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ON THE MAPPING CLASS GROUP ACTION ON THE COHOMOLOGY OF THE REPRESENTATION SPACE OF A SURFACE

INDRANIL BISWAS

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Abstract. The mapping class group of a d-pointed Riemann surface has a natural C^∞ action on any moduli space of parabolic bundles with the marked points as the parabolic points. We prove that under some numerical conditions on the parabolic data, the induced action of the mapping class group on the cohomology algebra of the moduli space of parabolic bundles factors through the natural epimorphism of the mapping class group onto the symplectic group.

1. Introduction

Consider an oriented surface with a finite number of punctures. Let $\mathfrak{R}$ be the space of gauge equivalence classes of flat unitary connections on it such that the holonomy around each puncture lies in a fixed conjugacy class of $U(n)$. Assume that $\mathfrak{R}$ satisfies some numerical conditions (§2). Using pull-back of connections by a diffeomorphism, the mapping class group $\mathfrak{M}$, for the marked surface, acts on $\mathfrak{R}$. In particular, we have a representation of $\mathfrak{M}$ in the cohomology ring of $\mathfrak{R}$. Let

$$\rho \ : \ \mathfrak{M} \ \longrightarrow \ H^*(\mathfrak{R}, \mathbb{Q})$$

denote the homomorphism ontained above.

Let X denote the compact surface. The group $\mathfrak{M}$ acts on $H_1(X, \mathbb{Z})$. Let

$$0 \ \longrightarrow \ \mathfrak{T} \ \longrightarrow \ \mathfrak{M} \ \longrightarrow \ G \ \longrightarrow \ 0$$

be the exact sequence of groups obtained from this action. The group G is a subgroup of the group of automorphisms of $H_1(X, \mathbb{Z})$. The group $\mathfrak{T}$ is usually called the Torelli group.

Using the main result of [BR] we construct a surjective algebra homomorphism $A \longrightarrow H^*(\mathfrak{R}, \mathbb{Q})$, where A is constructed from $H_1(X, \mathbb{Z})$. Our main result (Theorem 3.4) is the following

Theorem A. *The restriction of ρ to the Torelli group $\mathfrak{T}$ is a trivial representation. Moreover, the homomorphism $A \longrightarrow H^*(\mathfrak{R}, \mathbb{Q})$ is a homomorphism of G-modules.*

A similar result is proved for the space of flat $SU(n)$ connections.

In [BR] we constructed some cohomology classes $\{a_i\}$ on $X \times \mathfrak{R}$ such that the Künneth components of $\{a_i\}$ generate the algebra $H^*(\mathfrak{R}, \mathbb{Q})$. These classes are

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actually the characteristic classes of a certain $PGL(n,,\mathbb{C})$ bundle on $X \times \mathfrak{R}$. The proof of Theorem 3.4 is ultimately based on the fact that for any diffeomorphism, g, of X, the diffeomorphism of $X\times\mathfrak{R}$ induced by g leaves any a_i invariant (Lemma 2.7).

Using cup product and a symmetric invariant form on the Lie algebra, there is a natural symplectic form on $\mathfrak{R}$. It is easy to see that the diffeomorphisms of $\mathfrak{R}$ which are induced by $\mathfrak{M}$ actually preserve this symplectic form. Under certain conditions, for a symplectic diffeomorphism there are Floer homology groups [DS]. In [DS] the Floer homology of a symplectic diffeomorphism of above type for $SU(2)$ is identified with the Floer homology of the the mapping cylinder for the corresponding element of $\mathfrak{M}$. There is an obvious question of the possibility of a relationship between the action of $\mathfrak{M}$ on the usual cohomology of $\mathfrak{R}$ and the Floer homology.

After the work was completed the author came to know that M.S. Narasimhan anticipated the results obtained here.

