) = \eta_2(\pi_0, \psi) \in \{\pm 1\}$.

Now the main result of this section is:

Theorem 16.1. *Let $V_0 \subset V$ be hermitian spaces over a non-archimedean local field k with $\dim V = 3$ and $\dim V_0 = 2$. Let π_1 be an irreducible representation of $\mathrm{U}(V)$ belonging to a generic L-packet with Langlands parameter M. Let π_0 be a dihedral representation of $\mathrm{U}(V_0)$ with Langlands parameter $N = N_1 \oplus N_2$ with N_i conjugate symplectic. If $\mathrm{Hom}_{\mathrm{U}(V_0)}(\pi_1 \otimes \pi_0, \mathbb{C}) \neq 0$, then we have*

$$\begin{aligned} \epsilon(M \otimes N_1, \psi) &= \eta_1(\pi_0, \psi) \\ \epsilon(M \otimes N_2, \psi) &= \eta_2(\pi_0, \psi). \end{aligned}$$

Proof. We already know the desired result in all cases except when π_1 is a stable supercuspidal representation. To take care of this remaining case, we globalize everything in sight. More precisely,

(i) we first globalize the local fields $k_0 \subset k$ to global fields $F \subset E$ with F totally real and E totally complex; further we assume that E is unramified over F outside of the place defining k_0, as is known to be possible.

(ii) next, we globalize $V_0 \subset V$ to hermitian spaces $\mathbb{V}_0 \subset \mathbb{V}$ over F, keeping $\mathbb{V}_0$ quasi-split at all the finite places away from k, and $\mathbb{V}$ positive definite at all real spaces; this is possible by Corollary 15.5.

(iii) we then globalize the representation π_0 of $\mathrm{U}(V_0)$ to a dihedral automorphic representation Π_0 of $\mathrm{U}(\mathbb{V}_0)$ which is unramified outside the finite place of F corresponding to k_0. This is possible as it amounts to globalizing a character of $k^\times$ to a Grössencharacter on $\mathbb{A}_E^\times$ unramified at all the finite places different from k, cf. [P6, Lemma 3]. Further, we may ensure that the Grössencharacter on $\mathbb{A}_E^\times$ is not Galois invariant, so that the automorphic representation Π_0 on $\mathrm{U}(\mathbb{V}_0)$ is cuspidal. If $N_1 \neq N_2$, there is no issue about this, but if $N_1 = N_2$, one needs a short argument. Indeed, it suffices to note that the field E has many quadratic extensions which are *not* Galois over F and which are split over the place of E corresponding to k, i.e., there are quadratic characters of $\mathbb{A}_E^\times/E^\times$, trivial on $k^\times$, which are not Galois invariant. The usual trick of base changing F to a larger extension to absorb possible ramifications allows one to assume that the Grössencharacter on $\mathbb{A}_E^\times$ is unramified outside k. Multiplying such a non-invariant quadratic character of $\mathbb{A}_E^\times$ to the original (possibly invariant) Grössencharacter on $\mathbb{A}_E^\times$ will do the job.

(iv) we next globalize π_1 of $\mathrm{U}(V)$ to an automorphic representation Π_1 of $\mathrm{U}(\mathbb{V})$ keeping it unramified at all finite places of E away from k and with nonzero period integral:

$$\int_{\mathrm{U}(\mathbb{V}_0)\backslash \mathrm{U}(\mathbb{V}_0)(\mathbb{A})} f_0 f_1 \neq 0,$$

for some f_0 in Π_0, and f_1 in Π_1. This is possible by an application of relative trace formula exactly as in the proof of [PS, Theorem 4.1], though the result in this reference is proved only for a character on the subgroup; we grant ourselves such a generalization here. Further, we note that since π_1 is stable, all local components of Π_1 belong to generic L-packets.

Now by the theorem of Ginzburg, Jiang, and Rallis [GJR, Theorem 4.6], the non-vanishing of the period integral in (iii) above implies the nonvanishing of the central critical L-value:

$$L(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E) \neq 0,$$

where Π_0^E and Π_1^E denote base change of Π_0 and Π_1 to E, which are automorphic representations of $\mathrm{GL}_2(\mathbb{A}_E)$ and $\mathrm{GL}_3(\mathbb{A}_E)$ respectively. We note that the work of Ginzburg, Jiang, and Rallis is at the moment subject to the hypothesis that Π_0^E and Π_1^E are cuspidal, which is not the case here. However, it seems likely that their theorem can be strengthened to give what we need; again we grant ourselves this extension here.

