BESSEL MODELS FOR GSp(4)

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To Steve Gelbart

Abstract. Methods of theta correspondence are used to analyze local and global Bessel models for GSp_4 proving a conjecture of Gross and Prasad which describes these models in terms of local epsilon factors in the local case, and the non-vanishing of central critical L-value in the global case.

Contents

Key words and phrases. Bessel models, Weil representation, Theta correspondence, local epsilon factors, central critical L-value.

1. Introduction

In this paper we use the methods of theta correspondence to prove certain local and global conjectures of Gross and Prasad for the pair $(\mathrm{SO}(2), \mathrm{SO}(5))$ by reducing the question to simpler pairs for which the analogous question is known. These conjectures relate the existence of Bessel models (which are certain Fourier coefficients) to certain local epsilon factors in the local case, and to the non-vanishing of certain central critical L-value in the global case. Instead of $\mathrm{SO}(5)$ we will consider the related group GSp_4 in this paper. This gives first nontrivial evidence to the conjectures of Gross-Prasad in which the subgroup considered is neither reductive, nor unipotent. As a byproduct, we also obtain information about the one dimensional representations of $\mathrm{GL}_2(K)$ which appear as a quotient in representations of $\mathrm{GL}_4(k)$ when restricted to $\mathrm{GL}_2(K)$ where K is a quadratic extension of a local field k.

Among the earliest manifestations of the methods that we follow in this paper is the work of Waldspurger on Shimura correspondence in the late 70's relating period integral of automorphic forms on PGL_2 over tori to Fourier coefficients of automorphic forms on the metaplectic SL_2, both being related to twisted L-values at $1/2$.

Let us now explain the setup more precisely. Let W be a four dimensional symplectic vector space over a field k with a fixed basis $\{e_1, e_2, e_3, e_4\}$, and a symplectic form $\langle , \rangle$ on W such that $\langle e_1, e_3\rangle = -\langle e_3, e_1\rangle = 1$, $\langle e_2, e_4\rangle = -\langle e_4, e_2\rangle = 1$, and all other products among these basis vectors to be zero; thus the symplectic structure is given by the following skew-symmetric matrix:

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Let $W_1 = \langle e_1, e_2\rangle$ be a maximal isotropic subspace of W. Let $G = \mathrm{GSp}(W)$ denote the symplectic similitude group of W, and P the parabolic subgroup of G consisting of elements of G which take W_1 into itself. The group P is the so-called Siegel parabolic which has the Levi decomposition $P = MN$, where $M \cong \mathrm{GL}_2 \times \mathbb{G}_m$ is the group of pairs (g, λ) with

$$(g, \lambda) = \begin{pmatrix} g & \\ & \lambda \cdot {}^t g^{-1} \end{pmatrix},$$

and N is an abelian group which can be identified to the set of 2×2 symmetric matrices, $\mathrm{Sym}_2(k)$, over k. The inner-conjugation action of an element $(g, \lambda) \in \mathrm{GL}_2 \times \mathbb{G}_m$ on $n \in N$ is given by $\lambda^{-1} g n {}^t g$. It follows that the stabilizer in M of a non-degenerate symmetric matrix in N can be identified to the normalizer of a Cartan subgroup of GL_2.

Fix $\psi_0 : k \to \mathbb{C}^\times$ to be a nontrivial additive character of k. Any character of N is of the form $\psi(n) = \psi_0(\mathrm{tr}[sn])$ for some $s \in \mathrm{Sym}_2(k)$, and the corresponding subgroup $N(T) = N(T_s)$ of $\mathrm{GL}_2(k)$ is

$$N(T) = \{g \in \mathrm{GL}_2(k) \mid {}^t g s g = \det g \cdot s\},$$

which is considered a subgroup of $\mathrm{GSp}_4(k)$ via the embedding

$$g \mapsto \begin{pmatrix} g & \\ & \det g \cdot {}^t g^{-1} \end{pmatrix}.$$

Let π be an irreducible admissible representation of $\mathrm{GSp}_4(k)$. Let π_ψ denote the largest quotient of π on which N operates by ψ. Clearly π_ψ is a representation space for the subgroup M^ψ of M which stabilizes ψ. We will consider $\psi = \psi_0(sn)$ corresponding to a $s \in \mathrm{Sym}_2(k)$ with $\det s \neq 0$. For such ψ, M^ψ is isomorphic to the normalizer $N(T)$ of a Cartan subgroup T of $\mathrm{GL}_2(k)$. The question that we study in this paper is the structure of π_ψ as a module for T, called the Bessel model of π, both locally as well as globally for representations π of $\mathrm{GSp}_4(k)$.

We will work simultaneously with the rank 1 form of $\mathrm{GSp}_4(k)$, to be denoted by $\mathrm{GSp}_4^D(k)$, and defined using a quaternion division algebra D over k as

$$\left\{ g \in \mathrm{GL}_2(D) | g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\bar{g} = \lambda \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \lambda \in k^\times \right\}$$

where for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(D), {}^t\bar{g} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$, and where $a \to \bar{a}$ denotes the standard involution on D. The group $\mathrm{GSp}_4^D(k)$ contains the Siegel parabolic whose unipotent radical is the group of matrices

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

where $n \in D$ with $n + \bar{n} = 0$, and the Levi subgroup is isomorphic to $D^\times \times k^\times$ embedded in $\mathrm{GSp}_4^D(k)$ as

$$\begin{pmatrix} d & 0 \\ 0 & t\bar{d}^{-1} \end{pmatrix}$$

for $d \in D^\times$, and $t \in k^\times$.

Many of the results on GSp_4 in this paper are obtained by using theta lifting from GO_4. We recall the structure of four dimensional quadratic spaces over a general field k (of characteristic not 2), and of the connected component of identity of GO_4, denoted by GSO_4, as follows.

(1) The isomorphism class of a quadratic space of dimension 4 and trivial discriminant over a field k is given by a quaternion algebra D over k with a multiple of its (reduced) norm form. In this case, $\mathrm{GSO}_4(k) \cong [D^\times \times D^\times]/\Delta k^\times$, where $\Delta k^\times = k^\times$ is embedded in $D^\times \times D^\times$ as (a, a^{-1}).

(2) To a quadratic space of dimension 4 and non-trivial discriminant over a field k, defining a quadratic extension E of k, there is associated a quaternion algebra D_E over E with an involution i of the second kind. The quadratic space corresponding to D_E consists of hermitian elements, i.e, $\{x \in D_E | i(x) = x\}$, together with the norm form $\mathbb{N} : D_E \to E$ restricted to this subspace (where it takes values in k), or a scaling of this quadratic space by an element of $k^\times/\mathbb{N}E^\times$. In this case, $\mathrm{GSO}_4(k) \cong [D_E^\times \times k^\times]/\Delta E^\times$, where $\Delta E^\times = E^\times$ maps to $k^\times$ as $\mathbb{N}(a^{-1})$. (In the case of local fields, a quadratic space of dimension 4, and non-trivial discriminant always has a zero, so $D_E \cong \mathrm{M}_2(E)$.)

To summarize, for V a four dimensional quadratic space over a local field k, $\mathrm{GSO}(V)$, has the structure of one of the following groups:

(1) $\mathrm{GSO}(V^s) \cong [\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)]/\Delta k^\times$.

(2) $\mathrm{GSO}(V^a) \cong [D^\times \times D^\times]/\Delta k^\times$.

(3) $\mathrm{GSO}(V^d) \cong [\mathrm{GL}_2(E) \times k^\times]/\Delta E^\times$,

where $\Delta k^\times = k^\times$ sits as (t, t^{-1}) in $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$ and in $D^\times \times D^\times$ where D is the unique quaternion division algebra over k, and $\Delta E^\times = E^\times$ sits inside $\mathrm{GL}_2(E) \times k^\times$ via its natural embedding in $\mathrm{GL}_2(E)$, and in $k^\times$ by the inverse of the norm mapping; we have used V^s to denote the unique four dimensional split quadratic space, V^a to denote the unique anisotropic quadratic space of dimension 4, and V^d is one of the two quadratic spaces of rank 1 with discriminant algebra E, a quadratic field extension of k.

Notice that not all forms of $[\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)]/\Delta k^\times$ are represented by $\mathrm{GSO}(V)$ in $(1), (2), (3)$. Other forms of $[\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)]/\Delta k^\times$ are defined using skew-hermitian forms over D, which give rise to groups

(4) $[D^\times \times \mathrm{GL}_2(k)]/\Delta k^\times$.
(5) $[D_E^\times \times k^\times]/\Delta E^\times$, where E is a quadratic extension of k, and D_E the unique quaternion division algebra over E.

These groups will be used to do theta correspondence between $\mathrm{GSp}_4^D(k)$ and $\mathrm{GSO}_4^D(k)$, and will be discussed in greater detail later.

1.1. **Resumé of the main results.** We recall the following multiplicity 1 theorem of Novodvorsky extended in two ways. First we consider $\mathrm{GSp}_4(k)$ instead of his $\mathrm{PGSp}_4(k)$, and then we also consider rank 1 form of $\mathrm{GSp}_4(k)$. Both of these are standard extensions of the arguments in Novodvorsky's paper.

Theorem 1. *Let π be an irreducible admissible representation of either $\mathrm{GSp}_4(k)$, or $\mathrm{GSp}_4^D(k)$ with Siegel parabolic $P = MN$. Let K be a quadratic separable algebra over k, and χ a character of $K^\times$. Let $\psi : N \to \mathbb{C}^\times$ be a non-degenerate character of N centralized by $K^\times$, so that one can construct a one dimensional representation of $R = K^\times N$ which is χ on $K^\times$, and ψ on N, which will also be denoted by χ as ψ will be kept fixed in this paper. Then*

$$\dim \mathrm{Hom}_R(\pi, \chi) \leq 1.$$

Remark 1.1. If $\mathrm{Hom}_R(\pi, \chi) \neq 0$, then the representation π is said to have Bessel model for the character χ of $K^\times$.

Before proceeding further, recall that the Langlands parameter of a representation π of $\mathrm{GSp}_4(k)$ is a representation

$$\sigma_\pi : W_k' \to \mathrm{GSp}_4(\mathbb{C})$$

where W_k' is the Weil-Deligne group of k which we take to be $W_k' = W_k \times \mathrm{SL}_2(\mathbb{C})$. These have been constructed in a recent paper of Gan and Takeda [G-T1] who have also defined a notion of L-packets (of size 1 or 2) for representations of $\mathrm{GSp}_4(k)$ which is what we will use in this paper. Instead of working with the Langlands parameter of a representation of $\mathrm{GSp}_4(k)$, with values in $\mathrm{GSp}_4(\mathbb{C})$, it is more convenient to work with representations of W_k' into $\mathrm{GL}_4(\mathbb{C})$, which fix a symplectic form up to a similitude. The following lemma, implicit in many considerations about symplectic parameters, makes this possible.

Lemma 1.2. *For any group G, the natural homomorphism $\mathrm{GSp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathbb{C}^\times$, where the mapping from $\mathrm{GSp}_{2n}(\mathbb{C})$ to $\mathbb{C}^\times$ is the similitude character, gives rise* *to an injective map from conjugacy classes of homomorphisms from G to $\mathrm{GSp}_{2n}(\mathbb{C})$ to conjugacy classes of homomorphisms from G to $\mathrm{GL}_{2n}(\mathbb{C}) \times \mathbb{C}^\times$.*

Here is our main local theorem, proving the Gross-Prasad conjecture for Bessel models of $\mathrm{GSp}_4(k)$:

Theorem 2. *Let K be a quadratic separable algebra over a local field k of residue characteristic not 2, such that $K^\times \subset \mathrm{GL}_2(k)$ is contained in the centralizer of a non-degenerate character $\psi : N(k) \to \mathbb{C}^\times$. Let $\chi : K^\times \to \mathbb{C}^\times$ be a character. Let $\{\pi\}$ be an irreducible, admissible generic L-packet of representations of $\mathrm{GSp}_4(k)$ with Langlands parameter σ_π. Assume that the central character of $\{\pi\}$ is $\chi|_{k^\times}$. Let $\mathrm{GSp}_4^D(k)$ be the rank 1 form of $\mathrm{GSp}_4(k)$, and $\{\pi'\}$ an irreducible, admissible L-packet of representations of $\mathrm{GSp}_4^D(k)$ with Langlands parameter σ_π. (So $\{\pi'\}$ might be an empty set.) Then there is at most one representation $\pi \in \{\pi\}$ with $\mathrm{Hom}_{K^\times}(\pi_\psi, \chi) \neq 0$, and there is one if and only if $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1$. Similarly, there is at most one representation $\pi' \in \{\pi'\}$ with $\mathrm{Hom}_{K^\times}(\pi'_\psi, \chi) \neq 0$, and there is one if and only if $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = -1$. Furthermore, if $\{\pi\}$ or $\{\pi'\}$ consisted of more than one element, then the parameter σ_π with values in $\mathrm{GSp}_4(\mathbb{C})$ becomes a sum of two-dimensional representations $\sigma_\pi = \sigma_1 \oplus \sigma_2$ with $\det \sigma_1 = \det \sigma_2 = \chi|_{k^\times}$, and one can make precise which element of the L-packet, $\{\pi\}$ or $\{\pi'\}$ has a Bessel model for the character χ of $K^\times$.*

Here is the global theorem we prove.

Theorem 3. *Let D be a quaternion algebra over a number field F, with the adele ring $\mathbb{A}_F$. Let Π_1 and Π_2 be two automorphic representations of $D^\times(\mathbb{A}_F)$ with the same central characters, so that $\Pi_1 \boxtimes \Pi_2$ can be considered to be an automorphic representation on the corresponding orthogonal group $\mathrm{GSO}_4(\mathbb{A}_F)$ defined by the reduced norm on D. Let Π be the theta lift to $\mathrm{GSp}_4(\mathbb{A}_F)$ of $\Pi_1 \boxtimes \Pi_2$ on $\mathrm{GSO}_4(\mathbb{A}_F)$. Let E be a separable quadratic algebra over F, and χ a Grössencharacter on $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_F^\times$ is the central character of Π. Let $\psi : N(\mathbb{A}_F)/N(F) \to \mathbb{C}^\times$ be a character which is normalized by $\mathbb{A}_E^\times$, and hence (χ, ψ) gives rise to a character of $R(\mathbb{A}_F) = \mathbb{A}_E^\times N(\mathbb{A}_F)$ which we will abuse notation to denote simply by χ as ψ is considered fixed. Then the period integral on Π taking $f \in \Pi$ to*

$$\int_{R(F)\mathbb{A}_F^\times \backslash R(\mathbb{A}_F)} f(g)\chi^{-1}(g)dg,$$

is not identically zero if and only if the period integrals,

$$\int_{E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times} f_1(g)\chi^{-1}(g)dg,$$

and

$$\int_{E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times} f_2(g)\chi^{-1}(g)dg$$

on Π_1 and Π_2 respectively are not identically zero; in particular, by Waldspurger, if the period integral on $R(F)\mathbb{A}_F^\times \backslash R(\mathbb{A}_F)$ of functions in Π is not identically zero, then

$$L(\frac{1}{2}, BC_E(\Pi_1) \otimes \chi^{-1}) \neq 0,$$

and

$$L(\frac{1}{2}, BC_E(\Pi_2) \otimes \chi^{-1}) \neq 0,$$

where BC_E denotes the base change to $\mathrm{GL}_2(E)$ of an automorphic form on $\mathrm{GL}_2(F)$.

Further, for automorphic representations Π_1 and Π_2 on $\mathrm{GL}_2(\mathbb{A}_F)$, if $L(\frac{1}{2}, BC_E(\Pi_1) \otimes \chi^{-1}) \neq 0$, and $L(\frac{1}{2}, BC_E(\Pi_2) \otimes \chi^{-1}) \neq 0$, Waldspurger's theorem gives quaternion algebras D_1 and D_2 over F, and an automorphic representation of $(D_1^\times \times D_2^\times)/\Delta\mathbb{G}_m$ such that the corresponding period integrals on $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ are nonzero. Given D_1 and D_2, quaternion algebras over the number field F, let $D_1 \otimes D_2 \cong M_2(D)$. Taking the tensor product of canonical involutions on D_1 and D_2, we get an involution on $M_2(D)$ with fixed subspace of dimension 10, and hence there is a skew-hermitian form on a 2 dimensional vector space over D such that the corresponding $\mathrm{GSO}_4^D(k) = [D_1^\times \times D_2^\times]/\Delta\mathbb{G}_m$. Define $\mathrm{GSp}_4^D(k)$ using this D, and construct a representation of $\mathrm{GSp}_4^D(\mathbb{A}_F)$ via theta lifting. Then for this automorphic representation, say $\tilde{\Pi}$ on $\mathrm{GSp}_4^D(k)$, the corresponding period integral of functions f in $\tilde{\Pi}$,

$$\int_{R(F)\mathbb{A}_F^\times \backslash R(\mathbb{A}_F)} f(g)\chi^{-1}(g)dg,$$

is not identically zero (in particular $\tilde{\Pi}$ is not zero).

Remark 1.3. There is a considerable amount of *Geometric Algebra* especially using *exceptional isomorphisms* of low rank groups in this paper (such as $\mathrm{SO}_4(k)$ being related to $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$, SO(5) being related to PGSp(4), or SO(6) to SL(4), the structure of their inner forms which usually has different constructions for the two groups involved, and the relation of their subgroups under this isomorphism). This fits rather nicely to yield exactly what is needed for the similitude groups being considered (such as GSO(2), which is much preferred over SO(2)).

Remark 1.4. It may be noted that besides its intrinsic interest, as Bessel models are usually nonzero for some choice of $\chi : K^\times \to \mathbb{C}^\times$, they can be used in developing the theory of L-functions for $\mathrm{GSp}_4(k)$ as in the early work of Novodvorsky and Piatetski-Shapiro, extending considerably the scope of the theory of L-functions based on genericity hypothesis. See [F] and [Pi-S] for modern treatments of this idea.

One can bootstrap our results and techniques to deduce a theorem about the restriction of a representation of $\mathrm{GL}_4(k)$ to the subgroup $\mathrm{GL}_2(K)$ where K is a quadratic algebra over k, which we state now.

Theorem 4. *Let π be an irreducible, admissible, generic representation of $\mathrm{GL}_4(k)$ with central character ω_π. If π can be transferred to a representation of $\mathrm{GL}_2(D)$, let π' be the corresponding representation of $\mathrm{GL}_2(D)$. Let χ be a character of $K^\times$ such that $\chi^2|_{k^\times} = \omega_\pi$. Then if the character $\chi \circ \det$ of $\mathrm{GL}_2(K)$ appears as a quotient in π, or π', restricted to $\mathrm{GL}_2(K)$,*

1. *The Langlands parameter of π takes values in $\mathrm{GSp}_4(\mathbb{C})$ with similitude factor $\chi|_{k^\times}$.*
2. *The epsilon factor $\epsilon(\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1$.*

Conversely, assume that π is an irreducible, generic representation of $\mathrm{GL}_4(k)$ *such that:*

(1) The Langlands parameter of π takes values in $\mathrm{GSp}_4(\mathbb{C})$ *with similitude factor* $\chi|_{k^\times}$.

(2) The epsilon factor $\epsilon(\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1$.

From (1), *it follows by the work of Gan-Takeda that there exists an L-packet* $\{\pi^{\mathrm{GSp}}\}$ *on* $\mathrm{GSp}_4(k)$ *whose theta/Langlands lift to* $\mathrm{GL}_4(k)$ *is π. (By lemma 1, given* π and χ, *the L-packet* $\{\pi^{\mathrm{GSp}}\}$ *is unique.) If* $\pi' \neq 0$, *and the L-packet* $\{\pi^{\mathrm{GSp}}\}$ *has 2 elements, then exactly one of π or π' has a quotient on which* $\mathrm{GL}_2(K)$ *operates by χ unless $\pi = \tau \times \tau$ for a discrete series representation τ of* $\mathrm{GL}_2(k)$ *in which case both π and π' have a quotient on which* $\mathrm{GL}_2(K)$ *operates by χ for any character χ of $K^\times$ such that* $\omega_\tau = \chi|_{k^\times}$. *If* $\pi' \neq 0$, *and the L-packet* $\{\pi^{\mathrm{GSp}}\}$ *has only one element, then both π and π' have a quotient on which* $\mathrm{GL}_2(K)$ *operates by* $\chi \circ \det$.

Similarly, assume that π can be transferred to a representation π'' of $\mathcal{D}^\times$ where $\mathcal{D}$ is a quartic division algebra over k, and that K is a quadratic field extension of k embedded inside $\mathcal{D}$, and that the centralizer of K inside $\mathcal{D}$ is B_K for the quaternion division algebra B_K over K. Then a character χ of $K^\times$ thought of as a character of $B_K^\times$ appears in π'' restricted to $B_K^\times$ if and only if

(1) The Langlands parameter of π takes values in $\mathrm{GSp}_4(\mathbb{C})$ *with similitude factor* $\chi|_{k^\times}$.

(2) The epsilon factor $\epsilon(\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = -1$.

This theorem is proved in Section 11, where we formulate a general conjecture.

We construct an example to show that the question of the non-triviality of the corresponding global period integral is not merely one about L-values, and local conditions; this is given in Section 14; it should be contrasted with the case of Bessel periods, or the general Gross-Prasad conjectures, where local conditions, together with non-vanishing of a central critical L-value dictates the period integral to be nonzero.

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2. Bessel models for principal series representations

The aim of this section is to calculate the twisted Jacquet functor π_ψ for a principal series representation π of $\mathrm{GSp}_4(k)$ with respect to a non-degenerate character $\psi : N \to \mathbb{C}^\times$ given by a symmetric matrix $A \in M_2(k)$ as $\psi(X) = \psi_0(tr(AX))$ for $X \in N = \mathrm{Sym}_2(k)$.

We note that π_ψ is a module for the group

$$M_\psi \cong \{g \in \mathrm{GL}_2(k) | gA^tg = \det g \cdot A\}$$

considered as a subgroup of $\mathrm{GSp}_4(k)$ via

$$g \mapsto \begin{pmatrix} g & \\ & \det g \cdot {}^tg^{-1} \end{pmatrix}.$$

which is the normalizer in $\mathrm{GL}_2(k)$ of a certain torus $K_\psi^\times$.

Recall that we are using W to denote a four dimensional symplectic vector space over a field k with a fixed basis $\{e_1, e_2, e_3, e_4\}$, and a symplectic form $\langle , \rangle$ on W such that $\langle e_1, e_3\rangle = -\langle e_3, e_1\rangle = 1$, $\langle e_2, e_4\rangle = -\langle e_4, e_2\rangle = 1$, and all other products among these basis vectors to be zero; thus the symplectic structure is given by the following skew-symmetric matrix:

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

2.1. **Siegel Parabolic.** We begin with the case when π is induced from the Siegel parabolic P from an irreducible representation ρ of the Levi subgroup M of $P = MN$.

As usual, let P be the Siegel parabolic stabilizing the isotropic subspace $W = \{e_1, e_2\}$ with M the stabilizer of the isotropic subspaces $W = \{e_1, e_2\}$ and $W' = \{e_3, e_4\}$. The calculation of the twisted Jacquet functor will depend on the understanding of the double coset $P\backslash G/P$ with $G = \mathrm{GSp}_4(k)$, which is the same as $G\backslash[G/P \times G/P]$, or the orbit of $\mathrm{GSp}_4(k)$ on pairs of maximal isotropic subspaces. It is easy to see that there are three orbits of pairs of isotropic subspaces (W_1, W_2):

(1) $W_1 = W_2$; in this case we take $W_1 = W_2 = W$.
(2) $W_1 \cap W_2 = \{0\}$; in this case we take $W_1 = W$, and $W_2 = W'$.
(3) $W_1 \cap W_2$ is 1-dimensional; in this case we take $W_1 = W$, and $W_2 = \{e_1, e_4\}$.

