A Class of Stochastic Games with Ordered Field Property

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Abstract. It is shown that discounted general-sum stochastic games with two players, two states, and one player controlling the rewards have the ordered field property. For the zero-sum case, this result implies that, when starting with rational data, also the value is rational and that the extreme optimal stationary strategies are composed of rational components.

Key Words. Stochastic games, ordered field property, one-player controls rewards.

1. Introduction

A long time ago, Weyl (Ref. 1) proved the ordered field property for matrix games; i.e., if the payoffs of the matrix game are all elements of the same ordered field k, then also the value of the game is in k and the extreme optimal mixed actions have coordinates in the same field.

A similar property holds for LP-problems [cf. Dantzig (Ref. 2)], and an extension to bimatrix games was given by Vorob'ev (Ref. 3) and Kuhn (Ref. 4): a bimatrix game with all entries in k possesses a Nash equilibrium point with coordinates in k.

For β-discounted stochastic games, the ordered field property in general does not hold as was already shown by Shapley (Ref. 5). In recent years,

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several subclasses of stochastic games were studied, possessing the ordered field property:

(i) the one player controls transitions stochastic game [cf. Vrieze (Ref. 6) and Parthasarathy and Raghavan (Ref. 7)];
(ii) the switching control stochastic game [cf. Vrieze *et al.* (Ref. 8)];
(iii) the separable rewards and state-independent transitions stochastic games [cf. Sobel (Ref. 9) and Parthasarathy *et al.* (Ref. 10)];
(iv) the additive rewards and transitions stochastic games [cf. Raghavan *et al.* (Ref. 11)].

The main interest in subclasses of games with the ordered field property lies in the fact that the solution can be more easily computed as a consequence of the nice mathematical properties. Moreover, structured stochastic games often appear to be more suitable for modelling some practical problems.

In this paper, we introduce the following class of stochastic games:

(a) there are two players;
(b) the state space is $S=\{1,2\}$;
(c) the action sets are

$$\{1,2,\ldots,m_1\} \text{ and } \{1,2,\ldots,m_2\} \text{ for player I},$$

$$\{1,2,\ldots,n_1\} \text{ and } \{1,2,\ldots,n_2\} \text{ for player II};$$

(d) the reward functions are

$$a_1:\{1,2,\ldots,m_1\}\to\mathbb{R} \text{ for player I in state 1},$$

$$b_1:\{1,2,\ldots,m_1\}\to\mathbb{R} \text{ for player II in state 1},$$

$$a_2:\{1,2,\ldots,m_2\}\to\mathbb{R} \text{ for player I in state 2},$$

$$b_2:\{1,2,\ldots,m_2\}\to\mathbb{R} \text{ for player II in state 2};$$

(e) the transitions are: p_{ij} equals the probability that the system remains in state 1, if in state 1 player I chooses i and player II chooses j ($1-p_{ij}$ is the probability of moving to state 2); and q_{ij} equals the probability that the system moves to state 1, if in state 2 player I chooses i and player II chooses j ($1-q_{ij}$ is the probability of remaining in state 2).

Hence, the restrictions with regard to the general stochastic games are that there are only two states and that the rewards are governed by player I. We would get similar results if the rewards were governed by player II.