In the case at hand, Π_0^E is an Eisenstein series corresponding to a sum of two Grössencharacters $\Xi_1 + \Xi_2$, and therefore the L-function being considered factorizes as

$$L(s, [\Xi_1 + \Xi_2] \otimes \Pi_1^E) = L(s, \Xi_1 \otimes \Pi_1^E) \cdot L(s, \Xi_2 \otimes \Pi_1^E).$$

The nonvanishing of $L(\frac{1}{2}, \Pi_0^E \otimes \Pi_1^E)$ then implies the nonvanishing of both $L(\frac{1}{2}, \Xi_1 \otimes \Pi_1^E)$ and $L(\frac{1}{2}, \Xi_2 \otimes \Pi_1^E)$. The two L-functions being considered are both selfdual, and hence the corresponding global root numbers are 1:

$$\epsilon(\frac{1}{2}, \Xi_1 \otimes \Pi_1^E) = 1,$$

and

$$\epsilon(\frac{1}{2}, \Xi_2 \otimes \Pi_1^E) = 1.$$

By the multiplicity formula of Labesse-Langlands [LL] or Rogawski [Ro], since the representation Π_0 is automorphic, we have:

$$\eta_1(\Pi_0) := \prod_v \eta_1(\Pi_{0,v}, \Psi_v) = 1,$$

and

$$\eta_2(\Pi_0) := \prod_v \eta_2(\Pi_{0,v}, \Psi_v) = 1.$$

Here, Ψ is a character of $\mathbb{A}_E/E\mathbb{A}_F \to \mathbb{C}^\times$, with local components Ψ_v, and the values $\eta_i(\Pi_{0,v}, \Psi_v) = \pm 1$ are the labels for the members inside a Vogan packet defined in §6 and recalled at the beginning of this section.

In view of the above, we get that:

$$1 = \epsilon(\frac{1}{2}, \Xi_1 \otimes \Pi_1^E) = \prod_v \epsilon(\Xi_{1,v} \otimes \Pi_{1,v}^E) = \prod_v \eta_1(\Pi_{0,v}, \Psi_v), \tag{A}$$

and similarly,

$$1 = \epsilon(\frac{1}{2}, \Xi_2 \otimes \Pi_1^E) = \prod_v \epsilon(\Xi_{2,v} \otimes \Pi_{1,v}^E) = \prod_v \eta_2(\Pi_{0,v}, \Psi_v). \tag{B}$$

We note that at the places v of F split in E, the unitary groups $\mathrm{U}(\mathbb{V}_v)$ and $\mathrm{U}(\mathbb{V}_{0,v})$ become $\mathrm{GL}_3(F_v)$ and $\mathrm{GL}_2(F_v)$ respectively. At such places, the signs η_1 and η_2 are trivial (by definition); further, it is easy to see that if the place v of F splits into two places $\{v', v''\}$ of E, then $\epsilon(\Xi_{1,v'} \otimes \Pi_{1,v'}^E) \cdot \epsilon(\Xi_{1,v''} \otimes \Pi_{1,v''}^E) = 1$. This means that in the above product formulae $(A), (B)$, we can ignore places of F split in E. Since we have globalized Π_1 keeping it unramified at the finite places away from k, we know that the theorem being proved is known by the results of the previous sections. By the product formulae (A) and (B), our theorem is proved at this remaining place! $\square$

We end with a summary of the status of [GGP, Conjecture 17.3] for $\mathrm{U}(3) \times \mathrm{U}(2)$, as treated in the last 4 sections of this paper. If the L-parameter is $M \otimes N$, then we have:

(1) If M is endoscopic, [GGP, Conj. 17.3] is verified by Section 13.
(2) If M is Steinberg, [GGP, Conj. 17.3] is done by Prop. 14.1.
(3) If M is stable supercuspidal, and N corresponds to an irreducible principal series of $\mathrm{U}(2)$, or a twisted steinberg representation, [GGP, Conj. 17.3] is verified by Lemma 14.4, and the ensuing discussion.
(4) If M stable supercuspidal and N corresponds to a dihedral principal series representation, then [GGP, Conj. 17.3] is verified by Lemma 14.4 and the ensuing discussion, together with Theorem 16.1.
(5) If M stable supercuspidal and N (stable or dihedral) supercuspidal, then [GGP, Conj. 17.3] is partially verified by Theorems 15.1 and 16.1. More precisely, we show that the only representation in $\Pi_M \times \Pi_N$ which could possibly support an invariant form is the one corresponding to the distinguished character in [GGP,

Conj. 17.3]. However, we have not shown that this distinguished representation is actually distinguished!

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