As W_1 is chosen to be W in all the three cases, the stabilizer in $\mathrm{GSp}_4(k)$ of the pair of isotropic subspaces (W_1, W_2) is a subgroup of P which is the following subgroup H_i of P in the three cases:

(1) $H_1 = P$.
(2) $H_2 = M$.
(3) H_3 containing the unipotent group $\begin{pmatrix} x & y \\ y & 0 \end{pmatrix} \in \mathrm{Sym}^2(k) \subset N$.

From the Mackey theory, it follows that the representation $\pi = \mathrm{ind}_P^G\rho$ restricted to P is obtained by gluing the following three representations:

(1) ρ.
(2) $\mathrm{ind}_M^P\rho$.
(3) $\mathrm{ind}_{H_3}^P\rho|_{H_3}$.

(The discriminant function δ_P used for normalized induction is trivial on M_ψ as up to finite index M_ψ is the product of the center of G with its part in $[M, M]$; hence in light of the eventual answer, one can ignore δ_P in what follows.)

We observe that since the representation ρ of M is extended to P trivially across N, for a non-degenerate character ψ of N, $\rho_\psi = 0$ in case (i).

For case (iii), as

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x & y \\ y & 0 \end{pmatrix} = \begin{pmatrix} ax+by & ay \\ bx+cy & by \end{pmatrix},$$

$$\psi\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x & y \\ y & 0 \end{pmatrix}\right] = \psi_0(ax+2by),$$

it follows that if the character ψ of N were to be trivial on the subgroup $N \cap H_3$, $a = b = 0$, and hence ψ will not be a non-degenerate character. Noting that N is a normal subgroup of P, it follows that any character of N appearing in case (iii) is degenerate, and hence case (iii) does not contribute to the twisted Jacquet functor.

In case (ii), since $P \cong MN$ with N normal, we get $\mathrm{ind}_M^P\rho \cong \rho \otimes \mathrm{ind}_M^P \mathbf{1}$ as P-modules. Hence,

$$\left[\mathrm{ind}_M^P\rho\right]_\psi \cong \rho|_{M_\psi}$$

for any character ψ of N. Thus we find that the twisted Jacquet functor in the three cases is as follows:

(1) 0.
(2) $\rho|_{M_\psi}$.
(3) 0.

Therefore we have the following proposition.

Proposition 2.1. *For a principal series representation π of $G = \mathrm{GSp}_4(k)$ induced from a representation ρ of $P = MN$ of a Siegel parabolic, $\pi_\psi \cong \rho$ restricted to M_ψ.*

Analogously, for $\mathrm{GSp}_4^D(k)$, we have the following.

Proposition 2.2. *For a principal series representation π of $G = \mathrm{GSp}_4^D(k)$ induced from a representation ρ of $P = MN$ of the unique (up to conjugacy) parabolic of $\mathrm{GSp}_4^D(k)$ with $M \cong D^\times \times k^\times$, $\pi_\psi \cong \rho$ restricted to M_ψ.*

2.2. **Klingen Parabolic.** We next direct our attention to the calculation of the twisted Jacquet functor for representations induced from Klingen parabolic $Q = M'N'$ which we take to be the stabilizer of the isotropic line $\{e_1\}$. Once again, the restriction to P of a representation π of $\mathrm{GSp}_4(k)$ induced from a representation ρ of M' extended trivially across N' is obtained by gluing certain representations indexed by double cosets $P\backslash\mathrm{GSp}_4(k)/Q$ which is the same as the $\mathrm{GSp}_4(k)$-orbits of pairs (L, W) of a one dimensional subspace L of V, and a two dimensional isotropic subspace W of V. There are two orbits:

(1) $L \subset W$ in which case we take $L = \{e_1\}$, and $W = \{e_1, e_2\}$.
(2) $L \not\subset W$ in which case we take $L = \{e_3\}$, $W = \{e_1, e_2\}$.

In case (i), the part of the unipotent radical N of P which is contained in the unipotent radical N' of Q is the set of matrices,

$$\begin{pmatrix} 0 & y \\ y & z \end{pmatrix} \in \mathrm{Sym}^2(k) \subset N,$$

From a calculation as done in case (iii) of the principal series induced from Siegel parabolic, it is easy to see that there are no non-degenerate characters of N trivial on

$$\begin{pmatrix} 0 & y \\ y & z \end{pmatrix} \in \mathrm{Sym}^2(k) \subset N,$$

and therefore once again as N is normal in P, it follows that this double coset contributes nothing to the twisted Jacquet functor.

In case (ii), the stabilizer of the pair (L, W) with $L = \{e_3\}$, and $W = \{e_1, e_2\}$ is the subgroup

$$H = \begin{pmatrix} x_{11} & 0 & 0 & 0 \\ x_{21} & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix}.$$

There are embeddings of H in $Q = k^\times \times \mathrm{GL}_2(k) \times N'$ with image $k^\times \times B_2 \times \langle u \rangle$ where B_2 is the group of upper triangular matrices in $\mathrm{GL}_2(k)$, and $\langle u \rangle$ is a 1 parameter subgroup in N'. In the embedding of H in $P = k^\times \times \mathrm{GL}_2(k) \times N$, the one parameter subgroup $\langle u \rangle$ goes to the upper triangular unipotent subgroup in $\mathrm{GL}_2(k)$, the unipotent radical of B_2 goes inside N to a 1-dimensional subgroup that we denote by N_0, and the diagonal torus to the diagonal torus in $\mathrm{GL}_2(k)$.

By Mackey theory, it follows that the restriction of π to P contains $\mathrm{ind}_H^P \rho'$ where H can be taken to be $k^\times \times B_2 \times N_0$ and the representation ρ' is the restriction of ρ to the diagonal torus in $\mathrm{GL}_2(k)$ extended trivially across the unipotent subgroup of B_2 to all of B_2. We assume now that the representation ρ of $\mathrm{GL}_2(k)$ is infinite dimensional, so that (by Kirillov theory) its restriction to B_2 contains the representation of B_2 which is obtained by inducing from a character of the subgroup, ZU of B_2, consisting of central and unipotent elements of B_2.

As the action of $K^\times$ on $\mathrm{GL}_2(k)/P_1$ where P_1 is the subgroup of B_2 consisting of elements of the form

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix},$$

contains an open dense orbit, it follows that for $R = K^\times \cdot N$, $R \backslash P/H$ contains an open dense double coset which in the case of K a field is the unique double coset. Thus the representation $\mathrm{ind}_H^P \rho'$ restricted to R contains

$$\mathrm{ind}_{R\cap H}^R \tilde{\psi} = \mathrm{ind}_{k^\times N_0}^{K^\times N} \tilde{\psi},$$

where $\tilde{\psi}$ is the character of $k^\times N_0$ which is equal to the central character of π restricted to $k^\times$, and is the restriction of the character ψ of N to N_0, which can be checked to be nontrivial on N_0.

Thus its maximal quotient on which N operates by ψ is $\mathrm{ind}_{k^\times N}^{K^\times N} \omega\psi$ where ω is the central character of the representation ρ. We thus obtain the following conclusion.

Proposition 2.3. *For a principal series representation π of $G = \mathrm{GSp}_4(k)$ induced from an infinite dimensional irreducible representation ρ of $Q = M'N'$ of a Klingen parabolic, π_ψ restricted to $K^\times = M_\psi$ has each and every character of $K^\times$ with the same central character as that of ρ appearing with multiplicity one as a quotient.*

2.3. **Degenerate principal series coming from Klingen parabolic.** In this section we modify the argument of the previous section to calculate the ψ-Bessel model, for a non-degenerate character ψ of N, of a degenerate principal series representation of $\mathrm{GSp}_4(k)$ induced from a one dimensional representation ρ of the Klingen parabolic $Q = M'N'$. The analysis of the previous section gives the ψ-Bessel model of $\pi = \mathrm{ind}_Q^{\mathrm{GSp}_4(k)}\rho$ as the ψ-Bessel model of $\mathrm{ind}_H^P \rho|_H$ where $H = B_2 \times k^\times \times N_0$, with B_2 the lower triangular subgroup of $\mathrm{GL}_2(k)$, and N_0 the one parameter subgroup $\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$. If follows that if π has ψ-Bessel model for $s = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in \mathrm{GL}_2(k)$, then

$$\mathrm{tr}\left[\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}\right] \equiv 0, \text{ or } c = 0.$$

This means that if π has ψ-Bessel model corresponding to the symmetric matrix $s = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, c must be zero. For such symmetric matrices, the corresponding quadratic form is split, and hence we deduce that π has ψ-Bessel model only for K defined by a split quadratic algebra $K \cong k \oplus k$.

Fixing now the character $\psi\begin{pmatrix} x_{11} & x_{12} \\ x_{12} & x_{22} \end{pmatrix} = \psi_{00}(x_{12})$, of N which is trivial on N_0, and which is stabilized by the diagonal split torus T in $\mathrm{GL}_2(k)$ embedded in $\mathrm{GSp}_4(k)$ as

$$T = \begin{pmatrix} t_1 & 0 & 0 & 0 \\ 0 & t_2 & 0 & 0 \\ 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & t_1 \end{pmatrix},$$

it is easy to calculate the twisted Jacquet module for the character ψ of N of the induced representation $\mathrm{ind}_H^P \rho|_H$ with $H = B_2 \times k^\times \times N_0$, to conclude the following proposition.

Proposition 2.4. *Let ρ be a one dimensional representation of the Klingen parabolic Q of $\mathrm{GSp}_4(k)$, and $\pi = \mathrm{ind}_Q^{\mathrm{GSp}_4(k)}\rho$, the corresponding principal series representation. Then π has Bessel models for a quadratic algebra K if and only if $K = k \oplus k$, and in which case it has exactly one dimensional space of Bessel models for the character of $K^\times$ obtained by restriction of ρ to $K^\times$ which sits inside the Levi subgroup of Q.*

3. Theorem 2 for irreducible principal series

Let us begin by stating the Langlands parameters of principal series representations, and then do the relevant local epsilon factor calculations.

3.1. **Siegel parabolic.** Let $P = MN$ be a Siegel parabolic with $M \cong \mathrm{GL}_2(k) \times \mathbb{G}_m$. Let $\pi_1 \boxtimes \mu$ be an irreducible representation of M, giving rise to an irreducible principal series representation π of $\mathrm{GSp}_4(k)$ by parabolic induction. It is conventional to denote this principal series representation π by $\pi_1 \rtimes \mu$. The Langlands parameter of the representation π of $\mathrm{GSp}_4(k)$ is a representation

$$\sigma_\pi : W_k' \to \mathrm{GSp}_4(\mathbb{C})$$

where W_k' is the Weil-Deligne group of k which we take to be $W_k' = W_k \times \mathrm{SL}_2(\mathbb{C})$. Assuming that the Langlands parameter of the representation π_1 of $\mathrm{GL}_2(k)$ is σ_1, we have

$$\sigma_\pi = \mu\sigma_1 \oplus (\mu \oplus \mu \det \sigma_1).$$

We note that the Langlands parameter of an irreducible principal series representation of $\mathrm{GSp}_4(k)$ arising from parabolic induction of a representation of the Siegel parabolic takes values in (Levi subgroup of) the Klingen parabolic of $\mathrm{GSp}_4(\mathbb{C})$. (As a check on the Langlands parameter, one notes that twisting by a character $\chi : k^\times \to \mathbb{C}^\times$, thought of as a character of $\mathrm{GSp}_4(k)$ through the similitude character, takes the principal series representation $\pi_1 \rtimes \mu$ to $\pi_1 \rtimes \mu\chi$, and this on Langlands parameter is supposed to be twisting by χ.)

The central character ω_π of π is the same as the similitude character of σ_π which is $\mu^2 \det \sigma_1$. Therefore the characters χ of $K^\times$ appearing in Theorem 2 have the property that $\chi|_{k^\times} = \mu^2 \det \sigma_1$, and these are the only characters of $K^\times$ that we will consider in what follows.

By the standard properties of the local epsilon factors, it follows that for σ_π as above,

$$\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = \epsilon(\mu\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \epsilon(\mu \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \epsilon(\mu \det \sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})).$$

Since $\chi|_{k^\times} = \mu^2 \det \sigma_1$, it follows that for $V = \mu \otimes \mathrm{Ind}_K^k(\chi^{-1})$, $V^\vee \cong \mu \det \sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})$. Since for any representation V of W_k',

$$\epsilon(V) \cdot \epsilon(V^\vee) = \det V(-1),$$

it follows that,

$$\begin{aligned} \epsilon(\mu \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \epsilon(\mu \det \sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})) &= \det(\mu \otimes \mathrm{Ind}_K^k(\chi^{-1}))(-1) \\ &= \omega_{K/k}(-1)\chi(-1). \end{aligned}$$

Therefore,

$$\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = \epsilon(\mu\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \omega_{K/k}(-1)\chi(-1).$$

Therefore it follows from the theorem of Saito and Tunnell that

$$\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1$$

if and only if the character χ of $K^\times$ appears in the representation $\pi_1 \otimes \mu$ of $\mathrm{GL}_2(k)$, which by proposition 2.1 of the last section are exactly the characters of $K^\times$ for which π has Bessel models. (We note that $K^\times \cong M_\psi$ is included in $M = \mathrm{GL}_2(k) \times k^\times$ in such a way that the resulting map to $k^\times$ is the norm mapping.)

Furthermore, if $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = -1$, the representation π_1 of $\mathrm{GL}_2(k)$ is a discrete series representation, and if π_1' is the corresponding representation of $D^\times$, then χ appears in the representation π_1' restricted to $K^\times$. By proposition 2.2, the corresponding principal series representation of $\mathrm{GSp}_4^D(k)$ has Bessel model for χ, proving Theorem 2 in this case.

3.2. **Klingen parabolic.** Let $Q = MN$ be a Klingen parabolic with $M \cong k^\times \times \mathrm{GL}_2(k)$. Let $\mu \boxtimes \pi_1$ be an irreducible representation of M, giving rise to the principal series representation $\pi = \mu \rtimes \pi_1$ of $\mathrm{GSp}_4(k)$ by parabolic induction. Then the Langlands parameter of the representation $\pi = \mu \rtimes \pi_1$ of $\mathrm{GSp}_4(k)$ is a representation

$$\sigma_\pi : W'_k \to \mathrm{GSp}_4(\mathbb{C}).$$

Assuming that the Langlands parameter of the representation π_1 of $\mathrm{GL}_2(k)$ is σ_1, we have

$$\sigma_\pi = \sigma_1 \oplus \mu \cdot \sigma_1.$$

(This time twisting by $\chi : k^\times \to \mathbb{C}^\times$ takes $\mu \rtimes \pi_1$ to $\mu \rtimes \chi\pi_1$.) This parameter takes values in the Siegel parabolic of $\mathrm{GSp}_4(\mathbb{C})$, and for that reason it is better to write it as

$$\sigma_\pi = \sigma_1 \oplus (\mu \det \sigma_1) \cdot \sigma_1^\vee.$$

As the central character of π is equal to $\mu \det \sigma_1$, the characters χ of $K^\times$ appearing in Theorem 2 have the property that $\chi|_{k^\times} = \mu \cdot \det \sigma_1$, and these are the only characters that we will consider in what follows.

By standard properties of the local epsilon factors, for σ_π as above,

$$\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = \epsilon(\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \epsilon(\sigma_1 \otimes \mu \otimes \mathrm{Ind}_K^k(\chi^{-1})).$$

Since $\chi|_{k^\times} = \mu \cdot \det \sigma_1$, for $V = \sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})$, we have $V^\vee \cong \sigma_1 \otimes \mu \otimes \mathrm{Ind}_K^k(\chi^{-1})$, and therefore,

$$\begin{aligned}
\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) &= \epsilon(\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \epsilon(\sigma_1 \otimes \mu \otimes \mathrm{Ind}_K^k(\chi^{-1})) \\
&= \det(\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1}))(-1) \\
&= [\det(\sigma_1)^2 \cdot \det(\mathrm{Ind}_K^k(\chi^{-1}))^2](-1) \\
&= 1.
\end{aligned}$$

Therefore in this case Theorem 2 asserts that $\pi = \mu \rtimes \pi_1$ has Bessel model for all characters χ of $K^\times$ with $\chi|_{k^\times} = \mu \det \sigma_1$, and this is what Proposition 2.3 proves. (Note that the Klingen parabolic is not defined for the rank 1 form of $\mathrm{GSp}_4(k)$, so we do not need to consider $\mathrm{GSp}_4^D(k)$ here unlike in the case of principal series representations arising from the Siegel parabolic subgroup.)

4. Reducible principal series

It can be seen that if an irreducible representation of $\mathrm{GSp}_4(k)$ belongs to a generic L-packet, and is a sub-quotient of a principal series representation coming from the Siegel parabolic, then it also arises from theta correspondence with a representation of an orthogonal group in 4 variables, for which methods of theta correspondence to be developed in later sections work. It suffices then to restrict ourselves only to irreducible representation of $\mathrm{GSp}_4(k)$ belonging to a generic L-packet which is a sub-quotient of a principal series representation coming from the Klingen parabolic.

Let $Q = MN$ be a Klingen parabolic with $M \cong k^\times \times \mathrm{GL}_2(k)$. Let $\mu \boxtimes \pi_1$ be an irreducible representation of M, giving rise to the principal series representation π of $\mathrm{GSp}_4(k)$ by parabolic induction. Assume that the Langlands parameter of the representation π_1 of $\mathrm{GL}_2(k)$ is σ_1. It is known that if π_1 is a discrete series representation of $\mathrm{GL}_2(k)$, the principal series representation π is reducible if and only if

(1) $\mu = 1$, in which case π is a direct sum of two irreducible unitary representations of $\mathrm{GSp}_4(k)$ which form an L-packet.
(2) (Here we assume that π_1 is supercuspidal.) $\mu = \omega|\cdot|^{\pm 1}$ for a nontrivial quadratic character ω of $k^\times$, such that $\pi_1 \cong \pi_1 \otimes \omega$. We assume in what follows that $\mu = \omega \cdot |\cdot|$. In this case, there are exactly two components of the principal series representation $\omega|\cdot| \rtimes |\cdot|^{-1/2}\pi_1$ of $\mathrm{GSp}_4(k)$; one of the components is a discrete series representation with parameter σ_π which is

$$\sigma_\pi = \sigma_1 \otimes \mathrm{St}_2,$$

and the parameter of the other representation is

$$\sigma_\pi = |\cdot|^{-1/2}\sigma_1 \oplus |\cdot|^{1/2} \cdot \sigma_1.$$

The similitude character (necessary to define a $\mathrm{GSp}_4(\mathbb{C})$ valued parameter instead of just $\mathrm{GL}_4(\mathbb{C})$ valued parameter) of both the representations is $\omega \cdot \det \sigma_1$.

In case 1 above, the analysis of principal series done before tell us complete information about Bessel models for the sum of the two representations in the L-packet so obtained. In fact the two representations in the L-packet arise from theta lifting from $\mathrm{GSO}(V^s)$, and $\mathrm{GSO}(V^a)$ where V^s and V^a are the two split and anisotropic quadratic forms of dimension 4 over k, and hence complete information about Bessel models of the individual representations in such L-packet of representations of $\mathrm{GSp}_4(k)$ can be obtained by the method of theta correspondence developed later.

We now turn to case (2) in which case the representation of $\mathrm{GSp}_4(k)$ has parameter

$$\sigma_\pi = \sigma_1 \otimes \mathrm{St}_2.$$

We calculate the epsilon factor, $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1}))$, for this choice of σ_π. By generalities about epsilon factors,

$$\epsilon(V \otimes \mathrm{St}_n) = \epsilon(V)^n \det(-F, V^I)^{n-1},$$

where V^I denotes the subspace of V invariant under I. In our case, this formula gives

$$\begin{aligned} \epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) &= \epsilon(\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1}) \otimes \mathrm{St}_2) \\ &= \det(-F, V^I) \end{aligned}$$

where $V = \sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})$. If $V^I \neq 0$, as V is self-dual, so is V^I as a representation space for the cyclic group $\langle F \rangle$. Write $V^I = \sum \chi_i$. If a character χ_i with $\chi_i^2 \neq 1$ appears in V^I, then so does χ_i^{-1}; together $\{\chi_i, \chi_i^{-1}\}$ do not contribute anything to $\det(-F, V^I)$. A character χ_i with $\chi_i^2 = 1$, but $\chi_i \neq 1$ also does not contribute to $\det(-F, V^I)$. Therefore $\det(-F, V^I) = (-1)^r$ where r is the number of copies of the trivial representation of W_k in V. Assuming that σ_1 is irreducible, it follows that $\epsilon(\sigma_\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = -1$ if and only if σ_1 and $\mathrm{Ind}_K^k(\chi^{-1})$ are isomorphic.

By the calculation of the Bessel model for a principal series representation done in the last section, we already know that the principal series representation has Bessel models for all characters χ of $K^\times$ (with the usual restriction on the central character).

Thus by the exactness of the twisted Jacquet functor, either of the following two statements implies the other:

(1) The generic component of the principal series representation $\pi = Ps(\omega|\cdot|, \pi_1)$ has Bessel model for all characters of $K^\times$ whose restriction to $k^\times$ is the central character of π except χ and its Galois conjugate $\bar{\chi}$ if $\sigma_1 \cong \mathrm{Ind}_K^k(\chi)$.
(2) The other component of the principal series representation has Bessel model for exactly the two characters χ and its Galois conjugate $\bar{\chi}$ of $K^\times$ if $\sigma_1 \cong \mathrm{Ind}_K^k(\chi)$.

We will prove the assertion (2) in §8, as an application of seesaw duality.

5. The Steinberg representation

Let B denote the standard minimal parabolic of $\mathrm{GSp}_4(k)$, and P, Q respectively Siegel and Klingen parabolic subgroups. Let St_4 denote the Steinberg representation of $\mathrm{GSp}_4(k)$, as well as its Langlands parameter, which is the four dimensional irreducible representation of the $\mathrm{SL}_2(\mathbb{C})$ part of the Weil-Deligne group W_k'. By construction, St_4 is the alternating sum of certain representations induced from characters of $B, P, Q, \mathrm{GSp}_4(k)$ to $\mathrm{GSp}_4(k)$. Ignoring the trivial representation which does not contribute to the ψ-Bessel models for non-degenerate characters, Steinberg representation can be realized as the quotient of a representation induced from an irreducible representation of say $P = MN$ which is a twist of the Steinberg representation of M by a representation of $\mathrm{GSp}_4(k)$ which is induced from a character of Q.

Proposition 5.1. *Let K be a quadratic algebra, χ a character of $K^\times$ which is trivial on $k^\times$. Let ψ be a non-degenerate character of N, left invariant by $K^\times$ sitting inside M. Then the Steinberg representation of $\mathrm{GSp}_4(k)$ has Bessel model for χ if and only if χ is a non-trivial character of $K^\times$ in case K is a field, and for all characters of $K^\times$ if $K = k \oplus k$.*

Proof. The proposition is clear by combining propositions 2.1 and 2.4 in all cases except when $K = k \oplus k$, and χ is the trivial character. So the rest of the proof will be for this case only, which is subtler as it relies on an understanding of semi-simplicity of χ-Bessel models. (The proof below works for all characters of $K^\times$ in the case that $K = k \oplus k$.)

