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STABILITY OF THE PICARD BUNDLE

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Abstract

Let X be a non-singular algebraic curve of genus $g \geqslant 2$, $n \geqslant 2$ an integer, ξ a line bundle over X of degree $d > 2n(g-1)$ with $(n, d) = 1$ and $\mathcal{M}_\xi$ the moduli space of stable bundles of rank n and determinant ξ over X. It is proved that the Picard bundle $\mathcal{W}_\xi$ is stable with respect to the unique polarisation of $\mathcal{M}_\xi$.

1. *Introduction*

Let X be a non-singular projective algebraic curve of genus $g \geqslant 2$ defined over an algebraically closed field k; let n and d be integers with $d > 2n(g-1)$ and $(n, d) = 1$. Let $\mathcal{M}_{n,d}$ denote the moduli space of stable bundles of rank n and degree d over X. Then $\mathcal{M}_{n,d}$ is a smooth projective variety, and there exists a universal bundle $\mathcal{U}$ over $X \times \mathcal{M}_{n,d}$. Since $H^1(X, E) = 0$ for all $E \in \mathcal{M}_{n,d}$, the direct image sheaf $p_{2*}\mathcal{U}$ over $\mathcal{M}_{n,d}$ is locally free; the associated vector bundle $\mathcal{W}$ is called the *Picard bundle* over $\mathcal{M}_{n,d}$.

These Picard bundles have been studied for $n = 1$ in [**4**, **5**] and for general n in [**6**], where it was proved that, for $d > 2ng$, the Picard bundle is stable with respect to the generalised theta divisor on $\mathcal{M}_{n,d}$. The argument depends on pulling $\mathcal{W}$ back to an open subset of the Jacobian of a spectral curve Y over X, and then using the Abel–Jacobi map $Y \to \mathrm{Pic}^0(Y)$.

For $n \geqslant 2$ and ξ a line bundle over X of degree d, one can also consider the moduli space $\mathcal{M}_\xi$ of stable vector bundles of rank n and determinant ξ over X. Again (given the assumption that $(n, d) = 1$), $\mathcal{M}_\xi$ is a smooth projective variety, and there exist a universal bundle $\mathcal{U}_\xi$ over $X \times \mathcal{M}_\xi$ and a Picard bundle $\mathcal{W}_\xi$ over $\mathcal{M}_\xi$. However, there is no simple means of generalising the previous argument to prove the stability of $\mathcal{W}_\xi$. Some useful results on $\mathcal{M}_\xi$ have been obtained in [**2**], but stability is proved there only for $g = n = 2$.

Very recently, two of the authors [**1**] proved the stability of $\mathcal{W}_\xi$ for $g \geqslant 3$ and $n = 2$ using the Hecke correspondence of M. S. Narasimhan and S. Ramanan [**7**, **8**]. Extending the technique of [**1**], we prove the following theorem.

Theorem. *Let X be a non-singular projective algebraic curve of genus $g \geqslant 2$ defined over an algebraically closed field, let n and d be integers with $n \geqslant 2$, $d > 2n(g-1)$ and $(n, d) = 1$, and let ξ be a line bundle over X, of degree d. Then the Picard bundle $\mathcal{W}_\xi$ over $\mathcal{M}_\xi$ is stable with respect to the unique polarisation of $\mathcal{M}_\xi$.*

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In connection with this theorem, we recall that $\operatorname{Pic}\mathscr{M}_\xi \cong \mathbb{Z}$ (see [**9**, **3**]), so that $\mathscr{M}_\xi$ indeed has a unique polarisation. We denote the ample generator of $\operatorname{Pic}\mathscr{M}_\xi$ by Θ. Note that the condition $g \geqslant 3$ is not needed.

