

GENERALISED FORM OF A CONJECTURE OF JACQUET AND A LOCAL CONSEQUENCE

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ABSTRACT. Following the work of Harris and Kudla we prove a general form of a conjecture of Jacquet relating the non-vanishing of a certain period integral to non-vanishing of the central critical value of a certain L -function. As a consequence, we deduce a theorem relating the existence of $GL_2(k)$ -invariant linear forms on irreducible, admissible representations of $GL_2(\mathbb{K})$ for a commutative semi-simple cubic algebra \mathbb{K} over a non-archimedean local field k in terms of local epsilon factors which was proved only in some cases by the first author in his earlier work in [16]. This has been achieved by globalising a locally distinguished representation to a globally distinguished representation, a result of independent interest which is proved by an application of the relative trace formula.

1. INTRODUCTION

Let F be a number field and E a semisimple cubic algebra over F , i.e., let E be either a cubic field extension of F , or $E = F \oplus F'$ with a quadratic field extension F' of F , or $E = F \oplus F \oplus F$.

For a cuspidal automorphic representation Π of the group $GL_2(\mathbb{A}_E)$ with trivial central character restricted to \mathbb{A}_F^\times , where \mathbb{A}_F is the adèle ring of F , and $\mathbb{A}_E = E \otimes_F \mathbb{A}_F$, the period integral

$$\int_{\mathbb{A}_F^\times GL_2(F) \backslash GL_2(\mathbb{A}_F)} f(g) dg,$$

has been much studied in the split case $E = F \oplus F \oplus F$, where a famous question, asked by Jacquet, and now completely settled by Harris and Kudla in [7] completing their earlier proof for special cases given in [6], relates the non-vanishing of this period integral to the non-vanishing of the central critical L -value $L(\frac{1}{2}, \Pi_1 \otimes \Pi_2 \otimes \Pi_3)$ of an automorphic representation Π of $GL_2(\mathbb{A}_E) = GL_2(\mathbb{A}_F) \times GL_2(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$ of the form $\Pi_1 \otimes \Pi_2 \otimes \Pi_3$. It is natural to study this question in the general case of any semisimple cubic algebra E over a number field F . We recall that for an automorphic representation Π of $GL_2(\mathbb{A}_E)$, unramified outside a finite set S of places of E , the triple product L -function comprising of local factors outside S ,

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to be denoted by $L_3^S(s, \Pi)$, was shown to have meromorphic continuation and a certain functional equation by Piatetski-Shapiro and Rallis in [14], using a generalized version of Garrett's integral representation. In [9], Ikeda proved (again using the integral representation) that this result can be extended to the completed L -function defined as a product of local factors over all places (including the archimedean ones). The local factors occurring here are defined in terms of local zeta integrals.

On the other hand, Shahidi identified $L_3^S(s, \Pi)$ as a partial L -function which arises out of the Langlands-Shahidi method, and was able to get a complete L -function $L_3(s, \Pi)$ incorporating all the finite and infinite primes, having a functional equation of the form

$$L_3(s, \Pi) = \epsilon_3(s, \Pi) L_3(1-s, \Pi),$$

where

$$\epsilon_3(s, \Pi) = \prod_v \epsilon_{3,v}(s, \Pi_v),$$

and where $\Pi = \otimes_v \Pi_v$, Π_v being a representation of $GL_2(E \otimes_F F_v)$, and $\epsilon_{3,v}(s, \Pi_v)$ are the local epsilon factors associated to Π_v which are equal to 1 for almost all places v of F .

If D is a quaternion algebra over F we denote by D_E the algebra $D \otimes_F E$ and by Π^{D_E} the automorphic representation of $D^\times(\mathbb{A}_E)$ corresponding to Π under the correspondence of Jacquet and Langlands (if it exists).

The aim of this paper is to prove the following general form of the theorem of Harris and Kudla and to show that this general form can be applied to the study of invariant linear forms, completing the results of the first author in [16].

Theorem 1.1. *Let Π be a cuspidal irreducible automorphic representation of the adelic group $GL_2(\mathbb{A}_E)$ with central character that is trivial on \mathbb{A}_F^\times . Denote by $L_3(s, \Pi)$ the (completed) twisted tensor L -function of Π . Then $L_3(\frac{1}{2}, \Pi) \neq 0$ if and only if there exists a quaternion algebra D over F such that Π^{D_E} is defined and there exists a function f^{D_E} in the space of Π^{D_E} such that*

$$(1.1) \quad I(f^{D_E}) = \int_{\mathbb{A}_F^\times D^\times(F) \backslash D^\times(\mathbb{A}_F)} f^{D_E}(y) dy \neq 0.$$

We show in this paper that most of the proof given in [7] for the split case in fact goes through for a general semisimple cubic algebra. It should be noted that the proof in [7] uses the definition of the triple product L -function through the integral representation from [14, 9] and the fact that by the work of Ramakrishnan [20], this L -function in the split case coincides with the definition of Shahidi as well as with the one given in terms of representations of the Weil-Deligne group; this has not yet been established for general E (in particular, Ikeda's paper [9] excludes the case of a cubic field extension E/F in Lemma 2. 2 and its corollary).

We will see, however, that for questions of vanishing or nonvanishing at $1/2$ it makes no difference which definition of the L -function $L_3(s, \Pi)$ we choose to formulate our theorem.

We end this introduction by mentioning that in the split case $E = F \oplus F \oplus F$ more precise versions of Theorem 1.1 have been investigated, starting with the work of Harris and Kudla [6] and continued by Gross and Kudla [4], Boecherer and Schulze-Pillot [2], and in the thesis of Watson [24]. These more precise versions use the connection between the central critical value of the L -function $L_3(s, \Pi)$ with period integrals to prove rationality results and even explicit formulas for this central critical value under additional assumptions on Π ; the explicit formulas obtained have found applications to questions in quantum chaos in [24, 1]. It would be natural to extend these results in the context of our work to the case of general cubic algebras over a number field.

2. PROOF OF JACQUET'S CONJECTURE

Let \mathbf{G} denote the algebraic group over F whose group of F -points is the group

$$\{g \in GL_2(E) \mid \det(g) \in F^\times\}.$$

The group $\mathbf{G}(F)$ embeds into the symplectic similitude group $GSp_6(F)$ by viewing $GL_2(E)$ as the similitude group of the standard symplectic form on E^2 which can be embedded in $GSp_6(F)$ by choosing a F -basis of E and taking the trace of the symplectic form viewed as a symplectic form on F^6 ; the matrices from $GL_2(E)$ with determinant in F^\times yield then elements of the symplectic similitude group of this form on F^6 . This embedding carries over to the group of adelic points. Let \mathbf{G}_1 denote the algebraic group over F whose group of F -points is the group

$$\{g \in GL_2(E) \mid \det(g) = 1\}.$$

We consider the normalized Eisenstein series $E^\times(g, s, \Phi_s)$ on $GSp_6(\mathbb{A}_F)$ defined in [7, (1.5)] with a factorizable section Φ_s and an automorphic form $f \in \Pi$ and form the global zeta integral

$$(2.1) \quad Z(s, f, \Phi_s) = \int_{\mathbb{A}_F^\times \mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)} E^\times(\mathbf{g}, s, \Phi_s) f(\mathbf{g}) d\mathbf{g}.$$

We fix an additive character $\psi = \prod_v \psi_v$ of \mathbb{A}_F and suppose that f has a factorizable Whittaker function $W^\psi(g) = \prod_v W_v^\psi(g_v)$.

By Theorem 2.1 of [14], and the remark following it, the global zeta integral factors as a product $\prod_v Z_v(s, W_v^\psi, \Phi_{s,v})$ of local zeta integrals for sufficiently large $\text{Re}(s)$, where the local zeta integrals $Z_v(s, W_v^\psi, \Phi_{s,v}) = \int \Phi_{s,v}(\delta_v g) W_v^\psi(g) dg$ are absolutely convergent for large $\text{Re}(s)$ by [14, Proposition 3.2]. This factorization of the global zeta integral is obtained in [14] (following Garrett's method) by the usual unfolding trick; the element δ_v occurring in the last integral is a representative of the only open

orbit of $\mathbf{G}(F_v)$ acting on $P_v \backslash GSp_6(F_v)$, for P_v the Siegel parabolic in $GSp_6(F_v)$.