From the discussion above (and taking duals), it follows that the Steinberg representation of $\mathrm{GSp}_4(k)$ sits in an exact sequence of the form,

$$0 \to \mathrm{St}_4 \to Ps \to \pi \to 0$$

where Ps is a principal series representation of $\mathrm{GSp}_4(k)$ induced from the Siegel parabolic from an appropriate twist of the Steinberg representation of $M = \mathrm{GL}_2(k) \times k^\times$; and π is a representation of $\mathrm{GSp}_4(k)$ induced from a one dimensional representation of the Klingen parabolic (in fact π is a sub-representation of such a representation with quotient which is the one dimensional trivial representation of $\mathrm{GSp}_4(k)$, so does not affect the calculation of Bessel models). From an earlier observation that the discriminant δ_P is trivial on $K^\times$, it does not matter which twist of the Steinberg of $\mathrm{GL}_2(k)$ is used to construct the principal series Ps on $\mathrm{GSp}_4(k)$ above.

Taking the twisted Jacquet functor with respect to the character ψ of N, we get an exact sequence of T-modules where T is the diagonal split torus in $\mathrm{GL}_2(k)$,

$$0 \to \mathrm{St}_{4,\psi} \to Ps_\psi \to \pi_\psi \to 0.$$

From the calculation of the twisted Jacquet functor of principal series representation of $\mathrm{GSp}_4(k)$ induced from the Siegel parabolic, as well as that of a principal series representation of $\mathrm{GSp}_4(k)$ induced from a one dimensional representation of the Klingen parabolic done in earlier sections, we get an exact sequence of T-modules,

$$0 \to \mathrm{St}_{4,\psi} \to \mathrm{St}_2 \to \mathbb{C} \to 0 \tag{1}$$

where $\mathbb{C}$ is the one dimensional trivial representation of T, and St_2 denotes the Steinberg representation of $\mathrm{PGL}_2(k)$, now thought of as a T-module.

Since the Steinberg representation of $\mathrm{GL}_2(k)$ can be realized on the space of locally constant functions modulo constant functions on $\mathbb{P}^1(k)$, $f \to f(0) - f(\infty)$ is a T-equivariant map from St_2 to $\mathbb{C}$, giving rise to the following exact sequence to T-modules:

$$0 \to \mathcal{S}(k^\times) \to \mathrm{St}_2 \to \mathbb{C} \to 0. \tag{2}$$

As one knows that the Steinberg representation of $\mathrm{GL}_2(k)$ has a unique quotient on which T-operates trivially, the exact sequences (1) and (2) of T-modules must be the same, and therefore in particular $\mathrm{St}_{4,\psi}$ is isomorphic as a T-module to $\mathcal{S}(k^\times)$. This implies that St_4 has Bessel model for all characters of T which are trivial on the scalars. $\square$

We omit the simple check that this proposition proves Theorem 2 for the Steinberg representation, augmented by the following much simpler proposition.

Proposition 5.2. *The Steinberg representation of $\mathrm{GSp}_4^D(k)$ has Bessel model for a character χ of $K^\times$ if and only if K is a field, and χ is the trivial character of $K^\times$.*

Remark 5.3. We take the occasion here to emphasize that although the twisted Jacquet functor π_ψ is exact, it need not be a semi-simple representation of $K^\times$, unless K is a field in which case it is forced to be semi-simple as $K^\times/k^\times$ is compact.

6. Bessel model of the Weil representation

The aim of this section will be to calculate the twisted Jacquet functor of the Weil representation of a dual reductive pair (G_1, G_2) with respect to a character ψ of the unipotent radical N_2 of a maximal parabolic $P_2 = M_2N_2$ of the group G_2. We carry out the calculation of the twisted Jacquet functor only for the Siegel parabolic of a symplectic group, so $G_2 = \mathrm{Sp}(W)$. Recall that for any representation π of G_2, the twisted Jacquet functor π_ψ is the maximal quotient of π on which N_2 operates via ψ. If M_ψ denotes the maximal subgroup of M_2 which takes ψ to itself under the inner-conjugation action of M_2 on N_2, then π_ψ is a module for M_ψ, and therefore in the context of a dual reductive pair (G_1, G_2), for $G_1 \times M_\psi$.

We recall that in a famous work, Kudla calculated the standard Jacquet module of the Weil representation. We carry out the calculation of the twisted Jacquet functor only for the Siegel parabolic. Actually the simple calculation we perform in this section is known in the literature in both the local and global contexts, see e.g. [Ro1, Ra]. However, since we anyway will have to recall the notation and the results, we have preferred to give an independent co-ordinate free treatment which will be convenient for our purposes.

We now recall some elementary properties of the Weil representation for this purpose.

Let $W = W_1 \oplus W_1^\vee$ be a symplectic vector space over a local field k together with its natural symplectic pairing. Given a quadratic space $q : V \to k$, the Weil representation gives rise to a representation of $\mathrm{O}(V) \times \mathrm{Sp}(W)$ on $\mathcal{S}(V \otimes W_1^\vee)$. In this representation, elements of $S\mathrm{Hom}(W_1^\vee, W_1) = \{\phi \in \mathrm{Hom}(W_1^\vee, W_1) | \phi = \phi^\vee\} \cong \mathrm{Sym}^2 W_1$, which can be identified to the unipotent radical of the Siegel parabolic in $\mathrm{Sp}(W)$ stabilizing the isotropic subspace W_1, operates on $\mathcal{S}(V \otimes W_1^\vee)$ by

$$(n \cdot f)(x) = \psi((q \otimes q_n)x) f(x),$$

where $n \in S\mathrm{Hom}(W_1^\vee, W_1)$ gives rise to a quadratic form $q_n : W_1^\vee \to k$, which together with the quadratic form $q : V \to k$, gives rise to the quadratic form $q \otimes q_n : V \otimes W_1^\vee \to k$.

The Weil representation associated to the dual pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ is actually a representation of $\mathrm{G}[\mathrm{O}(V) \times \mathrm{Sp}(W)]$ where $\mathrm{G}[\mathrm{O}(V) \times \mathrm{Sp}(W)]$ is defined to be the group of pairs $(g_1, g_2) \in \mathrm{GO}(V) \times \mathrm{GSp}(W)$ such that the similitude factors for g_1 and g_2 are the same. We briefly recall this, referring to [H-K] for details on this.

The exact sequence

$$1 \to \mathrm{Sp}(W) \to \mathrm{G}[\mathrm{O}(V) \times \mathrm{Sp}(W)] \to \mathrm{GO}(V) \to 1,$$

has a natural splitting $\mathrm{GO}(V) \to \mathrm{G}[\mathrm{O}(V) \times \mathrm{Sp}(W)]$ depending on a complete polarization $W = W_1 \oplus W_1^\vee$ of the symplectic space W in which $g \in \mathrm{GO}(V)$ goes to $(g, \mu(g)) \in \mathrm{G}[\mathrm{O}(V) \times \mathrm{Sp}(W)]$ where $\mu(g)$ is the element of $\mathrm{GSp}(W)$ which acts by 1 on W_1 and by $\nu(g)$ on $W_1^\vee$ where $\nu(g)$ is the similitude factor of g. The Weil representation realized on $\mathcal{S}(V \otimes W_1^\vee)$ has the natural action of $\mathrm{GO}(V)$ operating as

$$L(h)\varphi(x) = |\nu(h)|^{-mn/4} \varphi(h^{-1}x),$$

where $m = \dim V$, $2n = \dim W$, and $\nu(h)$ is the similitude factor of h. The group $\mathrm{GL}(W_1)$ sits naturally inside $\mathrm{Sp}(W_1 \oplus W_1^\vee)$, and its action on $\mathcal{S}(V \otimes W_1^\vee)$ is given by

$$L(g)\varphi(x) = \chi_V(\det g) |\det g|^{m/2} \varphi(gx),$$

where χ_V is the quadratic character of $k^\times$ given in terms of the Hilbert symbol as $\chi_V(a) = (a, \mathrm{disc} V)$ with $\mathrm{disc} V$ the normalized discriminant of V.

For the element (g, λ) in $\mathrm{GSp}(W)$ with $g \in \mathrm{GL}(W_1)$, and $\lambda \in k^\times$, which is

$$(g, \lambda) = \begin{pmatrix} g & \\ & \lambda \cdot {}^t g^{-1} \end{pmatrix},$$

the action of $(g, \lambda) \times h \in \mathrm{G}[\mathrm{Sp}(W) \times \mathrm{O}(V)]$ becomes:

$$[(g, \lambda) \times h]\varphi(x) = \chi_V(\det g) |\det g|^{m/2} |\lambda|^{-mn/4} \varphi(h^{-1}gx). \qquad (*)$$

The inner-conjugation action of (g, λ) on $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, gives rise to the unipotent matrix with n replaced by $\lambda^{-1} g n {}^t g$. Therefore the stabilizer of a symmetric matrix n in $\mathrm{Hom}(W_1^\vee, W_1)$ consists of (g, λ) with $g n {}^t g = \lambda n$. Taking determinants, we have $(\det g)^2 = \lambda^n$. Therefore the equation $(*)$ for the action of $(g, \lambda) \times h \in \mathrm{G}[\mathrm{Sp}(W) \times \mathrm{O}(V)]$ with $g \in \mathrm{GO}(W_1)$ becomes:

$$[(g, \lambda) \times h]\varphi(x) = \chi_V(\det g) \varphi(h^{-1}gx). \qquad (**)$$

Assuming further that $n = \dim W_1$ is even, and the element $g \in \mathrm{GO}(W_1)$ in fact belongs to the connected component $\mathrm{GSO}(W_1)$ defined by $\det g = \lambda^{n/2}$, then

$\chi_V(\det g) = \chi_V(\lambda)^{n/2} = 1$, if λ is a similitude factor for V (as can be easily seen), simplifying the action $(**)$ to

$$[(g,\lambda)\times h]\varphi(x) = \varphi(h^{-1}gx). \qquad (***)$$

The Weil representation thus gives rise to a representation of the group $\mathrm{G}[\mathrm{O}(V)\times \mathrm{Sp}(W)]$; inducing this representation to $\mathrm{GO}(V)\times \mathrm{GSp}(W)$, we get, the 'big Weil representation', say Ω, of $\mathrm{GO}(V)\times \mathrm{GSp}(W)$. Given an irreducible representation π of $\mathrm{GO}(V)$, there exists a representation $\Theta(\pi)$ of $\mathrm{GSp}(W)$ of finite length, such that $\Theta(\pi)\otimes\pi$ is the maximal π-isotypic quotient of Ω. It is known that the representation $\Theta(\pi)$ of $\mathrm{GSp}(W)$ has a unique irreducible quotient $\theta(\pi)$. When one talks about the theta correspondence, one means the correspondence $\pi \to \theta(\pi)$; however, when one calculates Jacquet or twisted Jacquet functor of the Weil representation, it is invariably $\Theta(\pi)$ that one encounters. Thus most of the applications are restricted to the case when one can in fact prove that $\Theta(\pi) = \theta(\pi)$ which is the case for example when π is supercuspidal.

Lemma 6.1. *Let x be a vector in $V\otimes W_1^\vee$, considered as a homomorphism $x: W_1 \to V$, as well as the homomorphism on duals $x^\vee : V^\vee \to W_1^\vee$. Then for quadratic spaces $q_V : V \to k$, and $q_W : W_1^\vee \to k$, equivalently considered through homomorphisms $q_V : V \to V^\vee$, and $q_W : W_1^\vee \to W_1$, the trace of the map from W_1 to W_1 given as the compositum of maps,*

$$W_1 \xrightarrow{x} V \xrightarrow{q_V} V^\vee \xrightarrow{x^\vee} W_1^\vee \xrightarrow{q_W} W_1,$$

is the same as the value of the quadratic form $q_V\otimes q_W$ on the vector $x\in V\otimes W_1^\vee$, which is of course the same as the trace of the map from k to k, obtained as the compositum of maps:

$$k \xrightarrow{x} V\otimes W_1^\vee \xrightarrow{q_V\otimes q_W} V^\vee\otimes W_1 \xrightarrow{x^\vee} k.$$

Let $S\mathrm{Hom}[W_1^\vee, W_1]$ be the set of symmetric maps in $\mathrm{Hom}[W_1^\vee, W_1]$, i.e., $\phi \in \mathrm{Hom}[W_1^\vee, W_1]$ such that $\phi^\vee = \phi$. One can identify the dual of the k-vector space $S\mathrm{Hom}[W_1^\vee, W_1]$ to $S\mathrm{Hom}[W_1, W_1^\vee]$ via the natural pairing obtained by taking trace,

$$S\mathrm{Hom}[W_1, W_1^\vee]\times S\mathrm{Hom}[W_1^\vee, W_1] \to \mathrm{Hom}[W_1,W_1] \xrightarrow{tr} k.$$

Thus characters $\psi : N \to \mathbb{C}^\times$ can be identified to symmetric elements in $\mathrm{Hom}[W_1, W_1^\vee]$. As the map from W_1 to W_1 in the above lemma is the compositum of two maps, one from W_1 to $W_1^\vee$, and the other from $W_1^\vee$ to W_1, and that the first map is nothing but the restriction of the quadratic form on V to W_1 via the map $x : W_1 \to V$, the following corollary of the previous lemma is clear.

Corollary 6.2. *The twisted Jacquet functors of the Weil representation corresponding to the dual reductive pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ are nonzero exactly for the characters of the unipotent radical of the Siegel parabolic of $\mathrm{Sp}(W)$ which correspond to the 'restriction' of quadratic form on V to W_1 via a linear map $x : W_1 \to V$.*

We now note the following general lemma.

Lemma 6.3. *Let X be the k-rational points of an algebraic variety defined over a local field k. Let P be a locally compact totally disconnected group with $P = MN$*

for a normal subgroup N of P which we assume is a union of compact subgroups. Assume that P operates smoothly on $\mathcal{S}(X)$, and that the action of P restricted to M is given by an action of M on X. For a point $x \in X$, let $\ell_x : \mathcal{S}(X) \to \mathbb{C}$ be the linear functional given by $\ell_x(f) = f(x)$. Assume that for every point $x \in X$, N operates on ℓ_x by a character $\psi_x : N \to \mathbb{C}^\times$, i.e., $\ell_x(n \cdot f) = \psi_x(n)\ell_x(f)$ for all $n \in N$, and $f \in \mathcal{S}(X)$. Fix a character $\psi : N \to \mathbb{C}^\times$, and let M_ψ denote the subgroup of M which stabilizes the character ψ of N. The group M_ψ acts on the set of points $x \in X$ such that $\psi_x = \psi$. Denote this set of points in X by X_ψ which we assume to be closed in X. Then,

$$\mathcal{S}(X)_\psi \cong \mathcal{S}(X_\psi)$$

as M_ψ-modules.

Proof. We have an exact sequence of M_ψ-modules,

$$0 \to \mathcal{S}(X - X_\psi) \to \mathcal{S}(X) \to \mathcal{S}(X_\psi) \to 0.$$

Taking the ψ-twisted Jacquet functor is exact, and $\mathcal{S}(X - X_\psi)_\psi = 0$, so the assertion of the lemma follows. □

We apply this lemma to $X = V \otimes W_1^\vee$, but will need to twist the geometric action of $\mathrm{GL}(W_1)$ on $\mathcal{S}(V \otimes W_1^\vee)$ by $\chi_V(\det g)$ for an element $(h, g) \in \mathrm{O}(V) \times \mathrm{O}(W_1)$.

Corollary 6.4. *The twisted Jacquet functor of the Weil representation of the dual reductive pair $(\mathrm{O}(V), \mathrm{Sp}(W))$ for the character of the unipotent radical of the Siegel parabolic of $\mathrm{Sp}(W)$ which corresponds to a non-degenerate quadratic form on W_1, which we assume is obtained by restriction of the quadratic form on V via a linear map $x : W_1 \to V$ is the representation*

$$\chi_V(\det g) \otimes \mathrm{ind}_{\mathrm{O}(W_1^\perp) \times \Delta\mathrm{O}(W_1)}^{\mathrm{O}(V) \times \mathrm{O}(W_1)} \mathbb{C},$$

where $O(W_1^\perp)$ is the orthogonal group of the orthogonal complement of W_1 inside V, and $\Delta\mathrm{O}(W_1)$ represents the natural diagonal embedding of $\mathrm{O}(W_1)$ inside $\mathrm{O}(V) \times \mathrm{O}(W_1)$ as V contains W_1; the character $\chi_V(\det g)$ is for the element $(h, g) \in \mathrm{O}(V) \times \mathrm{O}(W_1)$.

Proof. Observe that $\mathrm{O}(V) \times \mathrm{O}(W_1)$ operates on the set of homomorphisms from W_1 to V, and in fact by Witt's theorem, this action is transitive on the set of homomorphisms from W_1 to V such that the quadratic form on V restricts to the quadratic form on W_1. The isotropy subgroup inside $\mathrm{O}(V) \times \mathrm{O}(W_1)$ of a fixed embedding of W_1 inside V is exactly $\mathrm{O}(W_1^\perp) \times \Delta\mathrm{O}(W_1)$, proving the claim. □

The previous analysis of twisted Jacquet functor in fact gives a representation space for $\mathrm{G}[\mathrm{O}(V) \times \mathrm{O}(W_1)]$ which we record as the following corollary, but before doing that let us note the following form of Witt's extension theorem for similitude groups.

Lemma 6.5. *Suppose W_1 is a nondegenerate subspace of a quadratic space V carrying the restricted quadratic form. Suppose ϕ belongs to $\mathrm{GO}(W_1)$ such that the similitude factor of ϕ arises as a similitude factor in $\mathrm{GO}(V)$. Then there is an element ϕ' in $\mathrm{GO}(V)$ taking W_1 into itself, and such that the restriction of ϕ' to W_1 is ϕ.*

Proof. Write $V = W_1 \oplus W_1^{\perp}$. Note that $\lambda \in k^{\times}$ is a similitude factor for the quadratic space V if and only if $V \cong \lambda \cdot V$. Since $\lambda \cdot V \cong \lambda \cdot W_1 \oplus \lambda \cdot W_1^{\perp}$, if λ is a similitude character for both V and W_1, we find that $W_1^{\perp} \cong \lambda \cdot W_1^{\perp}$, therefore the conclusion of the lemma. □

The group $\mathrm{G}[\mathrm{O}(V) \times \mathrm{O}(W_1)]$ operates on the set of embeddings of W_1 inside V which gives rise to a particular quadratic form on W_1. The action is transitive, and by Lemma 6.5, the stabilizer of a given embedding is $\mathrm{G}[\mathrm{O}(W_1^{\perp}) \times \Delta\mathrm{O}(W_1)]$. The subgroup $\mathrm{G}[\mathrm{O}(V) \times \mathrm{SO}(W_1)] = \mathrm{G}[\mathrm{O}(V) \times \mathrm{O}(W_1)] \bigcap [\mathrm{GO}(V) \times \mathrm{GSO}(W_1)]$ of $\mathrm{G}[\mathrm{O}(V) \times \mathrm{O}(W_1)]$ also operates transitively with stabilizer a given embedding being $\mathrm{G}[\mathrm{O}(W_1^{\perp}) \times \Delta\mathrm{SO}(W_1)]$. Thus using $(***)$, we obtain the following corollary.

Corollary 6.6. *The twisted Jacquet functor of the Weil representation of the dual reductive pair* $(\mathrm{O}(V), \mathrm{Sp}(W))$ *for the character of the unipotent radical of the Siegel parabolic of* $\mathrm{Sp}(W)$ *which corresponds to a non-degenerate quadratic form on* W_1*, which we assume is obtained by restriction of the quadratic form on* V *via a linear map* $x : W_1 \to V$ *is the representation*

$$\mathrm{ind}_{\mathrm{G}[\mathrm{O}(W_1^{\perp}) \times \Delta\mathrm{SO}(W_1)]}^{\mathrm{G}[\mathrm{O}(V) \times \mathrm{SO}(W_1)]} \mathbb{C},$$

where $\mathrm{O}(W_1^{\perp})$ *is the orthogonal group of the orthogonal complement of* W_1 *inside* V*, and* $\Delta\mathrm{O}(W_1)$ *represents the natural diagonal embedding of* $\mathrm{O}(W_1)$ *inside* $\mathrm{O}(V) \times \mathrm{O}(W_1)$ *as* V *contains* W_1*; the group* $\mathrm{G}[\mathrm{O}(W_1^{\perp}) \times \Delta\mathrm{SO}(W_1)]$ *is the subgroup of* $\mathrm{G}[\mathrm{O}(W_1^{\perp}) \times \mathrm{SO}(W_1) \times \mathrm{SO}(W_1)]$ *contained in* $\mathrm{G}[\mathrm{O}(V) \times \mathrm{SO}(W_1)]$ *consisting of the triples* $(g_1, g_2, g_3) \in \mathrm{GO}(W_1^{\perp}) \times \mathrm{GSO}(W_1) \times \mathrm{GSO}(W_1)]$ *with* $g_2 = g_3$ *(and the same similitude factors for* g_1, g_2, g_3*).*

The previous corollary together with the formalism of the Weil representation yields the following theorem as a simple consequence.

Theorem 5. *Let* π_1 *be an irreducible admissible representation of the group* $\mathrm{GSO}(V)$*. Assume that* $\pi_2 = \Theta(\pi_1)$ *is the theta lift of* π_1 *to* $\mathrm{GSp}(W)$*. Let* ψ *be a non-degenerate character of the unipotent radical* N *of the Siegel parabolic* $P = MN$ *of* $\mathrm{GSp}(W)$*. Assume that* ψ *corresponds to a quadratic form* q *on* W_1*, a maximal isotropic subspace of* W*. Then an irreducible representation* χ *of* $\mathrm{GSO}(W_1)$ *appears in* $\pi_{2,\psi}$ *as a quotient if and only if*

1. *(1)* (q, W_1) *can be embedded in the quadratic space* V*; let* $W_1^{\perp}$ *denote the orthogonal complement of* W_1 *sitting inside* V *through this embedding.*
2. *(2) The representation* $\chi^{\vee}$ *of* $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})]$ *appears as a quotient in the representation* π_1 *of* $\mathrm{GSO}(V)$ *restricted to* $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})]$*, where* $\chi^{\vee}$ *is obtained by pulling back the contragredient of* χ *under the natural map* $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})] \to \mathrm{GSO}(W_1)$*.*

Remark 6.7. It is a consequence of this theorem that if the representation $\chi^{\vee}$ of $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})]$ appears as a quotient in the representation π_1 of $\mathrm{GSO}(V)$ restricted to $G[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})]$, then $\pi_2 = \Theta(\pi_1)$ is nonzero. It is one of the standard ways by which one proves non-vanishing of local (or global) representations: by proving the non-vanishing of a particular Fourier coefficient; for example it proves that the theta lifting from $\mathrm{GSO}(4)$ to $\mathrm{GSp}(4)$ is always nonzero locally.

Remark 6.8. Theorem 5 roughly states that a representation π_1 of $\mathrm{GSO}(V)$ has a $\tilde{\chi}$-period for the subgroup $G[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})]$, where $\tilde{\chi}$ is obtained from a representation χ of $\mathrm{GSO}(W_1)$ extending trivially across $\mathrm{SO}(W_1^{\perp})$, if and only if $\pi_2 = \Theta(\pi_1)$, a representation of $\mathrm{GSp}(W_1 \oplus W_1^{\vee})$, has χ-Bessel model for the representation χ of $\mathrm{GSO}(W_1)$. This theorem has a certain symmetry in π_1 and π_2. However, we note an important asymmetry: one concludes from the theorem that as soon as a representation π_1 of $\mathrm{GSO}(V)$ has a nonzero $\tilde{\chi}$-period, $\Theta(\pi_1) \neq 0$; but it may happen that although a representation π_2 of $\mathrm{GSp}(W_1 \oplus W_1^{\vee})$ has a χ-Bessel model, $\Theta(\pi_2) = 0$.