2. The construction

Let X be a compact Riemann surface of genus $g \geq 2$. Fix a finite sub-set $S = \{p_1, p_2, \dots , p_d\} \subset X$; these points of S will be called *parabolic points*. To each parabolic point p_i, associate a conjugacy class of the unitary group $U(n)$, which is the same as an orbit of the adjoint action. The set of pairs $\bigcup_{i=1}^{d}(p_i, C_i)$ will be called the set of *parabolic data*. A *parabolic representation* of $\pi_1(X-S)$ is a representation in $U(n)$ such that the image of the positively oriented loop (anti-clock-wise) around p_i lies in C_i. Define

$$\mathfrak{R}^P \ := \ \mathrm{Hom}^{ir}_{\mathrm{par}}(\pi_1(X-S), U(n))/U(n)\,, \tag{2.1}$$

the space of equivalence classes of irreducible parabolic representations. Though $\pi_1(X-S)$ depends on a choice of a base point, their different identifications differ by conjugations; and hence the space $\mathfrak{R}^P$ is independent of the choice of base-point. $\mathfrak{R}^P$ has a natural C^∞-manifold structure.

From a theorem in [MS] (also [S]), the manifold $\mathfrak{R}^P$ can be identified with a moduli of parabolic stable bundles on X. This moduli space is denoted by $\mathfrak{U}^P$. The eigen-space decomposition of any element in a conjugacy class of $U(n)$ determines a flag-type of $\mathbb{C}^n$, along with weights determined by the eigenvalues. Let the flag-type $\{m_{i1}, \dots , m_{il_i}\}$, with $\sum_{j=1}^{l_i} m_{ij} = n$, and the real numbers $0 \leq \alpha_{i1} < \alpha_{i2} < \dots < \alpha_{il_i} < 1$ correspond to the conjugacy class C_i. So $\mathfrak{U}^P$ is the moduli of stable parabolic bundles E of parabolic degree zero with S as the set of parabolic points and flags

$$E_{p_i} = F_{p_i,1} \supset F_{p_i,2} \supset F_{p_i,2} \supset \dots \supset F_{p_i,l_i} \supset 0$$

at each p_i with $dim_{\mathbb{C}} F_{p_i,j} - dim_{\mathbb{C}} F_{p_i,j+1} = m_{ij}$, and the weight of $F_{p_i,j}$ being α_{ij}.

Assumption A. We impose the following two conditions on the parabolic data :

(i) Any parabolic semi-stable bundle E, of the above type, is actually parabolic stable. This condition is equivalent to the condition that any parabolic homomorphism is irreducible.

(ii) A universal parabolic bundle exists on $X \times \mathfrak{U}^P$. This condition is equivalent to the following condition: on $X \times \mathfrak{R}^P$ there is a C^∞-bundle V with a partial connection ∇ on the fibers of the projection $(X - S) \times \mathfrak{R}^P \to \mathfrak{R}^P$ such that ∇ is unitary flat and logarithmic singular on $X \times r$, for all $r \in \mathfrak{R}^P$, and holonomy around $p_i \times r$ in $X \times r$ lies in C_i, and the eigenvalues of the residue matrix are of the form $exp(2\pi i\theta)$ with $0 \leq \theta < 1$.

(A partial connection on a principal bundle on a manifold equipped with a foliation is defined to be a connection on the restriction of the bundle to the leaves such that it varies smoothly with the leaves. See [KT] for more details.)

Note that the assumptions imply that $d \geq 1$.

Let $\mathrm{Diff}^+(X, S)$ be the group of all orientation-preserving homomorphisms of X which preserve the set S point-wise. Define the mapping class group

$$\mathfrak{M}_g^d \ := \ \pi_0\Big(\mathrm{Diff}^+(X, S)\Big), \tag{2.2}$$

the group of connected components of $\mathrm{Diff}^+(X, S)$, which is known to coincide with the quotient of $\mathrm{Diff}^+(X, S)$ by diffeomorphisms which are homotopic to the identity map.

