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Cycles on the generic abelian threefold

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Abstract. H Clemens and C Schoen gave examples of three-folds where the group of codimension two cycles modulo algebraic equivalence has infinite rank. This paper provides yet another example of the same phenomenon.

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Let X be a smooth projective variety over C and denote by $R^2(X)$ the group of codimension two algebraic cycles homologically equivalent to zero modulo the subgroup of those cycles algebraically equivalent to zero. The first example of a variety X for which $R^2(X)\otimes Q \neq 0$ was given by Griffiths (see [5]). More recently, Clemens showed that $R^2(X)\otimes Q$ is infinite-dimensional where X is the generic quintic hypersurface in P^4 (see [3]).

We shall show in this paper that $R^2(X)\otimes Q$ is infinite dimensional where X is the generic abelian variety of dimension three. This result follows easily from a group-theoretic argument and the following basic result of Ceresa, which we now explain.

Denote by $\phi\colon C\to J(C)$ the embedding of a curve C (well-defined up to translations) in its Jacobian. Put $\psi(x) = -\phi(x)$, for $x\in C$ and consider the cycle $S(C) = [\phi(C)] - [\psi(C)]$ which is obviously trivial homologically. The theorem of Ceresa (see [2]) is that $S(C)$ is non-zero in the group $R^2(J(C))\otimes Q$ where C is the generic curve of genus three.

To construct cycles on a fixed three-dimensional abelian variety A we may proceed as follows. First choose an isogeny $h\colon B\to A$ with B principally polarized. Then $B\cong J(C)$ for some curve C of genus three (possibly degenerate) and thus B has the Ceresa cycle $S(C)$. Its image $h_*S(C)\in R^2(A)\otimes Q$ is non-zero if A is generic, thanks to the result of Ceresa. Keeping A fixed, there are plenty of choices of h and each of these choices gives a cycle on A. These cycles are linearly independent because they twist by different characters of $\mathrm{Gal}(\bar{F}/F)$ where F is a field of definition of A.

The formal argument follows.

Good references for all the facts concerning the moduli spaces that we have used in this paper are [4], [6] and [7].

Let N be a natural number $\geqslant 3$.

Denote by $M(N)$ (resp. $X(N)$) the moduli of curves (resp. principally polarized abelian varieties) of genus (resp. dimension) three with level N structure, defined over C, the field of complex numbers. These are smooth irreducible varieties of dimension six, and let $\mathscr{C}(N)\to M(N)$ and $\mathscr{A}(N)\to X(N)$ be the universal family of curves and abelian varieties respectively. We then have the $\mathrm{Sp}(6;Z/N)$-equivariant commutative diagram:

$$\text{I}\qquad \begin{array}{ccc} J(\mathscr{C}(N)) & \longrightarrow & \mathscr{A}(N) \\ \downarrow & & \downarrow \\ M(N) & \longrightarrow & X(N) \end{array}$$

where $J(\mathscr{C}(N))$ is the family of Jacobians of $\mathscr{C}(N)\to M(N)$. Note that $-1\in\mathrm{Sp}(6;Z/N)$ acts trivially on $X(N)$ and its action on $\mathscr{A}(N)$ is simply $x\to -x$ on the fibres of $\mathscr{A}(N)\to X(N)$.

Now $\mathrm{Sp}(6;Z/N)/\{\pm 1\}$ acts faithfully on $X(N)$, while the action of $\{\pm 1\}$ on $M(N)$ is non-trivial. Applying the global Torelli theorem, we see that

$$\{\pm 1\}\backslash M(N)\to X(N)$$

is a birational isomorphism (in fact it is an open immersion by Zariski's Main Theorem).

Let $C(N)$ and $A(N)$ be the generic fibres of $\mathscr{C}(N)\to M(N)$ and $\mathscr{A}(N)\to X(N)$. These are varieties over the functon fields $E(N)$ of $M(N)$ and $F(N)$ of $X(N)$ respectively. Furthermore $[E(N):F(N)]=2$. From I we get a $\mathrm{Sp}(6;Z/N)$-equivariant commutative diagram:

$$\text{II}\qquad \begin{array}{ccc} J(C(N)) & \longrightarrow & A(N) \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,E(N) & \longrightarrow & \mathrm{Spec}\,F(N) \end{array}$$

If N_1 divides N_2, there is a $\mathrm{Sp}(6;N_2)$-equivariant commutative diagram:

$$\begin{array}{ccc} \mathscr{C}(N_2) & \longrightarrow & \mathscr{C}(N_1) \\ \downarrow & & \downarrow \\ M(N_2) & \longrightarrow & M(N_1) \end{array}$$

so that if E is the union of all the $E(N)$, we get a curve $C\to\mathrm{Spec}\,E$ with the action of $\mathrm{Sp}(6;\hat{Z})$. Similarly we get an abelian variety $A\to\mathrm{Spec}\,F$, where F is the union of all the $F(N)$. Finally II now gives a $\mathrm{Spec}(6;\hat{Z})$-equivariant commutative diagram:

$$\text{III}\qquad \begin{array}{ccc} J(C) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathrm{Spec}\,E & \longrightarrow & \mathrm{Spec}\,F \end{array}$$