Remark 6.9. The considerations of this section can be pictorially represented by the following diagram where $M_\psi = \mathrm{O}(W_1)$, and $V = W_1 + W_1^{\perp}$, and a vertical line between representations denotes the appearance of the representation on the smaller group at the lower end of the line in the representation of the larger group at the upper end.

$$\begin{tikzcd}
\pi_2 = \Theta(\pi_1) \text{ on } \mathrm{Sp}(W) \arrow[rr, dash] \arrow[d, dash] & & \pi_1 \quad \text{on} \quad \mathrm{O}(V) \arrow[d, dash] \\
\chi \cdot \psi \quad \text{on} \quad M_\psi N & & \chi \times 1 \text{ on } \mathrm{O}(W_1) \times \mathrm{O}(W_1^{\perp}).
\end{tikzcd}$$

7. Applications

To be able to use Theorem 5, we need to understand the embedding of $\mathrm{O}(W_1)$ inside $\mathrm{O}(V)$ more concretely. For application to Theorem 2, we need it in the case when V is a four dimensional quadratic space, and W_1 is a two dimensional subspace of it, and for applications to Theorem 4, we need it in the case when V is a 6 dimensional quadratic space, and W_1 is a two dimensional subspace of it.

We begin with the case of a four dimensional quadratic space V of discriminant 1, so that it can be identified to the norm form of a four dimensional central simple algebra, say D, over k. Assume that the two dimensional non-degenerate subspace W_1 of $V = D$ is the norm form on a two dimensional sub-algebra K of D. Write $D = K \oplus K \cdot j$ where j is an element of $D^\times$ which normalizes $K^\times$ with $j^2 = a \in k^\times$. The group $D^\times \times D^\times$ operates on D by $(d_1, d_2)X = d_1 X \bar{d}_2$, and gives an identification of $[D^\times \times D^\times]/\Delta k^\times$ with $\mathrm{GSO}(D)$. Observe that the map $\iota : (x, y) \to (xy, x\bar{y})$ from $K^\times \times K^\times$ to itself gives an isomorphism of $(K^\times \times K^\times)/\Delta k^\times$ onto the subgroup $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^{\perp})]$ of $\mathrm{GSO}(W_1) \times \mathrm{GSO}(W_1^{\perp})$ consisting of pairs of elements of $K^\times$ with the same similitude factors for the two components. Since $x[K \oplus Kj]\bar{y} = x\bar{y}K \oplus xyKj$, the following diagram allows one to identify $(K^\times \times K^\times)/\Delta k^\times$ inside $(D^\times \times D^\times)/\Delta k^\times$ as the subgroup $G[SO(K) \times SO(K)]$ inside $GSO(D) = GSO(K \oplus K)$

$$\begin{tikzcd} & {[K^\times \times K^\times]/\Delta(k^\times)} \arrow[dl, "\cong"'] \arrow[dr] & \\ \mathrm{G}[\mathrm{SO}(K)\times \mathrm{SO}(K)] & & (D^\times\times D^\times)/(\Delta k^\times) \end{tikzcd}$$

Therefore a representation $\pi_1 \boxtimes \pi_2$ of $D^\times \times D^\times$ contains the restriction of the character (χ_1, χ_2) of $K^\times \times K^\times$ to the subgroup $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^\perp)]$ if and only if $\chi_1\bar{\chi}_2$ appears in π_1 and $\chi_1\chi_2$ appears in π_2. Taking $\chi_2 = 1$, we get the following corollary to Theorem 5.

Corollary 7.1. *Let $\pi_1 \boxtimes \pi_2$ be an irreducible admissible representation of $[D^\times \times D^\times]/k^\times \cong \mathrm{GSO}(V)$ where $V = D$ is a quaternion algebra over k equipped with its reduced norm as the quadratic form. Let ψ be a character of the unipotent radical of the Siegel parabolic of $\mathrm{GSp}(W)$ which corresponds to the non-degenerate quadratic space $\mathbb{N} : K \to k$ where K is a quadratic sub-algebra of D. Then for the representation $\Theta(\pi_1 \boxtimes \pi_2)$ of $\mathrm{GSp}(W)$, the twisted Jacquet functor, $\Theta_\psi(\pi_1 \boxtimes \pi_2)$ of $\mathrm{GSp}(W)$, contains the representation $\chi : K^\times \to \mathbb{C}^\times$ if and only if χ appears in both π_1 and π_2. (In particular, K is a field if D is a division algebra.)*

Similarly, for the case of the rank one form $\mathrm{Sp}_4^D(k)$ of the symplectic group defined using the quaternion division algebra D, we get the following result.

Corollary 7.2. *Let $\pi_1 \boxtimes \pi_2$ be a representation of $[D^\times \times \mathrm{GL}_2(k)]/k^\times \cong \mathrm{GSO}_4^D(k)$ where D is a quaternion division algebra over k. Let ψ be a character of the unipotent radical of the Siegel parabolic of $\mathrm{GSp}_4^D(k)$ which corresponds to the non-degenerate quadratic space $\mathbb{N} : K \to k$ where K is a quadratic sub-algebra of D. Then for the representation $\Theta(\pi_1 \boxtimes \pi_2)$ of $\mathrm{GSp}_4^D(k)$, the twisted Jacquet functor, $\Theta_\psi(\pi_1 \boxtimes \pi_2)$ of $\mathrm{GSp}_4^D(k)$, contains the representation $\chi : K^\times \to \mathbb{C}^\times$ if and only if χ appears in both π_1 and π_2.*

We next consider Bessel model of representations of $\mathrm{GSp}_4(k)$ which are obtained as theta lift from $\mathrm{GSO}(V)$ where V is a quadratic space of dimension 4 with non-trivial discriminant, in which case we recall that

$$\mathrm{GSO}(V) \cong [\mathrm{GL}_2(E) \times k^\times]/\Delta(E^\times),$$

for E a quadratic extension of k.

Let K and L be two distinct quadratic extensions of k, and let E be the third quadratic extension of k contained in KL. Considering K and L together with their norm forms, we have two 2-dimensional quadratic spaces, and $K \oplus L$ is a four dimensional quadratic space. It can be seen that $\mathrm{GSO}(K \oplus L) \cong (\mathrm{GL}_2(E) \times k^\times)/\Delta E^\times$ where $\Delta E^\times \cong E^\times$ sits inside $\mathrm{GL}_2(E)$ as scalar matrices, and inside $k^\times$ via the inverse of the norm mapping.

The group $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(L)]$ is the subgroup of $K^\times \times L^\times$ consisting of pairs $(x_1, x_2) \in K^\times \times L^\times$ with the same norm to $k^\times$.

The mapping from $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(L)]$ to $(\mathrm{GL}_2(E) \times k^\times)/\Delta E^\times$ obtained as the composition,

$$\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(L)] \to \mathrm{GSO}(K \oplus L) \cong (\mathrm{GL}_2(E) \times k^\times)/\Delta E^\times$$

fits in the following diagram of maps where ϕ_E denotes the natural inclusion of $(KL)^\times$ into $\mathrm{GL}_2(E)$, and i, i_K, i_L are inclusions of $k^\times$ in $k^\times, K^\times, L^\times$ respectively, and $\mathbb{N}_K$ and $\mathbb{N}_L$ are norm mappings from $(KL)^\times$ to $K^\times$ and $L^\times$ respectively:

$$\begin{tikzcd} & {[(KL)^\times \times k^\times]/\Delta(E^\times)} \arrow[dl, "(i_K\mathbb{N}_K, i_L\mathbb{N}_L)"'] \arrow[dr, "(\phi_E, i)"] & \\ \mathrm{G}[\mathrm{SO}(K)\times \mathrm{SO}(L)] & & (\mathrm{GL}_2(E)\times k^\times)/(\Delta E^\times) \end{tikzcd}$$

As the arrow on the left can be checked to be an isomorphism, it follows from this diagram that to check that a character of $\mathrm{G}[\mathrm{SO}(K)\times\mathrm{SO}(L)]$ appears in the restriction of a representation of $\mathrm{GSO}(K\oplus L)$, it is equivalent to check that its restriction to $[(KL)^\times \times k^\times]/\Delta E^\times$ now thought of as a subgroup of $[\mathrm{GL}_2(E)\times k^\times]/\Delta(E^\times)$ appears in the corresponding representation of $[\mathrm{GL}_2(E)\times k^\times]/\Delta(E^\times)$. Therefore we obtain the following theorem.

Theorem 6. *Let π_1 be an irreducible admissible representation of $\mathrm{GSp}_4(k)$ obtained from the theta lift of a representation π of $\mathrm{GO}_4(k)$ such that the normalized discriminant algebra associated to the four dimensional quadratic space is a quadratic field extension E of k. Assume that in the identification of $\mathrm{GSO}_4(k)$ with $(\mathrm{GL}_2(E)\times k^\times)/(\Delta E^\times)$, the restriction of π (from $\mathrm{GO}_4(k)$ to $\mathrm{GSO}_4(k)$) corresponds to the representation $\pi_2\boxtimes\mu$ of $\mathrm{GL}_2(E)\times k^\times$. Let ψ be a non-degenerate character of N, where N is the unipotent radical of the Siegel parabolic $P=MN$ stabilizing a maximal isotropic subspace W_1 of the four dimensional symplectic space W, corresponding to a quadratic form q on W_1 which defines a quadratic field extension $K\neq E$. (The case $K=E$ is easier to analyze but we do not do it here.) Then a character χ of $K^\times$ such that $\chi|_{k^\times}$ is the central character of π_1, appears in $\pi_{1,\psi}$ if and only if the character $\chi\circ\mathbb{N}:(KE)^\times \xrightarrow{\mathbb{N}} K^\times \xrightarrow{\chi} \mathbb{C}^\times$ of $(KE)^\times$ appears in the restriction of π_2 to $(KE)^\times$ which by the theorem of Saito and Tunnell is the case if and only if*

$$\begin{aligned} \omega_{KE/E}(-1)\omega_{\pi_2}(-1) &= \epsilon(\pi_2\otimes \mathrm{ind}_{KE}^{E}\chi^{-1}|_{KE}) \\ &= \epsilon(\pi_2\otimes \mathrm{Res}_E[\mathrm{ind}_K^k(\chi^{-1})]) \\ &= \epsilon(\mathrm{ind}_E^k(\pi_2)\otimes \mathrm{ind}_K^k(\chi^{-1})). \end{aligned}$$

Noting the generality that $\omega_{KE/E}=\omega_{K/k}\circ\mathbb{N}_{KE\to K}$, we have $\omega_{KE/E}(-1)=1$, and that $\omega_{\pi_2}(-1)=1$ as π_2 is a representation of $\mathrm{GL}_2(E)$ which extends to a representation of $(\mathrm{GL}_2(E)\times k^\times)/(\Delta E^\times)$, its central character restricted to E^1 is trivial, we get that

$$\epsilon(\mathrm{ind}_E^k(\pi_2)\otimes \mathrm{ind}_K^k(\chi^{-1}))=1$$

if and only if the character χ appears in the Bessel model of π as required by Theorem 2.

Remark 7.3. There is a form of this theorem for the rank 1 form $\mathrm{GSp}_4^D(k)$ of $\mathrm{GSp}_4(k)$ too in which one would be considering theta lifting from an orthogonal group in 4 variables defined using D and a skew-hermitian matrix in $\mathrm{GL}_2(D)$ whose discriminant in $k^\times/k^{\times 2}$ defines a quadratic extension E of k. In this case, the orthogonal similitude group turns out to be $(D_E^\times \times k^\times)/E^\times$ with D_E the unique quaternion division algebra over E. A similar analysis as done in the previous theorem confirms the relevant parts of Theorem 2 for such representations of $\mathrm{GSp}_4^D(k)$. We discuss symplectic and orthogonal groups arising out of hermitian and skew-hermitian forms over D, and the calculation of Bessel models in this context, in some detail in section 9.

8. A seesaw argument

In the previous sections we have used theta correspondence between $\mathrm{GSp}_4(k)$ and $\mathrm{GO}_4(k)$ to calculate Bessel models of certain representations of $\mathrm{GSp}_4(k)$. This method works well for all representations of $\mathrm{GSp}_4(k)$ which arise as theta lift from $\mathrm{GO}_4(k)$. Thus, it misses out on representations of $\mathrm{GSp}_4(k)$ which do not arise as theta lift from $\mathrm{GO}_4(k)$. Among the missed representations are those representations of $\mathrm{GSp}_4(k)$ with Langlands parameter of the form $\sigma_\pi \otimes \mathrm{St}_2$ for an irreducible representation σ_π of W_k which has a non-trivial self-twist (so the parameter σ_π takes values in $\mathrm{GO}_2(\mathbb{C})$). In this section, we will study Bessel model of the representations of $\mathrm{GSp}_4(k)$ whose Langlands parameter is of the form $\sigma_\pi \otimes \mathrm{St}_2$, by going over to $\mathrm{GSO}_6(k)$ (a group closely related to $\mathrm{GL}_4(k)$) via theta correspondence, where the representation obtained is what is called the generalized Steinberg representation. The question of χ-Bessel model on $\mathrm{GSp}_4(k)$ for a character χ of $K^\times$ becomes one of linear period on $\mathrm{GL}_4(k)$ for the corresponding character χ of the subgroup $\mathrm{GL}_2(K)$ of $\mathrm{GL}_4(k)$. This question on $\mathrm{GL}_4(k)$ also seems intractable, but what allows us to handle this case is the fact that corresponding to the generalized Steinberg, there is the companion Speh module on $\mathrm{GL}_4(k)$, which arises by theta lifting from $\mathrm{GL}_2(k)$, and a seesaw argument can be provided for $\mathrm{GL}_2(K)$-periods of the Speh module, which then implies the desired result about $\mathrm{GL}_2(K)$-period of the generalized Steinberg on $\mathrm{GL}_4(k)$. In fact, there is an extra twist to the argument. We use the dual pair $(\mathrm{GL}_2(k), \mathrm{GO}_6(k))$ to calculate $\mathrm{GL}_2(K)$-period for a representation of $\mathrm{GL}_4(k)$; but when we use the pair $(\mathrm{GSp}_4(k), \mathrm{GO}_6(k))$, we do not use the representation of $\mathrm{GO}_6(k)$ encountered for the pair $(\mathrm{GL}_2(k), \mathrm{GO}_6(k))$ but another one whose restriction to $\mathrm{GL}_4(k) \subset [\mathrm{GL}_4(k) \times k^\times]/\Delta(k^\times) = \mathrm{GSO}_6(k)$ is the same.

The section uses many details about theta correspondence which we borrow either from [G-T1], or from [Ro1], or directly from conversations with W-T Gan.

We begin by noting the following lemma about theta lifting between $\mathrm{SO}(4)$ and $\mathrm{GL}(2)$, cf. [Ro1]. (Only the part of the lemma asserting that a certain theta lift from $\mathrm{SO}(4)$ to $\mathrm{GL}(2)$ is one dimensional is what is used in the sequel; however, we have preferred to state the more complete result.)

Lemma 8.1. *For a character λ of $k^\times$, let*

$$\begin{aligned} \pi_1(\lambda) &= \lambda(\mathbb{N} \circ \det) \boxtimes \lambda^2, \\ \pi_2(\lambda) &= \lambda(\mathbb{N} \circ \det) \boxtimes \omega_K \lambda^2, \end{aligned}$$

be one dimensional representations of $\mathrm{GSO}(3,1) \cong [\mathrm{GL}_2(K) \times k^\times]/\Delta(K^\times)$ which are invariant under the action of $\mathrm{GO}(3,1)$. For the representations, $\pi_1(\lambda)$, $\pi_2(\lambda)$, exactly one extension to $\mathrm{GO}(3,1)$ participates in the theta correspondence with $\mathrm{GL}_2(k)^+$, the *subgroup of* $\mathrm{GL}_2(k)$ *with determinant in* $\mathbb{N}K^\times$. *One has,*

$$\begin{aligned} \Theta(\pi_1(\lambda)) = \theta(\pi_1(\lambda)) &= \mathrm{Ind}(\omega_K|\cdot|^{1/2}\lambda, |\cdot|^{-1/2}\lambda), \\ \Theta(\pi_2(\lambda)) = \theta(\pi_2(\lambda)) &= \lambda \circ \det|_{\mathrm{GL}_2^+(k)}, \end{aligned}$$

where $\mathrm{Ind}(\omega_K|\cdot|^{1/2}\lambda, |\cdot|^{-1/2}\lambda)$ *denotes the restriction of the corresponding principal series representation of* $\mathrm{GL}_2(k)$ *to* $\mathrm{GL}_2(k)^+$, *which we note is irreducible.*

We now prove the following theorem.

Theorem 7. *Let* π *be a supercuspidal representation of* $\mathrm{GL}_2(k)$, *with central character* ω_π, *which has a non-trivial self-twist by a quadratic character* ω_K *associated to a quadratic extension* K *of* k. *Let* $\mathrm{Sp}_2(\pi)$ *be the associated Speh module of* $\mathrm{GL}_4(k)$. *Let* $\chi : K^\times \to \mathbb{C}^\times$ *be a character such that* $\chi|_{k^\times} = \omega_\pi \cdot \omega_K$. *By abuse of notation, let* χ *also denote the character* $\chi \circ \det$ *of* $\mathrm{GL}_2(K)$ *given by* $\mathrm{GL}_2(K) \stackrel{\det}{\to} K^\times \stackrel{\chi}{\to} \mathbb{C}^\times$. *Then,*

$$\mathrm{Hom}_{\mathrm{GL}_2(K)}[\mathrm{Sp}_2(\pi), \chi] \neq 0, \text{ if and only if } \pi \cong \pi_\chi,$$

where π_χ *is the monomial representation of* $\mathrm{GL}_2(k)$ *arising from the character* χ *of* $K^\times$.

Proof. For K^-, the vector space K on which the quadratic form (the normform) is scaled by -1, let $V = K \oplus H \oplus K^-$, with H the hyperbolic plane, denote the 6 dimensional split quadratic space. The embedding $G[\mathrm{SO}(K+H) \times \mathrm{SO}(K^-)] \hookrightarrow \mathrm{GSO}(V)$ will be abbreviated to $G[\mathrm{SO}_4(k) \times \mathrm{SO}_2(k)] \hookrightarrow \mathrm{GSO}_6(k)$ in what follows.

The proof of the theorem will be based on the following seesaw diagram:

$$\begin{tikzcd} \mathrm{G}^+[\mathrm{SL}_2(k) \times \mathrm{SL}_2(k)] \arrow[d, dash] \arrow[dr, dash] & \mathrm{GSO}_6^+(k) \arrow[d, dash] \arrow[dl, dash] \\ \mathrm{GL}_2^+(k) & \mathrm{G}[\mathrm{SO}_4(k) \times \mathrm{SO}_2(k)], \end{tikzcd}$$

where G^+ denotes various similitude groups with similitude factor in $\mathbb{N}K^\times$.

In this diagram, we take the representation π of $\mathrm{GL}_2(k)$ (restricted to $\mathrm{GL}_2^+(k)$) on the lower left corner, whose theta lift $\Theta(\pi)$ to $\mathrm{GSO}_6^+(k)$ is, by Theorem 8.11 of [G-T1], the restriction to $\mathrm{GSO}_6^+(k)$, of the representation $\Theta(\pi) = \theta(\pi) = \mathrm{Sp}_2(\pi) \boxtimes \omega_\pi$ of $\mathrm{GSO}_6(k)$ under the identification

$$\mathrm{GSO}_6(k) = [\mathrm{GL}_4(k) \times k^\times]/\{(z, z^{-2}) : z \in k^\times\}.$$

Since $\mathrm{G}[\mathrm{SO}_4(k) \times \mathrm{SO}_2(k)]$ is a subgroup of $\mathrm{GSO}_4(k) \times \mathrm{GSO}_2(k)$, one can construct a representation of $\mathrm{G}[\mathrm{SO}_4(k) \times \mathrm{SO}_2(k)]$ by restricting one of $\mathrm{GSO}_4(k) \times \mathrm{GSO}_2(k)$ which we take to be $(\chi(\mathbb{N} \circ \det), \omega_K\chi^2) \times \chi^{-1}$ under the identification $\mathrm{GSO}_4(k) \cong [\mathrm{GL}_2(K) \times k^\times]/\Delta(K^\times)$; this is the representation on the right hand lower corner of the diagram.

The map,

$$\begin{array}{ccc} \dfrac{[\mathrm{GL}_2(K)\times k^\times]}{\Delta(k^\times)} & \longrightarrow & \dfrac{[\mathrm{GL}_2(K)\times k^\times]}{\Delta(K^\times)}\times K^\times = \mathrm{GSO}_4(k)\times\mathrm{GSO}_2(k) \\ (X,a) & \longrightarrow & ((X,a), a\det X), \end{array}$$

is an isomorphism onto the subgroup $\mathrm{G}[\mathrm{SO}_4(k)\times\mathrm{SO}_2(k)]$ of $\mathrm{GSO}_4(k)\times\mathrm{GSO}_2(k)$. Using this isomorphism

$$[\mathrm{GL}_2(K)\times k^\times]/\{(z,z^{-2}) : z\in k^\times\} \cong \mathrm{G}[\mathrm{SO}_4(k)\times\mathrm{SO}_2(k)]$$

the character $(\chi(\mathbb{N}\circ\det), \omega_K\chi^2)\times\chi^{-1}$ of $\mathrm{GSO}_4(k)\times\mathrm{GSO}_2(k)$ (restricted to $\mathrm{G}[\mathrm{SO}_4(k)\times\mathrm{SO}_2(k)]$) becomes the character $(\chi^\sigma\circ\det, \chi|_{k^\times}\cdot\omega_K)$ of $[\mathrm{GL}_2(K)\times k^\times]/\{(z,z^{-2}) : z\in k^\times\}$, where $\chi^\sigma = \chi(\mathbb{N}x)\chi^{-1}(x)$.

The embedding of $\mathrm{G}[\mathrm{SO}_4(k)\times\mathrm{SO}_2(k)]$ in $\mathrm{GSO}_6(k)$ can be identified to the natural embedding

$$[\mathrm{GL}_2(K)\times k^\times]/\{(z,z^{-2}) : z\in k^\times\} \hookrightarrow [\mathrm{GL}_4(k)\times k^\times]/\{(z,z^{-2}) : z\in k^\times\}.$$

For the character $(\chi(\mathbb{N}\circ\det), \omega_K\chi^2)\times\chi^{-1}$ of $\mathrm{G}[\mathrm{SO}_4(k)\times\mathrm{SO}_2(k)]$ to appear in the representation $\Theta(\pi) = \mathrm{Sp}_2(\pi)\boxtimes\omega_\pi$ of $\mathrm{GSO}_6(k)$, it is necessary that $\omega_\pi = \chi|_{k^\times}\omega_K$, which we assume is the case.

Let the theta lift of the character χ^{-1} of $\mathrm{GSO}_2(k) = K^\times$ to $\mathrm{GL}_2^+(k)$ be $(\pi_\chi^\vee)^+$, and let $\pi_\chi^\vee = \mathrm{ind}_{\mathrm{GL}_2^+(k)}^{\mathrm{GL}_2(k)}(\pi_\chi^\vee)^+$, an irreducible representation of $\mathrm{GL}_2(k)$.

