Invent. math. 168, 321–370 (2007)
DOI: 10.1007/s00222-007-0034-5

Inventiones mathematicae

Fake projective planes

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Oblatum 30-V-2006 & 4-I-2007
Published online: 13 February 2007 – © Springer-Verlag 2007

Dedicated to David Mumford

1. Introduction

1.1. A fake projective plane is a smooth compact complex surface which is not the complex projective plane but has the same Betti numbers as the complex projective plane. Such a surface is known to be projective algebraic and it is the quotient of the (open) unit ball B in $\mathbb{C}^2$ (B is the symmetric space of $\mathrm{PU}(2,1)$) by a torsion-free cocompact discrete subgroup of $\mathrm{PU}(2,1)$ whose Euler–Poincaré characteristic is 3. These surfaces have the smallest Euler–Poincaré characteristic among all smooth surfaces of general type. The first fake projective plane was constructed by David Mumford [Mu] using p-adic uniformization, and latter two more examples were found by M. Ishida and F. Kato in [IK] using a similar method. We have learnt from JongHae Keum that he has recently constructed an example which is birational to a cyclic cover of degree 7 of a Dolgachev surface (see 5.15 below). It is known that there are only finitely many fake projective planes [Mu], and an important problem in complex algebraic geometry is to determine them all.

It is proved in [Kl] and [Y] that the fundamental group of a fake projective plane is a torsion-free cocompact arithmetic subgroup of $\mathrm{PU}(2,1)$. It follows from Mostow's strong rigidity theorem [Mo] that the fundamental group of a fake projective plane determines it uniquely up to isomorphism. In this paper we will make use of the volume formula of [P], together with some number theoretic estimates, to list all torsion-free cocompact arithmetic subgroups (of $\mathrm{PU}(2,1)$) whose Euler–Poincaré characteristic is 3, see Sects. 5, 8 and 9. This list of course contains the fundamental groups of all fake projective planes. It provides several new examples of fake projective planes. In fact, we show that there are at least seventeen distinct nonempty classes of fake projective planes (see 1.4–1.5 below). We obtain these fake projective planes as quotients of the ball B by explicitly given torsion-free cocompact arithmetic subgroups of either $\mathrm{SU}(2,1)$ or $\mathrm{PU}(2,1)$. In Sect. 10, we use this explicit description of their fundamental groups to prove that for any fake projective plane P occurring in these seventeen classes, $H_1(P,\mathbb{Z})$ is nonzero, and if P is not one of the fake projective planes constructed in 9.3, its fundamental group embeds in $\mathrm{SU}(2,1)$. The latter result implies that the canonical line bundle K_P of P is divisible by 3, i.e., there is a holomorphic line bundle L on P such that $K_P = 3L$, see 10.4. $7L$ is very ample and it provides an embedding of P in $\mathbf{P}^{14}_{\mathbb{C}}$ as a smooth complex surface of degree 49.

We have also proved that besides the seventeen classes of fake projective planes mentioned above, there can exist at most four more classes, see 1.6.

We will now present a brief outline of our methods and results. We begin by giving a description of the forms of $\mathrm{SU}(2,1)$ over number fields used in this paper.

1.2. Let k be a real number field, v_o be a real place of k, and G be a simple simply connected algebraic k-group such that $G(k_{v_o}) \cong \mathrm{SU}(2,1)$, and for all other archimedean places v of k, $G(k_v) \cong \mathrm{SU}(3)$. From the description of absolutely simple simply connected groups of type ${}^2A_2$ (see, for example, [Ti1]), we see that k is totally real, and there is a totally complex quadratic extension ℓ of k, a division algebra $\mathcal{D}$ of degree $n|3$, with center ℓ, $\mathcal{D}$ given with an involution σ of the second kind such that $k = \{x \in \ell \mid x = \sigma(x)\}$, and a nondegenerate hermitian form h on $\mathcal{D}^{3/n}$ defined in terms of the involution σ such that G is the special unitary group $\mathrm{SU}(h)$ of h.

Let $k, \ell, \mathcal{D}$ be as above. We will now show that the k-group G is uniquely determined, up to a k-isomorphism, by $\mathcal{D}$ (i.e., the k-isomorphism class of G does not depend on the choice of the involution σ and the hermitian form h on $\mathcal{D}^{3/n}$). Let σ be an involution of $\mathcal{D}$ of the second kind with $k = \{x \in \ell \,|\, x = \sigma(x)\}$. Let h be a hermitian form on $\mathcal{D}^{3/n}$. For $x \in k^\times$, xh is again an hermitian form on $\mathcal{D}^{3/n}$, and $\det(xh) = x^3\det(h)$. Now since $N_{\ell/k}(\ell^\times) \supset k^{\times 2}$, $\det(\det(h)h)$, as an element of $k^\times/N_{\ell/k}(\ell^\times)$, is 1. Moreover, $\mathrm{SU}(h) = \mathrm{SU}(\det(h)h)$. Hence, it would suffice to work with hermitian forms of determinant 1.

If $\mathcal{D} = \ell$, and h is a hermitian form on ℓ^3 of determinant 1 such that the group $\mathrm{SU}(h)$ is isotropic at v_o, and is anisotropic at all other real places of k (or, equivalently, h is indefinite at v_o, and definite at all other real places), then being of determinant 1, its signature (or index) at v_o is -1, and at all other real places of k it is 3. Corollary 6.6 of [Sc, Chap. 10] implies that any two such hermitian forms on ℓ^3 are isometric, and hence they determine a unique G up to a k-isomorphism.

Now let us assume that $\mathcal{D}$ is a cubic division algebra with center ℓ, σ an involution of the second kind such that for the hermitian form h_0 on $\mathcal{D}$

defined by $h_0(x, y) = \sigma(x)y$, the group $\mathrm{SU}(h_0)$ is isotropic at v_o, and is anisotropic at every other real place of k. For $x \in \mathcal{D}^\times$, let $\mathrm{Int}(x)$ denote the automorphism $z \mapsto xzx^{-1}$ of $\mathcal{D}$. Let $\mathcal{D}^\sigma = \{z \in \mathcal{D} \mid \sigma(z) = z\}$. Then for all $x \in \mathcal{D}^\sigma$, $\mathrm{Int}(x) \cdot \sigma$ is again an involution of $\mathcal{D}$ of the second kind, and any involution of $\mathcal{D}$ of the second kind is of this form. Now for $x \in \mathcal{D}^\sigma$, given an hermitian form h' on $\mathcal{D}$ with respect to the involution $\mathrm{Int}(x) \cdot \sigma$, the form $h = x^{-1}h'$ is a hermitian form on $\mathcal{D}$ with respect to σ, and $\mathrm{SU}(h') = \mathrm{SU}(h)$. Therefore, to determine all the special unitary groups we are interested in, it is enough to work just with the involution σ, and to consider all hermitian forms h on $\mathcal{D}$, with respect to σ, of determinant 1, such that the group $\mathrm{SU}(h)$ is isotropic at v_o, and is anisotropic at all other real places of k. Let h be such a hermitian form. Then $h(x, y) = \sigma(x)ay$, for some $a \in \mathcal{D}^\sigma$. The determinant of h is $\mathrm{Nrd}(a)$ modulo $N_{\ell/k}(\ell^\times)$. As the elements of $N_{\ell/k}(\ell^\times)$ are positive at all real places of k, we see that the signatures of h and h_0 are equal at every real place of k. Corollary 6.6 of [Sc, Chap. 10] again implies that the hermitian forms h and h_0 are isometric. Hence, $\mathrm{SU}(h)$ is k-isomorphic to $\mathrm{SU}(h_0)$. Thus we have shown that $\mathcal{D}$ determines a unique k-form G of $\mathrm{SU}(2, 1)$, up to a k-isomorphism, namely $\mathrm{SU}(h_0)$, with the desired behavior at the real places of k. The group $G(k)$ of k-rational points of this G is

$$G(k) = \{z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \text{ and } \mathrm{Nrd}(z) = 1\}.$$

Let $\mathcal{D}$ and the involution σ be as in the previous paragraph. Let $\mathcal{D}^o$ be the opposite of $\mathcal{D}$. Then the involution σ is also an involution of $\mathcal{D}^o$. The pair $(\mathcal{D}^o, \sigma)$ determines a k-form of $\mathrm{SU}(2, 1)$ which is clearly k-isomorphic to the one determined by the pair $(\mathcal{D}, \sigma)$.

In the sequel, the adjoint group of G will be denoted by $\overline{G}$, and φ will denote the natural isogeny $G \to \overline{G}$.

1.3. Let Π be a torsion-free cocompact arithmetic subgroup of $\mathrm{PU}(2, 1)$ whose Euler–Poincaré characteristic is 3. The fundamental group of a fake projective plane is such a subgroup. Let $\varphi : \mathrm{SU}(2, 1) \to \mathrm{PU}(2, 1)$ be the natural surjective homomorphism. The kernel of φ is the center of $\mathrm{SU}(2, 1)$ which is a subgroup of order 3. Let $\widetilde{\Pi} = \varphi^{-1}(\Pi)$. Then $\widetilde{\Pi}$ is a cocompact arithmetic subgroup of $\mathrm{SU}(2, 1)$. The orbifold Euler–Poincaré characteristic $\chi(\widetilde{\Pi})$ of $\widetilde{\Pi}$ (i.e., the Euler–Poincaré characteristic in the sense of C.T.C. Wall, cf. [Se1, §1.8]) is 1. Hence, the orbifold Euler–Poincaré characteristic of any discrete subgroup of $\mathrm{SU}(2, 1)$ containing $\widetilde{\Pi}$ is a reciprocal integer.

Let k be the number field and G be the k-form of $\mathrm{SU}(2, 1)$ associated with the arithmetic subgroup Π. The field k is generated by the traces, in the adjoint representation of $\mathrm{PU}(2, 1)$, of the elements in Π, and G is a simple simply connected algebraic k-group such that for a real place, say v_o, of k, $G(k_{v_o}) \cong \mathrm{SU}(2, 1)$, and for all archimedean places $v \neq v_o$, $G(k_v)$ is isomorphic to the compact Lie group $\mathrm{SU}(3)$, and $\widetilde{\Pi}$ is an arithmetic subgroup of $G(k_{v_o})$ (i.e., it is commensurable with $\widetilde{\Pi} \cap G(k)$). Throughout this paper we will use the description of G and $\overline{G}$ given in 1.2. In particular, ℓ, $\mathcal{D}$ and h are as in there.

Let V_f (resp. V_∞) be the set of nonarchimedean (resp. archimedean) places of k. Let $\mathcal{R}_\ell$ be the set of nonarchimedean places of k which ramify in ℓ. The k-algebra of finite adèles of k, i.e., the restricted direct product of the k_v, $v \in V_f$, will be denoted by A_f.

The image Π of $\widetilde{\Pi}$ in $\overline{G}(k_{v_o})$ is actually contained in $\overline{G}(k)$ [BP, 1.2]. For all $v \in V_f - \mathcal{R}_\ell$, we fix a *parahoric* subgroup P_v of $G(k_v)$ which is *minimal* among the parahoric subgroups of $G(k_v)$ normalized by Π. For each of the finitely many $v \in \mathcal{R}_\ell$, we fix a maximal parahoric subgroup P_v of $G(k_v)$ normalized by Π (see 2.2). Then $\prod_{v \in V_f} P_v$ is an open subgroup of $G(A_f)$, see [BP, §1]. Hence, $\Lambda := G(k) \cap \prod_{v \in V_f} P_v$ is a *principal* arithmetic subgroup [P, 3.4] which is normalized by Π, and therefore also by $\widetilde{\Pi}$. Let Γ be the normalizer of Λ in $G(k_{v_o})$, and $\overline{\Gamma}$ be its image in $\overline{G}(k_{v_o})$. Then $\overline{\Gamma} \subset \overline{G}(k)$ [BP, 1.2]. As the normalizer of Λ in $G(k)$ equals Λ, $\Gamma \cap G(k) = \Lambda$. Since Γ contains $\widetilde{\Pi}$, its orbifold Euler–Poincaré characteristic $\chi(\Gamma)$ is a reciprocal integer.

In terms of the normalized Haar-measure μ on $G(k_{v_o})$ used in [P] and [BP], $\chi(\Gamma) = 3\mu(G(k_{v_o})/\Gamma)$ (see [BP, §4], note that the compact dual of the symmetric space B of $G(k_{v_o}) \cong \mathrm{SU}(2, 1)$ is $\mathbf{P}^2_{\mathbb{C}}$, and the Euler–Poincaré characteristic of $\mathbf{P}^2_{\mathbb{C}}$ is 3). Thus the condition that $\chi(\Gamma)$ is a reciprocal integer is equivalent to the condition that the covolume $\mu(G(k_{v_o})/\Gamma)$, of Γ, is one third of a reciprocal integer; in particular, $\mu(G(k_{v_o})/\Gamma) \leqslant 1/3$. Also, $\chi(\Gamma) = 3\mu(G(k_{v_o})/\Gamma) = 3\mu(G(k_{v_o})/\Lambda)/[\Gamma : \Lambda]$, and the volume formula of [P] can be used to compute $\mu(G(k_{v_o})/\Lambda)$ precisely, see 2.4 below. Proposition 2.9 of [BP] implies that $[\Gamma : \Lambda]$ is a power of 3. Now we see that if $\chi(\Gamma)$ is a reciprocal integer, then the numerator of the rational number $\mu(G(k_{v_o})/\Lambda)$ must be a power of 3.

1.4. In Sects. 4–5, and 7–9, we will determine all k, ℓ, $\mathcal{D}$, simple simply connected algebraic k-groups G so that for a real place v_o of k, $G(k_{v_o}) \cong \mathrm{SU}(2, 1)$, for all archimedean $v \neq v_o$, $G(k_v) \cong \mathrm{SU}(3)$, and (up to conjugation by an element of $\overline{G}(k)$) all collections $(P_v)_{v \in V_f}$ of parahoric subgroups P_v of $G(k_v)$ such that (i) for $v \in \mathcal{R}_\ell$, P_v is a maximal parahoric subgroup of $G(k_v)$, and $\prod_{v \in V_f} P_v$ is an open subgroup of $G(A_f)$, (ii) the principal arithmetic subgroup $\Lambda := G(k) \cap \prod_{v \in V_f} P_v$ considered as a (discrete) subgroup of $G(k_{v_o})$ is cocompact (by Godement compactness criterion, this is equivalent to the condition that G is anisotropic over k), and (iii) the image $\overline{\Gamma}$ in $\overline{G}(k_{v_o})$ of the normalizer Γ of Λ in $G(k_{v_o})$ contains a torsion-free subgroup Π of finite index whose Euler–Poincaré characteristic is 3. Then the orbifold Euler–Poincaré characteristic of Γ is a reciprocal integer.

It will turn out that for every $v \in V_f - \mathcal{R}_\ell$, P_v appearing in the preceding paragraph is a maximal parahoric subgroup of $G(k_v)$, it is in fact a *hyperspecial* parahoric subgroup if G is isotropic at v. In particular, if Π, Λ, Γ, and the parahoric subgroups P_v are as in 1.3, then for $v \in V_f - \mathcal{R}_\ell$, as P_v was assumed to be minimal among the parahoric subgroups of $G(k_v)$ normalized by Π, we conclude that it is the *unique* parahoric subgroup of $G(k_v)$ normalized by Π.

We will prove that there are exactly seventeen distinct $\{k, \ell, G, \mathscr{D}, (P_v)_{v\in V_f}\}$ with $\mathscr{D} \neq \ell$. Each of these seventeen sets determines a unique principal arithmetic subgroup Λ $(= G(k) \cap \prod_{v\in V_f} P_v)$, which in turn determines a unique arithmetic subgroup $\overline{\Gamma}$ of $\overline{G}(k_{v_o})$ (recall that $\overline{\Gamma}$ is the image in $\overline{G}(k_{v_o})$ of the normalizer Γ of Λ in $G(k_{v_o})$). For twelve of these seventeen, $k = \mathbb{Q}$, see Sect. 5; and there are two with $k = \mathbb{Q}(\sqrt{2})$, two with $k = \mathbb{Q}(\sqrt{5})$, and one with $k = \mathbb{Q}(\sqrt{6})$, see Sect. 9. We will show that the pair $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-7}))$ gives two of these Λ, and each of these two contains a torsion-free normal subgroup Λ^+ of index 7 which is the fundamental group of a fake projective plane. On the other hand, the pair $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-23}))$ gives us two principal arithmetic subgroups Λ, and the subgroup $\overline{\Gamma}$ determined by either of them is the fundamental group of a fake projective plane. The image $\overline{\Lambda}$ in $\overline{G}(k_{v_o})$ of each of the other thirteen Λ is the fundamental group of a fake projective plane.

1.5. We will now describe the class of fake projective planes associated to each of the seventeen Γs of 1.4. The orbifold Euler–Poincaré characteristic $\chi(\overline{\Gamma})$ of $\overline{\Gamma}$ equals $3\chi(\Gamma) = 3\chi(\Lambda)/[\Gamma : \Lambda]$, and we compute it precisely. Now if Π is a torsion-free subgroup of $\overline{\Gamma}$ of index $3/\chi(\overline{\Gamma})$, then $\chi(\Pi) = 3$, and if, moreover, $H^1(\Pi, \mathbb{C})$ vanishes (or, equivalently, the abelianization $\Pi/[\Pi, \Pi]$ is finite), then by Poincaré-duality, $H^3(\Pi, \mathbb{C})$ vanishes too, and hence, as $\chi(B/\Pi) = \chi(\Pi) = 3$, B/Π is a fake projective plane.

The class of fake projective planes given by Γ consists of the fake projective planes B/Π, where Π is a torsion-free subgroup of $\overline{\Gamma}$ of index $3/\chi(\overline{\Gamma})$ with $\Pi/[\Pi, \Pi]$ finite.

We observe that in principle, for a given Γ, the subgroups Π of $\overline{\Gamma}$ as above can all be determined in the following way: First find a "small" presentation of $\overline{\Gamma}$ using a "nice" fundamental domain in B (maximal arithmetic subgroups tend to have small presentation), and use this presentation to list all torsion-free subgroups of index $3/\chi(\overline{\Gamma})$ whose abelianization is finite. Note that the computations given below show that $3/\chi(\overline{\Gamma})$ is quite small; in fact, it equals 1, 3, 9 or 21.

1.6. Let h be a nondegenerate hermitian form on ℓ^3 (defined in terms of the nontrivial automorphism of ℓ/k) which is indefinite at v_o and definite at all other real places of k. Let $G = \mathrm{SU}(h)$, and $\overline{G}$ be its adjoint group. According to Proposition 8.8, if $\overline{G}(k_{v_o})$ contains a torsion-free cocompact arithmetic subgroup Π with $\chi(\Pi) = 3$, then, in the notation of 8.2, (k, ℓ) must be one of the following five: $\mathcal{C}_1$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$, or $\mathcal{C}_{21}$. Tim Steger has shown that (k, ℓ) cannot be $\mathcal{C}_8$, and together with Donald Cartwright he has shown that it cannot be $\mathcal{C}_{21}$ either. We do not expect that h described in terms of any of the three remaining pairs $(k, \ell) = \mathcal{C}_1$, $\mathcal{C}_{11}$ or $\mathcal{C}_{18}$, will give a fake projective plane. In any case, we know that the group G associated to the pair $\mathcal{C}_1$ can give at most two classes of fake projective planes and the group G associated to either of the pairs $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$ can give at most one.

1.7. The results of this paper show, in particular, that any arithmetic subgroup Γ of SU(2, 1), with $\chi(\Gamma) \leqslant 1$, must arise from a k-form G of SU(2, 1) as above, where the pair (k, ℓ) consists of $k = \mathbb{Q}$, and ℓ is one of the eleven imaginary quadratic fields listed in Proposition 3.5, or (k, ℓ) is one of the forty pairs $\mathcal{C}_1$–$\mathcal{C}_{40}$ described in 8.2. The covolumes, and hence the orbifold Euler–Poincaré characteristics, of these arithmetic subgroups can be computed using the volume formula given in 2.4 and the values of μ given in Proposition 3.5 and in 8.2. The surfaces arising as the quotient of B by one of these arithmetic subgroups will often be singular. However, as they have a small orbifold Euler–Poincaré characteristic, they may have interesting geometric properties.

2. Preliminaries

A comprehensive survey of the basic definitions and the main results of the Bruhat–Tits theory of reductive groups over nonarchimedean local fields is given in [Ti2].

2.1. Let the totally real number field k, and its totally complex quadratic extension ℓ, a real place v_o of k, and the k-form G of SU(2, 1) be as in 1.2. Throughout this paper, we will use the description of G given in 1.2 and the notations introduced in Sect. 1.

We shall say that a collection $(P_v)_{v\in V_f}$ of parahoric subgroups P_v of $G(k_v)$ is *coherent* if $\prod_{v\in V_f} P_v$ is an open subgroup of $G(A_f)$. If $(P_v)_{v\in V_f}$ is a coherent collection, then P_v is hyperspecial for all but finitely many v's.

We fix a coherent collection $(P_v)_{v\in V_f}$ of parahoric subgroups P_v of $G(k_v)$ such that for $v \in \mathcal{R}_\ell$, P_v is maximal (cf. 1.3). Let $\Lambda := G(k) \cap \prod_{v\in V_f} P_v$, and Γ be its normalizer in $G(k_{v_o})$. Note that as the normalizer of Λ in $G(k)$ equals Λ, $\Gamma \cap G(k) = \Lambda$. *We assume in the sequel that* $\chi(\Gamma) \leqslant 1$.

The Haar-measure μ on $G(k_{v_o})$ is the one used in [BP].

All unexplained notations are as in [BP] and [P]. Thus for a number field K, D_K denotes the absolute value of its discriminant, h_K its class number, i.e., the order of its class group $Cl(K)$. We shall denote by $n_{K,3}$ the order of the 3-primary component of $Cl(K)$, and by $h_{K,3}$ the order of the subgroup (of $Cl(K)$) consisting of the elements of order dividing 3. Then $h_{K,3} \leqslant n_{K,3} \leqslant h_K$.

For a number field K, $U(K)$ will denote the multiplicative-group of units of K, and K_3 the subgroup of $K^\times$ consisting of the elements x such that for every normalized valuation v of K, $v(x) \in 3\mathbb{Z}$.

We will denote the degree $[k : \mathbb{Q}]$ of k by d, and for any nonarchimedean place v of k, q_v will denote the cardinality of the residue field $\mathfrak{f}_v$ of k_v.

For a positive integer n, μ_n will denote the kernel of the endomorphism $x \mapsto x^n$ of GL_1. Then the center C of G is k-isomorphic to the kernel of the norm map $N_{\ell/k}$ from the algebraic group $R_{\ell/k}(\mu_3)$, obtained from μ_3 by Weil's restriction of scalars, to μ_3. Since the norm map $N_{\ell/k} : \mu_3(\ell) \to \mu_3(k)$ is onto, $\mu_3(k)/N_{\ell/k}(\mu_3(\ell))$ is trivial, and hence, the Galois cohomology group $H^1(k, C)$ is isomorphic to the kernel of the homomorphism $\ell^\times/\ell^{\times 3} \to k^\times/k^{\times 3}$ induced by the norm map. This kernel equals $\ell^\bullet/\ell^{\times 3}$, where $\ell^\bullet = \{x \in \ell^\times \mid N_{\ell/k}(x) \in k^{\times 3}\}$.

2.2. For $v \in V_f$, let the "type" Θ_v of P_v be as in [BP, 2.2], and Ξ_{Θ_v} be as in 2.8 there. We observe here, for later use, that for a nonarchimedean place v, Ξ_{Θ_v} is nontrivial if, and only if, G splits at v (then v splits in ℓ, i.e., $k_v \otimes_k \ell$ is a direct sum of two fields, each isomorphic to k_v) and P_v is an Iwahori subgroup of $G(k_v)$ (then Θ_v is the empty set), and in this case $\#\Xi_{\Theta_v} = 3$.