The group $\mathrm{Diff}^+(X, S)$ has a natural action on $\mathfrak{R}^P$. One way to see the action is the following. $\mathfrak{R}^P$ can be identified with gauge equivalence classes of flat $U(n)$ connection on $X - S$. Pulling-back of a connection using a diffeomorphism induces an action of $\mathrm{Diff}^+(X, S)$ on $\mathfrak{R}^P$. Moreover, pull-backs of a connection using two maps which are homotopic are gauge equivalent. In other words, for $g \in \mathrm{Diff}^+(X, S)$ the diffeomorphism of $\mathfrak{R}^P$ thus obtained depends only on the image of g in $\mathfrak{M}_g^d$. For $g \in \mathfrak{M}_g^d$ let, $\bar{g} \in \mathrm{Diff}(\mathfrak{R}^P)$ be the induced diffeomorphism. Let $\Phi : \mathfrak{M}_g^d \to \mathrm{Diff}(\mathfrak{R}^P)$ be the homomorphism defined by $g \mapsto \bar{g}^{-1}$.

$$\phi_k \ : \ \mathfrak{M}_g^d \ \longrightarrow \ \mathrm{Aut}(H^k(\mathfrak{R}^P, \mathbb{Q})) \tag{2.3}$$

is the homomorphism defined by $g \longmapsto (\Phi(g)^{-1})^*$.

Let $G_{\mathbb{Z}} \subset Aut(H_1(X, \mathbb{Z}))$ be the subgroup consisting of all those automorphisms of $H_1(X, \mathbb{Z})$ which preserve the symplectic structure on $H_1(X, \mathbb{Z})$ defined by cap-product. Clearly $G_{\mathbb{Z}}$ is isomorphic to $Sp(g, \mathbb{Z})$, the group of $2g \times 2g$ symplectic matrices. The group $\mathfrak{M}_g^d$ acts naturally on the homology group $H_1(X, \mathbb{Z})$, and the image of $\mathfrak{M}_g^d$ is contained in $G_{\mathbb{Z}}$. It is known that the homomorphism $\mathfrak{M}_g^d \to G_{\mathbb{Z}}$ is surjective. The *Torelli group* $\mathfrak{T}_g^d$ is defined by the exact sequence

$$1 \ \longrightarrow \ \mathfrak{T}_g^d \longrightarrow \ \mathfrak{M}_g^d \ \stackrel{\gamma}{\longrightarrow} \ G_{\mathbb{Z}} \ \longrightarrow \ 1. \tag{2.4}$$

Let (T, t_0) be a connected manifold with a base point t_0 and

$$\begin{array}{c} (X_T, S_T) \\ \big\downarrow f \\ T \end{array} \tag{2.5}$$

be a C^∞ family of d-pointed surfaces, along with diffeomorphism, denoted by b, of the pointed curve over $t_0 \in T$ with (X, S). ($S_T : T \to X_T \times X_T \times \ldots \times X_T$ is a C^∞ map satisfying the condition that $f \circ S_T = Id \times Id \times \ldots \times Id$.) Choosing a metric on X_T, we can assume that f is a C^∞ family of Riemann surfaces. Choose the metric so that b is a holomorphic isomorphism.

It is easy to check that choosing trivializations of the family over contractible sets in T, the homomorphism

$$\psi \ : \ \pi_1(T, t_0) \ \longrightarrow \ \mathfrak{M}_g^d$$

obtained actually does not depend upon the choices of trivializations. Assume that the homomorphism ψ lifts to a homomorphism $\bar{\psi} : \pi_1(T, t_0) \longrightarrow \mathrm{Diff}^+(X, S)$.

For each surface in the family, we can construct the space of parabolic representations as we did for (X, S). These spaces fit together to give a C^∞-fiber bundle

$$\bar{f} \; : \; \mathfrak{R}^P_T \; \longrightarrow \; T$$

with $\mathfrak{R}^P$ as fiber. Let $p : \tilde{T} \longrightarrow T$ be the universal cover. $\mathfrak{R}^P_T$ can be identified with the quotient of $\mathfrak{R}^P \times \tilde{T}$ by the action of $\pi_1(T, t_0)$ defined by the product of $\Phi \circ \psi$ and the deck transformations.

