ON THE $SL(2)$ PERIOD INTEGRAL

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Abstract. Let E/F be a quadratic extension of number fields. For a cuspidal representation π of $\mathrm{SL}_2(\mathbb{A}_E)$, we study the non-vanishing of the period integral on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)$. We characterise the non-vanishing of the period integral of π in terms of π being generic with respect to characters of $E\backslash\mathbb{A}_E$ which are trivial on $\mathbb{A}_F$. We show that the period integral in general is not a product of local invariant functionals, and find a necessary and sufficient condition when it is. We exhibit cuspidal representations of $\mathrm{SL}_2(\mathbb{A}_E)$ whose period integral vanishes identically while each local constituent admits an SL_2-invariant linear functional. Finally, we construct an automorphic representation π on $\mathrm{SL}_2(\mathbb{A}_E)$ which is abstractly $\mathrm{SL}_2(\mathbb{A}_F)$ distinguished but none of the elements in the global L-packet determined by π is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$.

Contents

1. Introduction

Let F be a number field and $\mathbb{A}_F$ its adèle ring. Let G be a reductive algebraic group over F and H a reductive subgroup of G over F. Assume that the center Z_H of H is contained in the center Z_G of G, a condition that holds in the cases we study in this paper. For an

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automorphic form ϕ on $G(\mathbb{A}_F)$ on which $Z_H(\mathbb{A}_F)$ acts trivially, the *period integral* of ϕ with respect to H is defined to be the integral (when convergent, which is the case if ϕ is cuspidal)

$$\mathcal{P}(\phi) = \int_{H(F)Z_H(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi(h)dh,$$

where dh is the natural measure on $H(F)\backslash H(\mathbb{A}_F)$, the so called Tamagawa measure. An automorphic representation π of $G(\mathbb{A}_F)$ is said to be globally distinguished with respect to H if this period integral is nonzero for some $\phi \in \pi$. More generally, if χ is a one-dimensional representation of $H(\mathbb{A}_F)$ trivial on $H(F)$ such that $Z_H(\mathbb{A}_F)$ acts trivially on $\phi(h)\chi(h)$, and

$$\int_{H(F)Z_H(\mathbb{A}_F)\backslash H(\mathbb{A}_F)} \phi(h)\chi(h)dh,$$

is nonzero for some $\phi \in \pi$, then π is said to be χ-distinguished.

The notion of distinguishedness has been extensively studied (and is being studied) both locally and globally for $G = \mathrm{GL}_n(\mathbb{A}_E)$, and $H = \mathrm{GL}_n(\mathbb{A}_F)$, E a quadratic extension of a number field F [F1, F2, H, H-L-R]. In this case (just as in the local case), distinguishedness implies that

$$\pi^\sigma \cong \pi^\vee,$$

where σ is the nontrivial automorphism of E/F, and is equivalent to the Asai L-function of π, say $L(s, r(\pi))$, having a pole at $s = 1$. From the factorisation of L-functions:

$$L(s, \pi \times \pi^\sigma) = L(s, r(\pi))L(s, r(\pi) \otimes \omega_{E/F}),$$

it follows that if $\pi = \otimes\pi_v$, v running over all places of E, is an automorphic representation of $G = \mathrm{GL}_n(\mathbb{A}_E)$, with all the local components π_v distinguished, then π is either distinguished, or $\omega_{E/F}$ distinguished, where $\omega_{E/F}$ is the quadratic character of $\mathbb{A}_F^*$ associated to the extension E/F. Thus local distinguishedness for GL_n 'almost' implies global distinguishedness. Observe furthermore that π cannot be both distinguished and $\omega_{E/F}$-distinguished (as it would contribute a pole of order 2 to the Rankin product L-function $L(s, \pi \otimes \pi^\sigma)$).

In an earlier work, the authors had studied the distinguishedness property for SL_2 in the local case. We carry out the global analysis of this case here. Since for GL_n as mentioned earlier, locally distinguished automorphic representations are either distinguished, or $\omega_{E/F}$ distinguished, one is led to ask whether locally distinguished automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ are globally distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. We show in this paper that this is not the case.

If π is an automorphic representation of $G(\mathbb{A}_F)$ for G a general reductive group over a number field F, then π factorizes as $\pi = \otimes \pi_v$, v running over all places of F. For an algebraic subgroup H of G defined over F, the period integral $\phi \mapsto \mathcal{P}(\phi)$ is an $H(\mathbb{A}_F)$ invariant form on π. If one knows that the space of $H(F_v)$ invariant forms on an irreducible representation of $G(F_v)$ is at most one dimensional for any place v of F, then the invariant form $\phi \mapsto \mathcal{P}(\phi)$ is a "product" of local invariant forms times a global constant which one expects to be intimately connected with special values of automorphic L-functions associated to π.

Recently, a very interesting case has been studied by Jacquet in [J] where the space of $H(F_v)$-invariant forms on an irreducible admissible representation of $G(F_v)$ is not always one dimensional but for which the functional $\phi \mapsto \mathcal{P}(\phi)$ is nevertheless expressible as a product of local factors. Jacquet's example is for the case: $(G, H) = (\mathrm{Res}_{E/F}\mathrm{GL}_3, \mathrm{U}_3)$. We have recently learnt that Jacquet has generalised this work to GL_n.