We recall that $G(k_v)$ contains a hyperspecial parahoric subgroup if, and only if, v is unramified in ℓ and G is quasi-split at v (i.e., it contains a Borel subgroup defined over k_v). Let $\mathcal{T}$ be the set of nonarchimedean places v of k which are unramified in ℓ and P_v is *not* a hyperspecial parahoric subgroup of $G(k_v)$, and $\mathcal{T}_0$ be the subset of $\mathcal{T}$ consisting of places where G is anisotropic. Then $\mathcal{T}$ is finite, and for any nonarchimedean $v \notin \mathcal{T}$, Ξ_{Θ_v} is trivial. We note that every place $v \in \mathcal{T}_0$ splits in ℓ since an absolutely simple anisotropic group over a nonarchimedean local field is necessarily of *inner* type A_n (another way to see this is to recall that, over a local field, the only central simple algebras which admit an involution of the second kind are the matrix algebras). We also note that every absolutely simple group of type A_2 defined and isotropic over a field K is quasi-split (i.e., it contains a Borel subgroup defined over K).

If v does not split in ℓ (i.e., $\ell_v := k_v \otimes_k \ell$ is a field), then G is quasi-split over k_v (and its k_v-rank is 1). In this case, if P_v is not an Iwahori subgroup, then it is a maximal parahoric subgroup of $G(k_v)$, and there are two conjugacy classes of maximal parahoric subgroups in $G(k_v)$. Moreover, if P' and P'' are the two maximal parahoric subgroups of $G(k_v)$ containing a common Iwahori subgroup I, then the derived subgroups of any Levi subgroups of the reduction mod $\mathfrak{p}$ of P' and P'' are nonisomorphic: if ℓ_v is an unramified extension of k_v, then the two derived subgroups are SU_3 and SL_2, and if ℓ_v is a ramified extension of k_v, then the two derived subgroups are SL_2 and PSL_2, see [Ti2, 3.5]. Hence, P' is not conjugate to P'' under the action of $(\mathrm{Aut}\, G)(k_v)$ $(= \overline{G}(k_v))$. In particular, *if an element of $\overline{G}(k_v)$ normalizes I, then it normalizes both P' and P'' also.* If v ramifies in ℓ, then P' and P'' are of same volume with respect to any Haar-measure on $G(k_v)$, since, in this case, $[P' : I] = [P'' : I]$.

2.3. By Dirichlet's unit theorem, $U(k) \cong \{\pm 1\} \times \mathbb{Z}^{d-1}$, and $U(\ell) \cong \mu(\ell) \times \mathbb{Z}^{d-1}$, where $\mu(\ell)$ is the finite cyclic group of roots of unity in ℓ. Hence, $U(k)/U(k)^3 \cong (\mathbb{Z}/3\mathbb{Z})^{d-1}$, and $U(\ell)/U(\ell)^3 \cong \mu(\ell)_3 \times (\mathbb{Z}/3\mathbb{Z})^{d-1}$, where $\mu(\ell)_3$ is the group of cube-roots of unity in ℓ. Now we observe that $N_{\ell/k}(U(\ell)) \supset N_{\ell/k}(U(k)) = U(k)^2$, which implies that the homomorphism $U(\ell)/U(\ell)^3 \to U(k)/U(k)^3$, induced by the norm map, is onto. The kernel of this homomorphism is clearly $U(\ell)^\bullet/U(\ell)^3$, where $U(\ell)^\bullet = U(\ell) \cap \ell^\bullet$, and its order equals $\#\mu(\ell)_3$.

The short exact sequence (4) in the proof of Proposition 0.12 in [BP] gives us the following exact sequence:

$$1 \to U(\ell)^\bullet/U(\ell)^3 \to \ell_3^\bullet/\ell^{\times 3} \to (\mathcal{P} \cap \mathcal{I}^3)/\mathcal{P}^3,$$

where $\ell_3^\bullet = \ell_3 \cap \ell^\bullet$, $\mathcal{P}$ is the group of all fractional principal ideals of ℓ, and $\mathcal{I}$ the group of all fractional ideals (we use multiplicative notation for the group operation in both $\mathcal{I}$ and $\mathcal{P}$). Since the order of the last group of the above exact sequence is $h_{\ell,3}$, see (5) in the proof of Proposition 0.12 in [BP], we conclude that

$$\#\ell_3^\bullet/\ell^{\times 3} \leqslant \#\mu(\ell)_3 \cdot h_{\ell,3}.$$

Now we note that the order of the first term of the short exact sequence of [BP, Proposition 2.9], for $G' = G$ and $S = V_\infty$, is $3/\#\mu(\ell)_3$.

The above observations, together with [BP, Proposition 2.9 and Lemma 5.4], and a close look at the arguments in [BP, 5.3 and 5.5] for $S = V_\infty$ and G of type ${}^2A_2$, give us the following upper bound (note that for our G, in [BP, 5.3], $n = 3$):

$$[\Gamma : \Lambda] \leqslant 3^{1+\#\mathcal{T}_0} h_{\ell,3} \prod_{v \in \mathcal{T} - \mathcal{T}_0} \#\Xi_{\Theta_v}. \tag{0}$$

We note also that [BP, Proposition 2.9] applied to $G' = G$ and $\Gamma' = \Gamma$, implies that the index $[\Gamma : \Lambda]$ of Λ in Γ is a power of 3.

2.4. As we mentioned in 1.3, $\chi(\Gamma) = 3\mu(G(k_{v_o})/\Gamma)$. Our aim here is to find a lower bound for $\mu(G(k_{v_o})/\Gamma)$. For this purpose, we first note that

$$\mu(G(k_{v_o})/\Gamma) = \frac{\mu(G(k_{v_o})/\Lambda)}{[\Gamma : \Lambda]}.$$

As the Tamagawa number $\tau_k(G)$ of G equals 1, the volume formula of [P] (recalled in [BP, §3.7]), for $S = V_\infty$, gives us

$$\mu(G(k_{v_o})/\Lambda) = D_k^4 \big(D_\ell/D_k^2\big)^{5/2} (16\pi^5)^{-d} \mathcal{E} = \big(D_\ell^{5/2}/D_k\big)(16\pi^5)^{-d} \mathcal{E};$$

where $\mathcal{E} = \prod_{v \in V_f} e(P_v)$, and

$$e(P_v) = \frac{q_v^{(\dim \overline{M}_v + \dim \overline{\mathcal{M}}_v)/2}}{\#\overline{M}_v(\mathfrak{f}_v)}.$$

We observe that if P_v is hyperspecial,

$$e(P_v) = \left(1 - \frac{1}{q_v^2}\right)^{-1}\left(1 - \frac{1}{q_v^3}\right)^{-1} \quad \text{or} \quad \left(1 - \frac{1}{q_v^2}\right)^{-1}\left(1 + \frac{1}{q_v^3}\right)^{-1}$$

according as v does or does not split in ℓ. If v ramifies in ℓ and P_v is a maximal parahoric subgroup of $G(k_v)$, then

$$e(P_v) = \left(1 - \frac{1}{q_v^2}\right)^{-1}.$$

Now let ζ_k be the Dedekind zeta-function of k, and $L_{\ell|k}$ be the Hecke L-function associated to the quadratic Dirichlet character of ℓ/k. Then as

$$\zeta_k(2) = \prod_{v \in V_f}\left(1 - \frac{1}{q_v^2}\right)^{-1},$$

and

$$L_{\ell|k}(3) = \prod{}'\left(1 - \frac{1}{q_v^3}\right)^{-1}\prod{}''\left(1 + \frac{1}{q_v^3}\right)^{-1},$$

where $\prod'$ is the product over those nonarchimedean places of k which split in ℓ, and $\prod''$ is the product over all the other nonarchimedean places v which do not ramify in ℓ, we see that

$$\mathcal{E} = \zeta_k(2)L_{\ell|k}(3)\prod_{v \in \mathcal{T}} e'(P_v);$$

where, for $v \in \mathcal{T}$,

- if v splits in ℓ, $e'(P_v) = e(P_v)\big(1 - \frac{1}{q_v^2}\big)\big(1 - \frac{1}{q_v^3}\big)$,
- if v does not split in ℓ but is unramified in ℓ, $e'(P_v) = e(P_v)\big(1-\frac{1}{q_v^2}\big)\big(1+\frac{1}{q_v^3}\big)$,
- if v ramifies in ℓ, $e'(P_v) = e(P_v)\big(1 - \frac{1}{q_v^2}\big)$.

Thus

$$(1) \qquad \mu(G(k_{v_0})/\Gamma) = \frac{D_\ell^{5/2}\zeta_k(2)L_{\ell|k}(3)}{(16\pi^5)^d[\Gamma : \Lambda]D_k}\prod_{v \in \mathcal{T}} e'(P_v)$$
$$\geqslant \frac{D_\ell^{5/2}\zeta_k(2)L_{\ell|k}r(3)}{3(16\pi^5)^d h_{\ell,3} D_k}\prod_{v \in \mathcal{T}} e''(P_v),$$

where, for $v \in \mathcal{T} - \mathcal{T}_0$, $e''(P_v) = e'(P_v)/\#\Xi_{\Theta_v}$, and for $v \in \mathcal{T}_0$, $e''(P_v) = e'(P_v)/3$.

2.5. Now we provide the following list of values of $e'(P_v)$ and $e''(P_v)$, for all $v \in \mathcal{T}$.

(i) v splits in ℓ and G splits at v:

(a) if P_v *is an Iwahori subgroup*, then

$$e''(P_v) = e'(P_v)/3,$$

and

$$e'(P_v) = (q_v^2 + q_v + 1)(q_v + 1);$$

(b) if P_v *is not an Iwahori subgroup* (note that as $v \in \mathcal{T}$, P_v is not hyperspecial), then

$$e''(P_v) = e'(P_v) = q_v^2 + q_v + 1;$$

(ii) v splits in ℓ and G is anisotropic at v (i.e., $v \in \mathcal{T}_0$):

$$e''(P_v) = e'(P_v)/3,$$

and

$$e'(P_v) = (q_v - 1)^2(q_v + 1);$$

(iii) v does not split in ℓ, then it is unramified in ℓ, and

$$e''(P_v) = e'(P_v) = \begin{cases} q_v^3 + 1 & \text{if } P_v \text{ is an Iwahori subgroup} \\ q_v^2 - q_v + 1 & \text{if } P_v \text{ is a non-hyperspecial} \\ & \text{maximal parahoric subgroup.} \end{cases}$$

2.6. As $\chi(\Gamma) \leqslant 1$, $\mu(G(k_{v_o})/\Gamma) \leqslant 1/3$. So from (1) in 2.4 we get the following:

$$1/3 \geqslant \mu(G(k_{v0})/\Gamma) \geqslant \frac{D_\ell^{5/2}\zeta_k(2)L_{\ell|k}(3)}{3(16\pi^5)^d h_{\ell,3} D_k} \prod_{v\in\mathcal{T}} e''(P_v). \tag{2}$$

We know from the Brauer–Siegel theorem that for all real $s > 1$,

$$h_\ell R_\ell \leqslant w_\ell s(s-1)\Gamma(s)^d \big((2\pi)^{-2d} D_\ell\big)^{s/2} \zeta_\ell(s), \tag{3}$$

where h_ℓ is the class number and R_ℓ is the regulator of ℓ, and w_ℓ is the order of the finite group of roots of unity contained in ℓ. Zimmert [Z] obtained the following lower bound for the regulator R_ℓ

$$R_\ell \geqslant 0.02 w_\ell e^{0.1d}.$$

Also, we have the following lower bound for the regulator obtained by Slavutskii [Sl] using a variant of the argument of Zimmert [Z]:

$$R_\ell \geqslant 0.00136 w_\ell \, e^{0.57d}.$$

We deduce from this bound and (3) that

$$\frac{1}{h_{\ell,3}} \geqslant \frac{1}{h_\ell} \geqslant \frac{0.00136}{s(s-1)} \left(\frac{(2\pi)^s e^{0.57}}{\Gamma(s)} \right)^d \frac{1}{D_\ell^{s/2} \zeta_\ell(s)}; \tag{4}$$

if we use Zimmert's lower bound for R_ℓ instead, we obtain

$$\frac{1}{h_{\ell,3}} \geqslant \frac{1}{h_\ell} \geqslant \frac{0.02}{s(s-1)} \left(\frac{(2\pi)^s e^{0.1}}{\Gamma(s)} \right)^d \frac{1}{D_\ell^{s/2} \zeta_\ell(s)}. \tag{5}$$

2.7. Lemma. *For every integer $r \geqslant 2$, $\zeta_k(r)^{1/2} L_{\ell|k}(r+1) > 1$.*

Proof. Recall that

$$\zeta_k(r) = \prod_{v \in V_f} \left(1 - \frac{1}{q_v^r}\right)^{-1},$$

and

$$L_{\ell|k}(r+1) = \prod{}' \left(1 - \frac{1}{q_v^{r+1}}\right)^{-1} \prod{}'' \left(1 + \frac{1}{q_v^{r+1}}\right)^{-1},$$

where $\prod'$ is the product over all finite places v of k which split over ℓ and $\prod''$ is the product over all the other nonarchimedean v which do not ramify in ℓ. Now the lemma follows from the following simple observation.

For any positive integer $q \geqslant 2$,

$$\left(1 - \frac{1}{q^r}\right)\left(1 + \frac{1}{q^{r+1}}\right)^2 = 1 - \frac{q-2}{q^{r+1}} - \frac{2q-1}{q^{2r+2}} - \frac{1}{q^{3r+2}} < 1.$$

2.8. Corollary. *For every integer $r \geqslant 2$,*

$$\zeta_k(r) L_{\ell|k}(r+1) > \zeta_k(r)^{1/2} > 1.$$

2.9. Remark. The following bounds are obvious from the Euler-product expression for the zeta-functions. For every integer $r \geqslant 2$,

$$\zeta(dr) \leqslant \zeta_k(r) \leqslant \zeta(r)^d,$$

where $\zeta(j) = \zeta_{\mathbb{Q}}(j)$. Now from the above corollary we deduce that

$$\zeta_k(2)L_{\ell|k}(3) > \zeta_k(2)^{1/2} \geqslant \zeta(2d)^{1/2} > 1. \tag{6}$$

2.10. Since $e''(P_v) \geqslant 1$ for all $v \in \mathcal{T}$, see 2.5 above, and D_ℓ/D_k^2 is an integer, so in particular, $D_k \leqslant D_\ell^{1/2}$, see, for example, Theorem A in the appendix of [P], bounds (2) and (3) lead to the following bounds by taking $s = 1+\delta$, with $0 < \delta \leqslant 2$, in (3)

$$\begin{aligned}(7)\quad D_k^{1/d} \leqslant D_\ell^{1/2d} &< \varphi_1(d, R_\ell/w_\ell, \delta)\\ &:= \left(\frac{\delta(1+\delta)}{\zeta(2d)^{1/2}(R_\ell/w_\ell)}\right)^{1/(3-\delta)d}\left(2^{3-\delta}\pi^{4-\delta}\Gamma(1+\delta)\zeta(1+\delta)^2\right)^{1/(3-\delta)},\end{aligned}$$

$$(8)\quad D_k^{1/d} \leqslant D_\ell^{1/2d} < \varphi_2(d, h_{\ell,3}) := \left[\frac{2^{4d}\pi^{5d}h_{\ell,3}}{\zeta(2d)^{1/2}}\right]^{1/4d},$$

and

$$D_\ell/D_k^2 < \mathfrak{p}(d, D_k, h_{\ell,3}) := \left[\frac{2^{4d}\pi^{5d}h_{\ell,3}}{\zeta(2d)^{1/2}D_k^4}\right]^{2/5}. \tag{9}$$

Using the bound $R_\ell/w_\ell \geqslant 0.00136e^{0.57d}$ due to Slavutskii, we obtain the following bound from (7):

$$\begin{aligned}(10)\quad D_k^{1/d} \leqslant D_\ell^{1/2d} &< f(\delta, d)\\ &:= \left[\frac{\delta(1+\delta)}{0.00136}\right]^{1/(3-\delta)d} \cdot [2^{3-\delta}\pi^{4-\delta}\Gamma(1+\delta)\zeta(1+\delta)^2e^{-0.57}]^{1/(3-\delta)}.\end{aligned}$$

2.11. As $\chi(\Lambda) = 3\mu(G(k_{v_o})/\Lambda)$,

$$\chi(\Gamma) = \frac{\chi(\Lambda)}{[\Gamma:\Lambda]} = \frac{3\mu(G(k_{v_o})/\Lambda)}{[\Gamma:\Lambda]}.$$

Now since $[\Gamma:\Lambda]$ is a power of 3 (see 2.3), if $\chi(\Gamma)$ is a reciprocal integer, the numerator of the rational number $\mu(G(k_{v_o})/\Lambda)$ is a power of 3.

We recall from 2.4 that

$$\mu(G(k_{v_o})/\Lambda) = \left(D_\ell^{5/2}/D_k\right)(16\pi^5)^{-d}\zeta_k(2)L_{\ell|k}(3)\prod_{v\in\mathcal{T}} e'(P_v).$$

Using the functional equations

$$\zeta_k(2) = (-2)^d\pi^{2d}D_k^{-3/2}\zeta_k(-1),$$

and

$$L_{\ell|k}(3) = (-2)^d \pi^{3d} (D_k/D_\ell)^{5/2} L_{\ell|k}(-2),$$

we can rewrite the above as:

$$\mu(G(k_{v_o})/\Lambda) = 2^{-2d} \zeta_k(-1) L_{\ell|k}(-2) \prod_{v \in \mathcal{T}} e'(P_v). \tag{11}$$

Hence we obtain the following proposition.

2.12. Proposition. *If the orbifold Euler–Poincaré characteristic $\chi(\Gamma)$ of Γ is a reciprocal integer, then the numerator of the rational number $2^{-2d} \zeta_k(-1) L_{\ell|k}(-2) \prod_{v\in\mathcal{T}} e'(P_v)$ is a power of 3. Moreover, as $e'(P_v)$ is an integer for all v, the numerator of $\mu := 2^{-2d} \zeta_k(-1) L_{\ell|k}(-2)$ is also a power of 3.*

3. Determining ℓ when $k = \mathbb{Q}$

We will assume in this, and the next section, that $k = \mathbb{Q}$. Then $\ell = \mathbb{Q}(\sqrt{-a})$, where a is a square-free positive integer.

We will now find an upper bound for D_ℓ.

3.1. Since $D_k = D_{\mathbb{Q}} = 1$, and $e''(P_v) \geqslant 1$, from (2), (5) and (6), taking $s = 1 + \delta$, we get the following:

$$D_\ell < (2\pi)^2 \left(\frac{5^2 \cdot \delta(1+\delta) \cdot \Gamma(1+\delta) \zeta(1+\delta)^2}{e^{0.1} \zeta(2)^{1/2}} \right)^{2/(4-\delta)}. \tag{12}$$

Letting $\delta = 0.34$, we find that $D_\ell < 461.6$. Hence we conclude that $D_\ell \leqslant 461$.

Thus we have established the following.

3.2. *If $\chi(\Gamma) \leqslant 1$ and $k = \mathbb{Q}$, then $D_\ell \leqslant 461$.*

3.3. We will now improve the upper bound for the discriminant of ℓ using the table of class numbers of imaginary quadratic number fields.

Inspecting the table of class numbers of $\ell = \mathbb{Q}(\sqrt{-a})$, with $D_\ell \leqslant 461$, in [BS], we find that $h_\ell \leqslant 21$, and hence, $h_{\ell,3} \leqslant n_{\ell,3} \leqslant 9$.

Since $D_{\mathbb{Q}} = 1$, $\zeta_{\mathbb{Q}}(2) = \zeta(2) = \pi^2/6$ and $\zeta(3) L_{\ell|\mathbb{Q}}(3) = \zeta_\ell(3) > 1$, (2) provides us the following bounds

$$\begin{aligned} 1 &\geqslant \frac{D_\ell^{5/2} L_{\ell|\mathbb{Q}}(3)}{2^5 \cdot 3 \cdot \pi^3 \cdot h_{\ell,3}} \prod_{v\in\mathcal{T}} e''(P_v) \geqslant \frac{D_\ell^{5/2} \zeta_\ell(3)}{2^5 \cdot 3 \cdot \pi^3 \cdot h_{\ell,3} \zeta(3)} \\ &> \frac{D_\ell^{5/2}}{2^5 \cdot 3 \cdot \pi^3 \cdot h_{\ell,3} \zeta(3)}. \end{aligned}$$

Hence, in particular, as $h_{\ell,3} \leqslant n_{\ell,3}$,

$$D_\ell < \left(2^5 \cdot 3 \cdot \pi^3 \cdot n_{\ell,3}\zeta(3)\right)^{2/5}.$$

The above leads to the following bounds once the value of $n_{\ell,3}$ is determined.

$n_{\ell,3}$	1	3	9
$D_\ell \leqslant$	26	40	63.

The last column of the above table implies that we need only consider $D_\ell \leqslant 63$.

3.4. We will further limit the possibilities for D_ℓ. If $40 < D_\ell \leqslant 63$, we observe that $n_{\ell,3} \leqslant 3$ from the table in Appendix. Hence, from the middle column of the above table we infer that D_ℓ can at most be 40.

For $26 < D_\ell \leqslant 40$, we see from the table in Appendix that unless $D_\ell = 31$, $n_{\ell,3} = 1$, and the first column of the above table shows that if $n_{\ell,3} = 1$, $D_\ell \leqslant 26$. Hence, the only possible values of D_ℓ are 31 or $D_\ell \leqslant 26$.

From the table in Appendix we now see that the possible values of $h_{\ell,3}$ and D_ℓ are the following (note that if $n_{\ell,3} = 3$, then $h_{\ell,3} = 3$ also).

$$h_{\ell,3} = 3: \quad D_\ell = 23, 31.$$
$$h_{\ell,3} = 1: \quad D_\ell = 3, 4, 7, 8, 11, 15, 19, 20, 24.$$

Now we recall that for $\ell = \mathbb{Q}(\sqrt{-a})$, $D_\ell = a$ if $a \equiv 3 \pmod 4$, and $D_\ell = 4a$ otherwise. Using this we can paraphrase the above result as follows.

3.5. Proposition. *Let $k = \mathbb{Q}$. Then $\ell = \mathbb{Q}(\sqrt{-a})$, where a is one of the following eleven integers,*

$$1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 11,\ 15,\ 19,\ 23,\ 31.$$

The following table provides the value of

$$\mu := \frac{D_\ell^{5/2}\zeta(2)L_{\ell|\mathbb{Q}}(3)}{16\pi^5} = -\frac{1}{48}L_{\ell|\mathbb{Q}}(-2)$$

(recall the functional equation $L_{\ell|\mathbb{Q}}(3) = -2\pi^3 D_\ell^{-5/2} L_{\ell|\mathbb{Q}}(-2)$).

a	1	2	3	5	6	7
$L_{\mathbb{Q}(\sqrt{-a})\mid\mathbb{Q}}(-2)$	$-1/2$	-3	$-2/9$	-30	-46	$-16/7$
μ	1/96	1/16	1/216	5/8	23/24	1/21

a	11	15	19	23	31
$L_{\mathbb{Q}(\sqrt{-a})\mid\mathbb{Q}}(-2)$	-6	-16	-22	-48	-96
μ	1/8	1/3	11/24	1	2.

3.6. The volume formula of [P] and the results of [BP] apply equally well to *noncocompact arithmetic* subgroups. So if we wish to make a list of all noncocompact arithmetic subgroups Γ of SU(2, 1) whose orbifold Euler–Poincaré characteristic $\chi(\Gamma)$ is $\leqslant 1$, we can proceed as above. If Γ is such a subgroup, then, associated to it, there is an absolutely simple simply connected algebraic group G defined and (by Godement compactness criterion) isotropic over a number field k such that $G(k \otimes_{\mathbb{Q}} \mathbb{R})$ is isomorphic to the direct product of SU(2, 1) with a compact semi-simple Lie group. But since G is k-isotropic, for every place v of k, G is isotropic over k_v, and hence, $G(k_v)$ is noncompact. In particular, for every archimedean place v of k, $G(k_v)$ is noncompact. This implies that $k = \mathbb{Q}$, G is an absolutely simple simply connected $\mathbb{Q}$-group of type A_2 of $\mathbb{Q}$-rank 1 (and hence G is quasi-split over $\mathbb{Q}$). Moreover, G splits over an imaginary quadratic extension $\ell = \mathbb{Q}(\sqrt{-a})$ of $\mathbb{Q}$. For a given positive integer a, there is a unique such G (up to $\mathbb{Q}$-isomorphism). The considerations of 3.1–3.4 apply again and imply that a has to be one of the eleven integers listed in Proposition 3.5.