The following Lemma is a generalization of Corollary 2.3 of [7]:

Lemma 2.1. *With notations as above one has $L_3(\frac{1}{2}, \Pi) \neq 0$ if and only if there is $f \in \Pi$ and a factorizable section Φ_s such that $Z(0, f, \Phi_s) \neq 0$.*

Proof. We prove the Lemma using the local factors given by the method of integral representation and discuss the other definitions for the local factors at bad places in the end.

We let S be any finite set of places of E containing all the archimedean places such that the additive character Ψ of \mathbb{A}_E obtained from the character ψ of \mathbb{A}_F by taking the trace map is unramified outside S and such that Π is unramified outside S .

For a factorizable section Φ_s and $f \in \Pi$ with factorizable Whittaker function W^ψ as above, we let $S_1 = S_1(f, \Phi_s) \supseteq S$ be a finite set of places of F such that for $v \notin S_1$, f is $GL_2(\mathcal{O}_v^E)$ fixed with $W_v^\psi(e) = 1$, where \mathcal{O}_v^E is the maximal compact subring of $E \otimes F_v$, and such that the section $\Phi_{s,v}$ is right $GSp_6(\mathcal{O}_v)$ -invariant with $\Phi_{s,v}(e) = 1$.

By Theorem 3.1 of [14] the local zeta integrals for the places $v \notin S_1$ are then equal to the local factors of the L -function $L_3(s, \Pi)$, with the argument shifted by $1/2$, so that we obtain

$$(2.2) \quad Z(s, f, \Phi_s) = L_3^{S_1}(s + \frac{1}{2}, \Pi) \prod_{v \in S_1} Z_v(s, W_v^\psi, \Phi_{s,v}).$$

This identity, at first valid for $\text{Re}(s)$ sufficiently large, extends to the complex plane by meromorphic continuation.

As a first step, we prove the assertion of the Lemma with $L_3(s, \Pi)$ replaced by the partial L -function $L_3^S(s, \Pi)$, where the exponent S on L_3 indicates the product of the local factors for the $v \notin S$.

If we have $L_3^S(\frac{1}{2}, \Pi) \neq 0$ we can choose f and the components $\Phi_{s,v}$ of a section Φ_s for the $v \notin S$ so that we can take $S_1 = S$ above and have

$$L_3^S(s + \frac{1}{2}, \Pi) = \prod_{v \notin S} Z_v(s, W_v^\psi, \Phi_{s,v}).$$

By [14, Proposition 3.3], [9, p.227] we can then choose the local data at the places in S in such a way that the local zeta integrals $Z_v(s, W_v^\psi, \Phi_{s,v})$ for $v \in S$ are $\neq 0$ at $s = 0$. With this choice of data f, Φ_s we obtain then

$$Z(0, f, \Phi_s) = \prod_{v \in S} Z_v(0, W_v^\psi, \Phi_{s,v}) \prod_{v \notin S} Z_v(0, W_v^\psi, \Phi_{s,v}) \neq 0,$$

since both factors are nonzero.

If $L_3^S(\frac{1}{2}, \Pi) = 0$ then for any choice of factorizable data we consider again the factorization

$$Z(s, f, \Phi_s) = \prod_v Z_v(s, W_v^\psi, \Phi_{s,v})$$

and choose a finite set $S_1 \supseteq S$ as above.

For the $v \in S_1 \setminus S$ by [14, App. 3 to Section 3], [9, p. 227] the local zeta integral $Z_v(s - 1/2, W_v^\psi, \Phi_{s,v})$ is a multiple of the local L -factor $L_v(s, \Pi)$ by a polynomial in q_v^s, q_v^{-s} , so that

$$\prod_{v \notin S} Z_v(s, W_v^\psi, \Phi_{s,v})$$

has a zero at $s = 0$ as well. Moreover, by [9, Lemma 2.1] and using the result of [11] for the Satake parameters of $\tilde{\Pi}_v$ at any place v , of an irreducible cuspidal automorphic representation $\tilde{\Pi}$ of GL_2 , we see that the local zeta integrals $Z_v(s, W_v^\psi, \Phi_{s,v})$ have no pole in the region $\text{Re } s > \frac{15}{34} - \frac{1}{2}$ (which contains the point $s = 0$), for any finite v .

The zero of $\prod_{v \notin S} Z_v(s, W_v^\psi, \Phi_{s,v})$ at $s = 0$ therefore cannot be cancelled by a pole in the other factors and we obtain $Z(s, f, \Phi_s) = 0$ at $s = 0$ for all choices of factorizable data as asserted.

To come back to $L_3(\frac{1}{2}, \Pi)$ itself, we notice that the local factors of $L_3(s, \Pi)$ are nonzero by definition. Moreover, by Ikeda's argument for the local zeta integrals used above, the local factors of $L_3(s, \Pi)$ have no poles at $s = \frac{1}{2}$. We therefore see that $L_3(\frac{1}{2}, \Pi) \neq 0$ is equivalent to $L_3^S(\frac{1}{2}, \Pi) \neq 0$, and the assertion of the Lemma follows.

We finally note that the argument given above for the transition between $L_3(\frac{1}{2}, \Pi)$ and $L_3^S(\frac{1}{2}, \Pi)$, goes through for the two other methods mentioned in the introduction for defining local L -factors at the places in S , so that the statement of the Lemma is true for these other definitions of local factors as well (even though we have at present no proof that the definitions do in fact coincide).

Indeed what we need is that the local factors of $L_3(s, \Pi)$ have no poles at $s = \frac{1}{2}$. This is clearly true for the local factors given in terms of representations of the Weil-Deligne group by the Kim-Shahidi estimate from [11] used above. For the local factors given by Shahidi in [22] it is also true, we check this for the case of a (local) cubic field extension: If Π_v is tempered, it is Proposition 7.2 of [22]. In the (hypothetical) case that Π_v is not tempered, it is a subquotient of a representation induced from a character $(\mu_1 | \cdot|^t, \mu_2 | \cdot|^{-t})$ of the Borel subgroup. In that case, Shahidi's construction in [22, Section 7] reduces the definition of the local factor by induction to local factors for characters. The latter are just the usual factors of Hecke L -functions with pole free region $\{s \mid \text{Re}(s) > 3t\}$, and the estimate $t < 5/34$ guarantees again that the local factor of L_3 is free of poles in that region. \square

It remains to check that in the general case one can convert the assertion of the last lemma about the global zeta integral into the assertion of the Theorem in the same way as in [7].

We recall some notation. We let D be a quaternion algebra over F and $\alpha \in F^\times$ and consider D equipped with the quadratic form $q_\alpha(x) = \alpha n(x)$

(where $n(x)$ denotes the reduced norm of x) as a quadratic space $V = V(D, \alpha)$ over F . We let $H = H_V = GO(V)$ be the similitude group of V (with similitude factor ν) and $H_1 = O(V) \subseteq H$.

We denote by $GS_{p_6}^{+,V}(\mathbb{A}_F)$ the adelic group of symplectic similitudes (over F) having positive similitude factor at all the (necessarily real) infinite places v of F where $D \otimes F_v$ is a division algebra. For $g \in GS_{p_6}^{+,V}(\mathbb{A}_F)$ and a test function $\varphi = \varphi^V \in S(V(\mathbb{A}_F)^3)$ the usual theta integral (defined by regularization if $D = M_2(F)$), is defined as

$$(2.3) \quad I(g, \varphi) = \int_{H_1(F) \backslash H_1(\mathbb{A}_F)} \theta(h_1 h, g; \varphi) dh_1,$$

where $h \in H(\mathbb{A}_F)$ with $\nu(h) = \nu(g)$ as in [7, (3.9)]. Moreover, the function $I(\cdot, \varphi)$ on $GS_{p_6}^{+,V}(\mathbb{A}_F)$ has a unique left $GS_{p_6}(F)$ -invariant extension to all of $GS_{p_6}(\mathbb{A}_F)$, which we will also denote by $g \mapsto I(g, \varphi)$.