The action of $\pi_1(T, t_0)$ on $X \times \mathfrak{R}^P \times \tilde{T}$ given by $\bar{\psi}$ on the first factor, by $\Phi \circ \psi$ on the second factor and by deck-transformations on the third factor is denoted by ρ. The quotient is the fiber product $X_T \times_T \mathfrak{R}^P_T$. It is easy to see that the direct image sheaf $R^k \bar{f}_* \mathbb{C}$, $k \geq 0$, has a structure of local system, which is known as the Gauss-Manin connection. Let $\mu_k : \pi_1(T, t_0) \to \mathrm{Aut}(H^k(\mathfrak{R}^P, \mathbb{Q}))$ be the holonomy of the Gauss-Manin connection. From the definition of Gauss-Manin connection, it is easy to check that

$$\mu_k \; = \; \phi_k \circ \psi \; . \tag{2.6}$$

Let (V, ∇) be a universal bundle on $X \times \mathfrak{R}^P$ as in Assumption A(ii). Let $(P(V), \nabla)$ be the associated projective bundle equipped with the induced partial connection. Let p_i, $i = 1, 2, 3$, be the projection of $X \times \mathfrak{R}^P \times \tilde{T}$ onto the i-th factor. Then we have

Lemma 2.7. *The action ρ of $\pi_1(T, t_0)$ on $X \times \mathfrak{R}^P \times \tilde{T}$ lifts canonically to an action on the pulled-back pair*

$$((p_1, p_2)^* P(V), (p_1, p_2)^* \nabla) \; \longrightarrow \; X \times \mathfrak{R}^P \times \tilde{T} \, .$$

Proof. Take any $\nu \in \mathfrak{R}^P$. It gives a bundle V_ν on X, equipped with a logarithmic connection ∇^ν on $X - S$. Since ν is irreducible (Assumption A(i)), the only C^∞-automorphisms of V_ν which preserve ∇^ν are multiplication by elements of $\mathbb{C}^*$. From this it implies that if (V, ∇) and $(\bar{V}, \bar{\nabla})$ are two universal bundles on $X \times \mathfrak{R}^P$, then there is a (complex) line-bundle $L \to \mathfrak{R}^P$ such that $V \otimes p_2^*(L)$ and $\bar{V}$ are canonically isomorphic. (For $\nu \in \mathfrak{R}^P$, the choice of scalar for an isomorphism between $V|_{X \times \nu}$ and $\bar{V}|_{X \times \nu}$ fits together to give L.) In particular, the projective bundles $P(V)$ and $P(\bar{V})$ are canonically isomorphic. For any $h \in \mathrm{Diff}^+(X, S)$, define $\bar{h} \in \mathrm{Diff}(X \times \mathfrak{R}^P)$ by $\bar{h}(x, \nu) = (h(x), (h^{-1})^* \nu)$. Now $(\bar{h}^* V, \bar{h}^* \nabla)$ is a universal bundle on $X \times \mathfrak{R}^P$. This follows from the obvious fact that for any $\nu \in \mathfrak{R}^P$, the flat bundle on $\bar{h}(X \times \nu)$ is obtained by pushing-forward, by the diffeomorphism h, the flat connection on $X \times \nu$. This implies that $(P(V), \nabla) = (P(\bar{h}^* V), \bar{h}^* \nabla)$. In particular this holds for any $h \in \bar{\psi}(\pi_1(T, t_0))$. Thus we have a lift of ρ to $((p_1, p_2)^* P(V), (p_1, p_2)^* \nabla)$. $\square$