We fix a coherent collection (P_p) of maximal parahoric subgroups P_p of $G(\mathbb{Q}_p)$ such that P_p is hyperspecial whenever $G(\mathbb{Q}_q)$ contains such a parahoric subgroup. Let $\Lambda = G(\mathbb{Q}) \cap \prod_p P_p$. (This Λ is a "Picard modular group".) From the volume formula of [P], recalled in 2.4, we obtain that

$$\begin{aligned}\chi(\Lambda) = 3\mu(G(\mathbb{R})/\Lambda) &= 3\frac{D_\ell^{5/2}\zeta_{\mathbb{Q}}(2)L_{\ell|\mathbb{Q}}(3)}{16\pi^5} = \frac{D_\ell^{5/2}L_{\ell|\mathbb{Q}}(3)}{32\pi^3}\\ &= -\frac{1}{16}L_{\ell|\mathbb{Q}}(-2) = 3\mu,\end{aligned}$$

where we have used the functional equation for the L-function $L_{\ell|\mathbb{Q}}$ recalled in 3.5, and the fact that $\zeta_{\mathbb{Q}}(2) = \zeta(2) = \pi^2/6$. (We note that the above computation of the orbifold Euler–Poncaré characteristic of Picard modular groups is independently due to Rolf-Peter Holzapfel, see [Ho, Sect. 5A.]) Now we can use the table of values of $L_{\ell|\mathbb{Q}}(-2)$ given in 3.5 to compute the precise value of $\chi(\Lambda)$ for each a.

Among all arithmetic subgroups of G contained in $G(\mathbb{Q})$, the above Λ has the smallest orbifold Euler–Poincaré characteristic. Its normalizer Γ in $G(\mathbb{R})$ has the smallest orbifold Euler–Poincaré characteristic among all discrete subgroups commensurable with Λ. Note that Λ has torsion.

4. Determination of G and the parahoric subgroups P_v

We continue to assume in this section that $k = \mathbb{Q}$. We will use the usual identification of a nonarchimedean place v of $\mathbb{Q}$ with the characteristic p of the residue field of $\mathbb{Q}_v$. Let ℓ be one of the eleven imaginary quadratic extensions of $\mathbb{Q}$ listed in Proposition 3.5. $\mathscr{R}_\ell$ will denote the set of rational primes which ramify in ℓ.

4.1. Let $\mathscr{D}$, the involution σ, the hermitian form h, and the k-group G, for $k = \mathbb{Q}$, be as in 1.2. As in 2.1 we fix a coherent collection (P_p) of parahoric subgroups of $G(\mathbb{Q}_p)$ such that for $p \in \mathscr{R}_\ell$, P_p is maximal. Let $\Lambda = G(\mathbb{Q}) \cap \prod_p P_p$, and Γ be the normalizer of Λ in $G(\mathbf{R})$. *We assume that Γ is cocompact and $\chi(\Gamma)$ is a reciprocal integer.*

We first show that $\mathscr{D}$ is a cubic division algebra. Assume, if possible, that $\mathscr{D} = \ell$. Then h is a hermitian form on ℓ^3. As the arithmetic subgroup Γ of $G(\mathbb{R})$ is cocompact, by Godement compactness criterion, h is an anisotropic form on ℓ^3. On the other hand, its signature over $\mathbb{R}$ is (2, 1). The hermitian form h gives us a quadratic form q on the six dimensional $\mathbb{Q}$-vector space $V = \ell^3$ defined as follows:

$$q(v) = h(v, v) \quad \text{for} \quad v \in V.$$

The quadratic form q is isotropic over $\mathbb{R}$, and hence by Meyer's theorem it is isotropic over $\mathbb{Q}$ (cf. [Se2]). This implies that h is isotropic and we have arrived at a contradiction.

4.2. Let $\mathscr{T}$ be the finite set of rational primes $p \notin \mathscr{R}_\ell$ such that P_p is not hyperspecial, and $\mathscr{T}_0$ be the subset of $\mathscr{T}$ consisting of p such that G is anisotropic over $\mathbb{Q}_p$. Since $\mathscr{D}$ must ramify at at least some nonarchimedean places of ℓ, $\mathscr{T}_0$ is nonempty. As pointed out in 2.2, *every $p \in \mathscr{T}_0$ splits in ℓ.* Theorem 4.4 lists all possible ℓ, $\mathscr{T}$, $\mathscr{T}_0$, and the parahoric subgroups P_p.

As $\zeta_{\mathbb{Q}}(2) = \zeta(2) = \pi^2/6$, using the functional equation

$$L_{\ell|\mathbb{Q}}(3) = -2\pi^3 D_\ell^{-5/2} L_{\ell|\mathbb{Q}}(-2),$$

we obtain the following from bound (1) for $k = \mathbb{Q}$:

$$\chi(\Gamma) = 3\mu(G(\mathbb{R})/\Gamma) \geqslant \frac{\mu}{h_{\ell,3}} \prod_{p\in\mathscr{T}} e''(P_p),$$

where μ is as in 3.5.

4.3. We recall here that given a square-free integer a, an odd prime p splits in $\ell = \mathbb{Q}(\sqrt{-a})$ if, and only if, p does not divide a, and $-a$ is a square modulo p; 2 splits in ℓ if, and only if, $-a \equiv 1 \pmod 8$; see [BS, §8 of Chap. 3]. A prime p ramifies in ℓ if, and only if, $p|D_\ell$; see [BS, §7 of Chap. 2 and §8 of Chap. 3].

Now using Proposition 3.5, the fact that the numerators of μ and $\mu(G(\mathbb{R})/\Lambda) = \mu \prod_{p\in\mathscr{T}} e'(P_p)$ are powers of 3 (Proposition 2.12), the value of μ given in 3.5, the values of $e'(P_p)$, $e''(P_p)$ given in 2.5, the value of $h_{\ell,3}$ given in 3.4, and the fact that $\chi(\Gamma) \leqslant 1$, we see by a direct computation that the following holds.

4.4. Theorem. *$\mathscr{T}_0$ consists of a single prime, and the pair (a, p), where $\ell = \mathbb{Q}(\sqrt{-a})$, and $\mathscr{T}_0 = \{p\}$, belongs to the set* $\{(1, 5), (2, 3), (7, 2), (15, 2), (23, 2)\}$. *Moreover, $\mathscr{T} = \mathscr{T}_0$.*

4.5. Since for $a \in \{1, 2, 7, 15, 23\}$, $\mathcal{T}_0$ consists of a single prime, for each a we get exactly two cubic division algebras, with center $\ell = \mathbb{Q}(\sqrt{-a})$, and they are opposite of each other. Therefore, each of the five possible values of a determines the $\mathbb{Q}$-form G of SU(2, 1) uniquely (up to a $\mathbb{Q}$-isomorphism), and for $q \notin \mathcal{R}_\ell$, the parahoric subgroup P_q of $G(\mathbb{Q}_q)$ uniquely (up to conjugation by an element of $\overline{G}(\mathbb{Q}_q)$, where $\overline{G}$ is the adjoint group of G).

We can easily compute $\mu(G(\mathbb{R})/\Lambda)$, which, using the volume formula given in 2.4 is seen to be equal to $\mu e'(P_p)$, where (a, p) is as in the preceding theorem, μ is as in Proposition 3.5, and (see 2.5 (*ii*)) $e'(P_p) = (p-1)^2(p+1)$. We find that $\mu(G(\mathbb{R})/\Lambda)$ equals 1, 1, 1/7, 1, and 3, for $a =$ 1, 2, 7, 15, and 23 respectively. This computation is clearly independent of the choice of maximal parahoric subgroups P_q in $G(\mathbb{Q}_q)$ for primes q which ramify in $\ell = \mathbb{Q}(\sqrt{-a})$.

In the sequel, the prime p appearing in the pair (a, p) will be called the prime *associated* to a, and we will sometimes denote it by p_a.

5. The fake projective planes arising from $k = \mathbb{Q}$

We will show in this section that there are exactly twelve finite classes (cf. 1.5) of fake projective planes with $k = \mathbb{Q}$. We will explicitly determine their fundamental groups.

We prove results in 5.2–5.4 for an arbitrary totally real number field k for applications in Sects. 8 and 9.

5.1. We will use the notation introduced in 1.2. In particular, k is a totally real number field of degree d, ℓ a totally complex quadratic extension of k, and v_o is a real place of k, G is a simple simply connected algebraic k-group, which is an *inner* form of SL_3 over ℓ, such that $G(k_{v_o}) \cong \mathrm{SU}(2, 1)$, and for all real places $v \neq v_o$, $G(k_v) \cong \mathrm{SU}(3)$. We recall (1.2) that there is a division algebra $\mathcal{D}$ of degree $n|3$, with center ℓ, $\mathcal{D}$ given with an involution σ of the second kind so that $\sigma|_\ell$ is the nontrivial k-automorphism of ℓ, and a nondegenerate hermitian form h on $\mathcal{D}^{3/n}$ defined in terms of the involution σ, such that G is the special unitary group $\mathrm{SU}(h)$ of the hermitian form h.

Let $\mathcal{T}_0$ be the finite set of nonarchimedean places of k where G is anisotropic. As pointed out in 2.2, *every place* $v \in \mathcal{T}_0$ *splits in* ℓ. If $\mathcal{D} = \ell$, then h is a hermitian form on ℓ^3, and G is isotropic at every nonarchimedean place of k, so in this case $\mathcal{T}_0$ is empty. Now we note that $\mathcal{T}_0$ is nonempty if $\mathcal{D}$ is a cubic division algebra since this division algebra must ramify at least at two nonarchimedean places of ℓ.

5.2. Let C be the center of G, $\overline{G}$ the adjoint group, and let $\varphi : G \to \overline{G}$ the natural isogeny. Let $\mathcal{P} = (P_v)_{v \in V_f}$ and $\mathcal{P}' = (P'_v)_{v \in V_f}$ be two coherent collections of parahoric subgroups such that for all $v \in V_f$, P'_v is conjugate to P_v under an element of $\overline{G}(k_v)$. For all but finitely many v, $P_v = P'_v$, and they are hyperspecial. Therefore, there is an element $g \in \overline{G}(A_f)$ such that $\mathcal{P}'$ is the conjugate of $\mathcal{P}$ under g. Let $\overline{P}_v$ be the stabilizer of P_v in $\overline{G}(k_v)$. Then $\overline{K} := \prod_{v \in V_f} \overline{P}_v$ is the stabilizer of $\mathcal{P}$ in $\overline{G}(A_f)$, and it is a compact-open subgroup of the latter. So the number of distinct $\overline{G}(k)$-conjugacy classes of coherent collections $\mathcal{P}'$ as above is the cardinality[1] of $\overline{G}(k)\backslash\overline{G}(A_f)/\overline{K}$.

As $\varphi : G \to \overline{G}$ is a central isogeny, $\varphi(G(A_f))$ contains the commutator subgroup of $\overline{G}(A_f)$. Moreover, as G is simply connected and $G(k_{v_o})$ is noncompact, by the strong approximation property [PlR, Theorem 7.12], $G(k)$ is dense in $G(A_f)$, i.e., for any open neighborhood Ω of the identity in $G(A_f)$, $G(k)\Omega = G(A_f)$. This implies that $\overline{G}(k)\overline{K}$ contains $\varphi(G(A_f))$, which in turn contains $[\overline{G}(A_f), \overline{G}(A_f)]$. Hence, $\overline{G}(k)\overline{K} = \overline{G}(k)[\overline{G}(A_f), \overline{G}(A_f)]\overline{K}$. Using this observation it is easy to see that $\overline{G}(k)\overline{K}$ is a subgroup, and the natural map from $\overline{G}(k)\backslash\overline{G}(A_f)/\overline{K}$ to the finite abelian group $\overline{G}(A_f)/\overline{G}(k)\overline{K}$ is bijective. We shall next show that this latter group is trivial if $h_{\ell,3} = 1$.

We begin by observing that for every $v \in V_\infty$, $H^1(k_v, C)$ vanishes since C is a group of exponent 3. Now since by the Hasse principle for simply connected semi-simple k-groups [PlR, Theorem 6.6] $H^1(k, G) \to \prod_{v \in V_\infty} H^1(k_v, G)$ is an isomorphism, we conclude that the natural map $H^1(k, C) \to H^1(k, G)$ is trivial, and hence the coboundary homomorphism $\delta : \overline{G}(k) \to H^1(k, C)$ *is surjective.*

Now we note that since for each nonarchimedean place v, $H^1(k_v, G)$ is trivial [PlR, Theorem 6.4], the coboundary homomorphism $\delta_v : \overline{G}(k_v) \to H^1(k_v, C)$ is surjective and its kernel equals $\varphi(G(k_v))$. Now let v be a nonarchimedean place of k which either does not split in ℓ, or it splits in ℓ and P_v is an Iwahori subgroup of $G(k_v)$, and $g \in \overline{G}(k_v)$. Then the parahoric subgroup $g(P_v)$ is conjugate to P_v under an element of $G(k_v)$, and hence, $\overline{G}(k_v) = \varphi(G(k_v))\overline{P}_v$, which implies that $\delta_v(\overline{P}_v) = \delta_v(\overline{G}(k_v)) = H^1(k_v, C)$. We observe also that for any nonarchimedean place v of k, the subgroup $\varphi(G(k_v))\overline{P}_v$ is precisely the stabilizer of the type Θ_v ($\subset \Delta_v$) of P_v under the natural action of $\overline{G}(k_v)$ on Δ_v described in [BP, 2.2]. Thus $\delta_v(\overline{P}_v) = H^1(k_v, C)_{\Theta_v}$, where $H^1(k_v, C)_{\Theta_v}$ is the stabilizer of Θ_v in $H^1(k_v, C)$ under the action of the latter on Δ_v through ξ_v given in [BP, 2.5]. It can be seen, but we do not need this fact here, that for any nonarchimedean place v of k which does not lie over 3 and P_v is a hyperspecial parahoric subgroup of $G(k_v)$, $\delta_v(\overline{P}_v)$ equals $H^1_{nr}(k_v, C)$, where $H^1_{nr}(k_v, C)$ ($\subset H^1(k_v, C)$) is the "unramified Galois cohomology" as in [Se3, Chap. II, §5.5].

The coboundary homomorphisms δ_v combine to provide an isomorphism

$$\overline{G}(A_f)/\overline{G}(k)\overline{K} \longrightarrow \mathcal{C} := \prod{}' H^1(k_v, C)/\Big(\psi(H^1(k, C)) \cdot \prod_v \delta_v(\overline{P}_v)\Big),$$

[1] This number is called the "class number" of $\overline{G}$ relative to $\overline{K}$ and is known to be finite, see for example, Proposition 3.9 of [BP].

where $\prod' H^1(k_v, C)$ denotes the subgroup of $\prod_{v\in V_f} H^1(k_v, C)$ consisting of the elements $c = (c_v)$ such that for all but finitely many v, c_v lies in $\delta_v(\overline{P}_v)$, and $\psi : H^1(k, C) \to \prod' H^1(k_v, C)$ is the natural homomorphism.

Andrei Rapinchuk's remark that $R_{\ell/k}(\mu_3)$ is a direct product of C and (the naturally embedded subgroup) μ_3 has helped us to simplify the following discussion.

$H^1(k, C)$ can be identified with $\ell^\times/k^\times{\ell^\times}^3$, and for any place v of k, $H^1(k_v, C)$ can be identified with $(k_v \otimes_k \ell)^\times/k_v^\times{(k_v \otimes_k \ell)^\times}^3$. Now let $\mathcal{S}$ be the finite set of nonarchimedean places of k which split in ℓ and P_v is an Iwahori subgroup of $G(k_v)$. If $v \notin \mathcal{S}$ is a nonarchimedean place which splits in ℓ, and w', w'' are the two places of ℓ lying over v, then the subgroup $\delta_v(\overline{P}_v)$ gets identified with

$$k_v^\times\left(\mathfrak{o}_{w'}^\times{\ell_{w'}^\times}^3 \times \mathfrak{o}_{w''}^\times{\ell_{w''}^\times}^3\right)/k_v^\times\left({\ell_{w'}^\times}^3 \times {\ell_{w''}^\times}^3\right),$$

where $\mathfrak{o}_{w'}^\times$ (resp., $\mathfrak{o}_{w''}^\times$) is the group of units of $\ell_{w'}$ (resp., $\ell_{w''}$), cf. [BP, Lemma 2.3(ii)] and the proof of Proposition 2.7 in there.

Now let I_k^f (resp., I_ℓ^f) be the group of finite idèles of k (resp., ℓ), i.e., the restricted direct product of the $k_v^\times$s (resp., $\ell_w^\times$s) for all nonarchimedean places v of k (resp., w of ℓ). We shall view I_k^f as a subgroup of I_ℓ^f in terms of its natural embedding. Then it is obvious that $\mathcal{C}$ is isomorphic to the quotient of I_ℓ^f by the subgroup generated by $I_k^f \cdot {(I_\ell^f)}^3 \cdot \ell^\times$ and all the elements $x = (x_w) \in I_\ell^f$ such that $x_w \in \mathfrak{o}_w^\times$ for every nonarchimedean place w of ℓ which lies over a place of k which splits in ℓ but is not in $\mathcal{S}$. From this it is obvious that $\mathcal{C}$ is a quotient of the class group $Cl(\ell)$ of ℓ, and its exponent is 3. This implies that $\mathcal{C}$ is trivial if $h_{\ell,3} = 1$.

Let us now assume that $\ell = \mathbb{Q}(\sqrt{-23})$, and $\mathcal{S} = \{2\}$. Then $h_{\ell,3} = 3$. But as either of the two prime ideals lying over 2 in $\ell = \mathbb{Q}(\sqrt{-23})$ generates the class group of ℓ, we see that $\mathcal{C}$ is again trivial. Thus we have proved the following.

5.3. Proposition. *Let $\mathcal{P} = (P_v)_{v\in V_f}$ and $\mathcal{P}' = (P'_v)_{v\in V_f}$ be two coherent collections of parahoric subgroups such that for every v, P'_v is conjugate to P_v under an element of $\overline{G}(k_v)$. Then there is an element in $\overline{G}(k)$ which conjugates $\mathcal{P}'$ to $\mathcal{P}$ if $h_{\ell,3} = 1$. This is also the case if $\ell = \mathbb{Q}(\sqrt{-23})$, and the set $\mathcal{S}$ of rational primes p which split in ℓ, and P_p is an Iwahori subgroup, consists of* 2 *alone.*

5.4. Let G, C, and $\overline{G}$ be as in 5.1 and 5.2. As before, let $\mathcal{T}_0$ be the finite set of nonarchimedean places of k where G is anisotropic.

We fix a coherent collection $(P_v)_{v\in V_f}$ of parahoric subgroups such that P_v is maximal for every v which splits in ℓ. Let $\Lambda = G(k) \cap \prod_v P_v$, Γ be the normalizer of Λ in $G(k_{v_o})$, and $\overline{\Gamma}$ be the image of Γ in $\overline{G}(k_{v_o})$. We know (see bound (0) in 2.3, and 2.2) that $[\Gamma : \Lambda] \leqslant 3^{1+\#\mathcal{T}_0}h_{\ell,3}$. From

[BP, Proposition 2.9] and a careful analysis of the arguments in 5.3, 5.5 and the proof of Proposition 0.12 of *loc. cit.* it can be deduced that, in fact, $[\Gamma : \Lambda] = 3^{1+\#\mathcal{T}_0}$, *if either* $h_{\ell,3} = 1$ (then $h_{k,3} = 1$, see [W, Theorem 4.10]), *or* $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-23}))$ *and* $\mathcal{T}_0 = \{2\}$. We briefly outline the proof below.

Let Θ_v ($\subset \Delta_v$) be the type of P_v, and $\Theta = \prod \Theta_v$. We have observed in 5.2 that the coboundary homomorphism $\delta : \overline{G}(k) \to H^1(k, C)$ is surjective. Using this fact we find that, for G at hand, the last term $\delta(\overline{G}(k))'_\Theta$ in the short exact sequence of [BP, Proposition 2.9], for $G' = G$, coincides with the subgroup $H^1(k, C)_\Theta$ of $H^1(k, C)$ defined in [BP, 2.8]. Also, the order of the first term of that short exact sequence is $3/\#\mu(\ell)_3$. So to prove the assertion about $[\Gamma : \Lambda]$, it would suffice to show that $H^1(k, C)_\Theta$ is of order $\#\mu(\ell)_3 3^{\#\mathcal{T}_0}$ if either $h_{\ell,3} = 1$, or $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-23}))$ and $\mathcal{T}_0 = \{2\}$.

As in 2.1, let $\ell^\bullet = \{x \in \ell^\times \mid N_{\ell/k}(x) \in {k^\times}^3\}$, and identify $H^1(k, C)$ with $\ell^\bullet/{\ell^\times}^3$. Let ℓ_3 (resp., $\ell^\bullet_{\mathcal{T}_0}$) be the subgroup of $\ell^\times$ (resp., $\ell^\bullet$) consisting of elements x such that for every normalized valuation w of ℓ (resp., every normalized valuation w of ℓ which does not lie over a place in $\mathcal{T}_0$), $w(x) \in 3\mathbb{Z}$. Let $\ell^\bullet_3 = \ell_3 \cap \ell^\bullet$. We can identify $H^1(k, C)_\Theta$ with the group $\ell^\bullet_{\mathcal{T}_0}/{\ell^\times}^3$, see 2.3, 2.7 and 5.3–5.5 of [BP]. We claim that the order of $\ell^\bullet_{\mathcal{T}_0}/{\ell^\times}^3$ is $\#\mu(\ell)_3 3^{\#\mathcal{T}_0}$. If $h_{\ell,3} = 1 = h_{k,3}$, from 2.3 above and [BP, Proposition 0.12] we see that $\#\ell^\bullet_3/{\ell^\times}^3 = \#\mu(\ell)_3$, and $U(k)/U(k)^3 \to k_3/{k^\times}^3$ is an isomorphism. Since the homomorphism $U(\ell)/U(\ell)^3 \to U(k)/U(k)^3$, induced by the norm map, is onto (2.3), given an element $y \in \ell^\times$ whose norm lies in k_3, we can find an element $u \in U(\ell)$ such that $uy \in \ell^\bullet$, i.e., $N_{\ell/k}(uy) \in {k^\times}^3$. Now it is easy to see that if $h_{\ell,3} = 1$, $\ell^\bullet_{\mathcal{T}_0}/\ell^\bullet_3$ is of order $3^{\#\mathcal{T}_0}$. This implies that $\#\ell^\bullet_{\mathcal{T}_0}/{\ell^\times}^3 = \#\mu(\ell)_3 3^{\#\mathcal{T}_0}$.

Let $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-23}))$ now. Then, as neither of the two prime ideals of $\ell = \mathbb{Q}(\sqrt{-23})$ lying over 2 is a principal ideal, we see that $\ell^\bullet_{\{2\}} = \ell^\bullet_3$. But since $\mathbb{Q}_3 = {\mathbb{Q}^\times}^3$, $\ell^\bullet_3 = \ell_3$, and therefore, $\ell^\bullet_{\{2\}}/{\ell^\times}^3 = \ell_3/{\ell^\times}^3$. The latter group is of order 3 ($= h_{\ell,3}$) since $\mathbb{Q}(\sqrt{-23})$ does not contain a nontrivial cube-root of unity, see the proof of Proposition 0.12 in [BP]. This proves the assertion that $[\Gamma : \Lambda] = 3^{1+\#\mathcal{T}_0}$ if either $h_{\ell,3} = 1$, or $(k, \ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-23}))$ and $\mathcal{T}_0 = \{2\}$.