By Lemma 8.1, the theta lift of the character $(\chi(\mathbb{N}\circ\det), \omega_K\chi^2)\times\chi^{-1}$ of $\mathrm{G}[\mathrm{SO}_4(k)\times\mathrm{SO}_2(k)]$ to $\mathrm{G}[\mathrm{SL}_2(k)\times\mathrm{SL}_2(k)]$ is the restriction of the representation $\chi|_{k^\times}\circ\det\boxtimes(\pi_\chi^\vee)^+$ of $\mathrm{GL}_2^+(k)\times\mathrm{GL}_2^+(k)$. The restriction of the representation $\chi|_{k^\times}\circ\det\boxtimes(\pi_\chi^\vee)^+$ of $\mathrm{GL}_2^+(k)\times\mathrm{GL}_2^+(k)$ to the diagonal $\mathrm{GL}_2^+(k)$ is a component of π_χ that we denote by π_χ^+. The proof of the theorem now follows by the seesaw identity after using the following form of the Frobenius reciprocity:

$$\mathrm{Hom}_{\mathrm{GL}_2(k)}[\pi_\chi, \pi] = \mathrm{Hom}_{\mathrm{GL}_2^+(k)}[\pi_\chi^+, \pi].$$

□

We will use the previous theorem to deduce a corollary about Bessel models. But before we can do that, we must note the following result about theta correspondence between $\mathrm{GSp}_4(k)$ and $\mathrm{GSO}_6(k)$ which is due to [G-T1], Theorem 8.4. (Only the part of the lemma asserting that a certain theta lift from $\mathrm{SO}(6)$ to $\mathrm{GSp}(4)$ is the non-generic component of a reducible principal series of $\mathrm{GSp}(4)$ is what is used in the sequel; however, we have preferred to state the more complete result.)

Lemma 8.2. *Let π be a supercuspidal representation of $\mathrm{GL}_2(k)$ which has a non-trivial self-twist by a quadratic character ω_K. Let $\mathrm{Sp}_2(\pi)$ be the associated Speh module, and $\mathrm{St}_2(\pi)$ the generalized Steinberg representation of $\mathrm{GL}_4(k)$. For the representations $\mathrm{Sp}_2(\pi)\boxtimes\omega_K\omega_\pi$ and $\mathrm{St}_2(\pi)\boxtimes\omega_K\omega_\pi$ of $\mathrm{GSO}(6) = [\mathrm{GL}_4(k)\times k^\times]/\Delta(k^\times)$, we have $\Theta(\mathrm{Sp}_2(\pi)\boxtimes\omega_K\omega_\pi) = \theta(\mathrm{Sp}_2(\pi)\boxtimes\omega_K\omega_\pi)$, the non-generic component of the reducible principal series representation of $\mathrm{GSp}_4(k)$ induced from the Klingen parabolic $Q = MN$ with $M = k^\times\times\mathrm{GL}_2(k)$, the representation $\omega_K|\cdot|\boxtimes|\cdot|^{-1/2}\pi$, and $\Theta(\mathrm{St}_2(\pi)\boxtimes\omega_K\omega_\pi) = \theta(\mathrm{St}_2(\pi)\boxtimes\omega_K\omega_\pi)$, the generic component of the same principal* *series. Further,* $\Theta(\mathrm{Sp}_2(\pi) \boxtimes \omega_\pi) = \theta(\mathrm{Sp}_2(\pi) \boxtimes \omega_\pi)$*, the irreducible principal series representation* $|\cdot| \boxtimes |\cdot|^{-1/2}\pi$ *.*

The next corollary is a consequence of Theorem 7, combined with Lemma 8.2, and Theorem 5 according to which the existence of a χ-invariant linear form for the subgroup $\mathrm{GL}_2(K)$ of $\mathrm{GL}_4(k)$ is the same as the existence of χ-Bessel model for the representation of $\mathrm{GSp}_4(k)$ which is the theta lift of the representation $\pi \boxtimes \mu$ of $\mathrm{GSO}_6(k) = [\mathrm{GL}_4(k) \times k^\times]/\Delta(k^\times)$ in which $\mu = \chi|_{k^\times}$ as we discuss in greater detail in section 11.

Corollary 8.3. *Let* π *be an irreducible supercuspidal representation of* $\mathrm{GL}_2(k)$ *with central character* ω_π *which has a nontrivial self-twist by a quadratic character* ω_K *associated to a quadratic extension* K *of* k*. Let* $\mathrm{Sp}_2(\pi)$ *be the irreducible non-generic component of the principal series representation of* $\mathrm{GSp}_4(k)$ *induced from the Klingen parabolic with Levi* $k^\times \times \mathrm{GL}_2(k)$ *the representation* $\omega_K \cdot |\cdot| \boxtimes |\cdot|^{-1/2}\pi$*. The irreducible representation* $\mathrm{Sp}_2(\pi)$ *of* $\mathrm{GSp}_4(k)$ *with Langlands parameter* $\sigma_\pi \otimes (|\cdot|^{1/2} \oplus |\cdot|^{-1/2})$ *has Bessel models for exactly those characters* χ *of* $K^\times$ *for which* $\pi_\chi \cong \pi$*.*

Since we know that the principal series representation of $\mathrm{GSp}_4(k)$ induced from a representation $\lambda \boxtimes \pi$ of $k^\times \times \mathrm{GL}_2(k)$, a Levi subgroup of the Klingen parabolic, has χ-Bessel model for every character χ of $K^\times$, which is unique, we know that when such a principal series has two irreducible sub-quotients, the two sub-quotients have Bessel models for complementary characters. We thus obtain the following corollary.

Corollary 8.4. *The generic representation of* $\mathrm{GSp}_4(k)$ *with parameter* $\sigma_\pi \otimes \mathrm{St}_2$ *has a Bessel model for a character* χ *of* $K^\times$ *if and only if*

$$\sigma_\pi \ncong \mathrm{Ind}_K^k \chi.$$

This is exactly the conclusion required by Theorem 2 for representations of $\mathrm{GSp}_4(k)$ with Langlands parameter $\sigma_\pi \otimes \mathrm{St}_2$ as discussed in 4.

Remark 8.5. Representations of $\mathrm{GSp}_4(k)$ with Langlands parameter $\sigma_\pi \otimes \mathrm{St}_2$ have Bessel models for all characters of $K^\times$ except the two characters χ for which $\sigma_\pi \cong \mathrm{Ind}_K^k \chi$. Theorem 2 in this case requires that these two missing characters appear in the Bessel model of the corresponding representation of the rank 1 form $\mathrm{GSp}_4^D(k)$ of $\mathrm{GSp}_4(k)$. Indeed, considerations of this section will prove this too for which instead of the 6 dimensional split quadratic space over k, we will use the unique anisotropic skew-hermitian space of dimension 3 over the quaternion division algebra (see next section for a description of this), and the representation of the isometry group coming from theta correspondence with the isometry group of the hermitian space of dimension 1 (the similitude group being $D^\times$). We leave the details to the interested reader.

9. Dual pairs involving division algebras

In this section we briefly recall the formalism of dual reductive pairs which involve quaternion division algebra; the final goal of this section will be to state the analogue of theorem 5 in this context.

Let D be a quaternion division algebra with its canonical involution $x \to \bar{x}$. Using this involution, right D-modules can be identified to left D-modules.

Let V be a right D-module, and $H : V \times V \to D$ a ϵ-hermitian form on V which is D-linear in the second variable, so that

(1) $H(v_1 d_1, v_2 d_2) = \bar{d}_1 H(v_1, v_2) d_2$.
(2) $\overline{H(v_1, v_2)} = \epsilon H(v_2, v_1)$. (This forces ϵ to be ± 1.)

If $\epsilon = 1$ (resp., $\epsilon = -1$), an ϵ-hermitian form is called hermitian (resp., skew-hermitian).

Let V_1 be a right D-module together with a ϵ_1-hermitian form linear in the second variable, and V_2 a left D-module together with a ϵ_2-hermitian form H_2 which is linear in the first variable. Then $V_1 \otimes_D V_2$ is a vector space over k together with a natural bilinear form $H = H_1 \otimes H_2$ given by

$$H(v_1 \otimes v_2, w_1 \otimes w_2) = \mathrm{tr}_{D/k}(H_1(v_1, w_1)\overline{H_2(v_2, w_2)}).$$

If $\epsilon_1 \epsilon_2 = -1$, as will always be the case in what follows, H will be a symplectic form on $V_1 \otimes_D V_2$. In this case, the isometry group G_1 of (V_1, H_1) to be denoted by $\mathrm{U}(V_1)$, and the isometry group G_2 of (V_2, H_2) to be denoted by $\mathrm{U}(V_2)$, form a dual reductive pair inside $\mathrm{Sp}(V_1 \otimes_D V_2)$. We let $\mathrm{GU}(V_1)$ and $\mathrm{GU}(V_2)$ denote the corresponding similitude groups.

It is known that to get a form of an orthogonal group, we need to take a skew-hermitian form, and that to get a form of the symplectic group, we need to take a hermitian form. Over a non-archimedean local field k, a non-degenerate hermitian form is uniquely determined by its dimension, whereas a non-degenerate skew-hermitian form is uniquely determined by its dimension and its discriminant which is an element of $k^\times / k^{\times 2}$. (It is curious that over $\mathbb{R}$, these assertions are interchanged: a skew-hermitian form is unique, whereas a hermitian form is determined by its signature.)

As an example of interest for our work, for $a \in D^\times$, let $D(a)$ denote the one dimensional right D-module which is D itself together with the form $H(d_1, d_2) = \bar{d}_1 a d_2$. This form is skew-hermitian if $a + \bar{a} = 0$, and hermitian if $a = \bar{a}$. Assuming a is such that $a + \bar{a} = 0$, it can be seen that $\mathrm{U}(D(a)) = K^1$, and $\mathrm{GU}(D(a)) = K^\times$ where K is the quadratic extension of k generated by a; we note in particular that $\mathrm{U}(D(a))$ is a form of $\mathrm{SO}(2)$, and not of $\mathrm{O}(2)$.

The following two examples play a role in this paper. (See for example the papers of T. Tsukamoto as well as that of I. Satake in J. Math. Soc. Japan, vol. 13, 1961, pages 387-400, and pages 401-409 for proofs.)

Example 9.1. The orthogonal group defined by the skew-hermitian form

$$\begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix},$$

for $a, b \in k^\times \setminus k^{\times 2}$ defines an orthogonal group in four variables which is,

(1) $\mathrm{GSO}_4^D(k) \cong [D^\times \times \mathrm{GL}_2(k)]/\Delta k^\times$ if $ab \in k^{\times 2}$; here $\Delta k^\times = k^\times$ is embedded in $D^\times \times \mathrm{GL}_2(k)$ as (a, a^{-1}).
(2) $\mathrm{GSO}_4^D(k) \cong [D_E^\times \times k^\times]/\Delta E^\times$ if $ab \notin k^{\times 2}$ and E is the quadratic extension of k given by $E = k(\sqrt{ab})$, and D_E is the unique quaternion division algebra over E. Here the mapping from $E^\times$ to $k^\times$ is the inverse of the norm mapping.

Example 9.2. The skew-hermitian form

$$\begin{pmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & \sqrt{ab} \end{pmatrix},$$

for $a, b, ab \in k^\times \setminus k^{\times 2}$, is, up to isomorphism, the *unique* an-isotropic skew-hermitian form in 3 variables over D. (Notice that elements such as $\sqrt{a}$ in $D^\times$ are well defined only up to conjugacy; but the skew-hermitian form above is, up to isomorphism, independent of these choices in $\sqrt{a}, \sqrt{b}, \sqrt{ab}$.) The corresponding orthogonal group $\mathrm{GSO}_6^D(k)$ is isomorphic to $[\mathcal{D}^\times \times k^\times]/\Delta k^\times$ where $\mathcal{D}$ is the unique division algebra over k with invariant $1/4$, and where $\Delta k^\times = k^\times$ is naturally included in $\mathcal{D}^\times$, and the map from $k^\times$ to $k^\times$ is $x \to x^{-2}$. (If $\mathcal{D}'$ is the division algebra with invariant $3/4$, then $\mathcal{D}'^\times \cong \mathcal{D}^\times$.)

Assume that H_1 is a skew-hermitian form on V_1, and H_2 is a hermitian form on V_2. Let $V_2 = W_2 \oplus W_2^\vee$ be a complete polarization of V_2. The Weil representation of $\mathrm{Sp}(V_1 \otimes_D V_2)$ is realized on the Schwartz space of functions on $V_1 \otimes_D W_2^\vee$ on which $\mathrm{U}(V_1)$ acts in the natural way. The polarization $V_2 = W_2 \oplus W_2^\vee$ gives rise to the parabolic P in $\mathrm{U}(V_2)$ stabilizing the subspace W_2 with $\mathrm{GL}(W_2)$ as the Levi subgroup, and the additive group of skew-hermitian forms on $W_2^\vee$ as N. Thus the character group of N can be identified to the additive group of skew-hermitian forms on W_2.

With these preliminaries, we state the analogue of Theorem 5 in this context; application of this result to theta lifting between $\mathrm{GSp}_4^D(k)$, and $\mathrm{GSO}_4^D(k)$ will not be explicitly stated.

Theorem 8. *Let π_1 be an irreducible admissible representation of $\mathrm{GU}(V_1)$, and π_2 that of $\mathrm{GU}(V_2)$. Assume that $\pi_2 = \Theta(\pi_1)$ is the theta lift of π_1 to $\mathrm{GU}(V_2)$. Let ψ be a non-degenerate character of the unipotent radical N of the Siegel parabolic $P = MN$ of $\mathrm{GU}(V_2)$ stabilizing a maximal isotropic subspace W_2 of V_2. Assume that ψ corresponds to a skew-hermitian form H on W_2. Then an irreducible representation χ of $\mathrm{GU}(W_2)$ appears in $\pi_{2,\psi}$ as a quotient if and only if*

(1) (H, W_2) can be embedded in the skew hermitian space V_1; let $W_2^\perp$ denote the orthogonal complement of W_2 sitting inside V_1 through this embedding.

(2) The representation $\chi^\vee$ of $\mathrm{G}[\mathrm{U}(W_2) \times \mathrm{U}(W_2^\perp)]$ appears as a quotient in the representation π_1 of $\mathrm{GU}(V_1)$ restricted to $\mathrm{G}[\mathrm{U}(W_2) \times \mathrm{U}(W_2^\perp)]$, where $\chi^\vee$ is obtained by pulling back the contragredient of χ under the natural map $\mathrm{G}[\mathrm{U}(W_2) \times \mathrm{U}(W_2^\perp)] \to \mathrm{GU}(W_2)$.

10. Concluding the proof of Theorem 2

We begin by observing that from what is called the *Standard modules conjecture*, which is a theorem for $\mathrm{GSp}_4(k)$, a generic representation cannot be a proper Langlands quotient, i.e., either it is already tempered (up to a twist), or it is a full induced representation.

For the full induced representation, analysis of principal series representations gives complete information about Bessel models, and if the principal series is irreducible, proves Theorem 2 in these cases.

If the representation is tempered but not discrete series, then the sum of the representations in its L-packet is obtained by inducing a unitary discrete series representation of a parabolic subgroup of $\mathrm{GSp}_4(k)$. This unitary principal series is irreducible except if the parabolic is the Klingen parabolic, and the representation is $1 \rtimes \pi$ for a discrete series representation π of $\mathrm{GL}_2(k)$. This principal series has two irreducible components which arise as theta lifts from compact orthogonal group $\mathrm{O}(4)$, and split orthogonal group $\mathrm{O}(2,2)$, for which methods of theta correspondence enable one to calculate Bessel models.

All L-packets of size > 1 for $\mathrm{GSp}_4(k)$, or in odd residue characteristic, all L-packets containing a supercuspidal representation arise as theta lift from an orthogonal group of a quadratic space of dimension 4, for which methods of theta correspondence give complete information about Bessel models; we indicate the calculation of necessary epsilon factors below.

It remains to deal with discrete series representations of $\mathrm{GSp}_4(k)$ which are not supercuspidal, and which is an L-packet by itself. There are two class of such representations:

(1) Steinberg, up to a twist.
(2) Representations of $\mathrm{GSp}_4(k)$ with parameter of the form $\sigma \otimes St_2$ where σ is a 2-dimensional irreducible monomial representation of W_k, and St_2 is the 2-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$.

Both these class of representations have been individually handled in sections 5, and 7, completing the proof of Theorem 2 in these cases.

Following Gan and Takeda in [G-T2], we now recall the Langlands parameter of representations of $\mathrm{GSp}_4(k)$ arising from theta correspondence with representations of $\mathrm{GO}_4(k)$, and then do the necessary epsilon factor calculation to verify Theorem 2 from results proved in the previous sections for such representations.

As recalled in the introduction, for a four dimensional quadratic space V, $\mathrm{GSO}(V)$ has the structure of one of the following groups:

(1) $\mathrm{GSO}(V^s) \cong [\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)]/\Delta k^\times$.
(2) $\mathrm{GSO}(V^a) \cong [D^\times \times D^\times]/\Delta k^\times$.
(3) $\mathrm{GSO}(V^d) \cong [\mathrm{GL}_2(E) \times k^\times]/\Delta E^\times$,

where $\Delta k^\times = k^\times$ sits as (t, t^{-1}), and $\Delta E^\times = E^\times$ sits inside $\mathrm{GL}_2(E) \times k^\times$ via its natural embedding in $\mathrm{GL}_2(E)$, and in $k^\times$ by the inverse of the norm mapping.

In cases (1) and (2), an irreducible representation of $\mathrm{GSO}(V)$ is a tensor product $\tau_1 \boxtimes \tau_2$ of two irreducible representations τ_1 and τ_2 which are both irreducible representations of $\mathrm{GL}_2(k)$ in case (1), and of $D^\times$ in case (2), and have the same central characters, and with Langlands parameters σ_1 and σ_2. The Langlands parameter of the representation of $\mathrm{GSp}(4)$ arising from theta correspondence from an irreducible representation of $\mathrm{GO}(V)$ which restricted to $\mathrm{GSO}(V)$ is $\tau_1 \boxtimes \tau_2$ in cases (1) and (2) is,

$$\sigma_1 \oplus \sigma_2.$$

In case (3), an irreducible representation of $\mathrm{GSO}(V^d)$ corresponds to an irreducible representation τ of $\mathrm{GL}_2(E)$ whose central character is invariant under $\mathrm{Gal}(E/k)$, together with a character χ of $k^\times$ such that the central character of τ can be considered to be the character of $E^\times$ obtained from the character χ of $k^\times$ through the norm mapping. (There are two possibilities for χ which are twists of each other by $\omega_{E/k}$.) In this case, the Langlands parameter of the representation $\mathrm{GSp}_4(k)$ arising from theta correspondence from this representation of $\mathrm{GO}(V^d)$ is,

$$\mathrm{Ind}_E^k \sigma.$$

where σ is the L-parameter of the representation τ of $\mathrm{GL}_2(E)$.

(This representation with values in $\mathrm{GL}_4(\mathbb{C})$ can be considered as a representation with values in $\mathrm{GSp}_4(\mathbb{C})$ in two non-conjugate ways depending on the two choices for χ; the corresponding representations of $\mathrm{GSp}_4(k)$ are obtained from distinct representations of $\mathrm{GSO}(V^d) \cong [\mathrm{GL}_2(E) \times k^\times]/\Delta E^\times$, which are same when restricted to $\mathrm{GL}_2(E)$.)

The epsilon factor $\epsilon(\sigma \otimes \mathrm{Ind}_K^k(\chi^{-1}))$ in cases (1) and (2) is simply the product of the epsilon factors, $\epsilon(\sigma_1 \otimes \mathrm{Ind}_K^k(\chi^{-1}))$ and $\epsilon(\sigma_2 \otimes \mathrm{Ind}_K^k(\chi^{-1}))$ which by the theorem of Saito and Tunnell can be easily interpreted in terms of the existence of the character χ of $K^\times$ in the representations τ_1, τ_2, making Theorem 2 a consequence of Corollaries 7.1 and 7.2. Similarly in case (3), Theorem 2 is equivalent to Theorem 6.

11. Theorem 4

In this section we use Theorem 5 to convert results about Bessel models for $\mathrm{GSp}_4(k)$ to results about χ-invariant linear forms on representations of $\mathrm{GL}_4(k)$ restricted to $\mathrm{GL}_2(K)$, where χ is a character of $K^\times$ thought of as a character of $\mathrm{GL}_2(K)$ through the determinant map. This is achieved by looking at Theorem 5 for the dual reductive pair $(\mathrm{Sp}_4(k), \mathrm{O}_6(k))$ where the group $\mathrm{O}_6(k)$ comes from a six dimensional quadratic space over k with discriminant of the split form in dimension 6. Thus $\mathrm{O}_6(k)$ is either split, or is a rank 1 form of it, and $\mathrm{GSO}_6(k)$ will be one of the following two groups:

(1) $[\mathrm{GL}_4(k) \times k^\times]/\{(z, z^{-2}) : z \in k^\times\}$,
(2) $[\mathrm{GL}_2(D) \times k^\times]/\{(z, z^{-2}) : z \in k^\times\}$.

It follows from these isomorphisms that an irreducible representation of $\mathrm{GSO}_6(k)$ corresponds to a pair (π, χ) of a representation π of $\mathrm{GL}_4(k)$ (or $\mathrm{GL}_2(D)$), and a character χ of $k^\times$ such that the central character ω_π of π is χ^2.

We will also have to use the duality correspondence between $\mathrm{GSp}_4^D(k)$ and $\mathrm{GSO}_6^D(k)$ $= \mathrm{GU}_3^D(k)$ defined using a skew-hermitian form over D of discriminant -1, giving rise to

(3) $\mathrm{GU}_3^D(k) \cong [\mathcal{D}^\times \times k^\times]/\Delta k^\times$ where the mapping from $\Delta k^\times = k^\times$ to $k^\times$ is $x \to x^{-2}$, and $\mathcal{D}$ is the unique division algebra over k of invariant $1/4$.

The following theorem of Gan and Takeda [G-T2] lies at the basis of our proof of Theorem 4; the last part of the theorem is due to Gan and Tantono [G-T3].

Theorem 9. *(1) The theta correspondence between* $\mathrm{GSp}_4(k)$ *and* $\mathrm{GSO}_6(k)$ *gives a correspondence between irreducible representations of* $\mathrm{GSp}_4(k)$ *and* $\mathrm{GL}_4(k)$ *and also between irreducible representations of* $\mathrm{GSp}_4(k)$ *and* $\mathrm{GL}_2(D)$ *which on Langlands parameters corresponds to the natural inclusion* $\mathrm{GSp}_4(\mathbb{C}) \hookrightarrow \mathrm{GL}_4(\mathbb{C})$.

(2) A representation π, *resp.* π', *of* $\mathrm{GL}_4(k)$, *resp.* $\mathrm{GL}_2(D)$ *can be lifted to* $\mathrm{GSp}_4(k)$ *if and only if the Langlands parameter of* π, *resp* π', *lies inside* $\mathrm{GSp}_4(\mathbb{C})$.

(3) If an L*-packet of* $\mathrm{GSp}_4(k)$ *has size 2, then exactly one of its members lifts to* $\mathrm{GL}_4(k)$, *and the other to* $\mathrm{GL}_2(D)$.