The strategy of the proof is as follows. Given a point $x \in X$ and an appropriate vector bundle F of determinant $\xi(x)$, and using the Hecke correspondence, we construct an embedding

$$\psi_{F,x} : \mathbf{P}(F_x^*) \longrightarrow \mathscr{M}_\xi;$$

we call the pair $(\mathbf{P}(F_x^*), \psi_{F,x})$ a *Hecke space*. We show that there is a homomorphism from the twisted restriction $\psi_{F,x}^*\mathscr{W}_\xi(-j)$ (for a suitable integer j) to the stable vector bundle $\Omega_{\mathbf{P}(F_x^*)}(1)$. Let $\mathscr{F}$ be a saturated subsheaf of $\mathscr{W}_\xi$ of rank r, with $0 < r < \operatorname{rk}\mathscr{W}_\xi$. We show that x and F can be chosen so that the restriction of $\mathscr{F}$ to the corresponding Hecke space is locally free outside some subvariety of codimension at least 2, and furthermore that the composition

$$\psi_{F,x}^*\mathscr{F}(-j) \longrightarrow \psi_{F,x}^*\mathscr{W}_\xi(-j) \longrightarrow \Omega_{\mathbf{P}(F_x^*)}(1)$$

is non-zero. A computation of degrees, involving the stability of $\Omega_{\mathbf{P}(F_x^*)}$, now gives the result.

NOTATION. Given any ample line bundle L over a smooth projective variety Y of dimension m, we recall that the degree of a torsion-free sheaf $\mathscr{F}$ with respect to L is defined by

$$\deg\mathscr{F} = c_1(\mathscr{F}).c_1(L)^{m-1}[Y].$$

When $\operatorname{Pic} Y \cong \mathbb{Z}$ and $c_1(L)$ is the ample generator of $\operatorname{Pic} Y$, we can write $c_1(\mathscr{F}) = \lambda.c_1(L)$ for some integer λ, and then

$$\deg\mathscr{F} = \lambda N, \qquad N = c_1(L)^m[Y].$$

When $Y = \mathscr{M}_\xi$, we shall always take $c_1(L) = \Theta$; when Y is a projective space, we shall always take $L = \mathscr{O}(1)$, so that $N = 1$.

We assume throughout that the hypotheses of the theorem hold.

2. *Hecke spaces*

In this section we recall some facts about the Hecke correspondence of [**7**, **8**]. Full details are contained in [**8**]; for the convenience of the reader, we include sketches of the proofs of the results of [**8**] that we need.

Let $x \in X$, and let F be a bundle over X of rank n and determinant $\xi(x)$. For each non-zero homomorphism $\phi : F \to k_x$ from F to the torsion sheaf k_x of length 1 supported at $\{x\}$, $\operatorname{Ker}\phi$ is a torsion-free sheaf over X. Since X is a non-singular curve, $\operatorname{Ker}\phi$ is locally free, and can be identified with a vector bundle E of rank n and determinant ξ. We have an exact sequence

$$0 \to E \to F \to k_x \to 0. \tag{1}$$

Moreover, the set of all such homomorphisms ϕ forms (up to scalar multiples) the projective space $\mathbf{P}(F_x^*)$. (We adopt the notation that $\mathbf{P}(F_x^*)$ stands for the projective space of lines in F_x^*.) For fixed F and x, the sequences (1) then form a family

$$0 \to \mathscr{E} \to p_1^*F \to \mathscr{O}_{\{x\}\times\mathbf{P}(F_x^*)}(1) \to 0 \tag{2}$$

over $X \times \mathbf{P}(F_x^*)$.

If all the bundles E arising in this family are stable, the universal property of $\mathcal{M}_\xi$ gives us a morphism

$$\psi_{F,x} : \mathbf{P}(F_x^*) \to \mathcal{M}_\xi$$

such that

$$\mathcal{E} \cong (\mathrm{id} \times \psi_{F,x})^* \mathcal{U}_\xi \otimes p_2^*(\mathcal{O}(-j)) \tag{3}$$

for some integer j. We shall call the pair $(\mathbf{P}(F_x^*), \psi_{F,x})$ a *Hecke space*.

REMARK. The *good Hecke cycles* of [**8**] arise from a more general form of this construction, that allows the point x to vary. We shall not need this, nor shall we need the fact that two distinct Hecke spaces have distinct images (compare [**8**, Theorem 5.13]).

In order to use this construction, we need to determine conditions on F which will ensure that all the bundles E in the family (2) are stable.

DEFINITION [**8**, Definition 5.1 and Remark 5.2]. Let l and m be integers. A vector bundle F over X is (l, m)-*stable* if, for every proper subbundle G of F,

$$\frac{\deg G + l}{\operatorname{rk} G} < \frac{\deg F + l - m}{\operatorname{rk} F}.$$

Note that $(0,0)$-stability is the same as ordinary stability. Apart from this, we shall use only $(0,1)$-stability.