5.5. In the rest of this section we will assume that $k = \mathbb{Q}$ and $\mathcal{D}$ is a cubic division algebra with center $\ell = \mathbb{Q}(\sqrt{-a})$ given with an involution σ of the second kind (cf. 4.1). Let G be the simple simply connected $\mathbb{Q}$-group such that

$$G(\mathbb{Q}) = \{ z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \ \text{ and } \ \mathrm{Nrd}(z) = 1\}.$$

5.6. Lemma. *$G(\mathbb{Q})$ is torsion-free if $a \neq 3$ or 7. If $a = 3$ (resp., $a = 7$), then the order of any nontrivial element of $G(\mathbb{Q})$ of finite order is 3 (resp., 7).*

Proof. Let $x \in G(\mathbb{Q})$ be a nontrivial element of finite order. Since the reduced norm of -1 in $\mathcal{D}$ is -1, $-1 \notin G(\mathbb{Q})$. Therefore, the order of x is odd, and the $\mathbb{Q}$-subalgebra $K := \mathbb{Q}[x]$ of $\mathcal{D}$ generated by x is a nontrivial field extension of $\mathbb{Q}$. Note that the degree of any field extension of $\mathbb{Q}$ contained in $\mathcal{D}$ is a divisor of 6. If $K = \ell$, then x lies in the center of G, and hence it is of order 3. But $\mathbb{Q}(\sqrt{-3})$ is the field generated by a nontrivial cube-root of unity. Hence, if $K = \ell$, then $a = 3$ and x is of order 3. Let us assume now that $K \neq \ell$. Then K cannot be a quadratic extension of $\mathbb{Q}$ since if it is a quadratic extension, $K \cdot \ell$ is a field extension of $\mathbb{Q}$ of degree 4 contained in $\mathcal{D}$, which is not possible. So K is an extension of $\mathbb{Q}$ of degree either 3 or 6. Since an extension of degree 3 of $\mathbb{Q}$ cannot contain a root of unity different from ± 1, K must be of degree 6, and so, in particular, it contains $\ell = \mathbb{Q}(\sqrt{-a})$. Note that the only roots of unity of odd order which can be contained in an extension of $\mathbb{Q}$ of degree 6 are the 7-th or the 9-th roots of unity.

For an integer n, let C_n be the extension of $\mathbb{Q}$ generated by a primitive n-th root ζ_n of unity. Then $C_7 = C_{14} \supset \mathbb{Q}(\sqrt{-7})$, and $C_9 = C_{18} \supset C_3 = \mathbb{Q}(\sqrt{-3})$, and $\mathbb{Q}(\sqrt{-7})$ (resp., $\mathbb{Q}(\sqrt{-3})$) is the only quadratic extension of $\mathbb{Q}$ contained in C_7 (resp., C_9). As $K \supset \mathbb{Q}(\sqrt{-a})$, we conclude that the group $G(\mathbb{Q})$ is torsion-free if $a \neq 3$ or 7, and if $a = 3$ (resp., $a = 7$), then the order of x is 9 (resp., 7). In particular, if $a = 3$ (resp., $a = 7$), then $K = \mathbb{Q}(\zeta_9)$ (resp., $K = \mathbb{Q}(\zeta_7)$). However, if $a = 3$, $N_{K/\ell}(\zeta_9) = \zeta_9^3 \neq 1$, and if $a = 7$, $N_{K/\ell}(\zeta_7) = 1$. This implies the last assertion of the lemma.

5.7. Let a be one of the following five integers: 1, 2, 7, 15, and 23, and let $p = p_a$ be the prime associated to a (see 4.4–4.5). Let $\ell = \mathbb{Q}(\sqrt{-a})$. Let $\mathcal{D}$ be a cubic division algebra with center ℓ whose local invariants at the two places of ℓ lying over p are nonzero and negative of each other, and the local invariant at all the other places of ℓ is zero. (There are two such division algebras, they are opposite of each other.) Then $\mathbb{Q}_p \otimes_{\mathbb{Q}} \mathcal{D} = (\mathbb{Q}_p \otimes_{\mathbb{Q}} \ell) \otimes_\ell \mathcal{D} = \mathfrak{D} \oplus \mathfrak{D}^o$, where $\mathfrak{D}$ is a cubic division algebra with center $\mathbb{Q}_p$, and $\mathfrak{D}^o$ is its opposite. $\mathcal{D}$ admits an involution σ of the second kind. Let the simple simply connected $\mathbb{Q}$-group G be as in 5.5. We may (and do) assume that σ is so chosen that $G(\mathbb{R}) \cong \mathrm{SU}(2,1)$. We observe that any other such involution of $\mathcal{D}$, or of its opposite, similarly determines a $\mathbb{Q}$-group which is $\mathbb{Q}$-isomorphic to G (1.2). As $\sigma(\mathfrak{D}) = \mathfrak{D}^o$, it is easily seen that $G(\mathbb{Q}_p)$ is the compact group $\mathrm{SL}_1(\mathfrak{D})$ of elements of reduced norm 1 in $\mathfrak{D}$.

We fix a coherent collection (P_q) of maximal parahoric subgroups P_q of $G(\mathbb{Q}_q)$ which are hyperspecial for every rational prime $q \neq p$ which does not ramify in ℓ. Let $\Lambda = G(\mathbb{Q}) \cap \prod_q P_q$, and let Γ be its normalizer in $G(\mathbb{R})$. Let $\overline{\Gamma}$ be the image of Γ in $\overline{G}(\mathbb{R})$.

5.8. Proposition. *If* $(a, p) = (23, 2)$, *then* $\overline{\Gamma}$ *is torsion-free.*

Proof. We assume that $(a, p) = (23, 2)$, and begin by observing that $\overline{\Gamma}$ is contained in $\overline{G}(\mathbb{Q})$, see, for example, [BP, Proposition 1.2]. Since $H^1(\mathbb{Q}, C)$ is a group of exponent 3, so is the group $\overline{G}(\mathbb{Q})/\varphi(G(\mathbb{Q}))$. Now as $G(\mathbb{Q})$ is torsion-free (5.6), any nontrivial element of $\overline{G}(\mathbb{Q})$ of finite order has order 3.

To be able to describe all the elements of order 3 of $\overline{G}(\mathbb{Q})$, we consider the connected reductive $\mathbb{Q}$-subgroup $\mathcal{G}$ of $\mathrm{GL}_{1,\mathcal{D}}$, which contains G as a normal subgroup, such that

$$\mathcal{G}(\mathbb{Q}) = \{ z \in \mathcal{D}^\times \mid z\sigma(z) \in \mathbb{Q}^\times \}.$$

Then the center $\mathcal{C}$ of $\mathcal{G}$ is $\mathbb{Q}$-isomorphic to $R_{\ell/\mathbb{Q}}(\mathrm{GL}_1)$. The adjoint action of $\mathcal{G}$ on the Lie algebra of G induces a $\mathbb{Q}$-isomorphism $\mathcal{G}/\mathcal{C} \to \overline{G}$. As $H^1(\mathbb{Q}, \mathcal{C}) = \{0\}$, we conclude that the natural homomorphism $\mathcal{G}(\mathbb{Q}) \to \overline{G}(\mathbb{Q})$ is surjective. Now given an element g of $\mathcal{G}(\mathbb{Q})$ whose image in $\overline{G}(\mathbb{Q})$ is an element of order 3, $\lambda := g^3$ lies in $\ell^\times$. Let $a = g\sigma(g) \in \mathbb{Q}^\times$. Then (i) $\lambda\sigma(\lambda) = a^3$. The field $L := \ell[X]/(X^3 - \lambda)$ admits an involution τ (i.e., an automorphism of order 2) whose restriction to the subfield ℓ coincides with $\sigma|_\ell$; τ is defined as follows: let x be the unique cube-root of λ in L, then $\tau(x) = a/x$. There is a unique embedding of L in $\mathcal{D}$ which maps x to g. In this embedding, $\tau = \sigma|_L$. The reduced norm of x (x considered as an element of $\mathcal{D}$) is clearly λ, and the image of g in $H^1(\mathbb{Q}, C) \subset \ell^\times/\ell^{\times 3}$ is the class of λ in $\ell^\times/\ell^{\times 3}$. Now if g stabilizes the collection (P_v), then its image in $H^1(\mathbb{Q}, C)$ must lie in the subgroup $H^1(\mathbb{Q}, C)_\Theta$, and hence, (ii) $w(\lambda) \in 3\mathbb{Z}$ for every normalized valuation w of ℓ not lying over 2 (cf. 5.4).

The conditions (i) and (ii) imply that $\lambda \in \ell_{\{2\}}^\bullet = \ell_3 = \bigcup_i \alpha^i \ell^{\times 3}$ (cf. 5.4), where $\alpha = (3 + \sqrt{-23})/2$. Since λ is not a cube in ℓ, $\lambda \in \alpha\ell^{\times 3} \cup \alpha^2\ell^{\times 3}$. But $\mathbb{Q}_2$ contains a cube-root of α (this can be seen using Hensel's Lemma), and hence for $\lambda \in \alpha\ell^{\times 3} \cup \alpha^2\ell^{\times 3}$, $L = \ell[X]/(X^3 - \lambda)$ is not embeddable in $\mathcal{D}$. (Note that L is embeddable in $\mathcal{D}$ if, and only if, $\mathbb{Q}_2 \otimes_{\mathbb{Q}} L$ is a direct sum of two field extensions of $\mathbb{Q}_2$, both of degree 3.) Thus we have shown that $\overline{G}(\mathbb{Q})$ does not contain any nontrivial elements of finite order which stabilize the collection (P_v). Therefore, $\overline{\Gamma}$ is torsion-free.

5.9. Examples of fake projective planes By Lemma 5.6 the subgroup Λ described in 5.7 is torsion-free if $(a, p) = (1, 5)$, $(2, 3)$, $(15, 2)$ or $(23, 2)$. Now let $(a, p) = (7, 2)$. Then $G(\mathbb{Q}_2)$ is the group $\mathrm{SL}_1(\mathfrak{D})$ of elements of reduced norm 1 in a cubic division algebra $\mathfrak{D}$ with center $\mathbb{Q}_2$ (cf. 5.7). The first congruence subgroup $G(\mathbb{Q}_2)^+ := \mathrm{SL}_1^{(1)}(\mathfrak{D})$ of $\mathrm{SL}_1(\mathfrak{D})$ is the unique maximal normal pro-2 subgroup of $G(\mathbb{Q}_2)$ of index $(2^3 - 1)/(2 - 1) = 7$ (see [Ri, Theorem 7(iii)(2)]). By the strong approximation property [PlR, Theorem 7.12], $\Lambda^+ := \Lambda \cap G(\mathbb{Q}_2)^+$ is a subgroup of Λ of index 7. Lemma 5.6 implies that Λ^+ is torsion-free since $G(\mathbb{Q}_2)^+$ is a pro-2 group. As $\mu(G(\mathbb{R})/\Lambda) = 1/7$ (see 4.5), $\mu(G(\mathbb{R})/\Lambda^+) = 1$, and hence the Euler–Poincaré characteristic of Λ^+ is 3.

Since Λ, and for $a = 7$, Λ^+ are congruence subgroups, according to [Ro, Theorem 15.3.1], $H^1(\Lambda, \mathbb{C})$, and for $a = 7$, $H^1(\Lambda^+, \mathbb{C})$ vanish. By Poincaré-duality, then $H^3(\Lambda, \mathbb{C})$, and for $a = 7$, $H^3(\Lambda^+, \mathbb{C})$ also vanish. For $a = 1,\ 2$, and 15, as $\mu(G(\mathbb{R})/\Lambda) = 1$ (4.5), the Euler–Poincaré characteristic $\chi(\Lambda)$ of Λ is 3, and for $a = 7$, $\chi(\Lambda^+)$ is also 3, we conclude that for $a = 1,\ 2$, and 15, $H^i(\Lambda, \mathbb{C})$ is 1-dimensional for $i = 0,\ 2$, and 4, and if $a = 7$, this is also the case for $H^i(\Lambda^+, \mathbb{C})$. Thus if B is the symmetric space of $G(\mathbb{R})$, then *for* $a = 1,\ 2$ *and* 15*,* B/Λ*, and for* $a = 7$*,* B/Λ^+*, is a fake projective plane.*

Let $\overline{\Lambda}$ (resp., $\overline{\Lambda}^+$) be the image of Λ (resp., Λ^+) in $\overline{G}(\mathbb{R})$. There is a natural faithful action of $\overline{\Gamma}/\overline{\Lambda}$ (resp., $\overline{\Gamma}/\overline{\Lambda}^+$), which is a group of order 3 (resp., 21), on B/Λ (resp., B/Λ^+). As $\overline{\Gamma}$ is the normalizer of Λ, and also of Λ^+, in $\overline{G}(\mathbb{R})$, $\overline{\Gamma}/\overline{\Lambda}$ (resp., $\overline{\Gamma}/\overline{\Lambda}^+$) is the full automorphism group of B/Λ (resp., B/Λ^+).

In 5.10–5.13, we will describe the classes of fake projective planes associated with each of the five pairs (a, p).

5.10. In this paragraph we shall study the fake projective planes arising from the pairs $(a, p) = (1, 5)$, $(2, 3)$, and $(15, 2)$. Let Λ and Γ be as in 5.7. Let $\Pi \subset \overline{\Gamma}$ be the fundamental group of a fake projective plane and $\widetilde{\Pi}$ be its inverse image in Γ. Then as $1 = \chi(\widetilde{\Pi}) = 3\mu(G(\mathbb{R})/\widetilde{\Pi}) = \mu(G(\mathbb{R})/\Lambda)$, $\widetilde{\Pi}$ is of index 3 ($= [\Gamma : \Lambda]/3$) in Γ, and hence Π is a torsion-free subgroup of $\overline{\Gamma}$ of index 3. Conversely, if Π is a torsion-free subgroup of $\overline{\Gamma}$ of index 3 such that $H^1(\Pi, \mathbb{C}) = \{0\}$ (i.e., $\Pi/[\Pi, \Pi]$ is finite), then as $\chi(\Pi) = 3$, B/Π is a fake projective plane, and Π is its fundamental group.

5.11. We will now study the fake projective planes arising from the pair $(7, 2)$. In this case, as in 5.7, let $\Lambda^+ = \Lambda \cap G(\mathbb{Q}_2)^+$, which is a torsion-free subgroup of Λ of index 7. We know that B/Λ^+ is a fake projective plane.

Now let $\widetilde{\Pi}$ be the inverse image in Γ of the fundamental group $\Pi \subset \overline{\Gamma}$ of a fake projective plane. Then as $\mu(G(\mathbb{R})/\Gamma) = \mu(G(\mathbb{R})/\Lambda)/9 = 1/63$, and $\mu(G(\mathbb{R})/\widetilde{\Pi}) = \chi(\widetilde{\Pi})/3 = 1/3$, $\widetilde{\Pi}$ is a subgroup of Γ of index 21, and hence $[\overline{\Gamma} : \Pi] = 21$. Conversely, if Π is a torsion-free subgroup of $\overline{\Gamma}$ of index 21, then as $\chi(\Pi) = 3$, B/Π is a fake projective plane if, and only if, $\Pi/[\Pi, \Pi]$ is finite. Mumford's fake projective plane is given by one such Π.

5.12. We finally look at the fake projective planes arising from the pair $(23, 2)$. In this case, $\mu(G(\mathbb{R})/\Gamma) = \mu(G(\mathbb{R})/\Lambda)/9 = 1/3$ (see 4.5). Hence, if $\widetilde{\Pi}$ is the inverse image in Γ of the fundamental group $\Pi \subset \overline{\Gamma}$ of a fake projective plane, then as $\mu(G(\mathbb{R})/\widetilde{\Pi}) = \chi(\widetilde{\Pi})/3 = 1/3 = \mu(G(\mathbb{R})/\Gamma)$, $\widetilde{\Pi} = \Gamma$. Therefore, the only subgroup of $\overline{\Gamma}$ which can be the fundamental group of a fake projective plane is $\overline{\Gamma}$ itself.

As $\overline{\Gamma}$ is torsion-free (Proposition 5.8), $\chi(\overline{\Gamma}) = 3$, and $\Lambda/[\Lambda, \Lambda]$, hence $\Gamma/[\Gamma, \Gamma]$, and so also $\overline{\Gamma}/[\overline{\Gamma}, \overline{\Gamma}]$ are finite, $B/\overline{\Gamma}$ is a fake projective plane and $\overline{\Gamma}$ is its fundamental group. Since the normalizer of $\overline{\Gamma}$ in $\overline{G}(\mathbb{R})$ equals $\overline{\Gamma}$, the automorphism group of $B/\overline{\Gamma}$ is trivial.

5.13. We recall that the hyperspecial parahoric subgroups of $G(k_v)$ are conjugate to each other under $\overline{G}(k_v)$, see [Ti2, 2.5]. Using the observations in 2.2 and Proposition 5.3 we see that if $a \neq 15$ (resp., $a = 15$), then up to conjugation by $\overline{G}(\mathbb{Q})$, there are exactly 2 (resp., 4) coherent collections (P_q) of maximal parahoric subgroups such that P_q is hyperspecial whenever q does not ramify in $\mathbb{Q}(\sqrt{-a})$, since if $a \neq 15$ (resp., $a = 15$), there is exactly one prime (resp., there are exactly two primes, namely 3 and 5) which ramify in $\ell = \mathbb{Q}(\sqrt{-a})$.

From the results in 5.10–5.12 we conclude that for each $a \in \{1, 2, 7, 23\}$, there are two distinct finite classes, and for $a = 15$, there are four distinct finite classes, of fake projective planes. Thus the following theorem holds.

5.14. Theorem. *There exist exactly twelve distinct classes of fake projective planes with $k = \mathbb{Q}$.*

5.15. Remark. To the best of our knowledge, only three fake projective planes were known before the present work. The first one was constructed by Mumford [Mu] and it corresponds to the pair $(a, p) = (7, 2)$; see 5.11. Two more examples have been given by Ishida and Kato [IK] making use of the discrete subgroups of $\mathrm{PGL}_3(\mathbb{Q}_2)$, which act simply transitively on the set of vertices of the Bruhat–Tits building of the latter, constructed by Cartwright, Mantero, Steger and Zappa. In both of these examples, (a, p) equals $(15, 2)$. We have learnt from JongHae Keum that he has recently constructed a fake projective plane which is birational to a cyclic cover of degree 7 of a Dolgachev surface. This fake projective plane admits an automorphism of order 7, so it appears to us that it corresponds to the pair $(7, 2)$, and its fundamental group is the group Λ^+ of 5.9 for a suitable choice of a maximal parahoric subgroup P_7 of $G(\mathbb{Q}_7)$.

6. Lower bound for discriminant in terms of the degree of a number field

6.1. Definition. We define $M_r(d) = \min_K D_K^{1/d}$, where the minimum is taken over all totally real number fields K of degree d. Similarly, we define $M_c(d) = \min_K D_K^{1/d}$ by taking the minimum over all totally complex number fields K of degree d.

It is well-known that $M_r(d) \geqslant (d^d/d!)^{2/d}$ from the classical estimates of Minkowski. The precise values of $M_r(d)$ for small values of d are known due to the work of many mathematicians as listed in [N]. For $d \leqslant 8$, the values of $M_r(d)$ are given in the following table.

d	2	3	4	5	6	7	8
$M_r(d)^d$	5	49	725	14641	300125	20134393	282300416.

An effective lower bound for $M_r(d)$, better than Minkowski's bound for d large, has been given by Odlyzko [O1]. We recall the following algorithm given in [O1, Theorem 1], which provides a useful estimate for $M_r(d)$ for arbitrary d.

6.2. Let $b(x) = [5 + (12x^2 - 5)^{1/2}]/6$. Define

$$g(x, d) = \exp\left[\log(\pi) - \frac{\Gamma'}{\Gamma}(x/2) + \frac{(2x-1)}{4}\left(\frac{\Gamma'}{\Gamma}\right)'(b(x)/2) + \frac{1}{d}\left\{-\frac{2}{x} - \frac{2}{x-1} - \frac{2x-1}{b(x)^2} - \frac{2x-1}{(b(x)-1)^2}\right\}\right].$$

Let $\alpha = \sqrt{\frac{14-\sqrt{128}}{34}}$. As we are considering only totally real number fields, according to [O1, Theorem 1], $M_r(d) \geqslant g(x, d)$ provided that $x > 1$ and $b(x) \geqslant 1 + \alpha x$.

Now let x_0 be the positive root of the quadratic equation $b(x) = 1 + \alpha x$. Solving this equation, we obtain $x_0 = \frac{\alpha+\sqrt{2-5\alpha^2}}{2(1-3\alpha^2)} = 1.01\ldots$. For a fixed value of d, define $\mathfrak{N}(d) = \limsup_{x \geqslant x_0} g(x, d)$.

6.3. Lemma. *For each $d > 1$, $M_r(d) \geqslant \mathfrak{N}(d)$, and $\mathfrak{N}(d)$ is an increasing function of d.*

Proof. It is obvious from our choice of x_0 that $M_r(d) \geqslant \mathfrak{N}(d)$. We will now show that $\mathfrak{N}(d)$ is an increasing function of d.

For a fixed value of $x > 1$, $g(x, d)$ is clearly an increasing function of d since the only expression involving d in it is

$$(1/d)\{-2/x - 2/(x-1) - (2x-1)/b(x)^2 - (2x-1)/(b(x)-1)^2\},$$

which is nonpositive. Now for a given d, and a positive integer n, choose a $x_n \geqslant x_0$ such that $g(x_n, d) \geqslant \mathfrak{N}(d) - 10^{-n}$. Then

$$\mathfrak{N}(d+1) = \limsup_{x \geqslant x_0} g(x, d+1) \geqslant g(x_n, d+1) \geqslant g(x_n, d) \geqslant \mathfrak{N}(d) - 10^{-n}.$$

Hence, $\mathfrak{N}(d+1) \geqslant \mathfrak{N}(d)$.

6.4. In the next section, we will use the lower bound for the root-discriminant $D_K^{1/d}$ of totally complex number fields K obtained by Odlyzko in [O2].

We will denote by $N_c(n_0)$ the entry for totally complex number fields given in the last column of Table 2 of [O2] for $n = n_0$. We recall from [O2] that for every number field K of degree $n \geqslant n_0$, the root-discriminant $D_K^{1/n} > N_c(n_0)$.

For small d, we will also use Table IV of [Ma]. This table was originally constructed by Diaz y Daiz.

7. Upper bounds for the degree d of k, D_k and D_ℓ

In this, and the next two sections, we will determine totally real number fields k of degree $d > 1$, their totally complex quadratic extensions ℓ, k-forms G of SU(2, 1) and coherent collections $(P_v)_{v \in V_f}$ of parahoric subgroups P_v of $G(k_v)$ such that for all $v \in \mathcal{R}_\ell$, P_v is maximal, and the image $\overline{\Gamma}$ in $\overline{G}(k_{v_o})$ (where v_o is the unique real place of k such that $G(k_{v_o}) \cong \mathrm{SU}(2, 1)$) of the normalizer Γ of $\Lambda := G(k) \cap \prod_{v \in V_f} P_v$ in $G(k_{v_o})$ contains a torsion-free subgroup Π of finite index with $\chi(\Pi) = 3$. *Then* $\chi(\Gamma)$ *is a reciprocal integer. In particular, it is* $\leqslant 1$.

In this section, we will use bounds (2), (3), (6), and (7)–(10) obtained in Sect. 2, the lower bound for the discriminant given in the preceding section, and Hilbert class fields, to prove that $d \leqslant 5$. We will also find good upper bounds for D_k, D_ℓ, and D_ℓ / D_k^2 for $d \leqslant 5$. Using these bounds, in the next section we will be able to make a complete list of (k, ℓ) of interest to us. It will follow then that d cannot be 5.