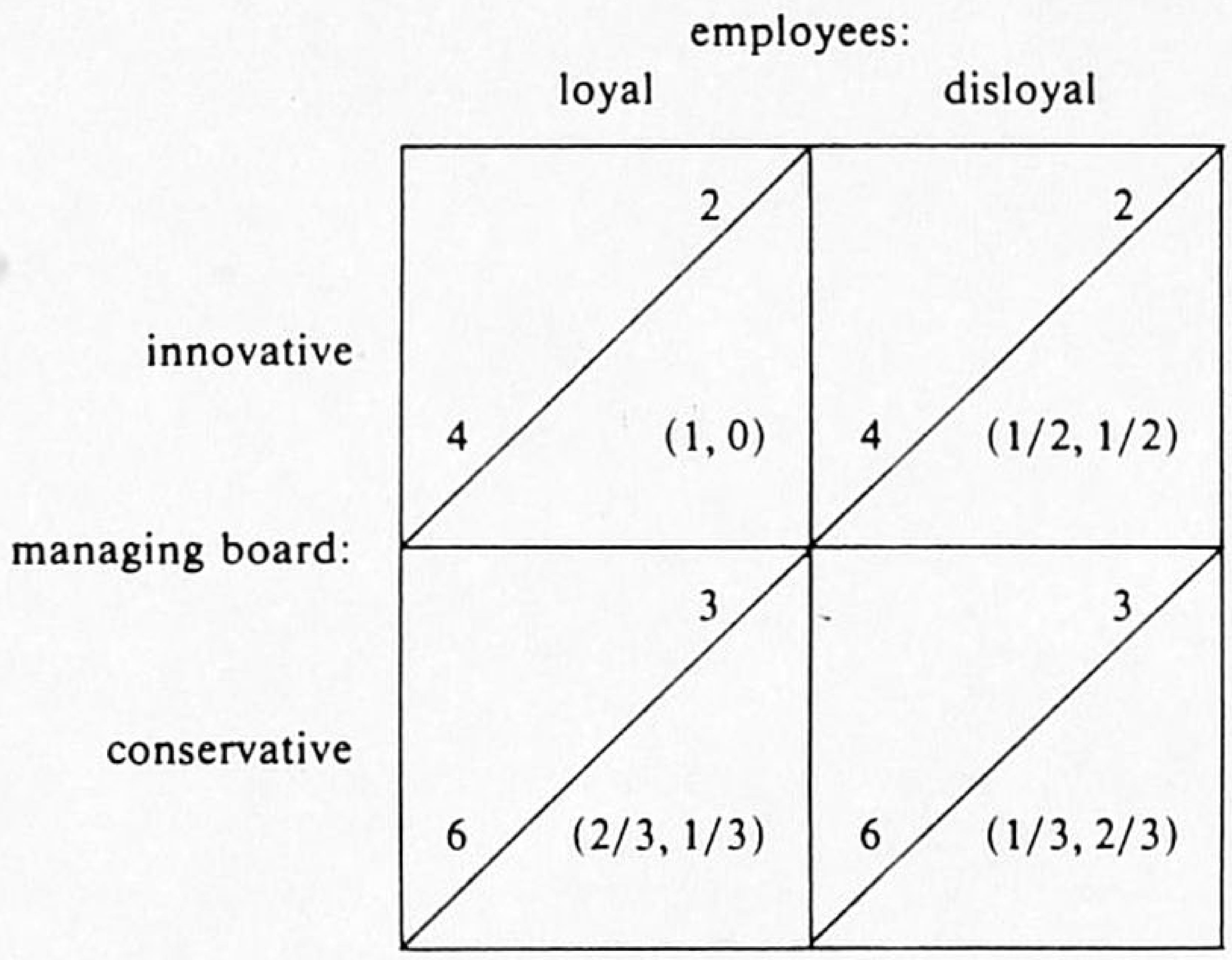

Fig. 1. Good state 1.