LEMMA 1. *If the bundle F in the sequence* (1) *is* $(0,1)$-*stable, then E is stable.*

Proof (see [**8**, Lemma 5.5]). Let G be a proper subbundle of E which contradicts the stability of E. Then the subbundle of F generated by G contradicts the $(0,1)$-stability of F.

REMARK. In fact, it is easy to verify that the condition of $(0,1)$-stability on F is precisely what is needed for $\psi_{F,x}$ to be defined. Thus the Hecke spaces correspond exactly to the pairs (F, x) with F $(0,1)$-stable.

LEMMA 2. *The* $(0,1)$-*stable bundles F form a non-empty open subset of the moduli space of stable bundles of rank n and determinant $\xi(x)$.*

Proof (see [**8**, Propositions 5.3 and 5.4]). For any stable bundle $F \in \mathcal{M}_{\xi(x)}$, the ranks and degrees for the quotients of F which contradict $(0,1)$-stability form a finite set. The openness of the subset of $(0,1)$-stable bundles follows from the properness of the Quot scheme.

To prove that this subset is non-empty, we estimate the dimension of its complement. F fails to be $(0,1)$-stable if and only if it has a subbundle G of rank $r < n$ and degree e such that $ne \geqslant r((d+1)-1)$; that is, $ne \geqslant rd$. Moreover,

$$h := \dim H^1(X, \mathrm{Hom}(F/G, G)) = r(d+1) - ne + r(n-r)(g-1).$$

The corresponding component of the subvariety of non-$(0,1)$-stable bundles in $\mathscr{M}_{\xi(x)}$ therefore has dimension at most

$$\big(r^2(g-1)+1\big)+\big((n-r)^2(g-1)+1\big)-g+(h-1)$$
$$=(n^2-nr+r^2-1)(g-1)+r(d+1)-ne.$$

This is smaller than $(n^2-1)(g-1)$, provided that

$$r(n-r)(g-1)>r(d+1)-ne.$$

Since $ne \geqslant rd$, this fails only if $g=2$, $r=n-1$ and $ne=rd$. Since $(n,d)=1$, this never happens.

LEMMA 3. *For any Hecke space, $\psi_{F,x}$ is an isomorphism onto its image.*

Proof. This follows from [**8**, Lemma 5.9]. We prove just the injectivity (sufficient for our purpose), following the proof of [**8**, Lemma 5.6].

It is sufficient to prove that, for any sequence (1),

$$\dim H^0(X,\mathrm{Hom}(E,F))=1.$$

For this, note first that any non-zero homomorphism $\phi : E \to F$ is of maximal rank; otherwise, the image of E in F would contradict $(0,1)$-stability. By taking determinants, we see further that ϕ is an isomorphism away from x.

Now fix $y \in X$, where $y \neq x$. If $\phi_1 : E \to F$ and $\phi_2 : E \to F$ are linearly independent, then there is a non-zero linear combination ϕ of ϕ_1 and ϕ_2 that is singular at y. This is a contradiction.

3. *Proof of the theorem*

Note first that, from (1), we obtain the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & k_x^{n-1} & \longrightarrow & F_x & \longrightarrow & k_x & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & E & \longrightarrow & F & \longrightarrow & k_x & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & F(-x) & = & F(-x) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{4}$$

Just as the sequences (1) combine to form a family (2), the diagrams (4) combine to form the following sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \Omega_{\{x\}\times\mathbf{P}(F_x^*)}(1) & \longrightarrow & F_x \otimes \mathcal{O}_{\{x\}\times\mathbf{P}(F_x^*)} & \longrightarrow & \mathcal{O}_{\{x\}\times\mathbf{P}(F_x^*)}(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{E} & \longrightarrow & p_1^*F & \longrightarrow & \mathcal{O}_{\{x\}\times\mathbf{P}(F_x^*)}(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & p_1^*(F(-x)) & = & p_1^*(F(-x)) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{5}$$

Here, the top sequence is the Euler sequence on $\{x\} \times \mathbf{P}(F_x^*)$, the kernel of which is indeed $\Omega_{\{x\}\times\mathbf{P}(F_x^*)}(1)$, where $\Omega_{\{x\}\times\mathbf{P}(F_x^*)}$ is the cotangent bundle.