7.1. Let $f(\delta, d)$ be the function occurring in bound (10). It is obvious that for $c > 1$, $c^{1/(3-\delta)d}$ decreases as d increases. Now for $\delta \geqslant 0.002$, as

$$\frac{\delta(1+\delta)}{0.00136} > 1,$$

$\inf_\delta f(\delta, d)$, where the infimum is taken over the closed interval $0.002 \leqslant \delta \leqslant 2$, decreases as d increases. A direct computation shows that $f(0.9, 20) < 16.38$. On the other hand, for $d \geqslant 20$, Lemma 6.3 gives us

$$M_r(d) \geqslant \mathfrak{M}(20) \geqslant g(1.43, 20) > 16.4,$$

where $g(x, d)$ is the function defined in 6.2. From these bounds we conclude that $d = [k : \mathbb{Q}] < 20$.

To obtain a better upper bound for d, we observe using Table 2 in [O2] that $M_r(d) > 17.8$ for $15 \leqslant d < 20$. But by a direct computation we see that $f(0.9, 15) < 17.4$. So the monotonicity of $f(\delta, d)$, as a function of d for a fixed δ, implies that *d cannot be larger than* 14.

7.2. Now we will prove that $d \leqslant 7$ with the help of Hilbert class fields. Let us assume, if possible, that $14 \geqslant d \geqslant 8$.

We will use the following result from the theory of Hilbert class fields. The Hilbert class field $L := H(\ell)$ of a totally complex number field ℓ is the maximal unramified abelian extension of ℓ. *Its degree over ℓ is the class number h_ℓ of ℓ, and $D_L = D_\ell^{h_\ell}$.*

We consider the two cases where $h_\ell \leqslant 63$ and $h_\ell > 63$ separately.

Case (a). $h_\ell \leqslant 63$: In this case $h_{\ell,3} \leqslant 27$, and from bound (8) we obtain

$$D_k^{1/d} < \varphi_2(d, h_{\ell,3}) < \varphi_3(d) := 27^{1/4d}(16\pi^5)^{1/4}.$$

The function $\varphi_3(d)$ decreases as d increases. A direct computation shows that $\varphi_3(d) \leqslant \varphi_3(8) < 9.3$. Hence, $D_k^{1/d} < 9.3$. On the other hand, from Table 2 in [O2] we find that, for $14 \geqslant d \geqslant 8$, $M_r(d) > 10.5$, so $D_k^{1/d} > 10.5$. Therefore, if $h_\ell \leqslant 63$, $d \leqslant 7$.

Case (b). $h_\ell > 63$: In this case, let L be the Hilbert class field of ℓ. Then $[L : \ell] = h_\ell$, $D_L = D_\ell^{h_\ell}$, and $2dh_\ell > 16 \times 63 > 1000$. From 6.4 we conclude that

$$D_\ell^{1/2d} = D_L^{1/2dh_\ell} \geqslant M_c(2dh_\ell) \geqslant N_c(1000) = 20.895,$$

where the last value is from Table 2 of [O2]. However, as $f(0.77, d) \leqslant f(0.77, 8) < 20.84$, bound (10) implies that $D_\ell^{1/2d} < 20.84$. Again, we have reached a contradiction. So we conclude that $d \leqslant 7$.

7.3. To find good upper bounds for d, D_k and D_ℓ, we will make use of improved lower bounds for R_ℓ/w_ℓ for totally complex number fields given in [Fr], Table 2. We reproduce below the part of this table which we will use in this paper.

$r_2 = d$	for $D_\ell^{1/2d} <$	$R_\ell/w_\ell \geqslant$
2	17.2	0.0898
3	24.6	0.0983
4	29.04	0.1482
5	31.9	0.2261
6	33.8	0.4240
7	34.4	0.8542.

We also note here that except for totally complex sextic fields of discriminants

$$-9747,\ -10051,\ -10571,\ -10816,\ -11691,\ -12167,$$

and totally complex quartic fields of discriminants

$$117,\ 125,\ 144,$$

R_ℓ/w_ℓ is bounded from below by $1/8$ for every number field ℓ, see [Fr, Theorem B$'$].

7.4. We consider now the case where $d = 7$. Bound (10) implies that $D_\ell^{1/14} < f(0.75, 7) < 22.1$. Using the lower bound for R_ℓ/w_ℓ given in the table above and bound (7), we conclude by a direct computation that

$$D_\ell^{1/14} < \varphi_1(7, 0.8542, 0.8) < 18.82.$$

On the other hand, the root-discriminant of any totally complex number field of degree $\geqslant 260$ is bounded from below by $N_c(260)$, see 6.4. From Table 2 in [O2] we find that $N_c(260) = 18.955$. So we conclude that the class number h_ℓ of ℓ is bounded from above by $260/2d = 260/14 < 19$, for otherwise the root-discriminant of the Hilbert class field of ℓ would be greater than 18.955, contradicting the fact that it equals $D_\ell^{1/14}$ (< 18.82).

As $h_\ell \leqslant 18$, $h_{\ell,3} \leqslant 9$. Now we will use bound (8). We see by a direct computation that $\varphi_2(7, 9) < 9.1$. Hence, $D_k^{1/7} \leqslant D_\ell^{1/14} < 9.1$. On the other hand, we know from 6.1 that $M_r(7) = 20134393^{1/7} > 11$. This implies that d cannot be 7. Therefore, $d \leqslant 6$.

7.5. Employing a method similar to the one used in 7.2 and 7.4 we will now show that d cannot be 6.

For $d = 6$, from bound (10) we get $D_\ell^{1/12} < f(0.71, 6) < 24$. Using the lower bound for R_ℓ/w_ℓ provided by the table in 7.3 and bound (7), we conclude by a direct computation that $D_\ell^{1/12} < \varphi_1(6, 0.424, 0.8) < 20$. From Table 2 in [O2] we find that $N_c(480) > 20$. Now, arguing as in 7.4, we infer that the class number h_ℓ of ℓ is bounded from above by $480/12 = 40$, which implies that $h_{\ell,3} \leqslant 27$. As $\varphi_2(6, 27) < 10$, bound (8) implies that $D_k^{1/6} \leqslant D_\ell^{1/12} < 10$. Now since $N_c(21) > 10$, we see that the class number of ℓ cannot be larger than $21/12 < 2$. Hence, $h_\ell = 1 = h_{\ell,3}$. We may now apply bound (8) again to conclude that $D_k^{1/6} < \varphi_2(6, 1) < 8.365$. Checking from the Table t66.001 of [1], we know that the two smallest discriminants of totally real sextics are 300125 as mentioned in 6.1, followed by 371293. As $371293^{1/6} > 8.47$, the second case is not possible and we are left with only one candidate, $D_k = 300125$. As $\mathfrak{p}(6, 300125, 1) < 1.3$, we conclude from bound (9) that $D_\ell/D_k^2 = 1$. Hence, if $d = 6$, $(D_k, D_\ell) = (300125, 300125^2)$ is the only possibility. From the tables in [1] we find that there is a unique totally real number field k of degree 6 with $D_k = 300125$. Moreover, the class number of this field is 1. Gunter Malle, using the procedure described in 8.1 below, has shown that there does not exist a totally complex quadratic extension ℓ of this field with $D_\ell = 300125^2$. Therefore d cannot be 6.

7.6. For $d = 5$, bound (10) implies that $D_\ell^{1/10} < f(0.7, 5) < 26.1$. It is seen from the table in 7.3 that $R_\ell/w_\ell \geqslant 0.2261$. Hence, $D_\ell^{1/10} < \varphi_1(5, 0.2261, 0.72) < 21.42$. As $N_c(2400) > 21.53$, arguing as in 7.4 we see that the class number h_ℓ of ℓ is bounded from above by $2400/10 = 240$. Hence, $h_{\ell,3} \leqslant 81 = 3^4$. Now we note that $\varphi_2(5, 81) < 10.43$, but $N_c(23) > 10.43$. So, $h_\ell < 23/10$, and therefore, $h_{\ell,3} = 1$. But then

$D_k^{1/5} \leqslant D_\ell^{1/10} < \varphi_2(5, 1) < 8.3649$. As $M_r(5)^5 \geqslant 14641$ and $\mathfrak{p}(5, 14641, 1) < 5.2$, we conclude from bound (9) that $D_\ell/D_k^2 \leqslant 5$.

7.7. Let now $d = 4$. In this case, k is a totally real quartic and ℓ is a totally complex octic containing k. Table 4 of [Fr] gives the lower bound $R_k \geqslant 41/50$ for the regulator. Since ℓ is a CM field which is a totally complex quadratic extension of k, we know that $R_\ell = 2^{d-1}R_k/Q$, where $Q = 1$ or 2 is the unit index of k (cf. [W]). We will now estimate w_ℓ, the number of roots of unity in ℓ.

We know that the group of roots of unity in ℓ is a cyclic group of even order, say m. Let ζ_m be a primitive m-th root of unity. As the degree of the cyclotomic field $\mathbb{Q}(\zeta_m)$ is $\phi(m)$, where ϕ is the Euler function, we know that $\phi(m)$ is a divisor of $2d = 8$. The following table gives the values of m and $\phi(m)$ for $\phi(m) \leqslant 8$.

m	2	4	6	8	10	12	14	16	18	20	24	30
$\phi(m)$	1	2	2	4	4	4	6	8	6	8	8	8.

If $\phi(m) = 8$, then $m = 16$, 20, 24 or 30, and $\mathbb{Q}(\zeta_m)$ equals ℓ. Note that $\mathbb{Q}(\zeta_{30}) = \mathbb{Q}(\zeta_{15})$. The class number of these four cyclotomic fields are all known to be 1 (see [W], pp. 230 and 432). So in these four cases, $h_{\ell,3} = 1$. Bound (8) implies that $D_k^{1/4} \leqslant D_\ell^{1/8} < \varphi_2(4, 1) < 8.3640$. As $M_r(4)^4 = 725$ and $\mathfrak{p}(4, 725, 1) < 21.3$, we conclude from bound (9) that $D_\ell/D_k^2 \leqslant 21$.

Assume now that $\phi(m) \neq 8$. Then $m \leqslant 12$. Hence, $w_\ell \leqslant 12$. So we conclude that except for the four cyclotomic fields dealt with earlier,

$$R_\ell/w_\ell \geqslant 2^3 R_k/12Q \geqslant R_k/3 \geqslant 41/150.$$

Applying bound (7), we conclude that $D_\ell^{1/8} < \varphi_1(4, 41/150, 0.69) < 21.75$ by a direct computation. From Table IV of [Ma], we know that totally complex number fields of degree $\geqslant 4000$ have unconditional root-discriminant lower bound 21.7825. It follows, as before, using the Hilbert Class field of ℓ, that the class number h_ℓ of ℓ is at most $4000/8 = 500$. Hence, $h_{\ell,3} \leqslant 3^5 = 243$. Bound (8) now gives that $D_\ell^{1/8} < \varphi_2(4, 243) < 11.8$. But from Table 2 of [O2] we find that $N_c(32) > 11.9$. So we conclude $h_\ell \leqslant 32/8 = 4$. Hence, $h_{\ell,3} \leqslant 3$. Applying bound (8) again we infer that $D_\ell^{1/8} < \varphi_2(4, 3) < 8.96$. As $N_c(18) > 9.2$, we conclude that $h_\ell < 18/8$. But then $h_{\ell,3} = 1$, and the argument in the preceding paragraph leads to the conclusion that $D_k^{1/4} \leqslant D_\ell^{1/8} < \varphi_2(4, 1) < 8.3640$ and $D_\ell/D_k^2 \leqslant 21$.

7.8. We consider now the case $d = 3$. Suppose that $D_\ell^{1/6} < 21.7$. Since according to Table IV of [Ma], $M_c(4000) \geqslant 21.7825$, we infer using the Hilbert class field of ℓ, that $h_\ell \leqslant 4000/6 < 667$. Then $h_{\ell,3} \leqslant 243 = 3^5$. It follows from bound (8) that $D_\ell^{1/6} < \varphi_2(3, 243) < 13.3$. From Table 2 of [O2] we find that $N_c(44) > 13.37$. Therefore, $h_\ell \leqslant 44/6 < 8$. Hence, $h_{\ell,3} \leqslant 3$. Now we observe that $\varphi_2(3,3) < 9.17$. But as $N_c(18) \geqslant 9.28$, $h_\ell < 18/6 = 3$, which implies that $h_{\ell,3} = 1$. We then deduce from bound (8) that $D_k^{1/3} \leqslant D_\ell^{1/6} < \varphi_2(3,1) < 8.3591$. Also since $D_k \geqslant 49$ (see 6.1), and $\mathfrak{p}(3,49,1) < 52.8$, we conclude from bound (9) that $D_\ell/D_k^2 \leqslant 52$.

We assume now that $D_\ell^{1/6} \geqslant 21.7$ (and $d = 3$). We will make use of a lower bound for R_ℓ/w_ℓ which is better than the one provided in 7.3. Table 4 of [Fr] gives that $R_k \geqslant 0.524$. Recall from 7.7 that $R_\ell/w_\ell = 2^{d-1}R_k/Qw_\ell \geqslant 2R_k/w_\ell \geqslant 2(0.524)/w_\ell$. From the table of values of the Euler function given in 7.7, we see that $\phi(m)$ is a proper divisor of 6 only for $m = 2, 4, 6$. So we conclude that $w_\ell \leqslant 6$ unless ℓ is either $\mathbb{Q}(\zeta_{14})$ or $\mathbb{Q}(\zeta_{18})$. Since both $\mathbb{Q}(\zeta_{14}) = \mathbb{Q}(\zeta_7)$ or $\mathbb{Q}(\zeta_{18}) = \mathbb{Q}(\zeta_9)$ are known to have class number 1 (cf. [W], pp. 229 and 412), the bounds obtained in the last paragraph apply to these two cases as well. Hence, it remains only to consider the cases where $w_\ell \leqslant 6$. So we assume now that $w_\ell \leqslant 6$. Then $R_\ell/w_\ell \geqslant 2(0.524)/6 > 0.17$.

Observe that bounds (2), (3) and (6) imply that

$$D_k^{1/d} > \xi(d, D_\ell, R_\ell/w_\ell, \delta) := \left[\frac{(R_\ell/w_\ell)\zeta(2d)^{1/2}}{\delta(\delta+1)}\right]^{1/d} \times \frac{(2\pi)^{1+\delta}}{16\pi^5\Gamma(1+\delta)\zeta(1+\delta)^2}(D_\ell^{1/2d})^{4-\delta}.$$

As $D_\ell^{1/6} \geqslant 21.7$, it follows from this bound by a direct computation that $D_k^{1/3} > \xi(3, 21.7^6, 0.17, 0.65) > 16.4$.

Recall now a result of Remak, stated as bound [Fr, 3.15],

$$R_k \geqslant \left[\frac{\log D_k - d\log d}{\{\gamma_{d-1}d^{1/(d-1)}(d^3-d)/3\}^{1/2}}\right]^{d-1},$$

where $d = 3$ and $\gamma_{d-1} = 2/\sqrt{3}$ as given in [Fr, p. 613]. Since $R_\ell = 2^{d-1}R_k/Q \geqslant 2R_k$, we obtain the following lower bound

$$R_\ell/w_\ell \geqslant r(D_k, w_\ell) := \frac{2}{w_\ell}\left[\frac{\log D_k - 3\log 3}{\{2(3^2-1)\}^{1/2}}\right]^2.$$

As in the argument in the last paragraph, we assume that $w_\ell \leqslant 6$. Then from the preceding bound we get the following:

$$R_\ell/w_\ell \geqslant r(16.4^3, 6) > 0.54.$$

We now use bound (7) to conclude that $D_\ell^{1/6} < \varphi_1(3, 0.54, 0.66) < 20.8 < 21.7$, contradicting our assumption that $D_\ell^{1/6} \geqslant 21.7$.

Therefore, $D_k^{1/3} \leqslant D_\ell^{1/6} < 8.3591$ and $D_\ell/D_k^2 \leqslant 52$.

7.9. Finally we consider the case $d = 2$. In this case, we know from 7.3 that $R_\ell/w_\ell \geqslant 1/8$ except in the three cases mentioned there. So bound (7) implies that $D_\ell^{1/4} < \varphi_1(2, 1/8, 0.52) < 28.96$. Hence, $D_\ell \leqslant 703387$. This bound holds for the three exceptional cases of 7.3 as well. Since quartics of such small discriminant are all known, we know the class number of all such fields explicitly. In particular, the number fields are listed in t40.001–t40.057 of [1], where each file contains 1000 number fields listed in ascending order of the absolute discriminants. There are altogether 5700 number fields in the files, the last one has discriminant 713808. So [1] is more than adequate for our purpose. Inspecting by hand, or using PARI/GP and a simple program, we find that the largest class number of an ℓ with $D_\ell \leqslant 703387$ is 64. The corresponding number field has discriminant 654400 with a defining polynomial $x^4 - 2x^3 + 27x^2 - 16x + 314$.

Once we know that $h_\ell \leqslant 64$, we find that $h_{\ell,3} \leqslant 27$. We may now apply bound (8) to conclude that $D_\ell^{1/4} < \varphi_2(2, 27) < 12.57$. Now since in Table 2 of [O2] we find that $N_c(38) > 12.73$, we infer that $h_\ell < 38/4 < 10$, which implies that $h_{\ell,3} \leqslant 9$. But $\varphi_2(2, 9) < 10.96$, and $N_c(26) > 11.01$. So $h_\ell < 26/4 < 7$, and hence $h_{\ell,3} \leqslant 3$. It follows from bound (8) that $D_k^{1/2} \leqslant D_\ell^{1/4} < \varphi_2(2, 3) < 9.5491$. From this we conclude that $D_k \leqslant 91$. As $D_k \geqslant 5$ (see 6.1) and $\mathfrak{p}(2, 5, 3) < 104.2$, bound (9) implies that $D_\ell/D_k^2 \leqslant 104$.

7.10. The results in 7.6–7.9 are summarized in the following table.

d	$D_k^{1/d} \leqslant D_\ell^{1/2d} \leqslant$	$h_{\ell,3} \leqslant$	$D_\ell/D_k^2 \leqslant$
5	8.3649	1	5
4	8.3640	1	21
3	8.3591	1	52
2	9.5491	3	104

8. (k, ℓ) with $d = 2,\ 3,\ 4,$ and 5

8.1. To make a list of all pairs (k, ℓ) of interest to us, we will make use of the tables of number fields given in [1]. In the following table, in the column under r_d (resp., c_d) we list the largest integer less than the d-th power (resp., an integer slightly larger than the $2d$-th power) of the numbers appearing in the second column of the table in 7.10. The column under x_d reproduces the numbers appearing in the last column of the table in 7.10. Therefore, we need only find all totally real number fields k of degree d, $2 \leqslant d \leqslant 5$, and totally complex quadratic extensions ℓ of each k, such that $D_k \leqslant r_d$, $D_\ell \leqslant c_d$, and moreover, $D_\ell/D_k^2 \leqslant x_d$. Thanks to a detailed computation carried out at our request by Gunter Malle, for each d, we know the exact number of pairs of (k, ℓ) satisfying these constraints. This number is listed in the last column of the following table. The data is obtained in the following way. The number fields k with D_k in the range we are interested in are listed in [1]. Their class numbers, and a set of generators of their group of units, are also given there. For $d = 2$, the quadratic extensions ℓ are also listed in [1]. Any quadratic extension of k is of the form $k(\sqrt{\alpha})$, with α in the ring of integers $\mathfrak{o}_k$ of k. For $d > 2$, the class number of any totally real k of interest turns out to be 1; hence, $\mathfrak{o}_k$ is a unique factorization domain. Now using factorization of small primes and explicit generators of the group of units of k, Malle listed all possible α modulo squares, and then for each of the α, the discriminant of $k(\sqrt{\alpha})$ could be computed. Using this procedure, Malle explicitly determined all totally complex quadratic extensions ℓ with D_ℓ satisfying the conditions mentioned above.

d	r_d	c_d	x_d	$\#(k, \ell)$
5	40954	17×10^8	5	0
4	4893	24×10^6	21	7
3	584	35×10^4	52	4
2	91	8320	104	52

Thus there are no (k, ℓ) with $d = 5$. For $2 \leqslant d \leqslant 4$, there are $52 + 4 + 7 = 63$ pairs (k, ℓ) satisfying the constraints on r_d, c_d and x_d imposed by the considerations in 7.6–7.9.

8.2. For each of the 63 potential pairs (k, ℓ) mentioned above, we know defining polynomials for k and ℓ, and also the values of D_k, D_ℓ, and $h_{\ell,3}$. It turns out that $h_{\ell,3} = 1$ or 3. We are able to further cut down the list of pairs (k, ℓ) such that there is a k-form of SU(2, 1), described in terms of the quadratic extension ℓ of k, which may provide an arithmetic subgroup Γ of SU(2, 1) with $\chi(\Gamma) \leqslant 1$, by making use of bound (9) for D_ℓ/D_k^2, and the fact that this number is an integer. We are then left with only 40 pairs. These are listed below.

In the lists below, there are only three pairs (k, ℓ) with $d = 3$. In the list provided by Malle there was a fourth pair with $(D_k, D_\ell, h_\ell) = (321, 309123, 1)$. Bound (9) for this pair gives us $D_\ell/D_k^2 < 2.7$, and therefore, $D_\ell \leqslant 2D_k^2$. But $309123 > 2 \times 321^2$, that is why the fourth pair with $d = 3$ does not appear in the lists below.

(k, ℓ)	k	ℓ
$\mathcal{C}_1$	$x^2 - x - 1$	$x^4 - x^3 + x^2 - x + 1$
$\mathcal{C}_2$	$x^2 - x - 1$	$x^4 - x^3 + 2x^2 + x + 1$
$\mathcal{C}_3$	$x^2 - x - 1$	$x^4 + 3x^2 + 1$
$\mathcal{C}_4$	$x^2 - x - 1$	$x^4 - x^3 + 3x^2 - 2x + 4$
$\mathcal{C}_5$	$x^2 - x - 1$	$x^4 - x^3 + 5x^2 + 2x + 4$
$\mathcal{C}_6$	$x^2 - x - 1$	$x^4 - 2x^3 + 6x^2 - 5x + 5$
$\mathcal{C}_7$	$x^2 - x - 1$	$x^4 + 6x^2 + 4$
$\mathcal{C}_8$	$x^2 - 2$	$x^4 + 1$
$\mathcal{C}_9$	$x^2 - 2$	$x^4 + 2x^2 + 4$
$\mathcal{C}_{10}$	$x^2 - 2$	$x^4 - 2x^3 + 5x^2 - 4x + 2$
$\mathcal{C}_{11}$	$x^2 - 3$	$x^4 - x^2 + 1$

(k, ℓ)	k	ℓ
$\mathcal{C}_{12}$	x^2-3	x^4+4x^2+1
$\mathcal{C}_{13}$	x^2-x-3	$x^4-x^3+4x^2+3x+9$
$\mathcal{C}_{14}$	x^2-x-3	$x^4-x^3+2x^2+4x+3$
$\mathcal{C}_{15}$	x^2-x-4	x^4-x^3-2x+4
$\mathcal{C}_{16}$	x^2-x-4	$x^4-x^3+5x^2+4x+16$
$\mathcal{C}_{17}$	x^2-x-5	$x^4-x^3-x^2-2x+4$
$\mathcal{C}_{18}$	x^2-6	x^4-2x^2+4
$\mathcal{C}_{19}$	x^2-6	x^4+9
$\mathcal{C}_{20}$	x^2-7	x^4-3x^2+4
$\mathcal{C}_{21}$	x^2-x-8	$x^4-x^3-2x^2-3x+9$
$\mathcal{C}_{22}$	x^2-11	x^4-5x^2+9
$\mathcal{C}_{23}$	x^2-14	$x^4-2x^3+9x^2-8x+2$
$\mathcal{C}_{24}$	x^2-x-14	$x^4-x^3-4x^2-5x+25$
$\mathcal{C}_{25}$	x^2-15	x^4-5x^2+25
$\mathcal{C}_{26}$	x^2-15	x^4-7x^2+16
$\mathcal{C}_{27}$	x^2-x-17	$x^4-x^3-5x^2-6x+36$
$\mathcal{C}_{28}$	x^2-19	x^4-9x^2+25
$\mathcal{C}_{29}$	x^2-x-19	x^4+9x^2+1
$\mathcal{C}_{30}$	x^2-22	$x^4-2x^3+11x^2-10x+3$
$\mathcal{C}_{31}$	x^3-x^2-2x+1	$x^6-x^5+x^4-x^3+x^2-x+1$
$\mathcal{C}_{32}$	x^3-x^2-2x+1	$x^6-x^5+3x^4+5x^2-2x+1$
$\mathcal{C}_{33}$	x^3-3x-1	x^6-x^3+1
$\mathcal{C}_{34}$	$x^4-x^3-4x^2+4x+1$	$x^8-x^7+x^5-x^4+x^3-x+1$
$\mathcal{C}_{35}$	x^4-5x^2+5	$x^8-x^6+x^4-x^2+1$
$\mathcal{C}_{36}$	x^4-4x^2+2	x^8+1
$\mathcal{C}_{37}$	x^4-4x^2+1	x^8-x^4+1
$\mathcal{C}_{38}$	$x^4-2x^3-7x^2+8x+1$	$x^8-3x^6+8x^4-3x^2+1$
$\mathcal{C}_{39}$	x^4-6x^2-4x+2	$x^8-4x^7+14x^6-28x^5+43x^4$ $-44x^3+30x^2-12x+2$
$\mathcal{C}_{40}$	$x^4-2x^3-3x^2+4x+1$	$x^8-4x^7+5x^6+2x^5-11x^4$ $+4x^3+20x^2-32x+16$

The relevant numerical values are given below, where μ is the expression $2^{-2d}\zeta_k(-1)L_{\ell|k}(-2)$.