By [7, Corollary 5.1] the assertion of Lemma 2.1 gives then, as a consequence of the Siegel-Weil theorem, that one has $L_3(\frac{1}{2}, \Pi) = 0$ if and only if for all choices of isometry class of $V = V(D, \alpha)$ as D and $\alpha \in F^*$ vary, test function $\varphi^V \in S(V(\mathbb{A}_F)^3)$ and $f \in \Pi$ one has

$$(2.4) \quad \int_{\mathbb{A}_F^\times \mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)} I(\mathbf{g}, \varphi^V) f(\mathbf{g}) d\mathbf{g} = 0;$$

in this step the structure of the cubic algebra E plays no role at all.

For each choice of V , let $\mathbf{H}(F) = \mathbf{H}_V(F) = \{h \in H(E) \mid \nu(h) \in F^\times\}$. We write \mathbf{H}_1 for the subgroup of \mathbf{H} where the similitude groups above are replaced by the respective orthogonal groups.

We write $\tilde{H} = (D^\times \times D^\times) \rtimes \langle \mathbf{t} \rangle$, where the involution \mathbf{t} acts on $D^\times \times D^\times$ by $(b_1, b_2) \mapsto (\bar{b}_2^{-1}, \bar{b}_1^{-1})$, and $\tilde{H}^0 = D^\times \times D^\times$. Analogously we have $\tilde{\mathbf{H}}, \tilde{\mathbf{H}}^0$ as covering groups of $\mathbf{H} = GO(V \otimes E)$, and $\mathbf{H}^0 = GSO(V \otimes E)$.

The groups Sp_6, H_1 form a dual reductive pair in Sp_{24} , and the groups $\mathbf{G}_1, \mathbf{H}_1$ form another dual reductive pair in this big symplectic group. For the latter pair, we obtain first in the usual way an embedding of $SL_2(E) \times O(V \otimes E)$ into the isometry group of the standard 8-dimensional symplectic space over E which is then transferred to an embedding of $\mathbf{G}_1 \times \mathbf{H}_1$ into Sp_{24} by taking the trace from E to F .

The proof in [7] proceeds then (for $E = F^3$) as follows: For a cuspidal automorphic form f on $\mathbf{G}(\mathbb{A}_F)$ one considers the lifted form on $\mathbf{H}(\mathbb{A}_F)$ given by

$$(2.5) \quad I(\mathbf{h}, \varphi; f) = \int_{\mathbf{G}_1(F) \backslash \mathbf{G}_1(\mathbb{A}_F)} \theta(\mathbf{h}, \mathbf{g}_1 \mathbf{g}; \varphi) f(\mathbf{g}_1 \mathbf{g}) d\mathbf{g}_1,$$

where $\mathbf{g} \in \mathbf{G}(\mathbb{A}_F)$ is arbitrary with $\det(\mathbf{g}) = \nu(\mathbf{h})$ and where φ is any Schwartz-Bruhat function on the space $V(\mathbb{A}_F)^3$.

One considers then the seesaw pair

$$(2.6) \quad \begin{array}{ccccc} I(\cdot, \varphi) & & GSp_6 & & \mathbf{H} & & I(\cdot, \varphi, f) \\ & & \downarrow & \swarrow & \downarrow & & \\ f & & \mathbf{G} & & H = GO(V) & & 1 \end{array},$$

where $\mathbf{H} = \mathbf{H}_V$ and $H = H_V$ depends on the choice of the quadratic space $V = V(D, \alpha)$.

The seesaw identity given in [12] yields in this situation

$$(2.7) \quad \int_{\mathbb{A}_F^\times \mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)} I(\mathbf{g}, \varphi^V) f(\mathbf{g}) d\mathbf{g} = \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} I(h, \varphi^V, f) dh$$

for all the possible choices of $V = V(D, \alpha)$.

From this we see that $L_3(\frac{1}{2}, \Pi) = 0$ if and only if for all choices of $V = V(D, \alpha)$, all $\varphi \in S(V(\mathbb{A}_F)^3)$, and all $f \in \Pi$, one has

$$(2.8) \quad \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} I(h, \varphi^V, f) dh = 0.$$

Finally, fixing one choice of $V = V(D, \alpha)$ (and omitting V from the notation) it is shown that

$$(2.9) \quad \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} I(h, \varphi, f) dh = \int_{(\mathbb{A}_F^\times \times \mathbb{A}_F^\times) \tilde{H}^0(F) \backslash \tilde{H}^0(\mathbb{A}_F)} \tilde{I}(h, \varphi, f) dh,$$

where $\tilde{I}(\cdot, \varphi, f)$ denotes the pullback of $I(\cdot, \varphi, f)$ to $\tilde{\mathbf{H}}(\mathbb{A}_F)$. Rewriting

$$(2.10) \quad \tilde{I}((\mathbf{b}_1, \mathbf{b}_2), \varphi, f) = \sum_r I^{1,r}(\mathbf{b}_1, \varphi, f) I^{2,r}(\mathbf{b}_2, \varphi, f)$$

for $\mathbf{b}_i \in D^\times(\mathbb{A}_E)$ and for suitable functions $I^{i,r}(\mathbf{b}_1, \varphi, f) \in \Pi^D$ one arrives at the assertion of the Theorem. (We recall that an automorphic function on a product of two groups is a finite sum of a product of automorphic functions on the two groups.)

The key tools in [7] in these steps are the analysis of the theta correspondence for similitude groups for the dual pair $(\mathbf{G} = (GL_2(F)^3)_0, \mathbf{H} = (GO(V)^3)_0)$ where the subscript 0 refers to the subgroup with equal similitude factors) and the use of the seesaw identity given above. Since the necessary facts about the theta correspondence for similitude groups were proved in [7] over an arbitrary number field, this analysis still applies if one replaces the pair $(GL_2(F)^3, D^\times(F)^3)$ by $(GL_2(E), D^\times(E))$ or $(GL_2(F) \times GL_2(F'), D^\times(F) \times D^\times(F'))$. It is then not difficult to check step by step that the proof carries over to our more general situation. For the sake of completeness we conclude this section by reconsidering two of the steps given above.

The first point that we wish to elaborate upon is about the case $D(F) = M_2(F)$, where as remarked earlier the theta integral giving the function $I(g, \varphi)$ on $GS p_6(\mathbb{A}_F)$ has to be defined by regularization. If F has at least one real place v_0 this can be done as described in [6, p. 620/621].

One considers then the differential operator z from the center of the universal enveloping algebra of $GS p_6(F_{v_0})$ (with $F_{v_0} = \mathbf{R}$) whose image under the Harish-Chandra isomorphism is the symmetric polynomial $(x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 - 5(x_1^2 + x_2^2 + x_3^2) + 21)/12$. This differential operator has the property that it maps to zero in the universal enveloping algebra of $SO(1, 1)$ under the homomorphism of universal enveloping algebras induced by the local theta correspondence, from which one can deduce that for any Schwartz function $\phi \in \mathcal{S}(M_{2,6}(\mathbb{R}))$ with Fourier transform $\hat{\phi} \in \mathcal{S}(M_{6,2}(\mathbb{R}))$, $\omega(z)\hat{\phi}$ vanishes on the set of matrices of rank ≤ 1 . As a consequence, $\theta(g, h, \omega(z)\phi)$ is of rapid decay along $O(V)(F) \backslash O(V)(\mathbb{A}_F)$, and hence it makes sense to define

$$(2.11) \quad I(g, \varphi) := \int_{H_1(F) \backslash H_1(\mathbb{A}_F)} \theta(g, h_1 h, \omega(z)\varphi) dh_1$$

where $h \in H(\mathbb{A}_F)$ with $\nu(h) = \nu(g)$ as above.