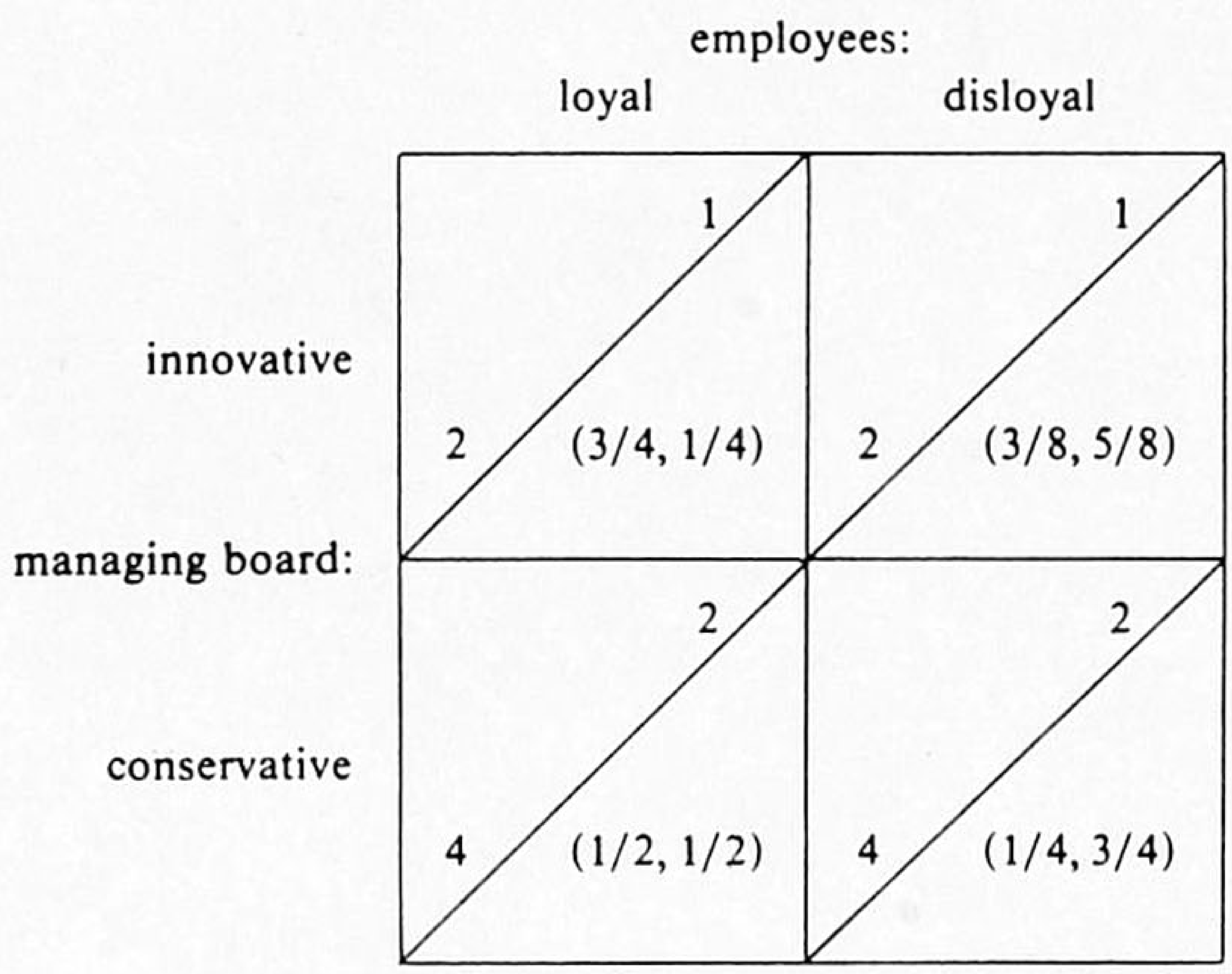

Fig. 2. Bad state 2.

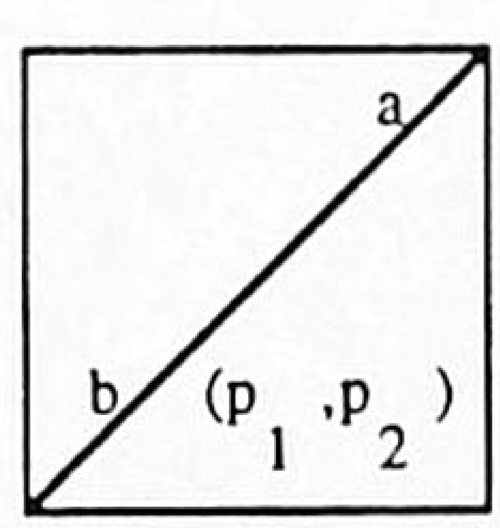

Fig. 3. Cell.

We consider the infinite-horizon game, and the stream of immediate payoffs will be evaluated by discounting with a factor $\beta \in (0, 1)$.

Example 1.1. As an example of a stochastic game of the above type, consider the following situation. The managing board of a company has to choose between a conservative policy management or a technically innovative management for the current period. The employees of the company may decide to follow the instructions of the managing board in a loyal way or they may be disloyal.

There are two market positions of the company: good and bad. The profit in a certain period depends only on the choice of the managing board and the market position. The earnings of both the managing board and the employees are certain portions of the profit. The disloyalty of the employees has delayed influence, in the sense that it has no influence on the profits in the current period, but it codetermines the market position of the company in the next period, together of course with the present market position and the decision of the managing board.

Then, this situation can be modelled as a stochastic game of the type at hand. An example is given in Figs. 1 and 2.