(k, ℓ)	D_k	D_ℓ	$\zeta_k(-1)$	$L_{\ell\|k}(-2)$	μ
$\mathcal{C}_1$	5	125	$1/30$	$4/5$	$1/600$
$\mathcal{C}_2$	5	225	$1/30$	$32/9$	$1/135$
$\mathcal{C}_3$	5	400	$1/30$	15	$1/2^5$
$\mathcal{C}_4$	5	1025	$1/30$	160	$1/3$
$\mathcal{C}_5$	5	1225	$1/30$	$1728/7$	$18/35$
$\mathcal{C}_6$	5	1525	$1/30$	420	$7/8$
$\mathcal{C}_7$	5	1600	$1/30$	474	$79/2^4\cdot 5$

(k,ℓ)	D_k	D_ℓ	$\zeta_k(-1)$	$L_{\ell\vert k}(-2)$	μ
$\mathcal{C}_8$	8	256	1/12	3/2	$1/2^7$
$\mathcal{C}_9$	8	576	1/12	92/9	$23/2^4\cdot 3^3$
$\mathcal{C}_{10}$	8	1088	1/12	64	1/3
$\mathcal{C}_{11}$	12	144	1/6	1/9	$1/2^5\cdot 3^3$
$\mathcal{C}_{12}$	12	2304	1/6	138	$23/2^4$
$\mathcal{C}_{13}$	13	1521	1/6	352/9	$11/3^3$
$\mathcal{C}_{14}$	13	2197	1/6	1332/13	111/104
$\mathcal{C}_{15}$	17	2312	1/3	64	4/3
$\mathcal{C}_{16}$	17	2601	1/3	536/9	67/54
$\mathcal{C}_{17}$	21	441	1/3	32/63	2/189
$\mathcal{C}_{18}$	24	576	1/2	2/3	1/48
$\mathcal{C}_{19}$	24	2304	1/2	23	23/32
$\mathcal{C}_{20}$	28	784	2/3	8/7	1/21
$\mathcal{C}_{21}$	33	1089	1	4/3	1/12
$\mathcal{C}_{22}$	44	1936	7/6	3	7/32
$\mathcal{C}_{23}$	56	3136	5/3	48/7	5/7
$\mathcal{C}_{24}$	57	3249	7/3	44/9	77/108
$\mathcal{C}_{25}$	60	3600	2	60/9	5/6
$\mathcal{C}_{26}$	60	3600	2	8	1
$\mathcal{C}_{27}$	69	4761	2	32/3	4/3
$\mathcal{C}_{28}$	76	5776	19/6	11	209/96
$\mathcal{C}_{29}$	77	5929	2	96/7	12/7
$\mathcal{C}_{30}$	88	7744	23/6	18	69/16
$\mathcal{C}_{31}$	49	16807	−1/21	−64/7	1/147
$\mathcal{C}_{32}$	49	64827	−1/21	−2408/9	$43/2^3\cdot 3^3$
$\mathcal{C}_{33}$	81	19683	−1/9	−104/27	$13/2^3\cdot 3^5$
$\mathcal{C}_{34}$	1125	1265625	4/15	128/45	$2/3^3\cdot 5^2$
$\mathcal{C}_{35}$	2000	4000000	2/3	12	$1/2^5$
$\mathcal{C}_{36}$	2048	16777216	5/6	411	$5\cdot 137/2^9$
$\mathcal{C}_{37}$	2304	5308416	1	46/3	$23/2^7\cdot 3$
$\mathcal{C}_{38}$	3600	12960000	8/5	160/3	1/3
$\mathcal{C}_{39}$	4352	18939904	8/3	96	1
$\mathcal{C}_{40}$	4752	22581504	8/3	928/9	29/27

8.3. Remark. The second table above lists the values of $\zeta_k(-1)$ and $L_{\ell|k}(-2)$. These were obtained with the help of PARI/GP and the functional equations

$$\zeta_k(2) = (-2)^d \pi^{2d} D_k^{-3/2} \zeta_k(-1),$$
$$L_{\ell|k}(3) = (-2)^d \pi^{3d} (D_k/D_\ell)^{5/2} L_{\ell|k}(-2).$$

The values have been rechecked using MAGMA. The latter software gives us precision up to more than 40 decimal places. On the other hand, we know from a result of Siegel [Si] that both $\zeta_k(-1)$ and $L_{\ell|k}(-2)$ are rational numbers. Furthermore, the denominator of $\zeta_k(-1)$ can be effectively estimated as explained in [Si]. Similar estimates for $L_{\ell|k}(-2)$ are given in [Ts]. In this way, we know that the values listed in the above table are exact. Alternatively, the values can also be obtained from the formulae in [Si] and [Ts], but the computations will be quite tedious.

Using Proposition 2.12, and the value of μ given in the second table of 8.2, we conclude the following at once.

8.4. *The pair (k, ℓ), with degree $d = [k : \mathbb{Q}] > 1$, can only be one of the following fifteen:* $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_8$, $\mathcal{C}_{10}$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$, $\mathcal{C}_{20}$, $\mathcal{C}_{21}$, $\mathcal{C}_{26}$, $\mathcal{C}_{31}$, $\mathcal{C}_{35}$, $\mathcal{C}_{38}$ *and* $\mathcal{C}_{39}$.

It is convenient to have the following concrete description provided to us by Tim Steger of the fifteen pairs occurring above. As before, in the sequel, ζ_n will denote a primitive n-th root of unity.

$$\begin{aligned}
\mathcal{C}_1 &= (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\zeta_5)),\\
\mathcal{C}_2 &= (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3)),\\
\mathcal{C}_3 &= (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_4)),\\
\mathcal{C}_4 &= \Big(\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{(-13+\sqrt{5})/2})\Big),\\
\mathcal{C}_8 &= (\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\zeta_8)),\\
\mathcal{C}_{10} &= \Big(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-7+4\sqrt{2}})\Big),\\
\mathcal{C}_{11} &= (\mathbb{Q}(\sqrt{3}), \mathbb{Q}(\zeta_{12})),\\
\mathcal{C}_{18} &= (\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3)),\\
\mathcal{C}_{20} &= (\mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{7}, \zeta_4)),\\
\mathcal{C}_{21} &= (\mathbb{Q}(\sqrt{33}), \mathbb{Q}(\sqrt{33}, \zeta_3)),\\
\mathcal{C}_{26} &= (\mathbb{Q}(\sqrt{15}), \mathbb{Q}(\sqrt{15}, \zeta_4)),\\
\mathcal{C}_{31} &= (\mathbb{Q}(\zeta_7 + \zeta_7^{-1}), \mathbb{Q}(\zeta_7)),\\
\mathcal{C}_{35} &= (\mathbb{Q}(\zeta_{20} + \zeta_{20}^{-1}), \mathbb{Q}(\zeta_{20})),\\
\mathcal{C}_{38} &= (\mathbb{Q}(\sqrt{3}, \sqrt{5}), \mathbb{Q}(\sqrt{3}, \sqrt{5}, \zeta_4)),\\
\mathcal{C}_{39} &= \Big(\mathbb{Q}(\sqrt{5+2\sqrt{2}}), \mathbb{Q}(\sqrt{5+2\sqrt{2}}, \zeta_4)\Big).
\end{aligned}$$

For the ℓ occurring in any of these fifteen pairs, $h_{\ell,3} = 1$.

8.5. We will now assume that the pair (k, ℓ) is one of the fifteen listed above; $\mathscr{D}$ and the k-group G be as in 1.2. Let $(P_v)_{v \in V_f}$ be a coherent collection of parahoric subgroups P_v of $G(k_v)$ such that for all $v \in \mathcal{R}_\ell$, P_v is maximal. Let $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$, and Γ be its normalizer in $G(k_{v_o})$. Let $\mathcal{T}$ be the set of nonarchimedean places of k which are unramified in ℓ

and P_v is *not* a hyperspecial parahoric subgroup, and $\mathcal{T}_0$ be the subset of $\mathcal{T}$ consisting of places where G is anisotropic. The places in $\mathcal{T}_0$ split in ℓ, cf. 2.2.

We first treat the case where $\mathcal{D}$ is a cubic division algebra. In this case, $\mathcal{T}_0$ is nonempty.

8.6. Proposition. *Assume that $\mathcal{D}$ is a cubic division algebra. If the orbifold Euler–Poincaré characteristic $\chi(\Gamma)$ of Γ is a reciprocal integer, then the pair (k, ℓ) must be one of the following six: $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_{10}$, $\mathcal{C}_{18}$, $\mathcal{C}_{31}$ and $\mathcal{C}_{39}$. Moreover, $\mathcal{T}_0$ consists of exactly one place $\mathfrak{v}$, and $\mathcal{T} = \mathcal{T}_0$. Except for the pairs $\mathcal{C}_3$ and $\mathcal{C}_{18}$, $\mathfrak{v}$ is the unique place of k lying over 2; for $\mathcal{C}_3$ it is the unique place of k lying over* 5, *and for $\mathcal{C}_{18}$ it is the unique place of k lying over* 3.

Proof. We recall from Sects. 1 and 2 that $\chi(\Gamma) = 3\mu(G(k_{v_o})/\Gamma)$, and $\mu(G(k_{v_o})/\Gamma) = \mu\cdot\prod_{v\in\mathcal{T}} e'(P_v)/[\Gamma : \Lambda]$, where, as before, $\mu = 2^{-2d}\zeta_k(-1)$ $L_{\ell|k}(-2)$. Moreover, $e'(P_v)$ is an integer for every v, and, as we have shown in 2.3, $[\Gamma : \Lambda]$, which is a power of 3, is at most $3^{1+\#\mathcal{T}_0}h_{\ell,3}\prod_{v\in\mathcal{T}-\mathcal{T}_0}\#\Xi_{\Theta_v}$. We note that $h_{\ell,3} = 1$ for the ℓ occurring in any of the fifteen pairs (k, ℓ) listed in 8.4. From 2.5(*ii*) we know that for $v \in \mathcal{T}_0$, $e'(P_v) = (q_v-1)^2(q_v+1)$. Now the proposition can be proved by a straightforward case-by-case analysis carried out for each of the fifteen pairs (k, ℓ), keeping in mind Proposition 2.12, the fact that every $v \in \mathcal{T}_0$ splits in ℓ, and making use of the values of $e'(P_v)$ and $\#\Xi_{\Theta_v}$ given in 2.5 and 2.2 respectively. We can show that unless (k, ℓ) is one of the six pairs $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_{10}$, $\mathcal{C}_{18}$, $\mathcal{C}_{31}$ and $\mathcal{C}_{39}$, $\mathcal{T}$, $\mathcal{T}_0$ are as in the proposition, and P_v is maximal for all $v \in V_f$, and is hyperspecial whenever $G(k_v)$ contains such a subgroup, at least one of the following two assertions will hold:

- The numerator of $\mu\cdot\prod_{v\in\mathcal{T}} e'(P_v)$ is divisible by a prime other than 3.
- $\mu\cdot\prod_{v\in\mathcal{T}} e'(P_v)/3^{\#\mathcal{T}_0}\prod_{v\in\mathcal{T}-\mathcal{T}_0}\#\Xi_{\Theta_v} > 1$.

8.7. Let k, ℓ, and G be as in 1.2 with $\mathcal{D} = \ell$. We assume here that $d = [k : \mathbb{Q}] > 1$, $h_{\ell,3} = 1$, and ℓ contains a root ζ of unity of order s. *We will now show that then given any coherent collection $(P_v^{\mathrm{m}})_{v\in V_f}$ of maximal parahoric subgroups, the principal arithmetic subgroup $\Lambda^{\mathrm{m}} := G(k)\cap\prod_{v\in V_f}P_v^{\mathrm{m}}$ contains an element of order s. In particular, Λ^{m} contains an element of order* 2. *Moreover, if for every nonarchimedean place v of k which does not split in ℓ, $\ell_v := k_v\otimes_k\ell$ contains a primitive cube-root of unity, then Λ^{m} contains an element of order* 3. (In several cases of interest, Tim Steger, using an argument different from the one employed below, showed that Λ^{m} contains elements of order 2 or 3.) For the proof, let $\ell' = k(\omega) = k[X]/(X^2+X+1)$, where ω is a primitive cube-root of unity, and Q be the quaternion division algebra with center k, which is unramified at all nonarchimedean places of k, and which is ramified at all real places of k if d is even, and if d is odd, it is ramified at all real places $v \neq v_o$. It is obvious that, as both ℓ and ℓ' are totally complex quadratic extension of k, they embed in Q. We will view Q as a ℓ-vector space of dimension 2 in terms of a fixed embedding of ℓ in Q (the action of ℓ on Q is by multiplication on the left). Then the reduced-norm-form on Q gives us an hermitian form h_0 on the two-dimensional ℓ-vector space Q. Now we choose $a \in k^\times$ so that the hermitian form $h_0 \perp \langle a \rangle$ is indefinite at v_o, and definite at all real places $v \neq v_o$. We may (and we do) assume that h is this form, see 1.2. We will view $G_0 := \mathrm{SU}(h_0)$ as a subgroup of $G = \mathrm{SU}(h)$ in terms of its natural embedding.

Let $c_\zeta \in G(k)$ be the element which on Q acts by multiplication on the left by ζ, and on the one-dimensional ℓ-subspace of the hermitian form $\langle a \rangle$ it acts by multiplication by ζ^{-2}. It is obvious that c_ζ is of order s, and it commutes with G_0.

We assume in this paragraph that ℓ does not contain a primitive cube-root of unity but for every nonarchimedean place v of k which does not split in ℓ, $\ell_v := k_v \otimes_k \ell$ contains a primitive cube-root of unity. Let c be the element of $G_0(k)$ which acts on Q by multiplication by ω on the right; c is of order 3. Let v be a nonarchimedean place of k which does not split in ℓ. Then $Q_v := k_v \otimes_k Q = \ell_v \otimes_k \ell' = \ell_v[X]/(X-\omega) \oplus \ell_v[X]/(X-\omega^2)$. Let $e_v = (1,0)$ and $f_v = (0,1)$ in $\ell_v[X]/(X-\omega) \oplus \ell_v[X]/(X-\omega^2) = Q_v$. Then e_v and f_v are eigenvectors of c for the eigenvalues ω and ω^2 respectively. As $e_v \cdot f_v = 0$, the reduced norm of both e_v and f_v in Q_v is zero. For $x \in k_v^\times$, let $s_v(x) \in G_0(k_v)$ be the element which maps e_v onto xe_v and f_v onto $x^{-1} f_v$. Then there is a 1-dimensional k_v-split torus S_v of G_0 such that $S_v(k_v) = \{\, s_v(x) \mid x \in k_v^\times \}$. It is obvious that c commutes with S_v.

As c and c_ζ are k-rational elements of finite order, they lie in P_v^{m} for all but finitely many $v \in V_f$. We assert that for every $v \in V_f$, c (resp., c_ζ) belongs to a conjugate of P_v^{m} under an element of $\overline{G}(k_v)$. This is obvious if v splits in ℓ since then the maximal parahoric subgroups of $G(k_v)$ form a single conjugacy class under $\overline{G}(k_v)$. If v does not split in ℓ, then both G and G_0 are of rank 1 over k_v, and as c commutes with the maximal k_v-split torus S_v of G described above, it fixes the apartment corresponding to S_v, in the building of $G(k_v)$, pointwise. On the other hand, as c_ζ commutes with all of G_0, it fixes pointwise the apartment corresponding to any maximal k_v-split torus of G contained in G_0. From these observations our assertion follows. Now Proposition 5.3 implies that a conjugate of c (resp., c_ζ) under the group $\overline{G}(k)$ lies in Λ^{m}.

We will now prove the following proposition where Γ is as in 2.1, $\Lambda = \Gamma \cap G(k)$, and $\overline{\Gamma}$ is the image of Γ in $\overline{G}(k_{v_o})$.

8.8. Proposition. *If $\mathcal{D} = \ell$, and $\overline{\Gamma}$ contains a torsion-free subgroup Π which is cocompact in $G(k_{v_o})$ and whose Euler–Poincaré characteristic is* 3, *then the pair (k, ℓ) can only be one of the following five: $\mathcal{C}_1$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$.*

Proof. It follows from 4.1 that $d > 1$, so (k, ℓ) can only be one of the fifteen pairs listed in 8.4. We will use the result proved in 8.7 to exclude the ten pairs not listed in the proposition.

Let $\widetilde{\Pi}$ be the inverse image of Π in $G(k_{v_o})$. As observed in 1.3, the orbifold Euler–Poincaré characteristic $\chi(\widetilde{\Pi})$ of $\widetilde{\Pi}$ is 1, hence the orbifold Euler–Poincaré characteristic $\chi(\Gamma)$ of Γ is a reciprocal integer. Moreover, $[\Gamma : \Lambda]$ is a power of 3. Let Λ^{m} be a maximal principal arithmetic subgroup of $G(k)$ containing Λ. From the volume formula (11) we see that $\mu(G(k_{v_o})/\Lambda^{\mathrm{m}})$ is an integral multiple $a\mu$ of $\mu = 2^{-2d}\zeta_k(-1)L_{\ell|k}(-2)$. We assume now that $\mathscr{D} = \ell$, and (k, ℓ) is one of the following ten pairs: $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_{10}$, $\mathcal{C}_{20}$, $\mathcal{C}_{26}$, $\mathcal{C}_{31}$, $\mathcal{C}_{35}$, $\mathcal{C}_{38}$ and $\mathcal{C}_{39}$. These are the pairs appearing in 8.4 excluding the five listed in the proposition. To each of these pairs we associate a prime p as follows. For all these pairs except $\mathcal{C}_3$ and $\mathcal{C}_{35}$, p is 2. For $\mathcal{C}_3$, p is 3. The totally complex number field ℓ in $\mathcal{C}_{35}$ equals $\mathbb{Q}(\zeta_{20})$. For the pair $\mathcal{C}_{35}$, p is 5. We observe that the denominator of μ, for each of the ten pairs, is prime to the corresponding p.

We will first treat the following nine pairs: $\mathcal{C}_2, \mathcal{C}_4, \mathcal{C}_{10}, \mathcal{C}_{20}, \mathcal{C}_{26}, \mathcal{C}_{31}, \mathcal{C}_{35}$, $\mathcal{C}_{38}$ and $\mathcal{C}_{39}$. As ℓ occurring in each of these pairs contains a root of unity of order p, and $h_{\ell,3} = 1$, it follows from 8.7 that Λ^{m} contains an element of order p. Hence, either Λ contains an element of order p, or its index in Λ^{m} is a multiple of p. This implies that either Γ contains an element of order p, or the numerator of $\mu(G(k_{v_o})/\Gamma) = \mu(G(k_{v_o})/\Lambda^{\mathrm{m}})[\Lambda^{\mathrm{m}} : \Lambda]/[\Gamma : \Lambda] = a\mu \cdot [\Lambda^{\mathrm{m}} : \Lambda]/[\Gamma : \Lambda]$ is a multiple of p. This in turn implies that either $\widetilde{\Pi}$ contains an element of order p, or the numerator of $\chi(\widetilde{\Pi}) = 3\mu(G(k_{v_o})/\widetilde{\Pi})$ is a multiple of p. Both these alternatives are impossible, the former because any element of finite order in $\widetilde{\Pi}$ is of order 3, whereas $p = 2$ or 5, and the latter because $\chi(\widetilde{\Pi}) = 1$.

We will now work with the remaining pair $\mathcal{C}_3$. In this pair, $k = \mathbb{Q}(\sqrt{5})$, and $\ell = \mathbb{Q}(\sqrt{5}, \zeta_4)$; ℓ does not contain a primitive cube-root of unity, but for every nonarchimedean place v of k which does not split in ℓ, $\ell_v := k_v \otimes_k \ell$ does contain a primitive cube-root of unity. (To see this, note that if K is a nonarchimedean local field which does not contain a primitive cube-root of unity, and whose residue field is of characteristic different from 3, then $K(\zeta_3)$ is its unique unramified extension of degree 2. Observe also that there is just one nonarchimedean place of k which ramifies in ℓ, it is the place $\mathfrak{v}$ lying over 2. As the residue field of $k_{\mathfrak{v}}$ is the field with 4 elements, $k_{\mathfrak{v}}$ contains a primitive cube-root of unity. On the other hand, the unique place of k lying over 3 splits in ℓ.) For $v \in V_f$, let P_v be as in 2.1. Then $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$. Using the volume formula (11), the values of $e'(P_v)$ given in 2.5, and the value of μ given in the second table in 8.2, it is easy to see that for all $v \in V_f$, P_v is a maximal parahoric subgroup and it is hyperspecial whenever $G(k_v)$ contains such a subgroup. Hence, Λ is a maximal principal arithmetic subgroup of $G(k)$, and $\chi(\Lambda) = 3\mu(G(k_{v_o})/\Lambda) = 3\mu$. We know from 5.4 that $[\Gamma : \Lambda] = 3$ since $h_{\ell,3} = 1$ and $\mathscr{T}_0$ is empty. Then $\chi(\Gamma) = \chi(\Lambda)/3 = \mu$. Since $\chi(\widetilde{\Pi}) = 1$, the index of $\widetilde{\Pi}$ in Γ is $1/\mu$, which is a power of 2 in the case presently under consideration. As ℓ does not contain a primitive cube-root of unity, the center of $G(k)$, and so also of Λ, is trivial, and therefore, $\Gamma = \Lambda \cdot C(k_{v_o})$, where $C(k_{v_o})$ is the center of $G(k_{v_o})$ which is a cyclic group of order 3. We conclude now that the image $\overline{\Gamma}$ of Γ in the adjoint group $\overline{G}(k_{v_o}) = \mathrm{PU}(2,1)$ coincides with the image $\overline{\Lambda}$ of Λ, and the index of Π in $\overline{\Lambda}$ is $1/\mu$. As Λ is a maximal principal arithmetic subgroup of $G(k)$, it follows from 8.7 that it, and so also $\overline{\Lambda}$, contains an element of order 3. But then any subgroup of $\overline{\Lambda}$ of index a power of 2, in particular, Π, contains an element of order 3. This contradicts the fact that Π is torsion-free. Thus we have proved the proposition.