Using this modified $I(g, \varphi)$ and writing $V = V_0$ in this case one obtains

$$(2.12) \quad \begin{aligned} & \int_{\mathbb{A}_F^\times \mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)} I(\mathbf{g}, \varphi^{V_0}) f(\mathbf{g}) d\mathbf{g} \\ &= \int_{\mathbb{A}_F^\times \mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)} \left(\int_{H_1(F) \backslash H_1(\mathbb{A}_F)} \theta(\mathbf{g}, h_1 h, \omega(z_H)\varphi) dh_1 \right) f(\mathbf{g}) d\mathbf{g} \\ &= \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} I(h, \omega(z_H)\varphi^{V_0}, f) dh \\ &= \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} r(z_H) I(h, \varphi^{V_0}, f) dh \\ &= c \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} I(h, \varphi^{V_0}, f) dh \end{aligned}$$

Here, in the first equality we take an element z_H of the center of the universal enveloping algebra of the Lie algebra \mathfrak{h} of $H(\mathbf{R})$ with $\omega(z) = \omega(z_H)$ on the space $S(V_0(\mathbb{A}_F)^3)$; such an element exists and can be computed explicitly as described in [19], [13, Proposition 5.1.1]. The second equality is the usual seesaw identity with the differentiated theta kernel $\theta(g, h, \omega(z_H)\varphi)$ which is of rapid decay. The third equality, in which we write $r(z_H)$ for the action of z_H on functions on $H(\mathbb{A}_F)$, is a consequence of the fact that

the map $\varphi \mapsto I(h, \varphi, f)$ from $S(V_0(\mathbb{A}_F)^3)$ to the space of functions on $\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)$ is $H(\mathbb{A}_F)$ -invariant.

Since the theta lift $I(h, \varphi^{V_0}, f)$ of the cusp form f to $H(\mathbb{A}_F)$ is cuspidal we obtain the last equality (with c being equal to the constant term of $r(D_H)$) from the well known fact that

$$\int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} \phi_1(h) X \phi_0(h) dh = - \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} X \phi_1(h) \phi_0(h) dh$$

for $X \in \mathfrak{h}$ and a pair consisting of a cusp form ϕ_0 and an arbitrary automorphic form ϕ_1 on $H(\mathbb{A}_F)$ with trivial central characters.

An explicit computation gives $c = 1$, and one obtains an identity of the usual shape

$$\int_{\mathbb{A}_F^\times \mathbf{G}(F) \backslash \mathbf{G}(\mathbb{A}_F)} I(\mathbf{g}, \varphi^{V_0}) f(\mathbf{g}) d\mathbf{g} = \int_{\mathbb{A}_F^\times H(F) \backslash H(\mathbb{A}_F)} I(h, \varphi^{V_0}, f) dh.$$

Thus although the seesaw identity is invoked only for certain elements of the space $S(V_0(\mathbb{A}_F)^3)$ of the form $\omega(z)\varphi$, these elements lift cusp forms on \mathbf{G} surjectively onto cusp forms on \mathbf{H} .

If all archimedean places of F are complex one can replace the argument above by a similar argument using a nonarchimedean place at which all representations are unramified principal series and replacing the differential operator used above by a suitable Hecke operator as in [23]. Explicitly, one uses the Hecke operator z whose image under the Satake isomorphism is the polynomial

$$x_1 x_2 + x_1 x_3 + x_2 x_3 - (y_2 + y_1)(x_1 + x_2 + x_3) + (y_4 + y_3 + y_2 + y_1 + 4)$$

where $x_j = q^{s_j} + q^{-s_j}$, $y_j = q^j + q^{-j}$ and where q is the order of the residue field. This z maps to zero in the Hecke algebra of $SO(1, 1)$ under the homomorphism of Hecke algebras induced by the local theta correspondence and can therefore be used for regularizing the theta integral (similarly as done above at the real place) as in [23]; its image in the Hecke algebra of $SO(2, 2)$ acts by the nonzero scalar $y_4 - y_3 - 2y_2 + y_1$ on the constant function 1, and the argument from above works in the same way.

Our second remark concerns the proof of (2.9) in [6, 7] relating the period integral on $GO(4)$ to one on $GSO(4)$ which we elaborate upon in the next section.

3. AUTOMORPHIC FORMS ON DISCONNECTED GROUPS

The period integral which naturally occurs from the consideration of seesaw pairs in the last section is on the group $GO(4)$, whereas the final theorem is about period integral on $GSO(4)$. (By 2.10, it is trivial to see that the integral on $GSO(4)$ is nonzero if and only if the integral on the corresponding quaternion algebra is nonzero.) Harris and Kudla give a proof of this in [6]. This is a subtle point. We take the occasion to

make some general comments about automorphic forms on disconnected groups. We begin with some generality.

Let G be a group containing a subgroup G_0 of index 2, and let σ be an involution in G such that $G = G_0 \rtimes \langle \sigma \rangle$.

An irreducible representation of G_0 which is invariant under σ can be extended in exactly two distinct ways to an irreducible representation of G . In general, there is no way of distinguishing between these two extensions. However, there are many situations in which one can distinguish between these two extensions. One context in which one has learnt to distinguish between the two extensions is in the problem of base change for GL_n , say over a local field. Here given a quadratic extension K/k of local fields with σ the nontrivial element of the Galois group, for an irreducible admissible representation π of $GL_n(K)$ which is Galois invariant, there is a natural extension $\tilde{\pi}$ of π to $GL_n(K) \rtimes \langle \sigma \rangle$ such that the character identity,

$$\Theta_{\tilde{\pi}}(g, \sigma) = \Theta_{\tau}(\text{Nm}g),$$

holds for a representation τ of $GL_n(k)$, g any σ -regular element of $GL_n(K)$ with $\text{Nm}g$ its norm. This natural extension is achieved via uniqueness of Whittaker, or degenerate Whittaker model through a generality that we discuss below.

In general, suppose G_0 has a subgroup H together with a character $\chi : H \rightarrow \mathbb{C}^\times$ such that the inner-conjugation action by σ takes H to H , and χ to χ . Assuming that χ appears in π with multiplicity 1, we define a *preferred* extension $\tilde{\pi}$ of π to be the representation of G for which the character $\tilde{\chi} : H \rtimes \langle \sigma \rangle \rightarrow \mathbb{C}^\times$ extended from χ trivially across $\langle \sigma \rangle$ appears (as a quotient) in $\tilde{\pi}$. Of course this notion of preferred extension depends on the G -conjugacy class of the triple (σ, H, χ) . It happens in many situations that there is more than one natural choice of G -conjugacy class of the triple (σ, H, χ) in which case it may be of interest to compare the various preferred extensions.

In our context, there exists a canonical extension of a representation of the group $GSO(4)$ over a local field to one of $GO(4)$ which we discuss now.

Let (q, V) be a quadratic space of dimension 4, and discriminant 1. It is easy to see that $GO(V)$ operates transitively on the set of vectors $V_{q \neq 0} = \{v \in V \mid q(v) \neq 0\}$. Fix $v_0 \in V$ such that $q(v_0) \neq 0$, and let W be the orthogonal complement of v_0 , and σ the reflection around W . Let $H = O(W)$ be the stabiliser in $GO(V)$ of the vector v_0 . It can be seen that an irreducible representation π of $GSO(V)$ which is invariant under the action of σ contains a unique, up to scaling, linear form $\ell : \pi \rightarrow \mathbb{C}$ fixed under $H_0 = SO(W)$. The preferred extension of π to $GO(V)$ is the one for which $H = O(W)$ operates trivially on ℓ .