Note also that $[E:F]=2$, and that III gives an isomorphism:

$$J(C)\to A_E.$$

Factoring $C\to\mathrm{Spec}\,E$ by the action of $-1\in\mathrm{Sp}(6;\hat{Z})$, we get a curve $C'\to\mathrm{Spec}\,F$ such that $C'_E\cong C$. Put $A'=J(C')$. Thus we have (non-isomorphic) abelian varieties A'

and A defined over F and isomorphisms

$$A'_E \leftarrow J(C) \rightarrow A_E.$$

Denote by $f\colon A'_E \rightarrow A_E$ the induced isomorphism and by σ the non-trivial element of $\mathrm{Gal}(E/F)$. The equivalence of III under $-1 \in \mathrm{Sp}(6; \hat{Z})$ is now equivalent to

$$\text{IV} \qquad f\circ(1_{A'} \times \sigma) = (i_A \times \sigma)\circ f.$$

In IV above and always, $i_Z\colon Z \rightarrow Z$ denotes the morphism $x \mapsto -x$ of an abelian variety Z. Occasionally i_Z will be abbreviated to i simply.

We are now ready to tackle the Galois action on the generic Ceresa cycle. We need first some notation and an elementary result.

The group of codimension k cycles modulo algebraic equivalence on a variety X will be denoted by $B^k(X)$. If X is defined over K and L is a field extension of K, then $\mathrm{Aut}(L/K)$ acts on $B^k(X_L)$.

Lemma. Let D be a curve of genus $g \geqslant 1$ over a field K. Denote by $\bar{K}$ an algebraic closure of K. The Ceresa cycle $S(D) \in B^{g-1}(J(D)_{\bar{K}})$ is invariant under the action of $\mathrm{Gal}(\bar{K}/K)$, and $i^*S(D) = -S(D)$.

Proof. A divisor R of degree $= -1$ on $D_{\bar{K}}$ defines an embedding

$$\phi_R\colon D_{\bar{K}} \rightarrow J(D)_{\bar{K}}$$

given by $\phi_R(x) = [x] + R$. If S is also a divisor of degree $= -1$, then ϕ_S is a translate of ϕ_R and therefore the cycle $\xi = \phi_R(D_{\bar{K}})$ as an element of $B^{g-1}(J(D)_{\bar{K}})$ does not depend on the choice of R at all. For all $g \in \mathrm{Gal}(\bar{K}/K)$, $g(\phi_R) = \phi_{gR}$ and this shows that ξ is invariant under the Galois action. Since the Ceresa cycle $S(D)$ is $\xi - i^*(\xi)$, it follows that $S(D)$ is invariant too. Also

$$i^*S(D) = i^*(\xi) - \xi = -S(D)$$

and this proves the lemma.

Now let $\bar{F} \supset E \supset F$ be an algebraic closure of F and let χ be the composite $\mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{Gal}(E/F) \cong \{\pm 1\}$. Let $f_F\colon A'_F \rightarrow A_F$ be extended from $f\colon A'_E \rightarrow A_E$ and put $\theta = f_F(S(C')) \in B^2(A_F)$.

PROPOSITION 1

For all $g \in \mathrm{Gal}(\bar{F}/F)$, $g^\theta = \chi(g)\theta$.

Proof. Abbreviating i_A, $i_{A'}$, etc. to i, and because $i\circ f_{\bar{F}} = f_{\bar{F}}\circ i$, we deduce from IV that for all $g \in \mathrm{Gal}(\bar{F}/F)$,

(a) $f_F\circ(1_{A'} \times g) = (1_A \times g)\circ f_F$ if $\chi(g) = 1$, and

(b) $f_F\circ(i_{A'} \times g) = (1_A \times g)\circ f_F$ if $\chi(g) = -1$. For an element Z of $B^k(A'_F)$ this gives

(c) $f_F(gZ) = gf_F(Z)$ if $\chi(g) = 1$

(d) $f_F(g(i^*Z)) = gf'_F(Z)$ if $\chi(g) = -1$.

Putting $Z = S(C')$ and applying the above lemma, the proposition follows.