Here, a cell such as the one shown in Fig. 3 means: payoff a to player I (the managing board), payoff b to player II (the employees), and with probability p_1 a jump to state 1 (good market position), and with probability $p_2 = 1 - p_1$ a jump to state 2 (bad market position).

2. Main Theorem

We will prove the following theorem.

Theorem 2.1. If all the data of a stochastic game of the above type are rational, then there exists a rational equilibrium point in stationary strategies.

Before proving this theorem we introduce first some notation and next give three lemmas.

Stationary strategies will be denoted by

$$\rho := (x_1, x_2), \ \ x_1 = (x_{11}, x_{12}, \ldots, x_{1m_1}), x_2 = (x_{21}, x_{22}, \ldots, x_{2m_2}), \ \ \text{for player I,}$$

and by

$$\sigma := (y_1, y_2), \ \ y_1 = (y_{11}, y_{12}, \ldots, y_{1n_1}), y_2 = (y_{21}, y_{22}, \ldots, y_{2n_2}), \ \ \text{for player II.}$$

Note that

$$x_{1i}, x_{2i}, y_{1j}, y_{2j} \geq 0$$

and that

$$\sum_{i=1}^{m_1} x_{1i} = \sum_{i=1}^{m_2} x_{2i} = \sum_{j=1}^{n_1} y_{1j} = \sum_{j=1}^{n_2} y_{2j} = 1.$$

Here, x_{1i} equals the probability with which player I, adopting ρ, will choose action i whenever the system is in state 1. The other components have analogous meaning.

MDP (x_1, x_2) or MDP (ρ) will denote the maximizing Markov decision problem with payoffs derived from b_{11}, b_{12} (etc.) faced by player II when player I adopts the stationary strategy $\rho = (x_1, x_2)$ [cf. Hordijk *et al.* (Ref. 12)]. MDP (σ) and MDP (y_1, y_2) have analogous meaning.

$C(x_1)$ will denote the carrier of x_1, i.e., the set of actions on which x_1 puts positive weight. Thus, $C(x_1) \subset \{1, 2, \ldots, m_1\}$. $C(x_2)$, $C(y_1)$, and $C(y_2)$ have analogous meaning.

Observe that an element of $C(x_1) \times C(x_2)$ can be interpreted as a pure stationary strategy.

$F_1(C(x_1), C(x_2)) := \{\sigma$: all elements of $C(x_1) \times C(x_2)$ are optimal for MDP $(\sigma)\}$. $F_2(C(y_1), C(y_2))$ is defined analogously. Thus, $f_1(C(x_1), C(x_2))$ contains those stationary strategies of player II, for which each pure strategy out of $C(x_1) \times C(x_2)$ is a best answer for player I with regard to his own payoffs.

Lemma 2.1. A pair of stationary strategies $((x_1, x_2), (y_1, y_2))$ forms an equilibrium point if and only if $(y_1, y_2) \in F_1(C(x_1), C(x_2))$ and $(x_1, x_2) \in F_2(C(y_1), C(y_2))$.

Proof. $((x_1, x_2), (y_1, y_2))$ is an equilibrium point if and only if (y_1, y_2) is optimal in MDP (x_1, x_2) and (x_1, x_2) is optimal in MDP (y_1, y_2), which is the case if and only if all elements of $C(y_1) \times C(y_2)$ are optimal in MDP (x_1, x_2) and all elements of $C(x_1) \times C(x_2)$ are optimal in MDP (y_1, y_2). These conditions correspond to $(y_1, y_2) \in F_1(C(x_1), C(x_2))$ and $(x_1, x_2) \in F_2(C(y_1), C(y_2))$. □

Let $\alpha_1(\alpha_2)$ be a nonempty subset of pure actions of player I in state 1 (state 2). First, we prove that $F_1(\alpha_1, \alpha_2)$ is either empty or a polytope with rational extreme points. Let $\gamma_1(\gamma_2)$ be a nonempty subset of pure actions of player II in state 1 (state 2). Next, we prove that $F_2(\gamma_1, \gamma_2)$ is either empty or consists of the union of a finite number of polytopes with rational extreme points.

Lemma 2.2. $F_1(\alpha_1, \alpha_2)$ is either empty or a polytope with rational extreme points.