(4) *If an L-packet $\{\pi\}$ of representations of $\mathrm{GSp}_4(k)$ has size one, then it lifts to a representation, say π', of $\mathrm{GL}_4(k)$; if the L-parameter of π' is relevant to $\mathrm{GL}_2(D)$, then π also lifts to $\mathrm{GL}_2(D)$.*

(5) *Let $\mathcal{D}$ be a division algebra of dimension 16 over k such that for the unitary group $\mathrm{U}_3^D(k)$ defined by a skew-hermitian form in 3 variables over D of discriminant -1, $\mathrm{GU}_3^D(k) \cong [\mathcal{D}^\times \times k^\times]/\Delta k^\times$ where the mapping from $\Delta k^\times = k^\times$ to $k^\times$ is $x \to x^{-2}$. Then the theta correspondence between $\mathrm{GSp}_4^D(k)$ and $\mathrm{GU}_3^D(k)$ gives an injection of representations of $[\mathcal{D}^\times \times k^\times]/\Delta k^\times$ with symplectic similitude parameter into irreducible representations of $\mathrm{GSp}_4^D(k)$.*

In this section, we will be looking at the embedding of the quadratic space underlying K (with its norm form as the quadratic form) in a six dimensional quadratic space, say $K \hookrightarrow K \oplus aK \oplus H$, a direct sum of quadratic spaces where aK is the same underlying vector space as K, but the quadratic form is scaled by a, and H is the two dimensional hyperbolic plane. The embedding of quadratic spaces gives an embedding of $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(aK \oplus H)]$ inside $\mathrm{GSO}(K \oplus aK \oplus H)$. We remind ourselves that

$$\mathrm{GSO}(aK \oplus H) \cong [\mathrm{GL}_2(K) \times k^\times]/K^\times,$$

where $\Delta K^\times = K^\times$ sits inside $[\mathrm{GL}_2(K) \times k^\times]$ as $(x, \mathbb{N}x^{-1})$. Therefore there is a natural embedding of $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(aK \oplus H)]$ inside $K^\times \times [\mathrm{GL}_2(K) \times k^\times]/\Delta K^\times$. We claim that under this embedding, the image of $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(aK \oplus H)]$ inside $K^\times \times [\mathrm{GL}_2(K) \times k^\times]/\Delta K^\times$ can be identified to $[\mathrm{GL}_2(K) \times k^\times]/\Delta k^\times$ where $k^\times$ sits naturally as the scalar matrices in $\mathrm{GL}_2(K)$, and in $k^\times$ through $t \to t^{-2}$. To prove this claim, note that there is a natural map from $[\mathrm{GL}_2(K) \times k^\times]/\Delta k^\times$ to $[\mathrm{GL}_2(K) \times k^\times]/\Delta K^\times$, and therefore to $K^\times \times [\mathrm{GL}_2(K) \times k^\times]/\Delta K^\times$ in which (X, t) goes to $t \det X$ in $K^\times$. It is easy to check that this map is injective, and its image is exactly $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(aK \oplus H)]$.

Using the identifications indicated above, the embedding of the group $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(aK \oplus H)]$ inside $\mathrm{GSO}(K \oplus aK \oplus H)$, becomes the standard embedding of $[\mathrm{GL}_2(K) \times k^\times]/\Delta k^\times$ inside $[\mathrm{GL}_4(k) \times k^\times]/\{(z, z^{-2}) : z \in k^\times\}$, or inside $[\mathrm{GL}_2(D) \times k^\times]/\{(z, z^{-2}) : z \in k^\times\}$ as the case may be, and further the natural map from $\mathrm{G}[\mathrm{SO}(K) \times \mathrm{SO}(aK \oplus H)] = [\mathrm{GL}_2(K) \times k^\times]/\Delta k^\times$ to $K^\times = \mathrm{GSO}(K)$ appearing in Theorem 5 is nothing but (X, t) going to $t \det X$ in $K^\times$, and thus Theorem 5 detects the appearance of one dimensional representations of $\mathrm{GL}_2(K)$ as a quotient of a representation of $\mathrm{GL}_4(k)$ which arise from theta lifting from $\mathrm{GSp}_4(k)$.

From Theorem 9 (due to Gan and Takeda), it follows that a representation of $\mathrm{GL}_4(k)$ arises as a theta lift from $\mathrm{GSp}_4(k)$ if and only if its Langlands parameter belongs to the symplectic similitude group $\mathrm{GSp}_4(\mathbb{C})$. By the remark following Theorem 5, as soon as a character of $\mathrm{GL}_2(K)$ appears as a quotient of a representation of $\mathrm{GL}_4(k)$, the representation of $\mathrm{GL}_4(k)$ arises from theta lifting from $\mathrm{GSp}_4(k)$, and therefore its parameter belongs to the symplectic similitude group. Further, the existence of χ-invariant linear form for the subgroup $\mathrm{GL}_2(K)$ of $\mathrm{GL}_4(k)$ is the same as the existence of χ-Bessel model for the representation of $\mathrm{GSp}_4(k)$ which is the theta lift of the representation $\pi \boxtimes \mu$ of $\mathrm{GSO}_6(k) = [\mathrm{GL}_4(k) \times k^\times]/\Delta(k^\times)$ in which $\mu = \chi|_{k^\times}$. Having proved the theorem about Bessel models for $\mathrm{GSp}_4(k)$, we deduce Theorem 4 about $\mathrm{GL}_4(k)$. For deducing Theorem 4 about other forms of $\mathrm{GL}_4(k)$, we will need to use theta correspondence between $\mathrm{GSp}_4(k)$ and the rank 1 form of $\mathrm{GO}_6(k)$ giving rise to $\mathrm{GL}_2(D)$, as well as theta correspondence between $\mathrm{GSp}_4^D(k)$ and $\mathrm{GU}_3^D(k)$ giving rise to $\mathcal{D}^\times$ for a division algebra of dimension 16 over k; we omit very analogous arguments in these cases.

We note, however, that, as usual, the methods of theta correspondence give results only for those irreducible representations of $\mathrm{GL}_4(k)$ which arise as $\Theta(\pi)$ with $\Theta(\pi) = \theta(\pi)$ for an irreducible representation π of $\mathrm{GSp}_4(k)$. For ensuring this, we will use the methods of theta correspondence only for supercuspidal representations of $\mathrm{GL}_4(k)$. Other representations of $\mathrm{GL}_4(k)$ for which there is a character of $\mathrm{GL}_2(K)$ appearing in it as a quotient, must arise from parabolic induction of an irreducible representation of the $(2,2)$ parabolic (as their parameter is in $\mathrm{GSp}_4(\mathbb{C})$, so cannot arise from parabolic induction of a supercuspidal representation of a Levi subgroup of the $(3,1)$ parabolic subgroup). If we are dealing with non-discrete series but generic representation of $\mathrm{GL}_4(k)$, we can assume that the representation is a full induced representation from an irreducible representation of the $(2,2)$ parabolic, and analyze separately the existence of χ-invariant linear form for the subgroup $\mathrm{GL}_2(K)$ of $\mathrm{GL}_4(k)$.

For the induced representation of $\mathrm{GL}_4(k)$ arising from the $(2,2)$ parabolic subgroup, Mackey theory will answer questions about restriction to a subgroup. This depends on the understanding of the double cosets,

$$\mathrm{GL}_2(K)\backslash\mathrm{GL}_4(k)/P_{(2,2)},$$

which we describe now.

To describe the double cosets $\mathrm{GL}_2(K)\backslash\mathrm{GL}_4(k)/P_{(2,2)}$, it will be convenient to let V be a two dimensional vector space over K thought of as a four dimensional vector space R_kV over k so that $\mathrm{GL}_2(K)$ as well as $\mathrm{GL}_4(k)$ operate on R_kV. With this notation, $\mathrm{GL}_2(K)\backslash\mathrm{GL}_4(k)/P_{(2,2)}$ can be identified to $\mathrm{GL}_2(K)$-orbits on the set of two dimensional k-subspaces W of R_kV which is easily seen to consist of two orbits, one represented by a W which is invariant under K, and the other which is not. It follows that the restriction to $\mathrm{GL}_2(K)$ of a principal series representation of $\mathrm{GL}_4(k)$ induced from a representation $\pi_1 \otimes \pi_2$ of $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k) = M$, which is a Levi subgroup of the $(2,2)$ parabolic $P_{(2,2)}$ with the Levi decomposition $P_{(2,2)} = M \times N$, is

$$(3)\quad 0 \to \mathrm{ind}_{\mathrm{GL}_2(k)}^{\mathrm{GL}_2(K)}(\pi_1 \otimes \pi_2) \to \pi|_{\mathrm{GL}_2(K)} \to \mathrm{ind}_{B(K)}^{\mathrm{GL}_2(K)}(|\cdot|_K^{1/2}\pi_1|_{K^\times} \otimes |\cdot|_K^{-1/2}\pi_2|_{K^\times}) \to 0,$$

where $B(K)$ is the Borel subgroup of $\mathrm{GL}_2(K)$ consisting of upper-triangular matrices with entries in K, and $|\cdot|_K^{1/2}\pi_1|_{K^\times} \otimes |\cdot|_K^{-1/2}\pi_2|_{K^\times}$ denotes the restriction of $|\cdot|_k^{1/2}\pi_1 \otimes |\cdot|_k^{-1/2}\pi_2$ to $K^\times \times K^\times$, which is then extended trivially across the unipotent radical of $B(K)$, and then induced to $\mathrm{GL}_2(K)$. (All inductions considered in this paper are normalized induction.)

It follows that if π has a χ-invariant form for a character $\chi : \mathrm{GL}_2(K) \overset{\det}{\to} K^\times \to \mathbb{C}^\times$, then either

(1) $\pi_1 \otimes \pi_2$ has a $\chi|_{k^\times}$-invariant linear form for $\mathrm{GL}_2(k)$, i.e., $\pi_1 \cong \pi_2^\vee \otimes \chi|_{k^\times}$. Or,
(2) π_1 and π_2 both contain the character χ of $K^\times \hookrightarrow \mathrm{GL}_2(k)$, in particular, $\omega_1 = \omega_2 = \chi|_{k^\times}$.

In both cases, it is easy to see that the parameter of the representation π lies inside $\mathrm{GSp}_4(\mathbb{C})$ with similitude character $\chi|_{k^\times}$, and that we further have (as consequence of the theorem due to Tunnell and Saito in case (2), and by generalities about epsilon factors in case (1)),

$$\epsilon(\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = \epsilon(\pi_1 \otimes \mathrm{Ind}_K^k(\chi^{-1})) \cdot \epsilon(\pi_2 \otimes \mathrm{Ind}_K^k(\chi^{-1})) = 1.$$

If π_1 and π_2 both contain the character χ of $K^\times$, then it follows from the exact sequence (3) that π carries a linear form on which $\mathrm{GL}_2(K)$ operates via χ. We now show that if $\pi_1 \cong \pi_2^\vee \otimes \chi|_{k^\times}$, but that one of π_1 or π_2 does not contain the character χ, then the unique (up to scaling) linear form on $\mathrm{ind}_{\mathrm{GL}_2(k)}^{\mathrm{GL}_2(K)}(\pi_1 \otimes \pi_2)$ on which $\mathrm{GL}_2(K)$ operates by χ extends to a linear form on π on which $\mathrm{GL}_2(K)$ operates by χ. For this one needs to prove that an exact sequence of the form

$$0 \to \mathbb{C}_\chi \to \pi' \to \mathrm{ind}_{B(K)}^{\mathrm{GL}_2(K)}(|\cdot|_K^{1/2}\pi_1|_{K^\times} \otimes |\cdot|_K^{-1/2}\pi_2|_{K^\times}) \to 0,$$

is a split extension under the condition that one of π_1 or π_2 does not contain the character χ. Since $\mathrm{ind}_{B(K)}^{\mathrm{GL}_2(K)}(|\cdot|_K^{1/2}\pi_1|_{K^\times} \otimes |\cdot|_K^{-1/2}\pi_2|_{K^\times})$ is a direct sum of (infinitely many) principal series representations, if we can create splittings over each principal series representations, we will be done. Therefore, it suffices to note the following result from [[P3], Corollary 5.9].

Lemma 11.1. *Let χ be a character of $K^\times$, and $\chi \circ \det$ the corresponding character of $\mathrm{GL}_2(K)$. Then if V is a principal series representation of $\mathrm{GL}_2(K)$ such that $\mathrm{Hom}_{\mathrm{GL}_2(K)}[V, \chi \circ \det] = 0$, we have $\mathrm{Ext}^1_{\mathrm{GL}_2(K)}[V, \chi \circ \det] = 0$.*

Completing proof of Theorem 4. Notice that if the representations π_1 and π_2 of $\mathrm{GL}_2(k)$ are discrete series representations with π_1^{JL} and π_2^{JL} the corresponding representations of $D^\times$, one can construct a representation $\pi^{JL} = \pi_1^{JL} \times \pi_2^{JL}$ of $\mathrm{GL}_2(D)$ by parabolic induction of the representation $\pi_1^{JL} \boxtimes \pi_2^{JL}$ of $D^\times \times D^\times$ which is a Levi subgroup in $\mathrm{GL}_2(D)$, and one can restrict the representation π^{JL} from $\mathrm{GL}_2(D)$ to $\mathrm{GL}_2(K)$, and get the analogous exact sequence,

(4)

$$0 \to \mathrm{ind}_{D^\times}^{\mathrm{GL}_2(K)}(\pi_1^{JL} \otimes \pi_2^{JL}) \to \pi|_{\mathrm{GL}_2(K)} \to \mathrm{ind}_{B(K)}^{\mathrm{GL}_2(K)}(|\cdot|_K^{1/2}\pi_1^{JL}|_{K^\times} \otimes |\cdot|_K^{-1/2}\pi_2^{JL}|_{K^\times}) \to 0.$$

It follows that if π has a $\chi \circ \det$-invariant form, then either

(1) $\pi_1^{JL} \otimes \pi_2^{JL}$ has a $\chi|_{k^\times}$-invariant linear form for $D^\times$, i.e., $\pi_1^{JL} \cong \pi_2^{JL\vee} \otimes \chi|_{k^\times}$. Or,
(2) π_1^{JL} and π_2^{JL} both contain the character χ, in particular, $\omega_1 = \omega_2 = \chi|_{k^\times}$.

It is clear that if condition (1) held for π_1, π_2, it will also hold for representations π_1^{JL}, π_2^{JL} of $D^\times$, and hence π^{JL} will have a χ-invariant linear form when restricted to $\mathrm{GL}_2(K)$. On the other hand, if the condition (2) held for π_1 and π_2, it will not hold for π_1^{JL}, π_2^{JL}. These conclusions together with the knowledge of L-packets for $\mathrm{GSp}_4(k)$ completes the proof of Theorem 4 on noting that conditions $(1), (2)$ simultaneously hold exactly when $\pi_1 = \pi_2$, with their central characters equal to $\chi|_{k^\times}$, in which case both π and π^{JL} have a χ-invariant linear form when restricted to $\mathrm{GL}_2(K)$.

Remark 11.2. We note a curious consequence of the proof above in the case $\pi_1 = \pi_2$, a supercuspidal representation of $\mathrm{GL}_2(k)$, in which case both the representation $\pi_1 \times \pi_1$ of $\mathrm{GL}_4(k)$, and the representation $\pi_1^{JL} \times \pi_1^{JL}$ of $\mathrm{GL}_2(D)$ have $\chi \circ \det$-invariant linear form for $\mathrm{GL}_2(K)$ for any character χ of $K^\times$, hence both lift to $\mathrm{GSp}_4(k)$, and the two lifts to $\mathrm{GSp}_4(k)$ have χ-Bessel models. Since $\theta(\pi_1 \times \pi_1)$ and $\theta(\pi_1^{JL} \times \pi_1^{JL})$ are respectively the generic and non-generic members of the unitary principal series representation $\mathbf{1} \rtimes \pi_1$ of $\mathrm{GSp}_4(k)$ induced from the Klingen parabolic, only one of these two have χ-Bessel model. It follows that either for $\theta(\pi_1 \times \pi_1)$ or for $\theta(\pi_1^{JL} \times \pi_1^{JL})$, there is a difference between Θ and θ, i.e, there must be a nontrivial extension between the two irreducible components of the unitary principal series representation $\mathbf{1} \rtimes \pi_1$ induced from the Klingen parabolic (which are $\theta(\pi_1 \times \pi_1)$ and $\theta(\pi_1^{JL} \times \pi_1^{JL})$). Extension between irreducible components of a reducible unitary principal series representation seems not to have been noticed earlier.

For later use, we note the following lemma which is clear from the analysis of the principal series representation arising out of the $(2,2)$ parabolic.

Lemma 11.3. *Let π_1 and π_2 be two representations of $\mathrm{GL}_2(k)$, of the same central characters, for k either an archimedean or a non-archimedean local field. Then the principal series representation $\pi_1 \times \pi_2$ of $\mathrm{GL}_4(k)$ has χ-Bessel model for all characters χ of $K^\times$ which appear in both π_1 and π_2. Thus if π_1 and π_2 are principal series representations of the same central characters, then $\pi_1 \times \pi_2$ has Bessel models for all characters of $K^\times$ whose restriction to $k^\times$ is the central character of π_1 and π_2.*

We end this section by formulating the following general conjecture, which is a modified form of a conjecture in [P2].

Conjecture 1. *Let $A \cong M_r(\mathcal{D})$, with $\mathcal{D}$ a central division algebra over k, be a central simple algebra over a local field k of dimension $4n^2$, and K a quadratic separable algebra over k which can be embedded in A. (The set of embeddings of K in A is unique by the Skolem-Noether theorem.) Let A^K be the centralizer of K in A which is a central simple algebra over K of dimension n^2. Let π be an irreducible, admissible representation of $A^\times$ such that the corresponding representation of $\mathrm{GL}_{2n}(k)$ is generic with central character ω_π. Let χ be a character of $K^\times$ such that $\chi^n|_{k^\times} = \omega_\pi$. Let $\det : (A^K)^\times \rightarrow K^\times$ denote the reduced norm map. If the character $\chi \circ \det$ of $(A^K)^\times$ appears as a quotient in π restricted to $(A^K)^\times$, then*

1. *The Langlands parameter of π takes values in $\mathrm{GSp}_{2n}(\mathbb{C})$ with similitude factor $\chi|_{k^\times}$.*
2. *The epsilon factor $\epsilon(\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = (-1)^r \omega_{K/k}(-1)^n \chi(-1)^n$.*

If π is a discrete series representation of $A^\times$, then these two conditions are necessary and sufficient for the character $\chi \circ \det$ of $(A^K)^\times$ to appear as a quotient in π restricted to $(A^K)^\times$. If π is not a discrete series representation, then one will need to consider not just the epsilon factor appearing in the condition (2) above, but other epsilon factors just as in (2) built out of irreducible sub-representations in the Langlands parameter of π which are of symplectic type with similitude factor $\chi|_{k^\times}$.

Remark 11.4. Multiplicity 1 of the trivial representation of $\mathrm{GL}_n(K)$ inside an irreducible admissible representation of $\mathrm{GL}_{2n}(k)$ was proved by J. Guo in [G], but the multiplicity 1 of more general characters of $\mathrm{GL}_n(K)$, or in our context, of even more general subgroups $(A^K)^\times$ of $A^\times$, seems not to have been addressed in the literature.

Remark 11.5. By generalities about epsilon factors (twisting by highly ramified characters) it can be seen that given π, a Galois representation of dimension $2n$,

$$\epsilon(\pi \otimes \mathrm{Ind}_K^k(\chi^{-1})) = \omega_{K/k}(-1)^n \chi(-1)^n,$$

for all but finitely characters χ of $K^\times$ with $\chi^n|_{k^\times} = \omega_\pi$. This makes the conjecture fit well with the fact that irreducible representations of $E^\times$ where E is a division algebra over k of dimension $4n^2$ are finite dimensional; the corresponding finiteness assertion for other odd values of r is not clear.

12. Discrete series over the reals

Our study of Bessel model for principal series representations in the p-adic case depended on two crucial, although rather elementary facts.

(1) If Y is a closed subspace of a p-adic manifold X, one has an exact sequence,

$$0 \to \mathcal{S}(X - Y) \to \mathcal{S}(X) \to \mathcal{S}(Y) \to 0.$$

(2) Twisted Jacquet functor is exact for any character θ of N.

Both these fail for real groups in general, necessitating extra work.

12.1. **Preliminaries.** We begin by setting up the notation.

Let M be a real analytic manifold, N a closed sub-manifold. We have a sequence of natural maps,

$$0 \to \mathcal{C}_c^\infty(M - N) \to \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(N) \to 0$$

which is exact except in the middle. If we denote $\mathcal{D}(M)$, resp $\mathcal{D}(N)$, resp $\mathcal{D}(M - N)$, the space of distributions on M, resp N, resp. $M - N$, then there is a natural map from $\mathcal{D}(M)$ to $\mathcal{D}(M - N)$, whose kernel is the space of distributions on M supported in N, which we denote by $\mathcal{D}_N(M)$:

$$0 \to \mathcal{D}_N(M) \to \mathcal{D}(M) \to \mathcal{D}(M - N).$$

Given a vector field X on M, it makes sense to differentiate functions on M by X, and hence also a distribution $\mathcal{D}$ on M by X, which we denote by $X\mathcal{D}$; clearly if a distribution $\mathcal{D}$ is supported on a closed sub-manifold N, then so is $X\mathcal{D}$. Thus from distributions $\mathcal{D}(N)$ thought of as distributions on M, one can create newer distributions on M supported on N by differentiating. It is known by the work of L. Schwartz that this way one constructs all distributions on M supported on N by iterated differentiation. Define a filtration $\mathcal{D}_N^d(M)$ on $\mathcal{D}_N(M)$ which consists of the space of distributions on M, supported on N, and which are obtained from the subspace $\mathcal{D}(N)$ of $\mathcal{D}(M)$ by differentiating by at most d vector fields on M. Observing that as $\mathcal{D}(N)$ is invariant under differentiation by vector fields *along* N, vector fields on M which are *transversal* to N need only be considered.

More precisely, let $X_1, \cdots, X_q$ be vector fields on M which are transversal to N at points of N in some neighborhood (in N) of a point of N, i.e., $T_x(M) = T_x(N) \oplus \mathbb{C}X_1 \oplus \cdots \oplus \mathbb{C}X_q$, where $T_x(M)$ is the tangent space to M at a point x of N, and $T_x(N)$ is the tangent space to N. This defines a filtration on $\mathcal{D}_N(M)$ which consists of the space of distributions on M, supported on N, and which are in the sum of the image of the natural maps

$$\begin{array}{rcl} \otimes^r\{X_1,\cdots,X_q\}\otimes\mathcal{D}(N) & \to & \mathcal{D}(M) \\ X_{i_1}\otimes\cdots\otimes X_{i_r}\otimes\mathcal{D} & \to & X_{i_1}\cdots X_{i_r}\mathcal{D}, \end{array}$$

for $r \leq d$. This filtration is the same as the filtration $\mathcal{D}_N^d(M)$ introduced earlier, and is therefore independent of the choice of vector fields $X_1,\cdots,X_q$ on M in the neighborhood of the point of N where these are transversal.

The following lemma identifies the successive quotients of this filtration. This lemma is a variant of lemma 2.4 of Shalika's paper [26] which itself is essentially due to Schwartz, but notice that unlike Shalika's paper, we do not assume that the transversal vector fields $X_1\cdots,X_q$, exist globally on N, nor do we need to assume that they span a Lie algebra of vector fields. In this lemma, we will in fact work more generally with distributions with coefficients in a vector bundle E over M which is modeled on a Fréchet space. In this context, let $\mathcal{D}(M,E)$ be the dual in the natural topology of $\mathcal{C}_c^\infty(M,E)$, the space of compactly supported $\mathcal{C}^\infty$ sections of E over M.