We shall now embed $\mathrm{Gal}(\bar{F}/F)$ in a larger group that acts on $B^k(A_{\bar{F}}) \otimes Q$. Recall that $X(N)$ is the quotient of the Siegel half-space $H = \{T \in M_3(C): T = {}^tT \text{ and } \operatorname{Im} T > 0\}$ by the action of $\Gamma(N)$, the principal congruence subgroup of level N in $\mathrm{Sp}(6; Z)$. Let $\tilde{\mathrm{S}}\mathrm{p}(6; R)$ be the subgroup of $\mathrm{GL}_6(R)$ generated by $\mathrm{Sp}(6; R)$ and the scalar matrices. Put $\tilde{\mathrm{S}}\mathrm{p}(6; Q) = \tilde{\mathrm{S}}\mathrm{p}(6; R) \cap \mathrm{GL}_6(Q)$. There is an action of $\tilde{\mathrm{S}}\mathrm{p}(6; R)/R^*$ on H. For every $g \in \tilde{\mathrm{S}}\mathrm{p}(6; Q)$ there is a natural number $\mathbf{a}$ and a commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{g} & H \\ \downarrow & & \downarrow \\ X(N\mathbf{a}) & \longrightarrow & X(N) \end{array}$$

where the vertical arrows are the quotient maps. Passing to the direct limit over the N and taking generic points we get, for each $g \in \tilde{\mathrm{S}}\mathrm{p}(6; Q)$ an automorphism $\rho_1(g)$ of $\operatorname{Spec} F$. And $\rho_1(g) = 1_F$ if $g \in Q^*$.

Moreover, if $g \in M_6(Z) \cap \tilde{\mathrm{S}}\mathrm{p}(6; Q)$ we get commutative diagrams:

$$\begin{array}{ccc} \mathscr{A}(N\mathbf{a}) & \longrightarrow & \mathscr{A}(N) \\ \downarrow & & \downarrow \\ X(N\mathbf{a}) & \longrightarrow & X(N) \end{array}$$

where the horizontal arrows are isogenies on the fibres. These induce:

V

$$\begin{array}{ccc} A_F & \xrightarrow{\rho_2(g)} & A_F \\ \downarrow & & \downarrow \\ \operatorname{Spec} F & \xrightarrow{\rho_1(g)} & \operatorname{Spec} F \end{array}$$

Denote by j: $\operatorname{Spec} \bar{F} \to \operatorname{Spec} F$ the given morphism and define

$$G = \{(\alpha, g) \in \operatorname{Aut} \operatorname{Spec} \bar{F} \times \tilde{\mathrm{S}}\mathrm{p}(6; Q) \mid \rho_1(g) \circ j = j \circ \alpha\},$$

$$S = \{(\alpha, g) \in G \mid g \in M_6(Z)\}$$

$$T = \{(\alpha, g) \in S \mid g \text{ is a scalar matrix and } \alpha = 1_{\bar{F}}\}.$$

Then we have an exact sequence:

(A) $1 \to \mathrm{Gal}(\bar{F}/F) \to G \to \tilde{\mathrm{S}}\mathrm{p}(6; Q) \to 1.$

Also

(B) $G = S^{-1} \cdot T$ and S^{-1} is a semi-group, and

(C) T is contained in the centre of G.

From (B) and (C) it follows that any homomorphism of S^{-1} to a group extends uniquely to a homomorphism of G. To define $\rho: G \to \operatorname{Aut} B^k(A_{\bar{F}}) \otimes Q$ it suffices therefore to give ρ on S^{-1} such that $\rho(\omega_1)\rho(\omega_2) = \rho(\omega_1 \omega_2)$ for all $\omega_1, \omega_2 \in S^{-1}$.

Now let $(\alpha, g) \in S$. Taking the fibre-product of

$$\begin{array}{ccc} \operatorname{Spec} \bar{F} & \xrightarrow{\alpha} & \operatorname{Spec} \bar{F} \\ j\downarrow & & \downarrow j \\ \operatorname{Spec} F & \xrightarrow{\rho_1(g)} & \operatorname{Spec} F \end{array}$$

with V, we get

$$\begin{array}{ccc} A_{\bar{F}} & \xrightarrow{s} & A_{\bar{F}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \bar{F} & \xrightarrow{\alpha} & \operatorname{Spec} \bar{F} \end{array}$$

so that the induced morphism $A_{\bar{F}} \to \alpha^* A_{\bar{F}}$ is an isogeny. From the easy isogeny lemma below it follows that

$$s^*: B^k(A_{\bar{F}}) \otimes Q \to B^k(A_{\bar{F}}) \otimes Q$$

is an isomorphism. We define $\rho(s^{-1}) = s^*$. That ρ is an action on S^{-1} follows from the fact: $\rho_2(g_1 g_2) = \rho_2(g_1)\rho_2(g_2)$ for $g_1, g_2 \in M_6(Z) \cap \tilde{\mathrm{S}}\mathrm{p}(6; Q)$. Modulo the lemma below, therefore, an action ρ of G on $B^k(A_{\bar{F}}) \otimes Q$ has been defined.