Proof. In the proof of this lemma (and in the proof of Lemma 2.3), we will rely frequently on properties of the set of functional equations which solve discounted stochastic games [cf. Shapley (Ref. 5)].

Let $\sigma = (y_1, y_2) \in F_1(\alpha_1, \alpha_2)$. Let $(V_1(\sigma), V_2(\sigma))$ be the optimal reward to player I in MDP(σ). Let $i_1 \in \alpha_1$ and $i_2 \in \alpha_2$. Then, the following equalities hold.

In state 1, we have

$$V_1(\sigma) = \sum_{j=1}^{n_1} y_{1j}[a_1(i_1) + \beta p_{i_1 j} V_1(\sigma) + \beta(1 - p_{i_1 j}) V_2(\sigma)]. \tag{1}$$

Rearranging terms yields

$$(1-\beta) V_1(\sigma) = a_1(i_1) + \beta[V_1(\sigma) - V_2(\sigma)]\left[\sum_{j=1}^{n_1} y_{1j} p_{i_1 j} - 1\right]. \tag{2}$$

Similarly,

$$(1-\beta) V_2(\sigma) = a_2(i_2) + \beta[V_1(\sigma) - V_2(\sigma)]\left[\sum_{j=1}^{n_2} y_{2j} q_{i_2 j} - 1\right]. \tag{3}$$

Subtracting Eq. (3) from Eq. (2) yields

$$a_2(i_2) - a_1(i_1) = [V_1(\sigma) - V_2(\sigma)]\left[\beta \sum_{j=1}^{n_1} y_{1j} p_{i_1 j} - \beta \sum_{j=1}^{n_2} y_{2j} q_{i_2 j} - 1\right],$$

or

$$V_1(\sigma) - V_2(\sigma) = [a_2(i_2) - a_1(i_1)]/N(y_1, y_2), \tag{4}$$

with

$$N(y_1, y_2) := \beta \sum_{j=1}^{n_1} y_{1j} p_{i_1 j} - \beta \sum_{j=1}^{n_2} y_{2j} q_{i_2 j} - 1 < 0.$$

For each $i \in \alpha_1$, but $i \neq i_1$, an equality similar to (1) holds, which after rearranging the terms leads to

$$(1-\beta) V_1(\sigma) = a_1(i) + \beta[V_1(\sigma) - V_2(\sigma)]\left[\sum_{j=1}^{n_1} y_{1j} p_{ij} - 1\right]. \tag{5}$$

An action $i \notin \alpha_1$ should not be better than an action $i \in \alpha_1$, which is the case if and only if

$$V_1(\sigma) \geq \sum_{j=1}^{n_1} y_{1j}[a_1(i) + \beta p_{ij} V_1(\sigma) + \beta(1 - p_{ij}) V_2(\sigma)],$$

or

$$(1-\beta) V_1(\sigma) \geq a_1(i) + \beta[V_1(\sigma - V_2(\sigma)]\left[\sum_{j=1}^{n_1} y_{1j} p_{ij} - 1\right]. \tag{6}$$

Now, subtracting each of Eqs. (5) and (6) from (2) and inserting (4) results in $m_1 - 1$ linear inequalities,

$$N(y_1, y_2)[a_1(i_1) - a_1(i)] \leq \beta[a_2(i_2) - a_1(i_1)]\left[\sum_{j=1}^{n_1} y_{1j}(p_{ij} - p_{i_1 j})\right]. \tag{7}$$

If $i \in \alpha_1$, $i \neq i_1$, then in (7) the equality sign holds.

Likewise, in state 2, the counterparts to (5) and (6) lead to $m_2 - 1$ linear inequalities for $i \neq i_2$,

$$N(y_1, y_2)[a_2(i_2) - a_2(i)] \leq \beta[a_2(i_2) - a_1(i_1)]\left[\sum_{j=1}^{n_2} y_{2j}(q_{ij} - q_{i_2 j})\right]. \tag{8}$$

If $i \in \alpha_2$, $i \neq i_2$, then in (8) the equality sign holds. Thus, if $\sigma = (y_1, y_2) \in F_1(\alpha_1, \alpha_2)$, then y_1 and y_2 satisfy (7) and (8).