Taking quotient by $\pi_1(T, t_0)$ of $(p_1, p_2)^* P(V) \longrightarrow X \times \mathfrak{R}^P \times \tilde{T}$ we get a projective bundle

$$P \; \longrightarrow \; X_T \times_T \mathfrak{R}^P_T \, . \tag{2.8}$$

We know that the residue of a logarithmic connection gives the parabolic structure (the details are in [MS], [S]). Given a universal bundle (V, ∇), for the i-th parabolic point p_i, we have vector bundles $F_{i,j} \to \mathfrak{R}^P \times \tilde{T}$, $1 \leq j \leq l_i$. We saw that any two universal bundles differ by tensoring with a pull-back of a line bundle on $\mathfrak{R}^P$. So, exactly as in 2.7, the action of $\pi_1(T, t_0)$ lifts to any $\mathrm{Hom}(F_{i,j}, F_{i,j-1})$. The quotient by this action is a bundle on $\mathfrak{R}^P_T$, which is denoted by $H^T_{i,j}$.

3. The theorem

For $t \in T$, mapping $(x, E) \in X_t \times \mathfrak{R}_t^P$ to $(x, \overset{n}{\wedge} E)$ we get a submersion

$$D \ : \ X_T \times_T \mathfrak{R}_T^P \ \longrightarrow \ X_T \times_T J_T\, ,$$

where $J_T \longrightarrow T$ is a family of Jacobians. Let e be the degree of E - it is constant over the family. For a pointed Riemann surface (Y, y) there is a unique Poincaré bundle on $Y \times J^e(Y)$. Using this fact, and taking the first parabolic point as the base point, we can construct a C^∞ bundle on the fiber product

$$\mathfrak{P}_T \ \longrightarrow \ X_T \times_T J_T$$

such that for any $t \in T$, the restriction $\mathfrak{P}_t$ is the Poincaré bundle on $X_t \times J^e(X_t)$. Any two such line bundles may not be C^∞ isomorphic; but they differ by tensoring with a pull-back of a C^∞ bundle on T. In any case, we fix one such bundle $\mathfrak{P}_T$. Let $\omega := c_1(\mathfrak{P}_T) \in H^2(X_T \times_T J_T, \mathbb{Q})$ be the 1-st Chern class.

Given a $PGL(n, \mathbb{C})$ bundle Q on a manifold M, the usual Chern-Weil characteristic classes are given by symmetric $PGL(n, \mathbb{C})$ invariant polynomials on the Lie algebra. In this case the ring of invariant polynomials is generated by $n-1$ elements $\{a_2, \ldots, a_n\}$ with a_i being a polynomial of degree i with rational coefficients.

Let $\mathfrak{L}_i := (R^i f_* \mathbb{Q})^*$ be the local system on T given by the i-th $\mathbb{Q}$-valued homology of fibers of the projection f. The projection of the fiber product $X_T \times_T \mathfrak{R}_T^P \to T$ is denoted by $\tilde{f}$. For $t \in T$, β_t is the inclusion of the fiber $X_t \times \mathfrak{R}_t^P := \tilde{f}^{-1}(t)$ in $X_T \times_T \mathfrak{R}_T^P$. Take any $\sigma \in H^j(X_T \times_T \mathfrak{R}_T^P, \mathbb{Q})$. For $\theta \in H_i(X_t, \mathbb{Q})$, taking the slant product of $\beta_t^* \sigma$ and θ we get $\langle \beta_t^* \sigma, \theta \rangle \in H^{j-i}(\mathfrak{R}_t^P, \mathbb{Q})$. Thus we get a set theoretic map from fibers of $\mathfrak{L}_i$ to the fibers of $R^{j-i} \bar{f}_* \mathbb{Q}$ - this map is denoted by $L(\sigma)$.

Lemma 3.1. *The map $L(\sigma)$ preserves the the local system structures of $\mathfrak{L}_i$ and $R^{j-i} \bar{f}_* \mathbb{Q}$ respectively.*