9. Five additional classes of fake projective planes

Until 9.5, (k, ℓ) will be one of the following six pairs (see Proposition 8.6).

$$\begin{aligned}
\mathcal{C}_2 &= (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3)),\\
\mathcal{C}_3 &= (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_4)),\\
\mathcal{C}_{10} &= \Big(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-7+4\sqrt{2}})\Big),\\
\mathcal{C}_{18} &= (\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3)),\\
\mathcal{C}_{31} &= (\mathbb{Q}(\zeta_7 + \zeta_7^{-1}), \mathbb{Q}(\zeta_7)),\\
\mathcal{C}_{39} &= \Big(\mathbb{Q}(\sqrt{5+2\sqrt{2}}), \mathbb{Q}(\sqrt{5+2\sqrt{2}}, \zeta_4)\Big).
\end{aligned}$$

Let $\mathfrak{v}$ be the unique place of k lying over $p := 2$ if $(k, \ell) \neq \mathcal{C}_3, \mathcal{C}_{18}$; if $(k, \ell) = \mathcal{C}_3$, let $\mathfrak{v}$ be the unique place of k lying over $p := 5$; and if $(k, \ell) = \mathcal{C}_{18}$, let $\mathfrak{v}$ be the unique place of k lying over $p := 3$. Let $q_{\mathfrak{v}}$ be the cardinality of the residue field of $k_{\mathfrak{v}}$.

9.1. Let $\mathcal{D}$ be a cubic division algebra with center ℓ whose local invariants at the two places of ℓ lying over $\mathfrak{v}$ are nonzero and negative of each other, and whose local invariant at all the other places of ℓ is zero. There are two such division algebras, they are opposite of each other. $k_{\mathfrak{v}} \otimes_k \mathcal{D} = (k_{\mathfrak{v}} \otimes_k \ell) \otimes_\ell \mathcal{D} = \mathfrak{D} \oplus \mathfrak{D}^o$, where $\mathfrak{D}$ is a cubic division algebra with center $k_{\mathfrak{v}}$, and $\mathfrak{D}^o$ is its opposite.

We fix a real place v_o of k, and an involution σ of $\mathcal{D}$ of the second kind so that $k = \{x \in \ell \mid \sigma(x) = x\}$, and if G is the simple simply connected k-group with

$$G(k) = \{z \in \mathcal{D}^\times \mid z\sigma(z) = 1 \ \text{ and } \ \mathrm{Nrd}(z) = 1\},$$

then $G(k_{v_o}) \cong \mathrm{SU}(2,1)$, and G is anisotropic at all real places of k different from v_o. Any other such involution of $\mathcal{D}$, or of its opposite, similarly determines a k-group which is k-isomorphic to G (cf. 1.2).

The set $\mathcal{T}_0$ of nonarchimedean places of k where G is anisotropic equals $\{\mathfrak{v}\}$. As $\sigma(\mathfrak{D}) = \mathfrak{D}^o$, it is easily seen that $G(k_\mathfrak{v})$ is the compact group $\mathrm{SL}_1(\mathfrak{D})$ of elements of reduced norm 1 in $\mathfrak{D}$. The first congruence subgroup $\mathrm{SL}_1^{(1)}(\mathfrak{D})$ of $\mathrm{SL}_1(\mathfrak{D})$ is known to be a pro-p group, and $\mathfrak{C} := \mathrm{SL}_1(\mathfrak{D})/\mathrm{SL}_1^{(1)}(\mathfrak{D})$ is a cyclic group of order $(q_\mathfrak{v}^3-1)/(q_\mathfrak{v}-1) = q_\mathfrak{v}^2+q_\mathfrak{v}+1$, see [Ri, Theorem 7(iii)(2)].

Let $(P_v)_{v\in V_f}$ be a coherent collection of maximal parahoric subgroups P_v of $G(k_v)$, $v \in V_f$, such that P_v is hyperspecial whenever $G(k_v)$ contains such a subgroup (recall that according to Proposition 8.6, $\mathcal{T} = \mathcal{T}_0$). Let $\Lambda = G(k) \cap \prod_{v\in V_f} P_v$. Let Γ be the normalizer of Λ in $G(k_{v_o})$. It follows from 5.4 that $[\Gamma : \Lambda] = 9$ since $\#\mathcal{T}_0 = 1$.

Then $\chi(\Lambda) = 3\mu(G(k_{v_o})/\Lambda) = 3\mu \cdot e'(P_\mathfrak{v})$, and (see 2.5 (*ii*)) $e'(P_\mathfrak{v}) = (q_\mathfrak{v} - 1)^2(q_\mathfrak{v} + 1)$. We list $q_\mathfrak{v}$, μ and $\chi(\Lambda)$ in the table given below.

(k,ℓ)	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_{10}$	$\mathcal{C}_{18}$	$\mathcal{C}_{31}$	$\mathcal{C}_{39}$
$q_\mathfrak{v}$	4	5	2	3	8	2
μ	1/135	1/32	1/3	1/48	1/147	1
$\chi(\Lambda)$	1	9	3	1	9	9

Let $\overline{G}$ be the adjoint group of G. Let $\overline{\Lambda}$ (resp., $\overline{\Gamma}$) be the image of Λ (resp., Γ) in $\overline{G}(k_{v_o})$.

We shall now prove the following lemma.

9.2. Lemma. *$G(k)$ is torsion-free except when (k,ℓ) is either $\mathcal{C}_2$ or $\mathcal{C}_{18}$, in which case any nontrivial element of $G(k)$ of finite order is central and hence is of order* 3.

Proof. Let $x \in G(k)$ ($\subset \mathcal{D}$) be a nontrivial element of finite order, say of order m. As the reduced norm of -1 is -1, $-1 \notin G(k)$, and so m is *odd*. Let L be the ℓ-subalgebra of $\mathcal{D}$ generated by x. Then L is a field extension of ℓ of degree 1 or 3. If $L = \ell$, then x is clearly central, and hence it is of order 3. As ℓ does not contain a nontrivial cube-root of unity unless (k,ℓ) is $\mathcal{C}_2$ or $\mathcal{C}_{18}$, to prove the lemma, we can assume that L is an extension of ℓ of degree 3. Then $[L : \mathbb{Q}] = 6d$, where $d = 2$, 3 or 4.

(i) $(k,\ell) = \mathcal{C}_2$ *or* $\mathcal{C}_{18}$: Then $d = 2$, $[L : \mathbb{Q}] = 12$, and ζ_3 is in ℓ. Hence, we can assume that m is a multiple of 3. Then as $\phi(m)$, where ϕ is the Euler function, must divide 12, we conclude that m is either 9 or 21. We assert that if $(k,\ell) = \mathcal{C}_2$ or $\mathcal{C}_{18}$, then $m = 9$. For if $m = 21$, then $L \cong \mathbb{Q}(\zeta_{21})$, and since 3 and 7 are the only primes which ramify in $\mathbb{Q}(\zeta_{21})$, whereas 5 ramifies in $k \subset L$, if $(k,\ell) = \mathcal{C}_2$, so m cannot be 21 in this case. Next we observe that if $(k,\ell) = \mathcal{C}_{10}$, $\mathcal{C}_{18}$, or $\mathcal{C}_{39}$, then as $7 \nmid D_\ell$, 7 does not ramify in ℓ, and hence the ramification index of L at 7 is at most 3. But the ramification index of $\mathbb{Q}(\zeta_7)$ at 7 is 6. So if $(k,\ell) = \mathcal{C}_{10}$, $\mathcal{C}_{18}$, or $\mathcal{C}_{39}$, then L cannot contain a nontrivial 7-th root of unity. We conclude, in particular, that if $(k,\ell) = \mathcal{C}_{18}$, $m = 9$.

Now let $(k,\ell) = \mathcal{C}_2$ or $\mathcal{C}_{18}$. Then, as ℓ contains ζ_3, and x^3 is of order 3, the latter is contained in ℓ. So any automorphism of L/ℓ will fix x^3, and hence it will map x to either x, or to x^4, or to x^7. Therefore, $\mathrm{Nrd}(x) = x^{12} = x^3 \neq 1$ and x cannot belong to $G(k)$.

(ii) $(k, \ell) = \mathcal{C}_3$: Then $\mathrm{SL}_1^{(1)}(\mathfrak{D})$ is a pro-5 group, and $\mathfrak{C}$ is a group of order 31. Since $\phi(31) = 30 > 6d = 12$, we conclude that m must be a power of 5. But ℓ, and hence L, contains ζ_4, so L contains ζ_{4m}. This is impossible since $\phi(4m)$ is not a divisor of 12.

(iii) $(k, \ell) = \mathcal{C}_{10}$, *or* $\mathcal{C}_{31}$, *or* $\mathcal{C}_{39}$: Then $\mathrm{SL}_1^{(1)}(\mathfrak{D})$ is a pro-2 group, and $\mathfrak{C}$ is a group of order 7 if $(k, \ell) = \mathcal{C}_{10}$ or $\mathcal{C}_{39}$, and is of order 73 if $(k, \ell) = \mathcal{C}_{31}$. Therefore, if $(k, \ell) = \mathcal{C}_{31}$, $m = 73$, but this is impossible since $\phi(73) = 72 > 6d = 18$. On the other hand, if $(k, \ell) = \mathcal{C}_{10}$ or $\mathcal{C}_{39}$, then $m = 7$. But this is impossible since, as we observed above, L does not contain a nontrivial 7-th root of unity.

9.3. Classes of fake projective planes arising from $\mathcal{C}_2$ and $\mathcal{C}_{18}$. We assume here that (k, ℓ) is either $\mathcal{C}_2$ or $\mathcal{C}_{18}$. Then ℓ contains a nontrivial cube-root of unity, and hence the center $C(k)$ of $G(k)$ is a group of order 3 which is contained in Λ. The naural homomorphism $\Lambda \to \overline{\Lambda}$ is surjective and its kernel equals $C(k)$. Hence, $\chi(\overline{\Lambda}) = 3\chi(\Lambda) = 3$. Lemma 9.2 implies that $\overline{\Lambda}$ is torsion-free. According to [Ro, Theorem 15.3.1], $H^1(\Lambda, \mathbb{C})$ vanishes which implies that so does $H^1(\overline{\Lambda}, \mathbb{C})$. By Poincaré-duality, $H^3(\overline{\Lambda}, \mathbb{C})$ also vanishes. We conclude that if B is the symmetric space of $G(k_{v_o})$, then $B/\overline{\Lambda}$ is a fake projective plane. Its fundamental group is $\overline{\Lambda}$. There is a natural faithful action of $\overline{\Gamma}/\overline{\Lambda}$ on $B/\overline{\Lambda}$. As the normalizer of $\overline{\Lambda}$ in $\overline{G}(k_{v_o})$ is $\overline{\Gamma}$, the automorphism group of $B/\overline{\Lambda}$ equals $\overline{\Gamma}/\overline{\Lambda}$.

Clearly, $[\overline{\Gamma} : \overline{\Lambda}] = [\Gamma : \Lambda] = 9$. Now let Π be a torsion-free subgroup of $\overline{\Gamma}$ of index 9. Then $\chi(\Pi) = 3$, and so if $H^1(\Pi, \mathbb{C}) = 0$ (or, equivalently, $\Pi/[\Pi, \Pi]$ is finite), then B/Π is a fake projective plane, and its fundamental group is Π. The set of these fake projective planes is the class associated with Γ.

9.4. Remark. Let $(k, \ell) = \mathcal{C}_2$, and $\mathfrak{D}$, Λ and $\overline{\Lambda}$ be as in 9.1. Then as $\mathrm{SL}_1(\mathfrak{D})/\mathrm{SL}_1^{(1)}(\mathfrak{D})$ is a cyclic group of order 21, $\mathrm{SL}_1(\mathfrak{D})$ contains a (unique) normal subgroup N of index 3 containing $\mathrm{SL}_1^{(1)}(\mathfrak{D})$. Let $\Lambda^+ = \Lambda \cap N$. Then since $\mathrm{SL}_1^{(1)}(\mathfrak{D})$ is a pro-2 group, Λ^+ is a torsion-free normal subgroup of Λ of index 3. It maps isomorphically onto $\overline{\Lambda}$.

It is not clear that in case $(k, \ell) = \mathcal{C}_{18}$, Λ contains a subgroup which maps isomorphically onto $\overline{\Lambda}$.

9.5. The class of fake projective planes arising from $\mathcal{C}_{10}$. We now assume that $(k, \ell) = \mathcal{C}_{10}$. Then Λ is torsion-free (9.2). Hence, $\overline{\Lambda} \cong \Lambda$, and therefore, $\chi(\overline{\Lambda}) = \chi(\Lambda) = 3$. Theorem 15.3.1 of [Ro] once again implies that $H^1(\Lambda, \mathbb{C})$, and so also $H^1(\overline{\Lambda}, \mathbb{C})$, vanishes. From this we conclude, as above, that if B is the symmetric space of $G(k_{v_o})$, then B/Λ is a fake projective plane. Its fundamental group is $\Lambda \cong \overline{\Lambda}$. There is a natural faithful action of $\overline{\Gamma}/\overline{\Lambda}$ on $B/\overline{\Lambda}$. As the normalizer of $\overline{\Lambda}$ in $\overline{G}(k_{v_o})$ is $\overline{\Gamma}$, the automorphism group of $B/\overline{\Lambda}$ equals $\overline{\Gamma}/\overline{\Lambda}$.

Since $[\Gamma : \Lambda] = 9$, $[\overline{\Gamma} : \overline{\Lambda}] = 3$, and as in 9.3, any torsion-free subgroup Π of $\overline{\Gamma}$ of index 3 with vanishing $H^1(\Pi, \mathbb{C})$ is the fundamental group of a fake projective plane, namely, that of B/Π. The set of these fake projective planes is the class associated with Γ.

9.6. The constructions in 9.3 and 9.5 give us five distinct classes of fake projective planes. To see this, note that the construction is independent of the choice of a real place of k since (in 9.3 and 9.5) k is a quadratic extension of $\mathbb{Q}$ and the nontrivial Galois automorphism of $k/\mathbb{Q}$ interchanges the two real places of k. On the other hand, if v is a nonarchimedean place of k which is unramified in ℓ, the parahoric P_v involved in the construction of Λ is hyperspecial, and the hyperspecial parahoric subgroups of $G(k_v)$ are conjugate to each other under $\overline{G}(k_v)$, see [Ti2, 2.5]. But if v is a nonarchimedean place of k which ramifies in ℓ, there are two possible choices of a maximal parahoric subgroup P_v of $G(k_v)$ up to conjugation. Hence, it follows from Proposition 5.3 that each of the pairs $\mathcal{C}_2$ and $\mathcal{C}_{10}$ gives two distinct classes of fake projective planes, and the pair $\mathcal{C}_{18}$ gives only one since in case $(k, \ell) = \mathcal{C}_2$ or $\mathcal{C}_{10}$, there is (just) one nonarchimedean place of k which ramifies in ℓ, and if $(k, \ell) = \mathcal{C}_{18}$, every nonarchimedean place of k is unramified in ℓ since $D_\ell = D_k^2$.

9.7. We will now show that *the pairs $\mathcal{C}_3$, $\mathcal{C}_{31}$ and $\mathcal{C}_{39}$ do not give rise to any fake projective planes*. For this purpose, let (k, ℓ) be one of these pairs here. Then $h_k = 1 = h_\ell$. We first recall that Λ is a torsion-free subgroup (9.2) and its Euler–Poincaré characteristic is 9. Therefore, $\chi(\overline{\Lambda}) = 9$. As $[\Gamma : \Lambda] = 9$, $[\overline{\Gamma} : \overline{\Lambda}] = 3$. Hence, the orbifold Euler–Poincaré characteristic $\chi(\overline{\Gamma})$ of $\overline{\Gamma}$ equals 3. So no proper subgroup of $\overline{\Gamma}$ can be the fundamental group of a fake projective plane. We will prove presently that $\overline{\Gamma}$ contains an element of order 3. This will imply that it cannot be the fundamental group of a fake projective plane either.

As before, let $\mathfrak{v}$ be the unique place of k lying over 5 if the pair is $\mathcal{C}_3$, and the unique place of k lying over 2 if the pair is either $\mathcal{C}_{31}$ or $\mathcal{C}_{39}$. Let $\mathfrak{v}'$ and $\mathfrak{v}''$ be the two places of ℓ lying over $\mathfrak{v}$. Recall that the cubic division algebra $\mathcal{D}$ ramifies only at $\mathfrak{v}'$ and $\mathfrak{v}''$. Hence, $\mathfrak{v}$ is the only nonarchimedean place of k where G is anisotropic, at all the other nonarchimedean places of k it is quasi-split. Let v' and v'' be the normalized valuations of ℓ corresponding to $\mathfrak{v}'$ and $\mathfrak{v}''$ respectively.

To find an element of $\overline{G}(k)$ of order 3 which normalizes Λ (and hence lies in $\overline{\Gamma}$) we proceed as follows. Since $h_\ell = 1$, there is an element $a \in \ell^\times$ such that $v'(a) = 1$, and for all the other normalized valuations v of ℓ, $v(a) = 0$. Let $\lambda = a/\sigma(a)$. Then $v'(\lambda) = 1$, $v''(\lambda) = -1$, for all normalized valuations $v \neq v'$, v'', of ℓ, $v(\lambda) = 0$, and $N_{\ell/k}(\lambda) = 1$. The field $L := \ell[X]/(X^3 - \lambda)$ admits an involution τ (i.e., an automorphism of order 2) whose restriction to the subfield ℓ coincides with $\sigma|_\ell$; τ is defined as follows: let x be the unique cube-root of λ in L, then $\tau(x) = x^{-1}$.

We assert that there is an embedding ι of L in $\mathcal{D}$ such that, in terms of this embedding, $\sigma|_L = \tau$. Since $k_{\mathfrak{v}} \otimes_k L = (k_{\mathfrak{v}} \otimes_k \ell) \otimes_\ell L$ is clearly a direct sum of two cubic extensions of $k_{\mathfrak{v}}$, L does embed in $\mathcal{D}$. Now to see that there is an embedding such that $\sigma|_L = \tau$, we can apply [PrR, Proposition A.2]. The existence of local embeddings respecting the involutions σ and τ need to be checked only at the real places of k, since at all the nonarchimedean places of k, G is quasi-split (see p. 340 of [PlR]). We will now show that for every real place v of k, there is an embedding ι_v of $k_v \otimes_k L$ in $k_v \otimes_k \mathcal{D}$ such that $\tau = \iota_v^{-1}\sigma\iota_v$. This will imply that there is an embedding ι of L in $\mathcal{D}$ with the desired property.

Let $y = x + \tau(x) = x + x^{-1}$. Then $L^\tau = k[y]$. As $y^3 = x^3 + x^{-3} + 3(x + x^{-1}) = \lambda + \sigma(\lambda) + 3y$, $y^3 - 3y - b = 0$, where $b = \lambda + \sigma(\lambda) \in k$. The discriminant of the cubic polynomial $Y^3 - 3Y - b$ is $27(4 - b^2) = 27\{4\lambda\sigma(\lambda) - (\lambda+\sigma(\lambda))^2\} = -27(\lambda - \sigma(\lambda))^2$. Since ℓ is totally complex, for any real place v of k, $k_v \otimes_k \ell = \mathbb{C}$, and $\lambda - \sigma(\lambda)$ is purely imaginary. So the discriminant $-27(\lambda - \sigma(\lambda))^2$ is positive in $k_v = \mathbb{R}$. Therefore, for any real place v of k, all the roots of $Y^3 - 3Y - b$ are in k_v. Hence, the smallest Galois extension of k containing L^τ is totally real, and so it is linearly disjoint from the totally complex quadratic extension ℓ of k. Moreover, $k_v \otimes_k L$ is a direct sum of three copies of $\mathbb{C}$, each of which is stable under τ. This implies the existence of an embedding ι_v of $k_v \otimes_k L$ in $k_v \otimes_k \mathcal{D}$, and hence of an embedding ι of L in $\mathcal{D}$, with the desired property. We use ι to identify L with a maximal subfield of $\mathcal{D}$.

For a nonzero element $h \in L^\tau$, which we will choose latter, we denote by σ_h the involution of $\mathcal{D}$ defined as follows

$$\sigma_h(z) = h\sigma(z)h^{-1} \text{ for } z \in \mathcal{D}.$$

Now let G (resp., $\mathcal{G}$) be the connected simple (resp., reductive) k-subgroup of $\mathrm{GL}_{1,\mathcal{D}}$ such that

$$G(k) = \left\{z \in \mathcal{D}^\times \mid z\sigma_h(z) = 1 \quad \text{and} \quad \mathrm{Nrd}(z) = 1\right\}$$
$$\left(\text{resp., } \mathcal{G}(k) = \left\{z \in \mathcal{D}^\times \mid z\sigma_h(z) \in k^\times\right\}\right).$$

G is a normal subgroup of $\mathcal{G}$, the center $\mathcal{C}$ of the latter is k-isomorphic to $R_{\ell/k}(\mathrm{GL}_1)$. The adjoint action of $\mathcal{G}$ on the Lie algebra of G induces a k-isomorphism of $\mathcal{G}/\mathcal{C}$ onto the adjoint group $\overline{G}$ of G.

As $x\sigma_h(x) = 1$, x is an element of $\mathcal{G}(k)$. Let g be its image in $\overline{G}(k)$. Since $x^3 = \lambda \in \ell$, g is an element of order 3. Let T be the centralizer of g in G. Then T is a maximal k-torus of G, and its group of k-rational points is $L^\times \cap G(k)$.

We will choose h so that (**1**) $G(k_{v_o})$ is isomorphic to $\mathrm{SU}(2,1)$, and for all real places $v \neq v_o$ of k, $G(k_v)$ is isomorphic to the compact group $\mathrm{SU}(3)$. This condition will clearly hold if for every real place v of k, h is a square in $k_v \otimes_k L^\tau$, or, equivalently, in every embedding of L^τ in $\mathbb{R}$,

h is positive. *It will imply that the group G defined here in terms of the involution σ_h of $\mathcal{D}$ is k-isomorphic to the group introduced in* 9.1; see 1.2. (**2**) For every nonarchimedean place v of k such that $G(k_v)$ contains a hyperspecial parahoric subgroup (this is the case if, and only if, $v \neq \mathfrak{v}$ and v is unramified in ℓ), g normalizes one.

The assertions in the next four paragraphs hold for an arbitrary nonzero $h \in L^\tau$. We will choose h in the fifth paragraph below.

We first observe that the reduced norm of x (x considered as an element of $\mathcal{D}$) is λ, and the image of g in $H^1(k, C) \subset \ell^\times/\ell^{\times 3}$, where C is the center of G, is the class of λ^{-1} in $\ell^\times/\ell^{\times 3}$. Now let $v \neq \mathfrak{v}$ be a nonarchimedean place of k which splits in ℓ. Then $G(k_v) \cong \mathrm{SL}_3(k_v)$, and hence every maximal parahoric subgroup of $G(k_v)$ is hyperspecial. As λ is a unit in both the embeddings of ℓ in k_v, g does normalize a maximal parahoric subgroup of $G(k_v)$, see [BP, 2.7 and 2.3(i)].

Let v now be a nonarchimedean place of k which does not split in ℓ, and $\ell_v := k_v \otimes_k \ell$ is an unramified field extension of k_v. If 3 does not divide $q_v + 1$ (for example, if v lies over 3), then g must normalize a hyperspecial parahoric subgroup of $G(k_v)$. To see this, we assume that g normalizes a non-hyperspecial maximal parahoric subgroup. The number of edges in the Bruhat–Tits building of $G(k_v)$ emanating from the vertex corresponding to this maximal parahoric subgroup is $q_v + 1$. Since g is a k-automorphism of G of order 3, and 3 does not divide $q_v + 1$, at least one of these edges is fixed by g. This implies that g normalizes a hyperspecial parahoric subgroup of $G(k_v)$.