Now conversely, let $\hat{y}_1$, $\hat{y}_2$ satisfy (7) and (8). Let $(Z_1(\hat{\sigma}), Z_2(\hat{\sigma}))$ be the total discounted reward associated with player II playing $\hat{\sigma} := (\hat{y}_1, \hat{y}_2)$ and player I playing the pure strategy (i_1, i_2). We shall prove that $\hat{\sigma} \in F_1(\alpha_1, \alpha_2)$. Then, (2) and (3) hold for $\hat{y}_1$, $\hat{y}_2$, $Z_1(\hat{\sigma})$, and $Z_2(\hat{\sigma})$. Now, via (4) and (2), we derive from (7) that (5) and (6) hold for $\hat{y}_1$, $\hat{y}_2$, $Z_1(\hat{\sigma})$, and $Z_2(\hat{\sigma})$ for respectively $i \in \alpha_1$, $i \neq i_1$, and for $i \neq \alpha_1$. Via (4) and (3), we can derive from (8) that the state 2 versions of (5) and (6) hold for $\hat{y}_1$, $\hat{y}_2$, $Z_1(\hat{\sigma})$, and $Z_2(\hat{\sigma})$. But, then we may conclude that

$$(Z_1(\hat{\sigma}), Z_2(\hat{\sigma})) = (V_1(\hat{\sigma}), V_2(\hat{\sigma})),$$

which means that not only (i_1, i_2) is a best answer for player I to $\hat{\sigma}$, but also that every element of $\alpha_1 \times \alpha_2$ is a best answer to $\hat{\sigma}$. Hence, $\hat{\sigma} \in F_1(\alpha_1, \alpha_2)$. Since (7) and (8) are linear relations with rational coefficients, this proves the lemma. □

Lemma 2.3. $F_2(\gamma_1, \gamma_2)$ is either empty or the union of at most three polytopes.

Proof. Let $\rho = (x_1, x_2) \in F_2(\gamma_1, \gamma_2)$. Let $(W_1(\rho), W_2(\rho))$ be the optimal reward to player II in MDP(ρ). Then, counterparts to (2), (3), (5), and (6) are valid if $W_i(\rho)$ replaces $V_i(\sigma)$, $i = 1, 2$. In particular,

$$(1-\beta) W_1(\rho) = \sum_{i=1}^{m_1} x_{1i} b_1(i) + \beta[W_1(\rho) - W_2(\rho)]\left[\sum_{i=1}^{m_1} p_{ij} x_{1i} - 1\right]. \tag{9}$$

Hence, the following counterpart to (4) is valid:

$$W_1(\rho) - W_2(\rho) = \left[\sum_{i=1}^{m_2} x_{2i} b_2(i) - \sum_{i=1}^{m_1} x_{1i} b_1(i)\right] \Big/ L(x_1, x_2), \tag{10}$$

with

$$L(x_1, x_2) := \beta \sum_{i=1}^{m_1} x_{1i} p_{ij_1} - \beta \sum_{i=1}^{m_2} x_{2i} q_{ij_2} - 1 < 0.$$

Subtracting (9) evaluated for $j \neq j_1$ from (9) evaluated for $j = j_1$ and inserting (10) gives

$$0 \leq \left[\sum_{i=1}^{m_2} x_{2i} b_2(i) - \sum_{i=1}^{m_1} x_{1i} b_1(i)\right]\left[\sum_{i=1}^{m_1} x_{1i}(p_{ij_1} - p_{ij})\right]. \tag{11}$$

Similarly,

$$0 \leq \left[\sum_{i=1}^{m_2} x_{2i} b_2(i) - \sum_{i=1}^{m_1} x_{1i} b_1(i)\right]\left[\sum_{i=1}^{m_2} x_{2i}(q_{ij_2} - q_{ij}\right]. \tag{12}$$

The equality sign holds in (12) if $j \in \gamma_2$, $j \not\in j_2$. Hence, if $\rho = (x_1, x_2) \in F_2(\gamma_1, \gamma_2)$, then x_1 and x_2 satisfy (11) and (12).

The proof of the converse is omitted, because it is similar to the converse part of the proof of Lemma 2.2.

Now, we have that $\rho = (x_1, x_2) \in F_2(\gamma_1, \gamma_2)$ if and only if x_1, x_2 satisfy (11) and (12).