Now suppose that F is $(0,1)$-stable. Then $\mathscr{E}$ is a family of stable bundles of degree $d > 2n(g-1)$ over X; hence, taking the direct image of (5) by p_2 and using (3), we obtain the following diagram of exact sequences over $\mathbf{P}(F_x^*)$.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & H^1(F(-x)) \otimes \mathcal{O} & = & H^1(F(-x)) \otimes \mathcal{O} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \Omega_{\mathbf{P}(F_x^*)}(1) & \longrightarrow & F_x \otimes \mathcal{O} & \longrightarrow & \mathcal{O}(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \psi_{F,x}^*\mathscr{W}_\xi(-j) & \longrightarrow & H^0(F) \otimes \mathcal{O} & \longrightarrow & \mathcal{O}(1) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & H^0(F(-x)) \otimes \mathcal{O} & = & H^0(F(-x)) \otimes \mathcal{O} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{6}$$

Now let $\mathscr{F}$ be a subsheaf of the Picard bundle $\mathscr{W}_\xi$ of rank r with $0 < r < \operatorname{rk} \mathscr{W}_\xi$, and such that the quotient $\mathscr{W}_\xi/\mathscr{F}$ is torsion-free. We must prove that

$$\frac{\deg \mathscr{F}}{r} < \frac{\deg \mathscr{W}_\xi}{\operatorname{rk} \mathscr{W}_\xi}. \tag{7}$$

Consider the projective bundle

$$P_x := \mathbf{P}(\mathscr{U}_\xi|_{\{x\}\times\mathscr{M}_\xi})$$

over $\mathscr{M}_\xi$. Each point of P_x corresponds to a pair (E, l) with $E \in \mathscr{M}_\xi$ and $l \in \mathbf{P}(E_x)$. We have

$$\mathbf{P}(E_x) \cong \mathbf{P}(\mathrm{Hom}(E, k_x)^*) \cong \mathbf{P}(\mathrm{Ext}^1(k_x, E)),$$

and hence a point of P_x also corresponds to a short exact sequence (1). Indeed, after identifying scalar multiples, there is a universal family of non-trivial sequences (1) with $E \in \mathscr{M}_\xi$ parametrised by P_x. Let H_x be the open subset of P_x such that F is $(0,1)$-stable, and let V denote the open subset of $(0,1)$-stable bundles in $\mathscr{M}_{\xi(x)}$. Note that, by Lemma 2, V is non-empty.

There are the following morphisms.

$$\begin{array}{ccc} H_x & \xrightarrow{q} & V \subset \mathscr{M}_{\xi(x)} \\ \downarrow p & & \\ \mathscr{M}_\xi & & \end{array}$$

The morphism q is surjective and, by Lemma 1, its fibre can be identified with $\mathbf{P}(F_x^*)$. In particular, H_x is an irreducible variety of dimension $\dim \mathscr{M}_\xi + n - 1$. On the other hand, the fibre of H_x over $E \in \mathscr{M}_\xi$ is an open subset of $\mathbf{P}(E_x)$. Since p is the restriction of the projection morphism $P_x \to \mathscr{M}_\xi$, the image of H_x in $\mathscr{M}_\xi$ is a non-empty open subset. Note also that the restriction of p to $\mathbf{P}(F_x^*)$ is precisely $\psi_{F,x}$, so, by Lemma 3, $\mathbf{P}(F_x^*)$ maps isomorphically to its image $P(F,x) := \psi_{F,x}(\mathbf{P}(F_x^*))$.

LEMMA 4. *Let $x_1, \ldots, x_m \in X$. There exists a non-empty open subset U of $\mathscr{M}_\xi$ such that, if $E \in U$, then*

1. *$\mathscr{F}$ is locally free at E;*
2. *the homomorphism of fibres $\mathscr{F}_E \to \mathscr{W}_{\xi,E}$ is injective;*
3. *for all x_i and for the generic line ℓ in E_{x_i}, the corresponding vector bundle F is $(0,1)$-stable and $\mathscr{F}$ is locally free at every point of $P(F, x_i)$ outside some subvariety of codimension at least 2.*

Proof. The first two conditions clearly define non-empty open subsets of $\mathscr{M}_\xi$.