Proof. Take a local section $s \in \Gamma(U, \mathfrak{L}_i)$ on a contractible open set $U \subset T$. Let $\bar{s} := L(\sigma)(s)$ be the smooth section, over U, of the vector bundle for $R^{j-i} \bar{f}_* \mathbb{C}$ (it is easy to see that the set-theoretic section is smooth). We need to prove that $\bar{s}$ is actually a flat section. As U is contractible, the fiber bundle $f|_U : X_U \to U$ is a trivial C^∞-fiber bundle. In other words, there is a diffeomorphism $\delta : X_U \to X \times U$ which commutes with the projections to U. Similarly, there is diffeomorphism $\delta' : \mathfrak{R}_U^P \to \mathfrak{R}^P \times U$. Since s is flat, there is $\hat{s} \in H_i(X, \mathbb{Q})$ such that $s = (\delta_*)^{-1}(\hat{s} \otimes 1_U)$, where 1_U is the element 1 in $H_0(U, \mathbb{Q}) = \mathbb{Q}$. Using δ and δ', σ gives $\hat{\sigma} \in H^j(X \times \mathfrak{R}^P, \mathbb{Q})$. Now it easy to check that $\bar{s} = \bar{f}_*\big((\delta')^*(\langle \hat{\sigma}, \hat{s} \rangle)\big)$. This implies that $\bar{s}$ is a flat section. $\square$

For $(z_1, \ldots, z_n) \in \mathbb{Q}^n$, taking

$$\sigma \ = \ z_1 . D^* \omega \ + \ \sum_{i=2}^{n} z_i . a_i(P)$$

in 3.1, where $a_i(P)$ are the characteristic classes of the projective bundle P obtained in (2.8), we have a homomorphism of local systems

$$L \ : \ \Big(\bigoplus_{i=0}^{2} \mathfrak{L}_i \Big) \otimes_{\mathbb{Q}} \mathbb{Q}^n \ \longrightarrow \ R^* \bar{f}_* \mathbb{Q} \ := \ \bigoplus_j R^j \bar{f}_* \mathbb{Q}$$

where $\mathbb{Q}^n$ is the constant local system. We noted that the Poincaré bundle is not unique, but any two of them differ by tensoring with a pull-back of a C^∞ bundle on T. This implies that, though $D^* \omega$ depends on the choice of the Poincaré bundle, the homomorphism $L(\omega) : \mathfrak{L}_i \to R^{2-i}\bar{f}_*\mathbb{Q}$ does not depend upon the choice of the Poincaré bundle.

The k-th Chern class $c_k(H^T_{i,j})$ gives a section of $R^{2k}\bar{f}_*\mathbb{Q}$. This way, we get a homomorphism from the constant local system $L : \mathbb{Q}^N \longrightarrow R^*\bar{f}_*\mathbb{Q}$, where N is the number of possible (non-trivial) triplets (i, j, k).

Define

$$V \ := \ \mathfrak{L}_1 \otimes \mathbb{Q}^n, \ W \ := \ \Big(\mathfrak{L}_0 \oplus \mathfrak{L}_2\Big) \otimes \mathbb{Q}^n \ \oplus \ \mathbb{Q}^N .$$

Consider the homomorphism of local systems

$$L \oplus L' \ : \ V \oplus W \ \longrightarrow \ \bigoplus_j R^j\bar{f}_*\mathbb{Q} .$$

Note that the image of V lies in $R^{\mathrm{ev}}\bar{f}_*\mathbb{Q}$ and the image of W lies in $R^{\mathrm{odd}}\bar{f}_*\mathbb{Q}$. So $L \oplus L'$ induces a homomorphism

$$\bar{L} \ : \ S(W) \otimes \wedge V \ \longrightarrow \ R^*\bar{f}_*\mathbb{Q} , \tag{3.2}$$

where S (resp. $\wedge$) is the symmetric (resp. exterior) algebra.