It can be seen that if (q, V) is the direct sum of 2 Hyperbolic spaces, then for an irreducible unramified representation π of $GSO(V)$, the preferred extension as defined in the previous paragraph is the unique spherical representation of $GO(V)$ containing π .

Similarly there exists a canonical extension of a σ -invariant automorphic representation of the group $GSO_4(\mathbb{A}_F)$ to one of $GO_4(\mathbb{A}_F)$ which we describe now, in greater generality.

Suppose G is an algebraic group over a number field F , containing an involution σ , and a subgroup G_0 such that $G = G_0 \rtimes \langle \sigma \rangle$. Let $\pi = \otimes_v \pi_v$ be an automorphic representation of G_0 realised on a space of functions on $G_0(F) \backslash G_0(\mathbb{A}_F)$ which is invariant under σ . Since the factorisation $\pi = \otimes_v \pi_v$ is unique up to isomorphism, each of the representations π_v is invariant under $\sigma_v (= \sigma, \text{ now thought of as an element of } G(F_v))$, i.e., $\sigma_v(\pi_v) \cong \pi_v$; this cannot be used to define an action of $G(F_v)$ on π_v as the action of σ_v is well-defined only up to a sign.

Recall that evaluating at the identity element gives rise to a linear form $e_\pi : \pi \rightarrow \mathbb{C}$ which is $G_0(F)$ -invariant. This linear form is clearly σ -invariant for the natural action of σ on π . To extend the automorphic representation $\pi = \otimes_v \pi_v$ of $G_0(\mathbb{A}_F)$ to $G(\mathbb{A}_F)$, first we extend the representation π_v of $G_0(F_v)$ to a representation $\tilde{\pi}_v$ of $G(F_v)$, keeping it spherical at almost all places, and such that the action of σ on $\pi = \otimes_v \pi_v$ is the same as that of $\otimes_v \sigma_v$ on it, but with no other constraints. (We will call the condition $\sigma = \otimes_v \sigma_v$, the *coherence condition*.) We keep the linear form $e_{\tilde{\pi}} : \tilde{\pi} = \otimes_v \tilde{\pi}_v \rightarrow \mathbb{C}$ same as e_π on $\pi = \tilde{\pi}$. Since e_π is σ as well as $G_0(F)$ -invariant, it is $G(F)$ -invariant, and thus we have constructed an automorphic representation of $G(\mathbb{A}_F)$.

In particular, if an automorphic representation $\pi = \otimes_v \pi_v$ of $G_0(\mathbb{A}_F)$ is σ -invariant, and the representations π_v of $G_0(F_v)$ have a canonical extension to representations $\tilde{\pi}_v$ of $G(F_v)$, there is a canonical extension of automorphic forms from $G_0(\mathbb{A}_F)$ to $G(\mathbb{A}_F)$ assuming that the coherence condition $\sigma = \otimes_v \sigma_v$ is satisfied. In the context of $GSO(4)$, this canonical extension is the one that appears as theta lift from $GL(2)$, cf. [8], the coherence condition being automatic here.

The following lemma relates the non-vanishing of period integrals on $GO_4(\mathbb{A}_F)$ to that on $GSO_4(\mathbb{A}_F)$.

Lemma 3.1. *Let E be a cubic semi-simple algebra over a number field F , and Π an automorphic representation on $GO_4(\mathbb{A}_E)$ obtained by theta lifting from $GL_2(\mathbb{A}_E)$. Then a nonzero $GSO_4(\mathbb{A}_F)$ -invariant linear form on Π is $GO_4(\mathbb{A}_F)$ -invariant.*

Proof : Because of the multiplicity 1 theorems in [15], and [16], it suffices to prove the corresponding local statement. Let \mathbb{K} be a cubic semi-simple algebra over a local field k , and π an irreducible representation of $GO_4(\mathbb{K})$ obtained by theta lifting from $GL_2(\mathbb{K})$. Then we prove that any $GSO_4(k)$ -invariant linear form on π is $GO_4(k)$ -invariant.

Identifying GSO_4 to the quotient of $D^* \times D^*$ by the scalar matrices (t, t^{-1}) , where D is a quaternion algebra over k , GO_4 becomes the corresponding quotient of $(D^* \times D^*) \rtimes \mathbb{Z}/2$ where $\mathbb{Z}/2$ operates by switching the 2 factors.

Under the identification of GSO_4 with $D^* \times D^*$ divided by a central subgroup, the representation π which is invariant under the switching of the factors, can be written as $\pi_1 \otimes \pi_1$ for an irreducible representation π_1 of $D^*(\mathbb{K})$. Any $GSO_4(k)$ -invariant linear form on $\pi_1 \otimes \pi_1$ is up to a scalar, of the form $\ell \otimes \ell : \pi_1 \otimes \pi_1 \rightarrow \mathbb{C}$ for a $D^*(k)$ -invariant linear form $\ell : \pi_1 \rightarrow \mathbb{C}$. Clearly, the linear form $\ell \otimes \ell : \pi_1 \otimes \pi_1 \rightarrow \mathbb{C}$ is invariant under the switching of the factors, and that completes the proof of the lemma.

4. GLOBALISATION OF LOCALLY DISTINGUISHED REPRESENTATIONS

In this section we prove a globalisation theorem which is the main technical tool for establishing local theorems from the corresponding global theorems.

Let k be a local field, G a reductive algebraic group over k , and H a closed algebraic subgroup of G defined over k . We abuse notation and use G, H to also denote the corresponding group of k -rational points. For a character $\chi : H \rightarrow \mathbb{C}^\times$, we define a representation π of G to be χ -distinguished by H if there exists a nonzero linear form $\ell : \pi \rightarrow \mathbb{C}$ on which H operates via the character χ .

Let $\underline{G}, \underline{H}$ now be algebraic groups defined over a number field F , and let χ be a one dimensional automorphic representation of $\underline{H}(\mathbb{A}_F)$. Let \underline{Z} be the identity component of the center of the algebraic group \underline{G} , and which we assume without loss of generality in the rest of this section to be contained in \underline{H} . We assume further that $\underline{H}/\underline{Z}$ has no F -rational character. We abuse notation to denote the restriction of χ to $\underline{Z}(F) \backslash \underline{Z}(\mathbb{A}_F)$ also by χ . An automorphic representation Π of $\underline{G}(\mathbb{A}_F)$ is said to be globally χ -distinguished by \underline{H} , if the period integral

$$\int_{\underline{H}(F)\underline{Z}(\mathbb{A}_F) \backslash \underline{H}(\mathbb{A}_F)} f(h) \chi^{-1}(h) dh,$$

is nonzero for some $f \in \Pi$. Observe that for the function $f(h) \chi^{-1}(h)$ on $\underline{H}(\mathbb{A}_F)$ to be $\underline{Z}(\mathbb{A}_F)$ -invariant, we must have the central character of Π restricted to $\underline{Z}(\mathbb{A}_F)$ be the same as χ restricted to $\underline{Z}(\mathbb{A}_F)$. We note that by the theorem of Borel and Harish-Chandra, under our assumption that $\underline{H}/\underline{Z}$ has no F -rational characters, $\underline{H}(F)\underline{Z}(\mathbb{A}_F) \backslash \underline{H}(\mathbb{A}_F)$ has finite volume. (Here \underline{H} need not be reductive.) Therefore the period integral makes sense, for instance, for f a cusp form, as cusp forms are known to be bounded on $\underline{G}(\mathbb{A}_F)$.

Here is the main theorem of this section.