For the third condition, let S denote the singular set of $\mathscr{F}$. It is clearly sufficient to prove that, for any fixed x and generic F, either $P(F,x) \cap S$ is empty or

$$\dim(P(F,x) \cap S) \leqslant n - 3.$$

One can reduce immediately to the case where S is irreducible. The inverse image $S' = p^{-1}(S)$ of S in H_x is then either empty or an irreducible variety of dimension $\dim S + n - 1$. Note that the fibre of S' over F can be identified with $P(F,x) \cap S$. If $q(S')$ is of positive codimension, then certainly $P(F,x) \cap S$ is empty for generic F. On the other hand, if $q(S')$ is dense in V, the generic fibre of the projection of S' to V is of dimension

$$\dim S + n - 1 - \dim V = \dim S + n - 1 - \dim \mathscr{M}_\xi \leqslant n - 3,$$

which is exactly what we wanted to prove.

Proof of the theorem. Fix points $x_1, \ldots, x_m \in X$ with $m > d/n$, and choose $E \in \mathcal{M}_\xi$ satisfying the conditions of Lemma 4. Let v be a non-zero element of $\mathcal{F}_E$. The image s of v in $\mathcal{W}_{\xi,E} = H^0(X, E)$ is non-zero. If $s(x_i) = 0$ for all i, then $s \in H^0(X, E(-x_1 - \ldots - x_m))$. But $E(-x_1 - \ldots - x_m)$ has negative degree, which contradicts the stability of E. Hence we can choose i such that $s(x_i) \neq 0$. For simplicity, we put $x := x_i$. By Lemma 4, we can also choose a line ℓ in E_x such that $s(x) \notin \ell$ and the corresponding F has the property that $\mathcal{F}$ is locally free on $P(F, x)$ outside some subvariety of codimension at least 2.

Note that ℓ is just the image of $F(-x)_x$ in E_x in the diagram (4), so we conclude that

$$s \notin H^0(F(-x)). \tag{8}$$

Let $\mathcal{F}_1$ denote the image of $\psi^*_{F,x}\mathcal{F}(-j)$ in $\psi^*_{F,x}\mathcal{W}_\xi(-j)$. By (8) and (6), the induced homomorphism

$$\mathcal{F}_1 \to \Omega_{\mathbf{P}(F^*_x)}(1) \tag{9}$$

has non-zero image. Now $\Omega_{\mathbf{P}(F^*_x)}$ is the cotangent bundle of $\mathbf{P}(F^*_x)$, so $\Omega_{\mathbf{P}(F^*_x)}(1)$ is stable of degree -1. Hence the image of (9) has degree less than or equal to -1. Moreover, from (6) the kernel of (9) is a subsheaf of a trivial sheaf, and therefore has degree less than or equal to 0. So

$$\deg \mathcal{F}_1 \leqslant -1.$$

On the other hand, $\mathcal{F}_1$ is isomorphic to $\psi^*_{F,x}\mathcal{F}(-j)$ away from some subvariety of codimension 2, so

$$\deg \psi^*_{F,x}\mathcal{F}(-j) = \deg \mathcal{F}_1.$$

By (6), the bundle $\psi^*_{F,x}\mathcal{W}_\xi(-j)$ has degree -1, so

$$\frac{\deg \psi^*_{F,x}\mathcal{F}(-j)}{r} \leqslant -\frac{1}{r} < -\frac{1}{\operatorname{rk} \mathcal{W}_\xi} = \frac{\deg \psi^*_{F,x}\mathcal{W}_\xi(-j)}{\operatorname{rk} \mathcal{W}_\xi}.$$

Finally, by Lemma 3, $\psi^*_{F,x}\Theta = \mathcal{O}(\delta)$ for some positive δ, so

$$\frac{\deg \mathcal{F}}{r} = \frac{N}{\delta}\frac{\deg \psi^*_{F,x}\mathcal{F}}{r} < \frac{N}{\delta}\frac{\deg \psi^*_{F,x}\mathcal{W}_\xi}{\operatorname{rk} \mathcal{W}_\xi} = \frac{\deg \mathcal{W}_\xi}{\operatorname{rk} \mathcal{W}_\xi}.$$

This completes the proof of (7), and hence of the theorem.

REMARK. Let L be a line bundle over X. Under the same hypotheses as for our theorem, the bundle $\mathcal{W}_\xi(L) := p_{2*}(\mathcal{U}_\xi \otimes p_1^*(L))$ over $\mathcal{M}_\xi$ is stable. For the proof, we just replace E and F in (1) by $E \otimes L$ and $F \otimes L$, and argue as before.

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