Let

$$L_0(x_1, x_2) := \sum_{i=1}^{m_2} x_{2i} b_2(i) - \sum_{i=1}^{m_1} x_{1i} b_1(i)$$

and

$$L_{1j}(x_1) := \sum_{i=1}^{m_1} x_{1i}(p_{ij_1} - p_{ij}), \qquad \text{for } j \in \{1, 2, \ldots, n_1\}, j \neq j_1,$$

$$L_{2j}(x_2) := \sum_{i=1}^{m_2} x_{2i}(q_{ij_2} - q_{ij}), \qquad \text{for } j \in \{1, 2, \ldots, n_2\}, j \neq j_2.$$

From the above characterization of $F_2(\gamma_1, \gamma_2)$, we get

$$\begin{aligned}
F_2(\gamma_1, \gamma_2) = \{&(x_1, x_2);\ L(x_1, x_2) = 0\} \\
&\cup \{(x_1, x_2);\ L(x_1, x_2) \leq 0,\ L_{1j}(x_1) = 0, \\
&\quad j \in \gamma_1, j \neq j_1,\ L_{1j}(x_1) \leq 0, j \not\in \gamma_1, \\
&\quad L_{2j}(x_2) = 0, j \in \gamma_2, j \neq j_2,\ L_{2j}(x_2) \leq 0, j \not\in \gamma_2\} \\
&\cup \{(x_1, x_2);\ L(x_1, x_2) \geq 0,\ L_{1j}(x_1) = 0, \\
&\quad j \in \gamma_1, j \neq j_1,\ L_{1j}(x_1) \geq 0, j \not\in \gamma_1, \\
&\quad L_{2j}(x_2) = 0, j \in \gamma_2, j \neq j_2,\ L_{2j}(x_2) \geq 0, j \not\in \gamma_2\}.
\end{aligned}$$

Thus, $F_2(\gamma_1, \gamma_2)$ is empty or the union of at most three polytopes with rational extreme points, since each of the linear functions L_0, L_1, L_2 has rational coefficients. □

Proof of Theorem 2.1. It is well known that there exist equilibrium points in stationary strategies [cf. Fink (Ref. 13)]. Let $((x_1, x_2), (y_1, y_2))$ be an equilibrium point. Then, by Lemma 2.1,

$$(y_1, y_2) \in F_1(C(x_1), C(x_2)) \quad \text{and} \quad (x_1, x_2) \in F_2(C(y_1), C(y_2)).$$

Suppose that either x_1 and/or x_2 is irrational. By Lemma 2.3, we have that (x_1, x_2) belongs to a polytope with rational extremes contained in $F_2(C(y_1), C(y_2))$. But then, there is some rational element (x_1^*, x_2^*) of $F_2(C(y_1), C(y_2))$ such that

$$C(x_1^*) = C(x_1) \quad \text{and} \quad C(x_2^*) = C(x_2).$$

So,

$$(x_1^*, x_2^*) \in F_2(C(y_1), C(y_2))$$

and

$$(y_1, y_2) \in F_1(C(x_1), C(x_2)) = F_1(C(x_1^*), C(x_2^*)).$$

Hence, by Lemma 2.1, $((x_1^*, x_2^*), (y_1, y_2))$ is an equilibrium point. If furthermore either y_1 or y_2 is irrational, then starting from $((x_1^*, x_2^*), (y_1, y_2))$, we can find a rational point (y_1^*, y_2^*) via Lemma 2.2 such that $((x_1^*, x_2^*), (y_1^*, y_2^*))$ is an equilibrium point, which now is rational. □

Notice from the proof of Theorem 2.1 that the order field property can be proved whenever Lemmas 2.2 and 2.3 hold.

The converse in an interesting question. Observe that in our case these lemmas hold because, in (2), (3), (5), and (6), the reward term does not depend on the strategy (y_1, y_2) and because of the fact that, in using (9) to obtain (11), the reward terms vanish.

When focusing on zero-sum stochastic games, we get an interesting result.

Theorem 2.2. For a two-state, zero-sum stochastic game with one player controlling the immediate rewards and all data rational, the following results hold:

(i) the value is rational;

(ii) the set of optimal stationary strategies for each player can be written as the Cartesian product of two polytopes with rational extreme points.