Proposition 3.3. *The homomorphism of local systems $\bar{L}$ is surjective.*

Proof. This is basically the main theorem (Theorem 1.4) of [BR]. This theorem says that the cohomology ring $H^*(\mathfrak{R}^P, \mathbb{Q})$ is generated, as an algebra, by the union of the image of the slant product of the 1-st Chern class of a universal bundle and the characteristic classes of the associated projective bundle and the Chern classes of the homomorphisms, of the form $\mathrm{Hom}(F_{i,j}, F_{i,j-1})$, of the flags. This implies that $\bar{L}$ gives surjective homomorphism of stalks. □

Since (2.5) is a family of oriented surfaces, the monodromy of $\mathfrak{L}_0 \oplus \mathfrak{L}_2$ is trivial. The monodromy of $\mathfrak{L}_1$ is as follows. Recall (2.5), and consider $\gamma \circ \psi : \pi_1(T, t_0) \longrightarrow G_{\mathbb{Z}}$. The group $G_{\mathbb{Z}}$ has a tautological action on $H_1(X, \mathbb{Q})$. It is easy to see that the monodromy of $\mathfrak{L}_1$, $\pi_1(T, t_0) \to Aut(H_1(X, \mathbb{Q}))$, is given by $\gamma \circ \psi$. Obviously, any element of $\mathfrak{M}^d_g$ can be realized as an element in the image of ψ, for some suitable family. So from the previous discussion and (2.6) we obtain

Theorem 3.4. (i) *The restriction of any ϕ_k (defined in (2.3)) to the Torelli group $\mathfrak{T}^d_g$ is a trivial representation. Equivalently, ϕ_k factors through γ (defined in (2.4)).*

(ii) *The action of $G_{\mathbb{Z}}$ on $\bigoplus_j H^j(\mathfrak{R}^P, \mathbb{Q})$ satisfies the condition that the canonical surjective homomorphism*

$$S(\mathbb{Q}^{N+2n}) \otimes \bigwedge \Big(H_1(X, \mathbb{Q}) \bigotimes \mathbb{Q}^n\Big) \ \longrightarrow \ H^*(\mathfrak{R}^P, \mathbb{Q}) ,$$

as in (3.2), commutes with the action of $G_{\mathbb{Z}}$, where $\mathbb{Q}^{N+2n}$ and $\mathbb{Q}^n$ are trivial $G_{\mathbb{Z}}$ modules.

Replacing $U(n)$ by $SU(n)$, let $\mathfrak{S}\mathfrak{R}^P$ be the obvious analogue of $\mathfrak{R}^P$. Define $J := \mathrm{Hom}_{\mathrm{par}}(\pi_1(X - S), U(1))$. Using the homomorphism $SU(n) \times U(1) \to U(n)$, we have a unramified Galois covering

$$F \ : \ \mathfrak{S}\mathfrak{R}^P \times J \ \longrightarrow \ \mathfrak{R}^P .$$

It is easy to check that F commutes with the actions of $\mathfrak{M}^d_g$ on $SU(n)$, $U(1)$ and $U(n)$. It is well-known that F induces an isomorphism between the cohomology ring $H^*(\mathfrak{S}\mathfrak{R}^P \times J, \mathbb{Q})$ and $H^*(\mathfrak{R}^P, \mathbb{Q})$. So Theorem 3.4(i) implies that the Torelli group $\mathfrak{T}_g^d$ acts trivially on $H^*(\mathfrak{S}\mathfrak{R}^P, \mathbb{Q})$. The restriction of ω, the first Chern class of a Poincaré bundle, to $\mathfrak{S}\mathfrak{R}^P$ vanishes. So Theorem 3.4(ii) implies

Theorem 3.5. *The canonical surjective homomorphism*

$$S(\mathbb{Q}^{N+2n}) \otimes \bigwedge \Big(H_1(X, \mathbb{Q}) \otimes \mathbb{Q}^{n-1} \Big) \longrightarrow H^*(\mathfrak{S}\mathfrak{R}^P, \mathbb{Q}) \,,$$

where $\mathbb{Q}^{n-1}$ corresponds to the characteristic classes of the projective universal bundle, commutes with the action of $G_{\mathbb{Z}}$.

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School of Mathematics, Tata Institute of Fundamental Research, Bombay 400005, India

Current address: Institut Fourier des Mathématiques, Université Grenoble I, BP 74, 38402 St. Martin d'Héres-cédex, France

E-mail address: `indranil@math.tifr.res.in`