Theorem 4.1. *Let \underline{H} be a closed subgroup of a reductive group \underline{G} , both defined over a number field F . Let \underline{Z} be the identity component of the center of \underline{G} , which we assume is contained in \underline{H} such that $\underline{H}/\underline{Z}$ has no F -rational character. Let $\chi = \prod_v \chi_v$ be a one dimensional automorphic representation of $\underline{H}(\mathbb{A}_F)$. Suppose that S is a finite set of non-Archimedean places of F , and π_v a supercuspidal representation of $\underline{G}(F_v)$ for all $v \in S$,*

which is χ_v -distinguished by $\underline{H}(F_v)$, i.e., $\text{Hom}_{\underline{H}(F_v)}(\pi_v, \chi_v) \neq 0$ for all $v \in S$. Let T be a finite set of places containing S and all the infinite places, such that \underline{G} is quasi-split at places outside T , and χ_v is unramified outside T , i.e., if $\underline{G}(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup of $\underline{G}(F_v)$, then χ_v is trivial on $\underline{H}(F_v) \cap \underline{G}(\mathcal{O}_v)$. Then there exists a global automorphic form $\Pi = \otimes \Pi_v$ of $\underline{G}(\mathbb{A}_F)$, necessarily cuspidal, such that $\Pi_v = \pi_v$ for $v \in S$, and Π_v is unramified at all finite places of F outside T , and an $f \in \Pi$ such that

$$\int_{\underline{H}(F)\underline{Z}(\mathbb{A}_F)\backslash\underline{H}(\mathbb{A}_F)} f(h)\chi(h)^{-1}dh \neq 0.$$

Before beginning the proof of this theorem, we gather together a few results which go into the proof. The first result we need is a lemma about when certain global points in an adelic space are discrete.

Lemma 4.2. *Let \underline{H} be a closed algebraic subgroup of a reductive group \underline{G} , both defined over a number field F . Assume that there exists a finite dimensional algebraic representation V of \underline{G} defined over F , and a vector $v \in V$ whose stabiliser in \underline{G} is \underline{H} . Then the orbit of $\underline{G}(F)$ passing through the identity element of $\underline{G}(\mathbb{A}_F)/\underline{H}(\mathbb{A}_F)$ is a discrete subset of $\underline{G}(\mathbb{A}_F)/\underline{H}(\mathbb{A}_F)$.*

Proof : The hypothesis of the lemma identifies $\underline{G}(\mathbb{A}_F)/\underline{H}(\mathbb{A}_F)$ to a subset of $V(\mathbb{A}_F)$ containing the vector $v \in V(F)$. Since the $\underline{G}(F)$ orbit of $v \in V(F)$ is contained in $V(F)$ which is a discrete subset of $V(\mathbb{A}_F)$, the lemma follows. (We remark that if $\phi : X' \rightarrow X$ is a continuous injective map of topological spaces, D a subset of X' for which $\phi(D)$ is discrete in X , then D is discrete in X' . We use this remark for $X' = \underline{G}(\mathbb{A}_F)/\underline{H}(\mathbb{A}_F)$, and $X = V(\mathbb{A}_F)$.)

For the following well-known lemma about algebraic groups see for instance Propositions 7.8 and 7.7 of [3].

Lemma 4.3. 1. *Let H be a closed subgroup of an algebraic group G , both defined over a field k . Assume that H has no nontrivial characters defined over k . Then there exists a finite dimensional algebraic representation V of G defined over k , and a vector $v \in V$ whose stabiliser in G is H . In particular, if Z , the identity component of the center of G , is contained in H , and if H/Z has no k -rational characters, then there exists a finite dimensional algebraic representation V of G defined over k , and a vector $v \in V$ whose stabiliser in G is H .*

2. *Let H be a reductive subgroup of a connected algebraic group G , both defined over a field k . Then there exists a finite dimensional algebraic representation V of G defined over k , and a vector $v \in V$ whose stabiliser in G is H .*

For a continuous function f on G belonging to $C_c^\infty(Z \backslash G, \chi)$ (these are functions f on G with $f(zg) = \chi(z)f(g)$ for $z \in Z$, $g \in G$, and compactly supported modulo Z), define the 'orbital integral' along H to be the function on G given by

$$g \mapsto \Phi(f, g) := \int_{Z \backslash H} f(hg) \chi(h)^{-1} dh.$$

Clearly $\Phi(f, g)$ is in the set $C_c^\infty(H \backslash G, \chi)$ of continuous functions ϕ on G which have the property that $\phi(hg) = \chi(h)\phi(g)$ for $h \in H$, $g \in G$ and are compactly supported modulo H . It is easy to see that the 'orbital integral' map is a surjection from $C_c^\infty(Z \backslash G, \chi)$ to $C_c^\infty(H \backslash G, \chi)$.

The following general lemma is a consequence of Schur orthogonality relations, cf. the Corollary in the appendix to [5]

Lemma 4.4. *Let k be a non-Archimedean local field, G a reductive algebraic group over k , and H a closed subgroup of G . Let Z be the identity component of the center of G , which we assume is contained in H . Let $\chi : H \rightarrow \mathbb{C}^\times$ be a character of H as well as its restriction to Z . Let (π, V) be a supercuspidal representation of G , and (π', V') its contragredient. Suppose $\ell : \pi \rightarrow \mathbb{C}$ be a linear form on π on which H acts by χ , and $v \in \pi$ is chosen so that $\ell(v) \neq 0$. For $v' \in V'$, let $f(g) = v'(gv)$ be a matrix coefficient of π . Then*

$$g \mapsto \int_{Z \backslash H} f(hg) \chi(h)^{-1} dh$$

is a nonzero smooth function on G with compact support in $H \backslash G$, i.e., belongs to $C_c^\infty(H \backslash G, \chi)$.

Finally, here is the statement of the relative trace formula that we will use; see [5], Theorem 2.

Theorem 4.5. *Assume that $f = \otimes f_v \in C_c^\infty(\underline{Z}(\mathbb{A}_F) \backslash \underline{G}(\mathbb{A}_F), \chi)$ is such that at some finite place v of F , f_v is the matrix coefficient of a supercuspidal representation π_v of $\underline{G}(F_v)$. Then,*

$$\sum_{\pi} \sum_{\phi \in \mathbb{B}_{\pi}} P(R(f)\phi) \overline{\phi(1)} = \sum_{\gamma \in \underline{H}(F) \backslash \underline{G}(F)} \prod_v \Phi_v(f_v, \gamma),$$

where π ranges over χ -distinguished cuspidal representations of $\underline{G}(\mathbb{A}_F)$ which have π_v as the local component at v , \mathbb{B}_{π} is an orthonormal basis for π , $R(f)\phi$ denotes the action of $f \in C_c^\infty(\underline{Z}(\mathbb{A}_F) \backslash \underline{G}(\mathbb{A}_F), \chi)$ on the space of automorphic forms (on which the center operates via χ) by convolution, and $P(R(f)\phi)$ denotes the period integral of $R(f)\phi$ along \underline{H} :

$$P(R(f)\phi) = \int_{\underline{H}(F) \backslash \underline{Z}(\mathbb{A}_F) \backslash \underline{H}(\mathbb{A}_F)} (R(f)\phi)(h) \chi(h)^{-1} dh.$$