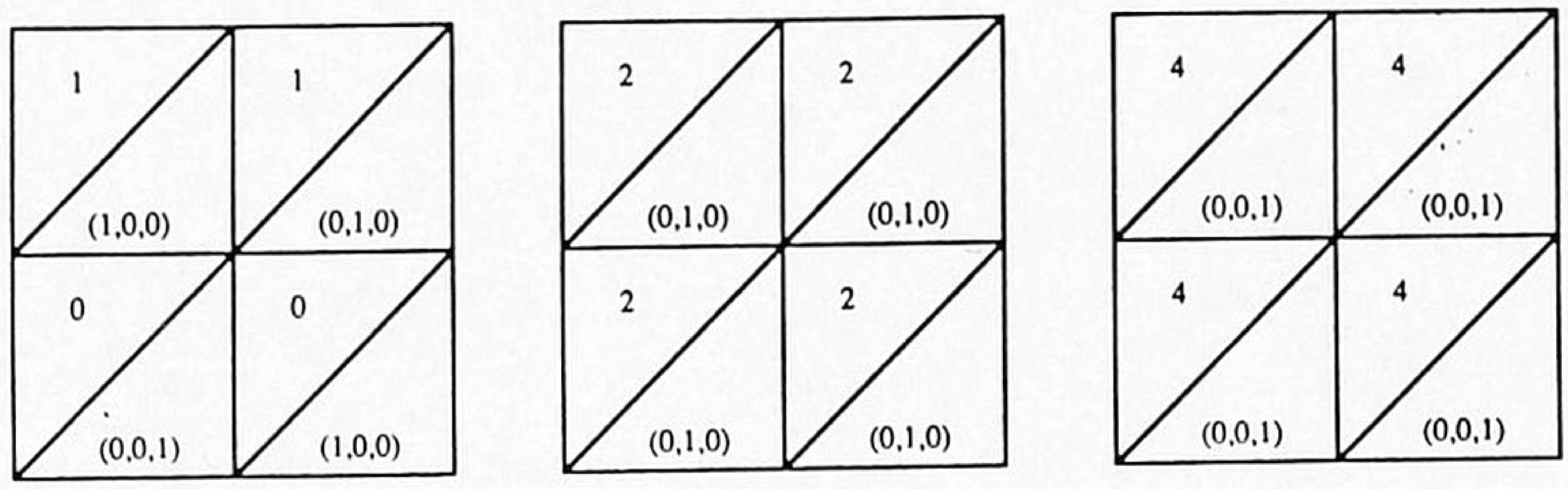

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Fig. 4. Example.

Proof. (i) Equilibrium strategies are optimal strategies in the case of zero-sum games. So, by Theorem 2.1, there exist optimal rational strategies for both players. The discounted payoff for such a pair of strategies is rational and equals the value of the game.

(ii) From Tijs and Vrieze (Ref. 14), the set of optimal stationary strategies for a two-state, zero-sum stochastic game can be written as $X_{s=1}^{2} O_k(s)$ for player k, where $O_k(s)$ is the set of optimal actions for player k in the matrix game

$$[G_s(V)] := [r_s(i,j) + \beta(u_{ij} V_1 + (1-u_{ij}) V_2)]_{i\in A_s, j\in B_s};$$

here, $u_{ij} = p_{ij}$, if $s=1$, and $u_{ij} = q_{ij}$, if $s=2$; $V=(V_1, V_2)$ is the value of the game; and A_s, B_2 are the action sets in state s. Since V is rational in our case, $[G_s(V)]$, $s=1, 2$, is a matrix game with rational entries. Hence, $O_k(s)$ is a polytope with rational extreme points [cf. Weyl (Ref. 1)]. $\square$

We conclude with an example which shows that, even for the zero-sum case, our result cannot be extended to stochastic games with three states for which one player controls the rewards. See Fig. 4. Clearly,

$$V_\beta(2) = 2/(1-\beta) \quad \text{and} \quad V_\beta(3) = 4/(1-\beta).$$

For state 1, one can compute that

$$V_\beta(1) = (7-\beta-\sqrt{17\beta^2-30\beta+17})/[2(2-\beta)(1-\beta)],$$

for all $\beta \geqslant 1/2$.

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