If v is a nonarchimedean place of k which does not lie over 3, then according to the main theorem of [PY], the set of points fixed by g in the Bruhat–Tits building of $G(k_v)$ is the Bruhat–Tits building of $T(k_v)$. We conclude from this that if T is anisotropic at v (i.e., $T(k_v)$ is compact), then as the building of $T(k_v)$ consists of a single point, g fixes a unique point in the building of $G(k_v)$. This implies that if T is anisotropic at v (and v does not lie over 3), then g normalizes a unique parahoric subgroup of $G(k_v)$; if v is unramified in L, this parahoric subgroup is the unique parahoric subgroup of $G(k_v)$ containing $T(k_v)$ ([Ti2, 3.6.1]). Since g is a k-rational automorphism of G of finite order, it normalizes an arithmetic subgroup of $G(k)$. Hence, for all but finitely many nonarchimedean places v of k, g normalizes a hyperspecial parahoric subgroup of $G(k_v)$. Thus, for all but finitely many v in the set of nonarchimedean places of k where T is anisotropic, the unique parahoric subgroup of $G(k_v)$ containing $T(k_v)$ is hyperspecial.

We assume now that v is a nonarchimedean place of k which does not split in ℓ, does not lie over 3, and $\ell_v := k_v \otimes_k \ell$ is an unramified field extension of k_v. Then ℓ_v contains all the cube-roots of unity, and $\ell_v \otimes_\ell L$ is either an unramified field extension of ℓ_v in which case $k_v \otimes_k L^\tau$ is an unramified field extension of k_v, or $\ell_v \otimes_\ell L$ is a direct sum of three copies of ℓ_v in which case $k_v \otimes_k L^\tau$ is either the direct sum of k_v and ℓ_v, or it is the direct sum of three copies of k_v. In case $k_v \otimes_k L^\tau$ is a field, the torus T is anisotropic over k_v and its splitting field is the unramified cubic extension $k_v \otimes_k L = \ell_v \otimes_\ell L$ of ℓ_v. This implies at once that the unique parahoric subgroup of $G(k_v)$ containing $T(k_v)$ is hyperspecial. This parahoric is normalized by g. On the other hand, if $k_v \otimes_k L^\tau = k_v \oplus \ell_v$, then T is isotropic over k_v. The apartment in the Bruhat–Tits building of $G(k_v)$ corresponding to this torus is fixed pointwise by g. In particular, g fixes a hyperspecial parahoric subgroup of $G(k_v)$.

Now let S be the set of all real places of k, and all the nonarchimedean places v such that (i) v does not lie over 3, (ii) $\ell_v := k_v \otimes_k \ell$ is an unramified field extension of k_v, and (iii) $k_v \otimes_k L^\tau$ is the direct sum of three copies of k_v. Then $k_v \otimes_k L = (k_v \otimes_k L^\tau) \otimes_k \ell$ is the direct sum of three copies of ℓ_v each of which is stable under σ_h. This implies that for all nonarchimedean $v \in S$, $T(k_v)$ is compact, i.e., T is anisotropic over k_v. We note that S does not contain the place of k lying over 2. As the smallest Galois extension of k containing L^τ is linearly disjoint from ℓ over k, there is a nonarchimedean place w of k such that $k_w \otimes_k L$ is an unramified field extension of k_w of degree 6. For this w, the field $k_w \otimes_k L = k_w \otimes_k L^\tau \otimes_k \ell$ is a quadratic extension of the subfield $k_w \otimes_k L^\tau$. Hence, by local class field theory, $N_{\ell/k}((k_w \otimes_k L)^\times)$ is a subgroup of index 2 of $(k_w \otimes_k L^\tau)^\times$. Using this, and the fact that as L is a quadratic extension of L^τ, by global class field theory $N_{L/L^\tau}(I_L) \cdot L^{\tau\times}$ is a subgroup of index 2 of I_{L^τ}, where I_L and I_{L^τ} are the idèle groups of L and L^τ respectively, and $N_{L/L^\tau} : I_L \to I_{L^\tau}$ is the norm map, we conclude that $N_{L/L^\tau}(I_L^S) \cdot L^{\tau\times} = I_{L^\tau}^S$, where I_L^S (resp., $I_{L^\tau}^S$) denotes the restricted direct product of $(k_v \otimes_k L)^\times$ (resp., $(k_v \otimes_k L^\tau)^\times$), $v \in S$. This implies that there is an element $h \in L^{\tau\times}$ which is positive in every embedding of L^τ in $\mathbb{R}$, and is such that for every nonarchimedean $v \in S$, we can find an isomorphism of $\ell_v \otimes_\ell \mathcal{D}$ with the matrix algebra $M_3(\ell_v)$ which maps $\ell_v \otimes_\ell L$ onto the subalgebra of diagonal matrices (the image of $T(k_v) \subset (\ell_v \otimes_\ell \mathcal{D})^\times$ under such an isomorphism is the group of diagonal matrices of determinant 1 whose diagonal entries are elements of ℓ_v of norm 1 over k_v), and which carries the involution σ_h into the standard involution of $M_3(\ell_v)$. We choose such an h. Then for every nonarchimedean $v \in S$, the unique parahoric subgroup of $G(k_v)$ containing $T(k_v)$ is hyperspecial, this parahoric subgroup is normalized by g.

From the discussion above, it follows that if $v \neq \mathfrak{v}$ is any nonarchimedean place of k which is unramified in ℓ, then g normalizes a hyperspecial parahoric subgroup of $G(k_v)$. We will now show that if v ramifies in ℓ, then g normalizes a conjugate of every maximal parahoric subgroup of $G(k_v)$.

For $(k, \ell) = \mathcal{C}_{39}$, since $D_\ell = D_k^2$, every nonarchimedean place of k is unramified in ℓ. If $(k, \ell) = \mathcal{C}_3$, the only place of k which ramifies in ℓ is the place v lying over 2, the residue field of k_v has 4 elements, so in the Bruhat–Tits building of $G(k_v)$, 5 edges emanate from every vertex. If $(k, \ell) = \mathcal{C}_{31}$, there is a unique nonarchimedean place v of k which ramifies in ℓ. It is the unique place of k lying over 7. The residue field of k_v has 7 elements, so in the Bruhat–Tits building of $G(k_v)$, 8 edges emanate from every vertex. We infer that if (k, ℓ) is either $\mathscr{C}_3$ or $\mathscr{C}_{31}$, g must fix an edge. This implies that g normalizes a conjugate of every maximal parahoric subgroup of $G(k_v)$.

As g normalizes an arithmetic subgroup of $G(k)$, it does normalize a coherent collection of parahoric subgroups of $G(k_v)$, $v \in V_f$. Since any two hyperspecial parahoric subgroups of $G(k_v)$ are conjugate to each other under an element of $\overline{G}(k_v)$, from the considerations above we conclude that g normalizes a coherent collection $(P'_v)_{v\in V_f}$ of maximal parahoric subgroups such that for every $v \in V_f$, P'_v is conjugate to P_v (P_v's are as in 9.1) under an element of $\overline{G}(k_v)$. Proposition 5.3 implies that a conjugate of g (in $\overline{G}(k)$) normalizes $(P_v)_{v\in V_f}$, and hence it normalizes Λ, and therefore lies in $\overline{\Gamma}$. This proves that $\overline{\Gamma}$ contains an element of order 3.

Combining the results of 8.6, 9.3, and 9.5–9.7 we obtain the following.

9.8. Theorem. *There exist exactly five distinct classes of fake projective planes with the underlying totally real number field k of degree > 1, a totally complex quadratic extension ℓ of k, and a cubic division algebra $\mathscr{D}$ with center ℓ. The pair $(k, \ell) = (\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{5}, \zeta_3))$ gives two of these five, the pair $(\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{-7+4\sqrt{2}}))$ also gives two, and the pair $(\mathbb{Q}(\sqrt{6}), \mathbb{Q}(\sqrt{6}, \zeta_3))$ gives one more.*

10. Some geometric properties of the fake projective planes

In the following, P will denote any of the fake projective planes constructed in 5.9–5.12 and 9.3, 9.5, and Π will denote its fundamental group. Let the pair (k, ℓ), the k-form G of $\mathrm{SU}(2, 1)$, and the real place v_o of k, be the ones associated to Π. Let $\overline{G}$ be the adjoint group of G, C the center of G, and $\varphi : G \to \overline{G}$ be the natural isogeny. Then Π is a torsion-free cocompact arithmetic subgroup of $\overline{G}(k_{v_o})$ ($\cong \mathrm{PU}(2, 1)$). Let $\widetilde{\Pi}$ be the inverse image of Π in $G(k_{v_o})$. Let Λ and Γ be as in 1.3. Then $\Lambda = \Gamma \cap G(k)$, and Γ is the normalizer of Λ in $G(k_{v_o})$.

10.1. Theorem. $H_1(P, \mathbb{Z}) = H_1(\Pi, \mathbb{Z}) = \Pi/[\Pi, \Pi]$ *is nontrivial.*

Proof. The image $\overline{\Gamma}$ of Γ, and hence the image Π of $\widetilde{\Pi}$, in $\overline{G}(k_{v_o})$ is contained in $\overline{G}(k)$, see [BP, Proposition 1.2]. There is a nonarchimedean place $\mathfrak{v}$ of k such that the group $G(k_{\mathfrak{v}})$ is the compact group $\mathrm{SL}_1(\mathfrak{D})$ of elements of reduced norm 1 in a cubic division algebra $\mathfrak{D}$ with center $k_{\mathfrak{v}}$ (cf. 5.7 and 9.1). We will view $\Pi \subset \overline{G}(k)$ as a subgroup of $\overline{G}(k_{\mathfrak{v}})$. We observe that $\overline{G}(k_{\mathfrak{v}})$ ($\cong \mathfrak{D}^\times/k_{\mathfrak{v}}^\times$) is a pro-solvable group, i.e., if we define the decreasing sequence $\{\mathscr{G}_i\}$ of subgroups of $\mathscr{G} := \overline{G}(k_{\mathfrak{v}})$ inductively as follows: $\mathscr{G}_0 = \mathscr{G}$, and $\mathscr{G}_i = [\mathscr{G}_{i-1}, \mathscr{G}_{i-1}]$, then $\bigcap \mathscr{G}_i$ is trivial, to see this use [Ri, Theorem 7(i)]. From this it is obvious that for any subgroup $\mathscr{H}$ of $\mathscr{G}$, $[\mathscr{H}, \mathscr{H}]$ is a proper subgroup of $\mathscr{H}$. We conclude, in particular, that $\Pi/[\Pi, \Pi]$ is nontrivial.

10.2. Remark. We can use the structure of $\mathrm{SL}_1(\mathfrak{D})$ to provide an explicit lower bound for the order of $H_1(P, \mathbb{Z})$.

In the following, P is one of the fake projective planes constructed in 5.9–5.12 and 9.5 (but *not* in 9.3).

10.3. Proposition. *The short exact sequence*

$$\{1\} \to C(k_{v_o}) \to \widetilde{\Pi} \to \Pi \to \{1\}$$

splits.

Proof. We know from 5.4 that $[\Gamma : \Lambda] = 9$. As observed in the proof of the preceding theorem, the image $\overline{\Gamma}$ of Γ, so the image Π of $\widetilde{\Pi}$, in $\overline{G}(k_{v_o})$ is contained in $\overline{G}(k)$. Hence, $\Gamma \subset G(\overline{k})$, where $\overline{k}$ is an algebraic closure of k. Now let x be an element of Γ. As $\varphi(x)$ lies in $\overline{G}(k)$, for every $\gamma \in \mathrm{Gal}(\overline{k}/k)$, $\varphi(\gamma(x)) = \varphi(x)$, and hence $\gamma(x)x^{-1}$ lies in $C(\overline{k})$. Therefore, $(\gamma(x)x^{-1})^3 = \gamma(x)^3x^{-3} = 1$, i.e., $\gamma(x)^3 = x^3$, which implies that $x^3 \in \Gamma \cap G(k) = \Lambda$.

Let $\overline{\Lambda}$ be the image of Λ in $\overline{G}(k_{v_o})$. Then $\overline{\Lambda}$ is a normal subgroup of $\overline{\Gamma}$ of index 3 (we have excluded the fake projective planes arising in 9.3 to ensure this). Now we observe that $\widetilde{\Pi} \cap \Lambda$ is torsion-free. This is obvious from Lemmas 5.6 and 9.2 if $\ell \neq \mathbb{Q}(\sqrt{-7})$, since then $G(k)$, and hence Λ, is torsion-free. On the other hand, if $\ell = \mathbb{Q}(\sqrt{-7})$, then any nontrivial element of finite order of Λ, and so of $\widetilde{\Pi} \cap \Lambda$, is of order 7 (Lemma 5.6), but as Π is torsion-free, the order of such an element must be 3. We conclude that $\widetilde{\Pi} \cap \Lambda$ is always torsion-free. Therefore, it maps isomorphically onto $\Pi \cap \overline{\Lambda}$. In particular, if $\Pi \subset \overline{\Lambda}$, then the subgroup $\widetilde{\Pi} \cap \Lambda$ maps isomorphically onto Π and we are done.

Let us assume now that Π is not contained in $\overline{\Lambda}$. Then Π projects onto $\overline{\Gamma}/\overline{\Lambda}$, which implies that $\Pi \cap \overline{\Lambda}$ is a normal subgroup of Π of index 3. We pick an element g of $\Pi - \overline{\Lambda}$ and let $\tilde{g}$ be an element of $\widetilde{\Pi}$ which maps onto g. Then $\tilde{g}^3 \in \widetilde{\Pi} \cap G(k) = \widetilde{\Pi} \cap \Lambda$, and $\bigcup_{0 \leqslant i \leqslant 2} \tilde{g}^i(\widetilde{\Pi} \cap \Lambda)$ is a subgroup of $\widetilde{\Pi}$ which maps isomorphically onto Π. This proves the proposition.

10.4. We note here that whenever the assertion of Proposition 10.3 holds, we get the geometric result that *the canonical line bundle K_P of P is three times a holomorphic line bundle*. To see this, we will use the following embedding of the open unit ball B as an $\mathrm{SU}(2,1)$-orbit in $\mathbf{P}^2_{\mathbb{C}}$ given in Kollár [Ko, 8.1]. We think of $\mathrm{SU}(2,1)$ as the subgroup of $\mathrm{SL}_3(\mathbb{C})$ which keeps the hermitian form $h(x_0, x_1, x_2) = -|x_0|^2 + |x_1|^2 + |x_2|^2$ on $\mathbb{C}^3$ invariant. We use the homogeneous coordinates $(x_0 : x_1 : x_2)$ on $\mathbf{P}^2_{\mathbb{C}}$. The affine plane described by $x_0 \neq 0$ admits affine coordinates $z_1 = x_1/x_0$ and $z_2 = x_2/x_0$, and the open unit ball $B = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\}$ in this plane is an $\mathrm{SU}(2,1)$-orbit. We identify B with the universal cover $\widetilde{P}$ of P. In the subgroup (of the Picard group) consisting of $\mathrm{SU}(2,1)$-equivariant line bundles on $\mathbf{P}^2_{\mathbb{C}}$, the canonical line bundle $K_{\mathbf{P}^2_{\mathbb{C}}}$ of $\mathbf{P}^2_{\mathbb{C}}$ equals $-3H$ for the hyperplane line bundle H on $\mathbf{P}^2_{\mathbb{C}}$ ([Ko], Lemma 8.3). Proposition 10.3

implies that Π can be embedded in SU(2, 1) as a discrete subgroup, and hence, $K_{\mathbf{P}^2_{\mathbb{C}}}|_{\widetilde{P}}$ and $-H|_{\widetilde{P}}$ descend to holomorphic line bundles K and L on the fake projective plane P. As $K = 3L$ and K is just the canonical line bundle K_P of P, the assertion follows.

10.5. Remark. It follows from Theorem 3(iii) of Bombieri [B] that three times the canonical line bundle K_P of P is very ample, and it provides an embedding of P in $\mathbf{P}^{27}_{\mathbb{C}}$ as a smooth surface of degree 81.

The above result can be improved whenever Proposition 10.3 holds. From 10.4, $K = K_P = 3L$ for some holomorphic line bundle L; L is ample as K is ample. From Theorem 1 of Reider [Re], $K + 4L = 7L$ is very ample. Kodaira Vanishing Theorem implies that $h^i(P, K + 4L) = 0$ for $i > 0$. It follows from Riemann–Roch, using the Noether formula for surfaces, that

$$h^0(P, 7L) = \frac{1}{2}c_1(7L)(c_1(7L) - c_1(3L)) + \frac{1}{12}\big(c_1^2(3L) + c_2(P)\big) = 15.$$

Let $\Phi : P \to \mathbf{P}^{14}_{\mathbb{C}}$ be the projective embedding associated to $7L$. The degree of the image is given by

$$\deg_\Phi(P) = \int_{\Phi(P)} c_1^2(H_{\mathbf{P}^{14}_{\mathbb{C}}}) = \int_P c_1^2(\Phi^* H_{\mathbf{P}^{14}_{\mathbb{C}}}) = c_1^2(7L) = 49.$$

Hence, holomorphic sections of $7L$ give an embedding of P as a smooth surface of degree 49 in $\mathbf{P}^{14}_{\mathbb{C}}$.

Appendix: Table of class numbers

The following table lists $(D_\ell, h_\ell, n_{\ell,3})$ for all complex quadratic extensions ℓ of $\mathbb{Q}$ with $D_\ell \leqslant 79$.

(3, 1, 1)	(4, 1, 1)	(7, 1, 1)	(8, 1, 1)	(11, 1, 1)
(15, 2, 1)	(19, 1, 1)	(20, 2, 1)	(23, 3, 3)	(24, 2, 1)
(31, 3, 3)	(35, 2, 1)	(39, 4, 1)	(40, 2, 1)	(43, 1, 1)
(47, 5, 1)	(51, 2, 1)	(52, 2, 1)	(55, 4, 1)	(56, 4, 1)
(59, 3, 3)	(67, 1, 1)	(68, 4, 1)	(71, 7, 1)	(79, 5, 1).

Acknowledgements. We thank Shigeaki Tsuyumine for helpful correspondence on the values of L-functions. He computed for us the precise values of $L_{\mathbb{Q}(\sqrt{-a})|\mathbb{Q}}(-2)$ given in 3.5 and several more. We thank Pierre Deligne, Tim Dokchitser, Igor Dolgachev, Jürgen Klüners, János Kollár, Ron Livné, Gunter Malle, Andrew Odlyzko, Chris Peters, Andrei Rapinchuk, Peter Sarnak, Tim Steger and Domingo Toledo for their interest in this work and for many useful conversations, comments and correspondence. We thank Anoop Prasad for checking our computations, and thank the referee whose comments have led to improvements in the exposition.

The first-named author would like to acknowledge support from the Institute for Advanced Study and the National Science Foundation (grant DMS-0400640). The second-named author would also like to acknowledge partial support from the National Science Foundation (grant DMS-0104089), and the hospitality of the Institute of Mathematical Research of the University of Hong Kong.

References

[B] Bombieri, E.: Canonical models of surfaces of general type. Publ. Math., Inst. Hautes Étud. Sci. **42**, 171–220 (1972)

[BP] Borel, A., Prasad, G.: Finiteness theorems for discrete subgroups of bounded covolume in semisimple groups. Publ. Math., Inst. Hautes Étud. Sci. **69**, 119–171 (1989)

[BS] Borevich, Z.I., Shafarevich, I.R.: Number theory. Academic Press, New York (1966)

[Fr] Friedman, E.: Analytic formulas for the regulator of a number field. Invent. Math. **98**, 599–622 (1989)

[Ho] Holzapfel, R.-P.: Ball and surface arithmetics. Vieweg & Sohn, Wiesbaden (1998)

[IK] Ishida, M.-N., Kato, F.: The strong rigidity theorem for non-archimedean uniformization. Tohoku Math. J. **50**, 537–555 (1998)

[Kl] Klingler, B.: Sur la rigidité de certains groupes fonndamentaux, l'arithméticité des réseaux hyperboliques complexes, et les 'faux plans projectifs'. Invent. Math. **153**, 105–143 (2003)

[Ko] Kollár, J.: Shafarevich maps and automorphic forms. Princeton University Press, Princeton (1995)

[Ma] Martinet, J.: Petits discriminants des corps de nombres. Number theory days, 1980 (Exeter, 1980), Lond. Math. Soc. Lect. Note Ser., vol. 56, pp. 151–193. Cambridge Univ. Press, Cambridge-New York, (1982)

[Mo] Mostow, G.D.: Strong rigidity of locally symmetric spaces. In: Ann. Math. Stud., vol. 78. Princeton University Press, Princeton (1973)

[Mu] Mumford, D.: An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$. Am. J. Math. **101**, 233–244 (1979)

[N] Narkiewicz, W.: Elementary and analytic theory of algebraic numbers, 3rd edn. Springer, New York (2000)

[O1] Odlyzko, A.M.: Some analytic estimates of class numbers and discriminants. Invent. Math. **29**, 275–286 (1975)

[O2] Odlyzko, A.M.: Discriminant bounds. http://www.dtc.umn.edu/~odlyzko/unpublished/index.html.

[PlR] Platonov, V.P., Rapinchuk, A.S.: Algebraic groups and Number theory. Academic Press, New York (1994)

[P] Prasad, G.: Volumes of S-arithmetic quotients of semi-simple groups. Publ. Math., Inst. Hautes Étud. Sci. **69**, 91–117 (1989)

[PrR] Prasad, G., Rapinchuk, A.S.: Computation of the metaplectic kernel. Publ. Math., Inst. Hautes Étud. Sci. **84**, 91–187 (1996)

[PY] Prasad, G., Yu, J.-K.: On finite group actions on reductive groups and buildings. Invent. Math. **147**, 545–560 (2002)

[Re] Reider, I.: Vector bundles of rank 2 and linear systems on algebraic surfaces. Ann. Math. **127**, 309–316 (1988)

[Ri] Riehm, C.: The norm 1 group of $\mathfrak{p}$-adic division algebra. Am. J. Math. **92**, 499–523 (1970)

[Ro] Rogawski, J.D.: Automorphic representations of unitary groups in three variables. In: Ann. Math. Stud., vol. 123. Princeton University Press, Princeton (1990)

[Sc] Scharlau, W.: Quadratic and hermitian forms. Springer, New York (1985)

[Se1] Serre, J.-P.: Cohomologie des groupes discrets. In: Ann. Math. Stud., vol. 70. Princeton University Press, Princeton (1971)

[Se2] Serre, J.-P.: A course in arithmetic. Springer, New York (1973)

[Se3] Serre, J.-P.: Galois cohomology. Springer, New York (1997)

[Si] Siegel, C.L.: Berechnung von Zetafunktionen an ganzzahligen Stellen. Nachr. Akad. Wiss. Gött., 87–102 (1969)

[Sl] Slavutskii, I.Sh.: On the Zimmert estimate for the regulator of an algebraic field. English translation of Mat. Zametki in Math. Notes **51**, 531–532 (1992)

[Ti1] Tits, J.: Classification of algebraic semisimple groups. Algebraic Groups and Discontinuous Subgroups. Proc. Am. Math. Soc. Symp. Pure Math. **9**, 33–62 (1966)
[Ti2] Tits, J.: Reductive groups over local fields. Proc. Am. Math. Soc. Symp. Pure Math. **33**(1), 29–69 (1979)
[Ts] Tsuyumine, S.: On values of L-functions of totally real algebraic number fields at integers. Acta Arith. **76**(4), 359–392 (1996)
[W] Washington, L.C.: Introduction to cyclotomic fields, 2nd edn. In: Grad. Texts Math., vol. 83. Springer, New York, (1997)
[Y] Yeung, S.-K.: Integrality and arithmeticity of co-compact lattices corresponding to certain complex two-ball quotients of Picard number one. Asian J. Math. **8**, 107–130 (2004)
[Z] Zimmert, R.: Ideale kleiner Norm in Idealklassen und eine Regulatorabschätzung. Invent. Math. **62**, 367–380 (1981)
[1] The Bordeaux Database, Tables obtainable from:
ftp://megrez.math.u-bordeaux.fr/pub/numberfields/