Proof of Theorem 4.1: The proof of this theorem is via the method of relative trace formula, and is a variation on the proof of Hakim-Murnaghan which is for $\underline{G} = GL_n$, and \underline{H} the subgroup of the fixed points of an involution. We repeat most of their proof, and observe that the choice of their $\underline{G}, \underline{H}$ is not really used in the proof, their explicit proof being replaced by the general lemmas 4.2 and 4.3 from the theory of Algebraic groups. It suffices to show that there exists $f = \otimes_v f_v \in C_c^\infty(\underline{Z}(\mathbb{A}_F) \backslash \underline{G}(\mathbb{A}_F), \chi)$ such that $\prod_v \Phi_v(f_v, \gamma)$ is nonzero for exactly one $\gamma \in \underline{H}(F) \backslash \underline{G}(F)$. We will do this by choosing f_v appropriately at each place v of F . For places in S , we choose f_v to be a matrix coefficient of π_v . Since the left and right translates of a matrix coefficient is again a matrix coefficient, lemma 4.4 allows us to assume that $\Phi_v(f_v, 1) \neq 0$. For places outside T , define $f_v(zk) = \chi_v(z)$ where $z \in \underline{Z}(F_v)$, and k an element of the hyperspecial maximal compact subgroup $\underline{G}(\mathcal{O}_v)$ of $\underline{G}(F_v)$, and define f_v to be 0 outside $\underline{Z}(F_v) \cdot \underline{G}(\mathcal{O}_v)$; this is well-defined because χ_v is unramified. For the remaining places of F we choose $f_v \in C_c^\infty(\underline{Z}(F_v) \backslash \underline{G}(F_v), \chi_v)$ arbitrarily so that $\Phi(f_v, 1)$ is nonzero. In this last choice, ensure that at some place, say an infinite place v_0 of F , the support of the function $\Phi(f_{v_0}, g)$ is a certain small neighborhood of the identity in $\underline{G}(F_{v_0}) / \underline{H}(F_{v_0})$. Since the 'orbital integral' map is easily seen to be a surjection from $C_c^\infty(\underline{Z}(F_{v_0}) \backslash \underline{G}(F_{v_0}), \chi_{v_0})$ to $C_c^\infty(\underline{H}(F_{v_0}) \backslash \underline{G}(F_{v_0}), \chi_{v_0})$, this is possible to achieve. Finally, we see that $\prod_v \Phi_v(f_v, \gamma)$, can be assumed to be nonzero for exactly one $\gamma \in \underline{H}(F) \backslash \underline{G}(F)$, in fact for $\gamma = 1$, using lemma 4.2, and the next lemma whose trivial proof we will omit. This completes the proof of the theorem.

Lemma 4.6. *Let X and Y be locally compact Hausdorff topological spaces, and $D \subset X \times Y$ be a discrete subset containing a point (x_0, y_0) . Let $K \subset X$ be a compact set containing x_0 . Then there exists an open set $U \subset Y$, such that $D \cap (K \times U) = (x_0, y_0)$.*

Remark : Because the globalisation theorem is useful in a variety of contexts, we have tried to state theorem 4.1 quite generally, without restricting \underline{H} to be reductive. In particular this result contains as a special case the well-known result that locally generic supercuspidal representations can be embedded as components of a generic automorphic representation. It can also be used to globalise local representations with particular Fourier-Jacobi, or Bessel models. We will discuss an example involving an \underline{H} which is neither unipotent, nor reductive.

Example : Let $G = GL_4 \times GL_2$, $H = \Delta GL_2 \cdot N$ where ΔGL_2 is the embedding of GL_2 in G given by

$$g \in GL_2 \rightarrow \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \times g \in GL_4 \times GL_2,$$

and N is the unipotent radical of the standard $(2, 2)$ parabolic in GL_4 ,

$$N = \left\{ \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} \mid X \in M_2(k) \right\}.$$

An additive character ψ_0 of k defines an additive character ψ of N by $\psi(X) = \psi_0(\text{tr} X)$. This extends to a character χ of H by $\chi(g, n) = \psi(n)$. A χ -distinguished representation of $GL_4 \times GL_2$ of the form $\pi_1 \otimes \pi_2$ is nothing but a representation π_1 of GL_4 whose maximal quotient $\pi_{1,\psi}$ on which N operates via ψ which is a representation space of GL_2 has π_2^\vee in its quotient. Theorem 4.1 allows us to globalise this to construct nonvanishing period integrals of the form:

$$\int_{GL_2(F)\mathbb{A}_F^\times \backslash GL_2(\mathbb{A}_F) \times M_2(F) \backslash M_2(\mathbb{A}_F)} f_1 \begin{pmatrix} g & gX \\ 0 & g \end{pmatrix} f_2(g) \psi(X) dg dX$$

where f_1 is a cusp form belonging to an automorphic representation Π_1 on $GL_4(\mathbb{A}_F)$ with P as the $(2, 2)$ maximal parabolic; the function f_2 belongs to a cuspidal automorphic representation Π_2 of $GL_2(\mathbb{A}_F)$. If the non-vanishing of this integral can be related to the non-vanishing of the central critical value $L(\Lambda^2 \Pi_1 \otimes \Pi_2^\times, \frac{1}{2})$, as is expected, then following the method of the next section, we will obtain a local result about the constituents of the twisted Jacquet functor $\pi_{1,\psi}$, as conjectured in [17] in terms of epsilon factors, and verified there for nonsupercuspidal representations of GL_4 .

5. A LOCAL CONSEQUENCE

Let \mathbb{K} be a commutative semisimple cubic algebra over a non-archimedean local field k . In this section we study conditions under which irreducible, admissible representations of $GL_2(\mathbb{K})$ have $GL_2(k)$ -invariant linear forms. The results are expressed in terms of certain epsilon factors. The case $\mathbb{K} = k \oplus k \oplus k$, has been previously studied in [15] in the odd residue characteristic and completed in [18]. In fact the method of proof given in [18] works also for the case when \mathbb{K} is of the form $K \oplus k$ where K is a quadratic field extension of k , but does not seem to work when \mathbb{K} is a cubic field extension of k . For the case when $\mathbb{K} = K \oplus k$ with K a quadratic field extension of k , or \mathbb{K} a cubic field extension of k , such a result was proved in [16] exactly for those cases for which the representation of $GL_2(\mathbb{K})$ was not supercuspidal; the local methods employed there were quite inadequate to handle supercuspidal representations in these cases. Our present method proves such a result in complete generality.

We begin by fixing some notation. Let D_k be the unique quaternion division algebra over k , and let $D_{\mathbb{K}} = D_k \otimes_k \mathbb{K}$. If π is a discrete series representation of $GL_2(\mathbb{K})$ (by which we will always mean an irreducible representation which has a twist whose matrix coefficients are square integrable modulo centre), we associate an irreducible, admissible representation π' of $D_{\mathbb{K}}^\times$ as follows. If $\mathbb{K} = K \oplus k$, where K is a quadratic field extension of k , then

$\mathrm{GL}_2(\mathbb{K}) = \mathrm{GL}_2(K) \times \mathrm{GL}_2(k)$ and the discrete series representation π of $\mathrm{GL}_2(\mathbb{K})$ is the tensor product $\pi_1 \otimes \pi_2$ of a discrete series representation π_1 of $\mathrm{GL}_2(K)$ and a discrete series representation π_2 of $\mathrm{GL}_2(k)$. In this case $D_{\mathbb{K}}^{\times} = \mathrm{GL}_2(K) \times D_k^{\times}$. We define the representation π' of $D_{\mathbb{K}}^{\times}$ to be $\pi_1 \otimes \pi'_2$ where π'_2 is the representation of D_k^{\times} associated to the discrete series representation π_2 of $\mathrm{GL}_2(k)$ by the Jacquet-Langlands correspondence. If \mathbb{K} is a cubic field extension of k then $D_{\mathbb{K}}$ is the unique quaternion division algebra over the field \mathbb{K} and we let π' be the representation of $D_{\mathbb{K}}^{\times}$ associated to the discrete series representation π of $\mathrm{GL}_2(\mathbb{K})$ (by the Jacquet-Langlands correspondence).

In this paper we will be dealing only with *generic* representations of $\mathrm{GL}_2(\mathbb{K})$ by which we will mean any infinite dimensional representation of $\mathrm{GL}_2(\mathbb{K})$ if \mathbb{K} is a cubic field extension of k , and if $\mathbb{K} = K \oplus k$ to be the tensor product of an infinite dimensional representation of $\mathrm{GL}_2(K)$ with an infinite dimensional representation of $\mathrm{GL}_2(k)$.

The main results proved in [16] were the following.