Isogeny lemma. If $f: X \to Y$ is an isogeny of abelian varieties, $f^*: B^k(Y) \otimes Q \to B^k(X) \otimes Q$ is an isomorphism.

Proof. In fact $(1/d) f_*$ is the inverse of f^*, where d is the degree of the isogeny. The projection formula gives $f_* f^* Z = \mathrm{d}Z$. And $f^* f_* W$, being the sum of the translates of W by the elements of the kernel, is algebraically equivalent to $\mathrm{d}W$, and this proves the lemma.

Theorem. *$B^2(A_{\bar{F}}) \otimes Q$ and $R^2(A_{\bar{F}}) \otimes Q$ are infinite-dimensional.*

$R^2(A_{\bar{F}}) \otimes Q$, consisting of homologically trivial cycles, has finite codimension in $B^2(A_{\bar{F}}) \otimes Q$, and so both the assertions of the theorem are equivalent.

Choose a sequence $r_1, r_2, \ldots \in \mathrm{Sp}(6; Q)$ which form a system of coset-representatives for $\mathrm{Sp}(6, Z) \backslash \mathrm{Sp}(6, Q)$ and then lift the r_i to $s_i \in G$. The infinite-dimensionality of $B^2(A_{\bar{F}}) \otimes Q$ follows from the linear independence of $\rho_1(s_1)\theta$, $\rho(s_2)\theta, \ldots$, with θ as in Proposition 1.

From Proposition 1 it follows that $\rho(h)\rho(g)\theta = \chi^g(h)\rho(g)\theta$ for all $h \in \mathrm{Gal}(\bar{F}/F)$, $g \in G$, where $\chi^g(h) = \chi(g^{-1}hg)$. We shall show that the χ^{s_i} are distinct characters of $\mathrm{Gal}(\bar{F}/F)$, from which the linear independence of the $\rho(s_i)\theta$ follows.

To this end we shall define a closed analytic subset $R(\eta)$ of the Siegel half-space H for any character $\eta: \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$ and then show that the $R(\chi^{r_i})$ are all distinct.

A character $\eta = \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$ determines a quadratic extension L of F, which gives for some $N \geqslant 3$ a quadratic extension $L(N)$ of $F(N)$ such that $L(N) \cdot F = L$. Now the $L(N)$ gives a double covering $Y(N) \to X(N)$ and let $R(N) \subset X(N)$ be its branch-locus. Finally let R be the inverse image of $R(N)$ in the projection $H \to X(N)$. That R is independent of the particular choice of N and $L(N)$ is clear. Thus we put $R = R(\eta)$. It is also immediate that $R(\eta^g) = h^{-1}R(\eta)$ where $g \in G$ and h is the image of g in $\widetilde{\mathrm{Sp}}(6; Q)$.

If we take $\eta = \chi$, we have explicitly $L(N) = E(N)$ and $R(\chi)$ is the locus of the hyperelliptic Jacobians (and also the degenerate Jacobians) in the Siegel half-space H. Thus to finish the proof of the theorem, we only need the following.

Lemma. $\pi = \{g \in \mathrm{Sp}(6; R) \doteq gR(\chi) = R(\chi)\}$ *equals* $\mathrm{Sp}(6; Z)$.

Proof. Clearly π is a *closed* subgroup of $\mathrm{Sp}(6; R)$, and its Lie algebra, being stable under the adjoint action of $\pi \supset \mathrm{Sp}(6; Z)$, must be zero or all of Lie $\mathrm{Sp}(6; R)$. In the latter case, $\pi = \mathrm{Sp}(6; R)$ and therefore $R(\chi) = \phi$ or $R(\chi) = H$, which is absurd. In the former, π is discrete and since $\mathrm{Sp}(6; Z)$ is a maximal discrete subgroup of $\mathrm{Sp}(6; R)$, we deduce that $\pi = \mathrm{Sp}(6; Z)$.

This completes the proof of the theorem.

What the above argument gives more generally is the following statement: if $v \in B^2(A_{\bar{F}}) \otimes Q$ is not invariant under $\mathrm{Gal}(\bar{F}/F)$, then the G-orbit of v spans an infinite-dimensional subspace. After all, the $R(\eta)$ can be defined for any irreducible representation η of $\mathrm{Gal}(\bar{F}/F)$, and all that was used is the fact that $R(\eta)$ is non-empty if η is not the trivial representation. In other words, if L is a finite extension of F, the branch locus (as a subset of H) is non-empty—this is a consequence of the congruence subgroup theorem for $\mathrm{Sp}(2n; Z)$, see e.g. [1].

The following interesting question has been raised by Clemens: Is $B^2(A_{\bar{F}}) \otimes Q$ a finitely generated G-module?

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