Lemma 12.1. *For a Fréchet vector bundle E on a manifold M, the space of distributions $\mathcal{D}_N(M,E)$ on M supported on a closed subanifold N comes equipped with a natural filtration $\mathcal{D}_N^d(M,E)$ such that the successive quotients, $\mathcal{D}_N^d(M,E)/\mathcal{D}_N^{d-1}(M,E)$ can be identified to a certain space of distributions on N, which is $\mathcal{D}(N,\mathrm{Sym}^d(TM/TN)^\vee\otimes E)$, where TM/TN is the quotient of the tangent bundle TM of M restricted to N by the tangent bundle TN of N, and $\mathrm{Sym}^d(TM/TN)^\vee$ represents the dual bundle of the symmetric power bundle.*

Remark 12.2. We will apply this lemma in the context where a Lie group G operates transitively on a manifold M, and R is a subgroup of G operating transitively on a closed submanifold N. Let $\circ$ be a point on N, with stabilizer H in G. The Lie algebra $\mathfrak{g}$ of G gives rise to vector fields on M, and if $\{X_1,\cdots,X_q\}$ is a set of generators of $\mathfrak{g}/(\mathfrak{r}+\mathfrak{h})$ where $\mathfrak{r}$ is the Lie algebra of R, and $\mathfrak{h}$ that of H, then the vector fields on M corresponding to X_i form a set of transversal vector fields to N in a neighborhood of $\circ$, and TM/TN can be realized as a homogeneous vector bundle on N corresponding to the representation $\mathfrak{g}/(\mathfrak{r}+\mathfrak{h})$ of the stabilizer (in R, i.e. $H\cap R$) of the previously chosen point $\circ$ in N.

The most important result for us will be the following form of the Frobenius reciprocity, cf. Theorem 5.3.3.1 of [Wa].

Proposition 12.3. *Let V be a real analytic manifold on which a Lie group H acts transitively with $H_\circ$ as the stabilizer of a point $\circ$ in V. Let E be a homogeneous vector bundle on V given by a representation of $H_\circ$ on a (possibly infinite dimensional) Fréchet space $E_\circ$. Let $\phi : H \to \mathbb{C}^\times$ be a character on H. Then if $\mathcal{D}(V,E)$ is the dual in the natural topology of $\mathcal{C}_c^\infty(V,E)$, the space of compactly supported $\mathcal{C}^\infty$ sections of E, then,*

$$\mathcal{D}(V,E)^{H,\phi}\cong E_\circ^{\vee,(H_\circ,\phi)}$$

where $E_\circ^{\vee,(H_\circ,\phi)}=\{e\in E_\circ^\vee \mid h\cdot e=\phi(h)e\ \forall h\in H_\circ\}$, and where $E_\circ^\vee$ is the space of continuous linear forms on $E_\circ$.

12.2. **Discrete series for $\mathrm{GSp}_4(\mathbb{R})$ and inner forms.** First we describe the discrete series representations of $\mathrm{GL}_2(\mathbb{R})$. Let $\eta = |\cdot|^s \mathrm{sgn}^\epsilon$, $\epsilon = 0, 1$, be any quasi-character of $\mathbb{R}^\times$. Then for any positive integer k, we have the following exact sequence of representations of $\mathrm{GL}_2(\mathbb{R})$

$$0 \longrightarrow \delta(\eta, k) \longrightarrow \eta|\cdot|^{k/2}\mathrm{sgn}^{k+1} \times \eta|\cdot|^{-k/2} \longrightarrow \zeta(\eta, k) \longrightarrow 0.$$

The representation $\zeta(\eta, k)$ is finite dimensional of dimension k, and the representation $\delta(\eta, k)$ is essentially square-integrable; it is discrete series if η is unitary.

We now deal with the group $\mathrm{Sp}_4(\mathbb{R})$. For every pair of integers (p, t) with $p > t > 0$ there is a collection of four discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ with the same infinitesimal character as that of $F(p, t)$, a finite dimensional irreducible representation of $\mathrm{Sp}_4(\mathbb{R})$ which in the standard notation has highest weight $(p-2)e_1 + (q-1)e_2$. We will denote these by $X(p, t), X(p, -t), X(t, -p), X(-t, -p)$. The representations $X(p, -t)$ and $X(t, -p)$ are generic, and the representations $X(p, t)$ and $X(-t, -p)$ are holomorphic and anti-holomorphic representations.

These discrete series representations appear in the principal series representations of $\mathrm{Sp}_4(\mathbb{R})$ obtained from the Siegel parabolic subgroup with $\mathrm{GL}_2(\mathbb{R})$ as the Levi subgroup. We state the following exact sequences from the paper [Mu] of Muic; in these sequences we use the standard notation of denoting $\pi \rtimes 1$ for the representation of $\mathrm{Sp}_4(\mathbb{R})$ obtained by inducing the representation π of $\mathrm{GL}_2(\mathbb{R})$ which is the Levi subgroup of the Siegel parabolic; the representation $L(\pi \rtimes 1)$ denotes the Langlands quotient, and V_p denotes the unique irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ of dimension p. Then we have the following exact sequences

$$0 \to X(p, -t) \oplus X(t, -p) \to \delta(|\cdot|^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1 \to L(\delta(|\cdot|^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1) \to 0;$$

$$0 \to F(p, t) \oplus L(\delta(|\cdot|^{\frac{p-t}{2}}\mathrm{sgn}^t, p+t) \rtimes 1) \to |\cdot|^t\mathrm{sgn}^t \rtimes V_p \to L(\delta(|\cdot|^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1) \to 0;$$

$$0 \to X(p, t) \oplus X(-t, -p) \to \zeta(|\cdot|^{\frac{p+t}{2}}\mathrm{sgn}^t, p-t) \rtimes 1 \to F(p, t) \to 0.$$

Now we have the following.

Lemma 12.4. *The principal series representation $\chi \rtimes V_p$ induced from a finite dimensional representation of the Klingen parabolic subgroup of $\mathrm{Sp}_4(\mathbb{R})$ has no Bessel models for non-degenerate characters of the unipotent radical of the Siegel parabolic for which the corresponding centralizer in the Levi is the compact torus $\mathbb{S}^1$.*

Proof. It is a simple consequence of the Bruhat theory that we omit. The corresponding statement for non-archimedean fields was proved earlier. □

We would have liked to use this lemma to conclude that Bessel models for any composition factor of $\chi \rtimes V_p$ are also zero. Although one would like to believe this to be a consequence of generalities (exactness of Bessel models), but that is not available in the literature anywhere, lacking which we resort to the result according to which by an appropriate choice of inducing data, any subquotient of a principal series representation can in fact be arranged to be a quotient, proving the following.

Lemma 12.5. *Sub-quotients of the representation $\chi \rtimes V_p$ of $\mathrm{Sp}_4(\mathbb{R})$ arising from finite dimensional representations of the Klingen parabolic have no Bessel models for non-degenerate characters of the unipotent radical of the Siegel parabolic for which the corresponding centralizer in the Levi is the compact torus $\mathbb{S}^1$.*

We now prove a few simple results about contragredients which allow one to turn questions about submodules to questions about quotient modules for which conclusions on Bessel models are easier to achieve.

Lemma 12.6. *Let $\alpha \in \mathrm{GSp}_{2n}(\mathbb{R})$ be an element of similitude factor -1. Then the automorphism of $\mathrm{Sp}_{2n}(\mathbb{R})$ induced by the inner-conjugation action of α takes an irreducible representation π of $\mathrm{Sp}_{2n}(\mathbb{R})$ to its contragredient $\pi^\vee$.*

Proof. It suffices to prove that the representations π^α and $\pi^\vee$ have the same characters. But one knows that the character $\Theta_{\pi^\vee}$ of $\pi^\vee$ is related to the character Θ_π of π by

$$\Theta_\pi(g^{-1}) = \Theta_{\pi^\vee}(g).$$

Therefore it suffices to note that $\alpha g \alpha^{-1}$ and g^{-1} are conjugate in $\mathrm{Sp}_{2n}(\mathbb{R})$ which is well-known. □

Corollary 12.7. *An irreducible representation π of $\mathrm{Sp}_4(\mathbb{R})$ has a Bessel model for a character ψ_t of $N \cong \left\{ n = \begin{pmatrix} a & b \\ b & c \end{pmatrix} | a, b, c \in \mathbb{R} \right\}$ given by $\psi_t(n) = e^{2\pi i t(a+c)}$ if and only if $\pi^\vee$ has a Bessel model for the character ψ_{-t}.*

Corollary 12.8. *If $0 \to \pi_1 \to \pi_2$ is an exact sequence of $\mathrm{Sp}_4(\mathbb{R})$ representations of finite length with $(\pi_2^\vee)_\psi = 0$, then $(\pi_1)_{\psi^{-1}} = 0$.*

These lemmas and corollaries, together with the exact sequences recalled earlier from [Mu] relating principal series and discrete series representations, reduce the study of Bessel models for discrete series representations of $\mathrm{Sp}_4(\mathbb{R})$ to principal series representations of $\mathrm{Sp}_4(\mathbb{R})$ induced from the Siegel parabolic.

The group $\mathrm{GSp}_4(\mathbb{R})$ contains $\mathbb{R}^\times \cdot \mathrm{Sp}_4(\mathbb{R})$ as a subgroup of index 2, and every discrete series representation of $\mathrm{GSp}_4(\mathbb{R})$ is obtained by inducing a discrete series representation of $\mathbb{R}^\times \cdot \mathrm{Sp}_4(\mathbb{R})$ which thus can be parametrized as $X(p, t; \xi)$ with ξ a character of $\mathbb{R}^\times$ such that $\xi|_{\pm 1}$ is the central character of the representation $X(p, t)$ of $\mathrm{Sp}_4(\mathbb{R})$. The action of $\mathrm{GSp}_4(\mathbb{R})$ on $\mathrm{Sp}_4(\mathbb{R})$ interchanges $X(p, t)$ with $X(-t, -p)$, and $X(p, -t)$ with $X(t, -p)$.

Given (p, t) with $p > t > 0$, and a character $\xi : \mathbb{R}^\times \to \mathbb{C}^\times$, let Π_1 be the generic representation of $\mathrm{GSp}_4(\mathbb{R})$ with central character ξ, and let Π_2 be the other discrete series representation of $\mathrm{GSp}_4(\mathbb{R})$ with the same infinitesimal character. Let Π_3 be the unique discrete series representation of $\mathrm{GSp}_4^{\mathbb{H}}(\mathbb{R})$ with the same infinitesimal and central character.

12.3. **The result.** For a given representation π, the Bessel functional is a continuous linear functional on the space of smooth vectors V_π^∞ in V_π which comes equipped with its Fréchet topology satisfying appropriate invariance equations with respect to the Bessel subgroup. Explicitly, let χ be a character of $\mathbb{C}^\times$ given by $\chi(re^{i\theta}) = \chi_1(r)e^{in\theta}$, for some quasi-character χ_1 of $\mathbb{R}_+^\times$. Given n and χ as above, we set $n(\chi) = n$. We identify $\mathbb{C}^\times$

with a subgroup of $\mathrm{GL}_2(\mathbb{R})$, and $\mathbb{H}^\times$, by sending $z = a + ib \mapsto t(z) := \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Define a subgroup R of $\mathrm{GSp}_4(\mathbb{R})$ by setting

$$R = \left\{ b(z; r, s, t) := \begin{pmatrix} t(z) & \\ & t(z) \end{pmatrix} \begin{pmatrix} 1 & & s & r \\ & 1 & r & t \\ & & 1 & \\ & & & 1 \end{pmatrix}; r, s, t \in \mathbb{R}, z \in \mathbb{C}^\times \right\}.$$

We now define a character χ_R of R by setting

$$\chi_R(b(z; r, s, t)) = \chi(z) e^{2\pi i (s+t)}.$$

There is a closely related subgroup $R_{\mathbb{H}}$ of $\mathrm{GSp}_4^{\mathbb{H}}(\mathbb{R})$. One defines a similar character of $R_{\mathbb{H}}$, again denoted by χ_R. We say a continuous functional λ on V_π^∞ is a χ-Bessel functional if it satisfies

$$\lambda(\pi(r)v) = \chi_R(r)\lambda(v),$$

for all $v \in V_\pi^\infty$ and $r \in R$. We define χ-Bessel functionals for representations of $\mathrm{GSp}_4^{\mathbb{H}}(\mathbb{R})$ similarly.

The exact sequences contained in the following lemma reduce questions about Bessel models for discrete series representations to similar questions for principal series representations.

In the following, we let $|\cdot|^s V_n$ denote the n-dimensional irreducible representation of $\mathbb{H}^\times = \mathbb{R}^+ \times \mathrm{SU}_2(\mathbb{R})$ on which $\mathbb{R}^+$ operates by $|x|^{2s}$.

Lemma 12.9. *For the discrete series representation Π_3 of $\mathrm{GSp}_4^{\mathbb{H}}(\mathbb{R})$, there are exact sequences of $\mathrm{GSp}_4^{\mathbb{H}}(\mathbb{R})$ representations as follows:*

$$0 \to \Pi_3 \to (|\cdot|^{\frac{p-t}{2}} V_{p+t}) \rtimes 1 \to L((|\cdot|^{\frac{p-t}{2}} V_{p+t}) \rtimes 1) \to 0,$$

$$0 \to L(|\cdot|^{\frac{p-t}{2}} V_{p+t}) \rtimes 1) \to (|\cdot|^{\frac{p+t}{2}} V_{p-t}) \rtimes 1 \to F(p, t) \to 0.$$

In the previous lemma as well as in an earlier lemma for $\mathrm{GSp}_4(\mathbb{R})$, it is useful to identify those principal series representations of a real group which contain (or by dualizing, have quotients) finite dimensional representations. The following lemma, whose simple proof is omitted, does exactly that.

Lemma 12.10. *Let G be the real points of a reductive algebraic group defined over $\mathbb{R}$. Let $P = MN$ be the real points of a parabolic defined over $\mathbb{R}$. Let F_λ be the finite dimensional irreducible representation of G of highest weight λ, containing the highest weight module V_λ for M with highest weight λ. (We assume having chosen positive system of roots for M as well as G in the usual way.) Let ρ_P denote half the sum of roots in N, thought of as a character $\rho_P : M \to \mathbb{R}^\times$, taking positive values. Then there is a natural inclusion,*

$$0 \to F_\lambda^\vee \to \mathrm{Ind}_P^G(V_\lambda^\vee \otimes \rho_P^{-1}),$$

and on taking duals, a surjection

$$\mathrm{Ind}_P^G(V_\lambda \otimes \rho_P) \to F_\lambda \to 0.$$

In the following theorem we are interested in the existence of Bessel functionals for the representations Π_i.

Theorem 10. *Let χ be a character of $\mathbb{C}^\times$ as above, and let $\{\Pi_1, \Pi_2, \Pi_3\}$ be the Vogan packet consisting of discrete series representations associated with a pair of integers (p, t) with $p > t > 0$, in such a way that $\chi|_{\mathbb{R}^\times}$ is the same as the central character of Π_1. Then exactly one of the representations Π_i, $1 \leq i \leq 3$, has a χ-Bessel model. More precisely*

(1) Π_1 has the model if and only if $|n(\chi)| > p + t$;
(2) Π_2 has the model if and only if $|n(\chi)| < p - t$; and
(3) Π_3 has the model if and only if $p - t < |n(\chi)| < p + t$.

In each case the space of the functionals is one dimensional.

A few remarks are in order. The theorem is of course the Gross-Prasad conjecture for discrete series representations of $\mathrm{GSp}_4(\mathbb{R})$ though we will not check the condition on local epsilon factors. From considerations of central characters, we note that the parity of $n(\chi)$ is opposite that of $p+t$ and $p-t$. Theorem 10 completes the work [TB]. We recall that [TB] proved the existence of Bessel functionals using global theta correspondence.

Proof of Theorem 10. By the lemma above, and the earlier exact sequences for $\mathrm{Sp}_4(\mathbb{R})$, we are reduced to calculating Bessel models for principal series representations of $\mathrm{Sp}_4(\mathbb{R})$, and $\mathrm{GSp}_4^{\mathbb{H}}(\mathbb{R})$ induced from the Siegel parabolic, which is what we will be doing now.

Our result will follow from the following claim:

Claim. Suppose the Π is a quotient of the $\mathrm{Ind}(\pi|P, G)$ with π an irreducible representation of $\mathrm{GL}_2(\mathbb{R})$. Then if Π has a χ-Bessel functional, there is a continuous functional λ on V_π^∞ satisfying $\lambda(\pi(t(z))v = \chi(z)\lambda(v)$ for all $v \in V_\pi^\infty$, $z \in \mathbb{C}^\times$; such linear forms will be called Waldspurger functional.

Suppose π acts on a space V_π. By the definition of an induced representation, a Bessel functional on $\mathrm{Ind}(\pi|P, G)$ defines a distribution T on the space of V_π valued Schwartz functions on $G = \mathrm{GSp}_4(\mathbb{R})$ satisfying

(1) $T(L_p F) = T(\pi(p)^{-1}F)$, for $p \in P$
(2) $T(R_r F) = \theta(n)\chi_n(t)T(F)$, for $r = nt \in R$.

Consider the Bruhat decomposition of G as $P \times P$ double cosets written as

$$\mathrm{GSp}_4(\mathbb{R}) = P \cup Pw_1P \cup Pw_2P,$$

with Pw_2P the unique open cell. The element w_1 can be represented by the following matrix

$$w_1 = \begin{pmatrix} 1 & & & \\ & & & 1 \\ & & 1 & \\ & -1 & & \end{pmatrix}.$$

We will show that if T is nonzero, it restricted to the open cell is nonzero too, and hence by Frobenius reciprocity, it happens only if the inducing representation π of $\mathrm{GL}_2(\mathbb{R})$ has a Waldspurger functional for the character χ_n.

Step 1. First step is to show that T restricted to the open set $Pw_1P \cup Pw_2P$ is non-zero. If it were zero, then T would be supported on P. We will show that there are no distributions supported on P satisfying the invariance properties. In fact we do not need the entire group $P \times R$; $P \times N$ is sufficient.

Note that the tangent space to G/P at the point P can be identified to $\mathfrak{g}/\mathfrak{p}$ as a P-module. The bilinear pairing,

$$\begin{aligned} \mathfrak{n} \times \mathfrak{g}/\mathfrak{p} &\rightarrow \mathbb{C} \\ (X, Y) &\rightarrow \mathrm{tr}(ad_X \cdot ad_Y), \end{aligned}$$

is a perfect pairing of $\mathfrak{p}$ modules. Therefore the tangent space of G/P at P can be identified to $\mathfrak{n}^\vee$ as a P-module, in particular as an N-module. Observe that as $\mathfrak{n}$ is abelian, this implies that the tangent space to G/P at the point P is the 3-dimensional trivial representation of N.

From lemma 1, it follows that in the space of distributions on G/P with values in the vector bundle on it arising from a representation of $\mathrm{GL}_2(\mathbb{R})$, those distributions supported at the point P do not carry any Bessel distributions.

Step 2. We now consider the restriction of T to the open set $Pw_1P \cup Pw_2P$. We would like to show that the restriction of T to Pw_2P is non-zero. We show that there are no distributions supported on Pw_1P satisfying the invariance properties. Here too we just need to use $P \times R$.

The orbit of P passing through w_1P has dimension 2, and is a homogeneous space for the Bessel subgroup R; this is crucial for our analysis. Denote the orbit by V. In this case, the normal bundle $T_x(G/P)/T_x(V)$ is a 1 dimensional representation space for the stabilizer $R_\circ$, a subgroup of R. We claim that the action of $R_\circ$ on $T_x(G/P)/T_x(V)$ is trivial. For this, we just need to note that $T_x(G/P)/T_x(V)$ being 1 dimensional, the action is given by a character $\mu : R_\circ \rightarrow \mathbb{R}^\times$. But $R_\circ$ is a subgroup of $\mathbb{S}^1 \times N$, from which it is clear that μ being algebraic must be trivial.

From lemma 1 combined with Frobenius reciprocity, it follows that in the space of distributions on $G/P - eP$ with values in the vector bundle on it arising from a representation of $\mathrm{GL}_2(\mathbb{R})$, those distributions supported on the submanifold Pw_1P do not carry Bessel distributions.

Thus Bessel distributions arise only through the open orbit, and arise only if the inducing representation π of $\mathrm{GL}_2(\mathbb{R})$ has a Waldspurger functional for the character χ_n (by Frobenius reciprocity).

From the work of Wallach in [Wa2], it follows that indeed when a character χ_n appears in π, then Bessel functional can be defined by a process of analytic continuation; in fact Wallach considers parabolic induction only from finite dimensional representations, but in our context extension of his argument to discrete series poses no essential difficulties.

The theorem now follows from the following elementary lemma. □

Lemma 12.11. *If π is a finite dimensional irreducible representation of $G = \mathrm{SL}_2(\mathbb{R})$, or $\mathrm{SU}_2(\mathbb{R})$ of dimension m, then π has characters χ_n of $\mathbb{S}^1 \hookrightarrow G$ exactly for $|n| < m$, and $n \equiv (m-1) \mod 2$.*

13. The global correspondence for the dual pair $(\mathrm{GSp}, \mathrm{GO})$

We now turn to the global setting. Let F be a number field and let $W, \langle\,.\,\rangle$ (resp. $V, (\,.\,)$) be a non-degenerate symplectic (resp. orthogonal) vector space over F with $\dim_F W = 2n$ (resp. $\dim_F V = m$). Let $G = \mathrm{GSp}(W)$ and $H = \mathrm{GO}(V)$. Also let $\mathbb{W} = V \otimes W$ and $\langle\langle\,.\,\rangle\rangle = (\,.\,) \otimes \langle\,.\,\rangle$, so that G and H form a dual reductive pair in the similitude group $\mathrm{GSp}(\mathbb{W})$. If ν denotes the similitude character for the various groups involved, let

$$R = \{(g,h) \in G \times H \mid \nu(g) = \nu(h)\},$$

so there is a natural homomorphism $i : R \to \mathrm{Sp}(\mathbb{W})$. Note that if we let $G_1 = \mathrm{Sp}(W)$ and $H_1 = \mathrm{O}(V)$, then $G_1 \times H_1 \subset R$.

From now on assume that $m = \dim_F V$ is even, and fix a non-trivial character ψ of $\mathbb{A} = \mathbb{A}_F$ trivial on F. Let $W = W_1 \oplus W_1^\vee$ denote a complete polarization of the symplectic space W. Let $\omega = \omega_\psi$ denote the usual action of $G_1(\mathbb{A})$ on the Schwartz-Bruhat space $\mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$ of $(V \otimes W_1^\vee)(\mathbb{A})$. For $h \in H(\mathbb{A})$ and $\varphi \in \mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$, let

$$L(h)\varphi(x) = |\nu(h)|^{-mn/4}\varphi(h^{-1}x).$$

Since $(\det h)^2 = \nu(h)^m$, these operators are unitary with respect to the natural pre-Hilbert space structure on the Schwartz-Bruhat functions. Note that the actions of $G_1(\mathbb{A})$ and $H_1(\mathbb{A})$ on $\mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$ commute, and are the usual ones associated to the dual pair (G_1, H_1).

For $(g,h) \in R(\mathbb{A})$ and $\varphi \in \mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$, let

$$\theta(g,h;\varphi) = \sum_{x \in (V \otimes W_1^\vee)(F)} \omega(g,h)\varphi(x).$$

It is then well-known that $\theta(g,h;\varphi)$ is invariant under $R(F)$. For $\varphi \in \mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$ and a cusp form $f \in \mathcal{A}_0(H)$, consider the integral

$$\theta(f;\varphi)(g) = \int_{H_1(F)\backslash H_1(\mathbb{A})} \theta(g, h_1 h;\varphi) f(h_1 h)\, dh_1$$

where $h \in H(\mathbb{A})$ is any element such that $\nu(g) = \nu(h)$ and dh_1 is a Haar measure on $H_1(F)\backslash H_1(\mathbb{A})$.