- (1) **Multiplicity one theorem:** For an irreducible, admissible representation π of $\mathrm{GL}_2(\mathbb{K})$, the space of $\mathrm{GL}_2(k)$ -invariant linear forms on π is at most one-dimensional.
- (2) **Dichotomy principle:** Let π be an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{K})$ such that the central character of π restricted to $k^{\times} \subseteq \mathbb{K}^{\times}$ is trivial. Then either there exists a $\mathrm{GL}_2(k)$ -invariant linear form on π which is unique up to scalars or the representation π of $\mathrm{GL}_2(\mathbb{K})$ is a discrete series representation and there exists a D_k^{\times} -invariant linear form on the representation π' of $D_{\mathbb{K}}^{\times}$ which is also unique up to scalars. Moreover, only one of the two possibilities occurs.
- (3) **Theorem about epsilon factors:** Let π be an irreducible, admissible, generic representation of $\mathrm{GL}_2(\mathbb{K})$ such that the central character of π restricted to $k^{\times} \subseteq \mathbb{K}^{\times}$ is trivial. Then for ψ a non-trivial additive character of k , $\epsilon(M_{\mathbb{K}}^k \sigma_{\pi}, \psi) \cdot \omega_{\mathbb{K}/k}(-1)$ is independent of ψ and takes the value $+1$ if and only if the representation π of $\mathrm{GL}_2(\mathbb{K})$ has a $\mathrm{GL}_2(k)$ -invariant linear form, and takes the value -1 if and only if the representation π of $\mathrm{GL}_2(\mathbb{K})$ is a discrete series representation and the representation π' of $D_{\mathbb{K}}^{\times}$ has a D_k^{\times} -invariant linear form.

In the theorem on epsilon factors, $M_{\mathbb{K}}^k \sigma_{\pi}$ is a certain 8-dimensional representation of the Deligne-Weil group of k defined in [16], which, for \mathbb{K} a cubic field extension of k , is associated by the process of ‘tensor induction’ to the 2-dimensional representation σ_{π} of the Deligne-Weil group of \mathbb{K} where σ_{π} is the Langlands parameter of the irreducible admissible representation π of $\mathrm{GL}_2(\mathbb{K})$. We have used $\omega_{\mathbb{K}/k}$ for the following quadratic character of k^{\times} .

(a) If $\mathbb{K} = K \oplus k$ with K a quadratic field extension of k then $\omega_{\mathbb{K}/k} = \omega_{K/k}$ where $\omega_{K/k}$ is the quadratic character of k^\times associated by local classfield theory to K .

(b) If \mathbb{K} is a cubic field extension of k then $\omega_{\mathbb{K}/k}$ will be the trivial character of k^\times if \mathbb{K} is Galois over k , and will be the quadratic character $\omega_{L/k}$ of k^\times if \mathbb{K} is not Galois over k and L is the unique quadratic extension of k contained in the Galois closure of \mathbb{K} .

Of these three theorems, only the first two were proved in complete generality in [16], and the third theorem on epsilon factors was checked case-by-case only for non-supercuspidal representations.

Remark : In [16], the epsilon factor used was that of an eight dimensional representation $M_{\mathbb{K}}^k \sigma_\pi$ of the Weil-Deligne group W'_k . In the present work, this Galois theoretic epsilon factor is changed to the one associated to π by the Langlands-Shahidi method, as in [21]. It is natural to expect that the two epsilon factors are the same, but it has not been proven yet except in the case $E = F \oplus F \oplus F$, which has been done by Ramakrishnan [20].

Proof of the Theorem on epsilon factors: The proof of the theorem on epsilon factors is exactly along the same lines as given in [18], except that we globalise a locally distinguished representation by means of theorem 4.1, thus via the methods of relative trace formula instead of the Burger-Sarnak principle that we employed in [18].

Given a commutative semisimple cubic algebra \mathbb{K} over k , realise this as the local component of a cubic field extension E of a number field F at a place v_0 , i.e., $k = F_{v_0}$, and $\mathbb{K} = E \otimes_F F_{v_0}$. We assume that both F and E are totally real. Assume that π has a $GL_2(k)$ -invariant linear form. For π non-supercuspidal, the theorem on epsilon factors is proved by direct calculation in [16]. We therefore assume in the rest of the proof that π is a supercuspidal representation of $GL_2(\mathbb{K})$. Then by theorem 4.1, π can be realised as the local component of an automorphic representation Π of $GL_2(\mathbb{A}_E)$, and there exists an $f \in \Pi$ such that

$$\int_{GL_2(F)\mathbb{A}_F^\times \backslash GL_2(\mathbb{A}_F)} f(h)dh \neq 0,$$

with the further property that Π is unramified at any finite place of E which is not lying over v_0 . By theorem 1.1, this means that $L_3(\frac{1}{2}, \Pi) \neq 0$. Therefore, in particular, the sign $\epsilon_3(\frac{1}{2}, \Pi)$ in the functional equation is 1. Before proceeding further, define a character $\omega_{E/F} : \mathbb{A}_F^\times \rightarrow \pm 1$ to be the sign of the permutation representation of the Galois group $\text{Gal}(\bar{F}/F)$ on the set $\text{Gal}(\bar{F}/F)/\text{Gal}(\bar{F}/E)$. Clearly $\omega_{E/F}$ has local components $\omega_{\mathbb{K}/k}$ where k is a completion of F , and $\mathbb{K} = k \otimes_F E$. In particular,

$\prod_v \omega_{E_v/F_v}(-1) = 1$ where the product is over all the places v of F , and $E_v = E \otimes_F F_v$.
Since

$$\epsilon_3\left(\frac{1}{2}, \Pi\right) = \prod_v \epsilon_3\left(\frac{1}{2}, \Pi_v\right) = \prod_v \epsilon_3\left(\frac{1}{2}, \Pi_v\right) \omega_{E_v/F_v}(-1) = 1,$$

(where v belongs to the set of places of F), if we can prove that all the factors $\epsilon_3(\frac{1}{2}, \Pi_v) \omega_{E_v/F_v}(-1)$ are equal to 1, except at v_0 , we would be able to deduce that $\epsilon_3(\frac{1}{2}, \Pi_{v_0}) \omega_{\mathbb{K}/k}(-1) = 1$ too. But $\epsilon_3(\frac{1}{2}, \Pi_v) \omega_{E_v/F_v}(-1) = 1$ for all finite places v , $v \neq v_0$, as Π_v is an unramified representation, for which this is an easy calculation as done in proposition 8.7 of [16]. (As an erratum to [16], we note that the factor $\omega_{E_v/F_v}(-1)$ is missing in the statement of the proposition 8.7 of [16].) For v Archimedean, again $\epsilon_3(\frac{1}{2}, \Pi_v) \omega_{E_v/F_v}(-1) = 1$. (We note that since we have assumed that all the real places of F split in E , $\omega_{E_v/F_v}(-1) = 1$, and that by the theorem on triple products at infinity, cf. [15], $\epsilon_3(\frac{1}{2}, \Pi_v) = 1$.) Thus we conclude that if π_{v_0} has a $GL_2(k)$ -invariant linear form, then $\epsilon_3(\frac{1}{2}, \Pi_{v_0}) \omega_{\mathbb{K}/k}(-1) = 1$. On the other hand if π does not have a $GL_2(k)$ -invariant linear form, then by the dichotomy principle proved in [16], π is a discrete series representation, and there exists a D_k^\times -invariant linear form on the representation π' of $D_{\mathbb{K}}^\times$ associated to π by the Jacquet-Langlands correspondence. Globalise π' to a distinguished automorphic form Π' on $(D \otimes_F E)^\times$ where D is the quaternion division algebra over F , which remains a division algebra at the place v_0 , and which is a division algebra at exactly one place at infinity of F , but at no other finite or infinite place of F besides these two. We can assume again that Π' is unramified at all finite places of E except the one over v_0 . This time $\epsilon_3(\frac{1}{2}, \Pi_v) \omega_{E_v/F_v}(-1) = -1$ at the infinite place of F where D has remained a division algebra, and is equal to 1 at all the other places besides v_0 . Thus, $\epsilon_3(\frac{1}{2}, \Pi_{v_0}) \omega_{\mathbb{K}/k}(-1) = -1$. This completes the proof of the theorem on epsilon factors.

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