It is easy to check that the integral defining $\theta(f;\varphi)$ is absolutely convergent and is independent of the choice of h. One can also check that $\theta(f;\varphi)$ is left-invariant under

$$\{\gamma \in G(F) \mid \nu(\gamma) = \nu(\gamma'), \text{ for some } \gamma' \in H(F)\}.$$

As far as the central characters are concerned, it's not hard to see that if the central character of f is χ, then the central character of $\theta(f;\varphi)$ is $\chi.\chi_V^n$, where $\chi_V(x) = (x, (-1)^{m/2}\det V)$ is the quadratic character associated to V, and therefore for n even, the central character of $\theta(f;\varphi)$ is χ.

Remark 13.1. One usually defines $\theta(f;\varphi)(g)$ by integration on the quotient $H_1(F)\backslash H_1(\mathbb{A})$ for $H_1 = O(V)$. However, if f belongs to an automorphic representation of $GO(V)(\mathbb{A})$ which does not remain irreducible when restricted to $GSO(V)(\mathbb{A})$, then the space of automorphic functions on $GSp(W)$ defined by

$$\theta^0(f;\varphi)(g) = \int_{H_{1,0}(F)\backslash H_{1,0}(\mathbb{A})} \theta(g, h_1 h;\varphi) f(h_1 h)\, dh_1$$

with $H_{1,0} = SO(V)$, is the same space of functions as those obtained as $\theta(f;\varphi)(g)$. We will use this well-known observation, and use θ^0 instead of θ in what follows.

13.1. **Global Bessel Models.** We recall the notion of Bessel model introduced by Novodvorsky and Piatetski-Shapiro [N-PS]. For a symmetric matrix $S \in \mathrm{GL}_2(F)$, define a subgroup $T = T_S$ of $\mathrm{GL}_2(F)$ by

$$T = \{g \in \mathrm{GL}_2(F) \mid {}^t gSg = \det g.S\}.$$

We consider T as a subgroup of $\mathrm{GSp}_4(F)$ via

$$t \mapsto \begin{pmatrix} t & \\ & \det t.\,{}^t t^{-1} \end{pmatrix}.$$

Let us denote by U the subgroup of $\mathrm{GSp}_4(F)$ defined by

$$\mathrm{U} = \left\{ u(X) = \begin{pmatrix} I_2 & X \\ & I_2 \end{pmatrix} \mid X = {}^t X \right\}.$$

Finally, we define a subgroup R of $\mathrm{GSp}_4(F)$ by $R = T\mathrm{U}$.

Let ψ be a non-trivial character of $F\backslash\mathbb{A}$. For a symmetric matrix $S \in \mathrm{GL}_2(F)$, define a character ψ_S on $\mathrm{U}(\mathbb{A})$ by $\psi_S(u(X)) = \psi(\mathrm{tr}\,(SX))$ for $X = {}^t X \in \mathrm{M}_2(\mathbb{A})$; as S will be fixed throughout, we abbreviate ψ_S to ψ. Let χ be a character of $T(F)\backslash T(\mathbb{A})$. Denote by $\chi \otimes \psi$ the character of $R(\mathbb{A})$ defined by $(\chi \otimes \psi)(tu) = \chi(t)\psi(u)$ for $t \in T(\mathbb{A})$ and $u \in \mathrm{U}(\mathbb{A})$.

Let π be an automorphic cuspidal representation of $\mathrm{GSp}_4(\mathbb{A})$ realized on a space V_π of automorphic functions. We assume that

$$\chi|_{\mathbb{A}^\times} = \omega_\pi. \tag{5}$$

Then for $\varphi \in V_\pi$, we define a function $B(\varphi, g)$ on $\mathrm{GSp}_4(\mathbb{A})$ by

$$B(\varphi, g) = \int_{Z_{\mathbb{A}} R_F \backslash R_{\mathbb{A}}} (\chi \otimes \psi)(r)^{-1}.\varphi(rg)\, dr. \tag{6}$$

We say that π has a global Bessel model of type (S, χ, ψ) if for some $\varphi \in V_\pi$, the function $B(\varphi, g)$ is non-zero. In this case, the $\mathbb{C}$-vector space of functions on $\mathrm{GSp}_4(\mathbb{A})$ spanned by $\{B(\varphi, g) \mid \varphi \in V_\pi\}$ is called the space of the global Bessel model of π. We abbreviate $B(\varphi, e)$ to be $B(\varphi)$.

Let $\mu : W_1 \to V$ be a homomorphism of vector spaces such that the quadratic form on V restricted to W_1 via μ is the quadratic form on W_1 with respect to which the Fourier coefficients is being calculated on $\mathrm{GSp}(W)$, i.e., the symmetric matrix S in the notation above, but now we prefer to do things in a co-ordinate free way. Let $\mathrm{GO}(W_1)^+$ be the subgroup of $\mathrm{GO}(W_1)$ consisting of those elements for which the similitude factor is the similitude factor of an element of $\mathrm{GO}(V)$. (It is understood that the quadratic form on W_1 arises from a $\mu : W_1 \to V$ which is fixed.) In our applications, $\mathrm{GO}^+(W_1) = \mathrm{GO}(W_1)$.

A map $\mu : W_1 \to V$ will be identified to a (F-valued) point of $V \otimes W_1^\vee$, also denoted by μ, and therefore for a function $f \in \mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$, it makes sense to consider $f(\mu)$, as well as $L(h)f(\mu)$ for any $h \in [\mathrm{GO}(V) \times \mathrm{GL}(W_1)](\mathbb{A})$. Let $\mathrm{O}(W_1^\perp)$ be the subgroup of $\mathrm{O}(V)$ acting trivially on $\mu : W_1 \to V$. It is a standard calculation that in the summation defining the theta function, $\theta(\varphi) = \sum_{\mu:W_1\to V} \varphi(\mu)$, only those μ's contribute to the Fourier coefficient we are looking at for which the quadratic form on V restricts to the desired quadratic form on W_1. Since such embeddings $\mu : W_1 \to V$ are conjugate under $\mathrm{SO}(V)$ with stabilizer $\mathrm{SO}(W_1^\perp)$, for an automorphic form f on $\mathrm{GSO}(V)(\mathbb{A})$, $\varphi \in \mathcal{S}((V \otimes W_1^\vee)(\mathbb{A}))$, and χ an automorphic form on $\mathrm{GSO}(W_1)(\mathbb{A})$

$$B_{\chi,\mu}(\theta^0(f;\varphi)) = \int_{\mathrm{SO}(W_1^\perp)(\mathbb{A})\backslash \mathrm{SO}(V)(\mathbb{A})} \Lambda_\mu(f,\chi)(h) L(h)\varphi(\mu)\, dh,$$

where $h \in \mathrm{SO}(V)(\mathbb{A})$, and

$$\begin{aligned}\Lambda_\mu(f,\chi)(h) &= \int_{\mathbb{A}^\times \mathrm{GSO}(W_1)\backslash \mathrm{GSO}(W_1)(\mathbb{A})} \left[\int_{\mathrm{SO}(W_1^\perp)\backslash \mathrm{SO}(W_1^\perp)(\mathbb{A})} f(\delta h(g)h) d\delta\right] \chi(g) dg \\ &= \int_{\mathbb{A}^\times \mathrm{G}[\mathrm{SO}(W_1^\perp)\times \mathrm{SO}(W_1)](F)\backslash \mathrm{G}[\mathrm{SO}(W_1^\perp)\times\mathrm{SO}(W_1)](\mathbb{A})} f(\delta h(g)h)\chi(g)\, d\delta\, dg,\end{aligned}$$

where $h(g) \in \mathrm{GSO}(V)(\mathbb{A})$ has similitude factor $\nu(g)$, preserves the embedding $\mu : W_1 \to V$, and acts as g on W_1; we have $(\delta, g) \in \mathrm{G}[SO(W_1^\perp) \times SO(W_1)] \subset \mathrm{GSO}(W_1^\perp) \times \mathrm{GSO}(W_1)$. For sake of explicitness, we record the following simple lemma needed for the last equality above.

Lemma 13.2. *Let G be an algebraic group over a number field F, and N a normal subgroup, with $H = N\backslash G$. Then for appropriate choice of Haar measures, the following holds for appropriate choice of functions f on $G(F)\backslash G(\mathbb{A})$*

$$\int_{H(F)\backslash H(\mathbb{A})} \int_{N(F)\backslash N(\mathbb{A})} f(ng) dn d\bar{g} = \int_{G(F)\backslash G(\mathbb{A})} f(g) dg.$$

The following theorem is now immediate by standard arguments; it may be noted that the statement of this theorem is identical to Theorem 5

Theorem 11. *Let π_1 be an irreducible cuspidal automorphic representation of $\mathrm{GSO}(V)(\mathbb{A})$, and π_2 that of $\mathrm{GSp}(W)(\mathbb{A})$. Assume that $\pi_2 = \Theta(\pi_1)$ is the theta lift of π_1 to $\mathrm{GSp}(W)$. Let ψ be a non-degenerate character of the unipotent radical N of the Siegel parabolic $P = MN$ of $\mathrm{GSp}(W)$. Assume that ψ corresponds to a quadratic form q on W_1, a maximal isotropic subspace of W. Then for a cuspidal automorphic representation χ of $\mathrm{GSO}(W_1)$, the period integral (on $\mathrm{GSO}(W_1)\mathbb{A}^\times\backslash \mathrm{GSO}(W_1)(\mathbb{A})$) of χ against the ψ-th Fourier coefficient of π_2 is not identically zero if and only if*

(1) (q, W_1) can be embedded in the quadratic space V; let $W_1^\perp$ denote the orthogonal complement of W_1 sitting inside V through this embedding.
(2) For $\tilde{\chi}$ the automorphic representation on $\mathrm{G}[\mathrm{SO}(W_1) \times \mathrm{SO}(W_1^\perp)]$ which is obtained by pulling back the automorphic representation χ under the natural map $\mathrm{G}[\mathrm{SO}(W_1)\times \mathrm{SO}(W_1^\perp)] \to \mathrm{GSO}(W_1)$, the period integral of $\tilde{\chi}$ against the automorphic forms in π_1 restricted to $\mathrm{G}[\mathrm{SO}(W_1)\times\mathrm{SO}(W_1^\perp)]$ is not identically zero.

Just as in the local case, the following diagram allows one to identify $(E^\times \times E^\times)/\Delta F^\times$ inside $(D^\times \times D^\times)/\Delta F^\times$ as the subgroup $\mathrm{G}[SO(E) \times SO(E)]$ inside $\mathrm{GSO}(D) = \mathrm{GSO}(E \oplus E)$

$$\begin{tikzcd}
& {[E^\times \times E^\times]/\Delta(F^\times)} \arrow[dl, "\cong"'] \arrow[dr] & \\
\mathrm{G}[\mathrm{SO}(E) \times \mathrm{SO}(E)] & & (D^\times \times D^\times)/(\Delta F^\times).
\end{tikzcd}$$

Therefore the integral

$$\int_{\mathbb{A}^\times \mathrm{G}[\mathrm{SO}(W_1^\perp)\times \mathrm{SO}(W_1)](F)\backslash \mathrm{G}[\mathrm{SO}(W_1^\perp)\times \mathrm{SO}(W_1)](\mathbb{A})} f(\delta h(g))\chi(g)\, d\delta\, dg,$$

becomes a product of two toral integrals on $E^\times \mathbb{A}_F^\times \backslash \mathbb{A}_E^\times$ on which the theorem of Waldspurger applies, yielding Theorem 3 of the introduction. In the case where the dual pair involves division algebras one can prove a similar theorem. The proof carries over in an essentially verbatim manner.

Corollary 13.3. *Let $\pi_1 = \otimes \pi_{1,v}$, and $\pi_2 = \pi_{2,v}$ be two cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ with the same central character $\omega : \mathbb{A}_F^\times/F^\times \to \mathbb{C}^\times$. Let K be a quadratic field extension of F. Then there are Grössencharacters $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ such that $\chi|_{\mathbb{A}_F^\times/F^\times} = \omega$, and such that*

$$\begin{aligned} L(\frac{1}{2}, \pi_1 \otimes \mathrm{Ind}(\chi^{-1})) &\neq 0, \text{ and} \\ L(\frac{1}{2}, \pi_2 \otimes \mathrm{Ind}(\chi^{-1})) &\neq 0. \end{aligned}$$

Proof. : If $\pi_1 = \pi_2$, this is part of Waldspurger's theorem. Therefore assume that $\pi_1 \neq \pi_2$. In this case, $\pi_1 \boxtimes \pi_2$ gives rise to an automorphic form on $\mathrm{GSO}(2,2) = [\mathrm{GL}_2(F)\times \mathrm{GL}_2(F)]/\Delta(F^\times)$. By a theorem due to B. Roberts, the theta lift $\Theta(\pi_1 \boxtimes \pi_2)$ to $\mathrm{GSp}_4(\mathbb{A}_F)$ is nonzero. Assume that v is a place of F which is inert in K, so that K_v is a quadratic field extension of F_v. Let $\chi_v : K_v^\times \to \mathbb{C}^\times$ be a character which appears in both $\pi_{1,v}$ and $\pi_{2,v}$. (As $\pi_{1,v}$ and $\pi_{2,v}$ contain all but finitely many characters of $K_v^\times$ with a given central character, this is possible.) Therefore, $\Theta_v(\pi_{1,v} \boxtimes \pi_{2,v})$ has Bessel models for the character χ_v. It follows by a globalization theorem along the lines of [P3] that there is a character $\chi : \mathbb{A}_K^\times/K^\times$, with χ_v as the local component at v such that $\Theta(\pi_1 \boxtimes \pi_2)$ has a nonzero χ-Bessel period integral. Thus from the above theorem,

$$\begin{aligned} L(\frac{1}{2}, \pi_1 \otimes \mathrm{Ind}(\chi^{-1})) &\neq 0, \text{ and} \\ L(\frac{1}{2}, \pi_2 \otimes \mathrm{Ind}(\chi^{-1})) &\neq 0, \end{aligned}$$

completing the proof of the corollary. □

Remark 13.4. As the existence of global Bessel model depends on the non-vanishing of an L-function at the center of symmetry, one can construct examples of cuspidal representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ which have local Bessel models at all primes of $\mathbb{Q}$, but do not have global Bessel model. This should be contrasted with what one expects for Whittaker models of automorphic representations of $\mathrm{GSp}_4(\mathbb{A}_\mathbb{Q})$ where existence of local Whittaker models is supposed to be necessary and sufficient for the existence of global Whittaker model.

14. An Example

Let Π be an automorphic cuspidal representation of $\mathrm{GL}_4(\mathbb{A}_F)$, K a quadratic algebra over F, $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$, thought of as a character on $\mathrm{GL}_2(\mathbb{A}_K)$. If the period integral

$$\int_{\mathbb{A}_F^\times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}_K)} f(g)\chi^{-1}(g)dg,$$

is not identically zero, then by Theorem 11, one knows that the theta lift of Π to $\mathrm{GSp}_4(\mathbb{A}_F)$ is nonzero, and the automorphic form so obtained on $\mathrm{GSp}_4(\mathbb{A}_F)$ has a global χ-Bessel model. By Theorem 13.1 of [G-T2], and the theorem due to Ginzburg, Jiang, Rallis in [J-G-R] about Bessel models for $\mathrm{GSp}_4(\mathbb{A}_F)$, it follows that,

(1) $L(s, \Lambda^2(\Pi) \otimes \chi^{-1}|_{\mathbb{A}_F^\times})$ has a pole at $s = 1$.

(2) $L(\frac{1}{2}, \Pi \otimes \mathrm{Ind}_K^F \chi^{-1}) \neq 0$.

However, we construct an example here to show that these global conditions together with the necessary local condition,

(3) $\mathrm{Hom}_{\mathrm{GL}_2(K_v)}(\Pi_v, \chi_v) \neq 0$,

are not adequate to ensure that the period integral

$$\int_{\mathbb{A}_F^\times \mathrm{GL}_2(K)\backslash \mathrm{GL}_2(\mathbb{A}_K)} f(g)\chi^{-1}(g)dg,$$

is not identically zero.

In the example constructed below, if the period integral were nonzero, then the theta lift to $\mathrm{GSp}_4(\mathbb{A}_F)$ would be a nonzero generic cuspidal *irreducible* automorphic representation, with a χ-Bessel model. In particular for the theta lift, $\theta(\Pi) = \otimes_v \theta(\Pi_v)$ each of $\theta(\Pi_v)$ will have χ_v-Bessel models. However, we will ensure that at some place, say v_0 of F, the representation of $\mathrm{GL}_4(F_{v_0})$ is of the form $\tau \times \tau$ for a supercuspidal representation τ of $\mathrm{PGL}_2(F_{v_0})$ such that the character χ_{v_0} of $K_{v_0}^\times$ does not appear in the restriction of τ to $K_{v_0}^\times$. In that case, $\theta(\Pi_{v_0})$ which is the generic member of the principal series representation $1 \rtimes \tau$ of $\mathrm{GSp}_4(F_{v_0})$ coming from the Klingen parabolic does not carry the character χ_{v_0} of $K_{v_0}^\times$ in its Bessel model by our calculations in earlier sections, contradicting our assumption of nonzero period integral.

It suffices to construct an automorphic representation Π on $\mathrm{GL}_4(\mathbb{A}_F)$, a quadratic field extension K of F, and a character $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ with properties $(1), (2), (3)$, which has the further property that at some place, say v_0 of F which remains inert in K, $\Pi_{v_0} = \tau \times \tau$ for a supercuspidal representation τ of $\mathrm{PGL}_2(F_{v_0})$ such that

(4) $\mathrm{Hom}_{K_{v_0}^\times}(\tau, \chi_{v_0}) = 0$.

Let us begin with F a totally real number field, K a totally imaginary quadratic extension of F, v_0 a place of F which is inert in K, τ a supercuspidal representation of $\mathrm{PGL}_2(F_{v_0})$, and χ_{v_0} a character of $K_{v_0}^\times$ which does not appear in τ. Let χ be a Grössencharacter $\chi : \mathbb{A}_K^\times/K^\times \to \mathbb{C}^\times$ which is trivial on $\mathbb{A}_F^\times$ is trivial, and whose restriction to $K_{v_0}^\times$ is χ_{v_0}.

Let E be a quadratic extension of F which is totally real and for which the place v_0 of F splits in two places v_1, v_2 of E. Let D_E be a quaternion division algebra over E

such that the invariants of D_E are $1/2, 1/2$ at the two places v_1, v_2 of E, and zero at all the other finite places. We also assume that D_E remains a division algebra at all the infinite places.

By the globalization theorem of [P-S], there is an automorphic representation Λ of $\mathbb{A}_E{}^\times \backslash D_E^\times(\mathbb{A}_E)$ with local components τ^{JL}, τ^{JL} at the two places v_1, v_2, unramified at all the other non-archimedean places, and such that the period integral

$$\int_{\mathbb{A}_E^\times (KE)^\times \backslash \mathbb{A}_{KE}^\times} f(g)\chi_{KE}^{-1}(g)dg,$$

is nonzero for some function f in the space of Λ, where χ_{KE} is the Grössencharacter on $\mathbb{A}_{KE}^\times$ which is obtained by taking the norm mapping to $\mathbb{A}_K^\times$ and composing with χ defined on $\mathbb{A}_K^\times$. (Note that since the character χ_{v_0} of $K_{v_0}^\times$ does not belong to τ, it belongs to τ^{JL}.)

Note the general identity of group representations for H a subgroup of index two in G:

$$\Lambda^2\left(\mathrm{Ind}_H^G X\right) \cong \mathrm{Ind}_H^G\left(\Lambda^2(X)\right) \oplus M(X),$$

where $M(X)$ is the multiplicative, or twisted tensor induction. It follows that if X is two dimensional, and the determinant of X is trivial,

$$\Lambda^2\left(\mathrm{Ind}_H^G X\right) \cong \mathbf{1} \oplus \omega \oplus M(X),$$

where ω is the nontrivial character of G trivial on H.

As the representation Λ of $D_E^\times(\mathbb{A}_E)$ has trivial central character, it follows from the above that the representation $\Pi = \mathrm{Ind}_E^F \Lambda$ of $\mathrm{GL}_4(\mathbb{A}_F)$ (automorphic induction due to Arthur and Clozel after going from $D_E^\times(\mathbb{A}_E)$ to $\mathrm{GL}_2(\mathbb{A}_E)$ by the Jacquet-Langlands correspondence) is symplectic in the sense that

$$L(s, \Lambda^2(\Pi))$$

has a pole at $s = 1$. (We will also need to appeal to a theorem of D. Ramakrishnan according to which the (Asai) representation $M(X)$ is modular for X two dimensional.)

Because of the nonvanishing of the period integral of Λ along a torus in $D_E^\times(\mathbb{A}_E)$, it follows from a theorem of Waldspurger that

$$L(\frac{1}{2}, \Lambda \otimes \mathrm{Ind}_{KE}^E \chi_{KE}^{-1}) \neq 0.$$

By generality about L-functions, for $\Pi = \mathrm{Ind}_E^F \Lambda$,

$$L(s, \Pi \otimes \mathrm{Ind}_K^F \chi^{-1}) = L(s, \Lambda \otimes \mathrm{Ind}_{KE}^E \chi_{KE}^{-1}).$$

In particular,

$$L(\frac{1}{2}, \Pi \otimes \mathrm{Ind}_K^F \chi^{-1}) \neq 0.$$

Observing that the local conditions,

(3) $\mathrm{Hom}_{\mathrm{GL}_2(K_v)}(\Pi_v, \chi_v) \neq 0$,

are automatically satisfied at all the other places of F outside of v_0 as by construction the local representations are principal series representations where this follows from lemma 11.3; at the place v_0, this condition is satisfied by the analysis of χ-invariant linear forms for the subgroup $\mathrm{GL}_2(K)$ of $\mathrm{GL}_4(k)$ done for principal series representations of $\mathrm{GL}_4(k)$

coming from the $(2,2)$ parabolic in section 11. This completes the construction of the desired example.

Remark 14.1. The question considered in this section is closely related to the existence of Shalika models considered recently by Jacquet-Martin [J-M], as well as Gan-Takeda [G-T1]. We recall that Gan-Takeda have constructed a counter-example to the existence of global Shalika periods for $\mathrm{GL}_2(D)$ even when all the natural local and global conditions are met. Our construction of the counter-example is very similar to that of Gan-Takeda; however, we note that the Gan-Takeda counter-example works for $\mathrm{GL}_2(D)$, whereas ours actually works (for a slightly different question) for $\mathrm{GL}_4(k)$. The counter-example here as well as in the work of Gan-Takeda are based on exploiting the difference between theta liftings $\Theta(\tau \times \tau)$ and $\theta(\tau \times \tau)$ from $\mathrm{GO}_6(k)$ to $\mathrm{GSp}_4(k)$. The representation $\tau \times \tau$ of $\mathrm{GL}_4(k)$ has nontrivial χ-period for the subgroup $\mathrm{GL}_2(K)$, so the representation $\Theta(\tau \times \tau)$ of $\mathrm{GSp}_4(k)$ has χ-Bessel model; but the representation $\theta(\tau \times \tau)$ of $\mathrm{GSp}_4(k)$ does not have χ-Bessel model. In this example, the representation $\Theta(\tau \times \tau)$ of $\mathrm{GSp}_4(k)$ is a nontrivial extension of $\theta(\tau \times \tau)$ by $\theta(\tau^{JL} \times \tau^{JL})$, which are the two irreducible components of the unitary principal series $1 \rtimes \tau$ coming from the Klingen parabolic.

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