In an earlier work [A-P], the authors analyzed the situation for $G = \mathrm{Res}_{E/F}\mathrm{SL}_2$ and $H = \mathrm{SL}_2$ locally and found that multiplicity one fails for the space of H-invariant forms on an irreducible admissible representation of G which can be even supercuspidal. We analyze in this paper whether the period integral is factorizable in this case. We find that this is so if the automorphic representation π is not monomial, and also in the case when it is monomial and comes from 3 quadratic extensions of E of which only one is Galois over F; in other cases, the period integral is not factorisable. What is most appealing about this result is that it is the exact global analogue (interpreted via Galois theory) of the local results obtained in [A-P] about the dimension of the space of $SL_2(k)$-invariant forms for a representation of $SL_2(K)$ where K is a quadratic extension of a non-archimedean local field k.

In trying to understand representations of $\mathrm{GL}_2(\mathbb{A}_E)$ which are distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$, we are naturally led to investigate a related concept, which we call *pseudo-distinguishedness*. They are studied in section 7.

In the final section, we construct an automorphic representation $\pi = \otimes \pi_v$ of $\mathrm{SL}_2(\mathbb{A}_E)$ which is abstractly $\mathrm{SL}_2(\mathbb{A}_F)$ distinguished but none of the elements in the global L-packet determined by π is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$.

The main results proved in this paper are theorems 4.2, 5.2, 6.8 and 8.2. In section 2 we take up some preliminary results about the structure of L-packets for SL_2. In particular, a rather simple proof is provided for the stability of non-monomial representations for SL_2 and more generally of primitive representations for SL_n, see Lemma 2.5, and Remark 2.7.

We end the introduction by mentioning that the examples constructed in this paper of automorphic representations $\pi = \otimes \pi_v$ of $\mathrm{SL}_2(\mathbb{A}_E)$ which are abstractly $\mathrm{SL}_2(\mathbb{A}_F)$ distinguished but which are not globally distinguished, or have the much stronger property of having no member in its L-packet which is globally distinguished seem a little ad hoc. Perhaps there is a certain multiplicity formula in the spirit of Labesse-Langlands, cf. [L-L], which determines when a member of an L-packet determined by π has non-vanishing period integral on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)$; this we have not been able to achieve here.

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2. Some lemmas about L-packets on SL_2

In this section we recall for reader's convenience some standard results about SL_2. We supply the proofs to emphasize their elementary nature (cf. [La]).

For an irreducible representation π of a normal subgroup N of a group G and an element g of G, we let ${}^g\pi$ denote the representation $n \mapsto \pi(gng^{-1})$ of N.

Lemma 2.1. *If $\pi_1 = \otimes \pi_{1,v}$ and $\pi_2 = \otimes \pi_{2,v}$ are two cuspidal automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ which are in the same L-packet, i.e., π_1 and π_2 are irreducible subrepresentations of an automorphic form $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_F)$, then there exists $g \in \mathrm{GL}_2(F)$ such that $\pi_2 \cong {}^g\pi_1$. Thus, $\mathrm{GL}_2(F)$ acts transitively on the set of isomorphism classes of automorphic representations of SL_2 in a given L-packet.*

Proof. Since π_1 and π_2 are cuspidal automorphic, they have Whittaker models with respect to characters $\psi_i : \mathbb{A}_F/F \longrightarrow \mathbb{C}^*$. As is well known, any two non-trivial characters of $\mathbb{A}_F/F$ differ by a scaling from F^*. That is, there exists $f \in F^*$ such that $\psi_2(x) = \psi_1(fx)$ for all $x \in \mathbb{A}_F/F$. From the uniqueness of the Whittaker model with respect to GL_2, it follows that if $\pi_{1,v}$ has a Whittaker model with respect to $\psi_{1,v}$ and $\pi_{2,v}$ for $\psi_{2,v}$, and if $\psi_{2,v}(x) = \psi_{1,v}(f_v x)$ for some $f_v \in F_v$, then $\pi_{2,v} \cong {}^g\pi_{1,v}$ where g is any element of $\mathrm{GL}_2(F_v)$ with $\det g = f_v$. This completes the proof of the lemma. □

Corollary 2.2. *Let π be an irreducible representation of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in the restriction of a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_F)$. Then π is automorphic if and only if π has a Whittaker model with respect to a non-trivial character $\psi : \mathbb{A}_F/F \longrightarrow \mathbb{C}^*$.*

Proof. Clearly if π is cuspidal automorphic, it has a Whittaker model. Conversely, fix π_1 to be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in $\tilde{\pi}$, and suppose that π_1 has a Whittaker model with respect to a non-trivial character $\psi_1 : \mathbb{A}_F/F \longrightarrow \mathbb{C}^*$. Since the set of non-trivial characters of $\mathbb{A}_F/F$ is parametrized by F^* as before, there exists $a \in F^*$ such that $\psi(x) = \psi_1(ax)$ for all $x \in \mathbb{A}_F/F$. Since $\mathrm{GL}_2(F)$ operates on the set of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in $\tilde{\pi}$, by conjugating π_1 by $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, we can assume that $\psi = \psi_1$. By the uniqueness of Whittaker model (for GL_2) $\pi \cong \pi_1$, hence π is automorphic. □

Corollary 2.3. *For an irreducible representation π of $\mathrm{SL}_2(\mathbb{A}_F)$ contained in a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_F)$, the following are equivalent.*

(1) *π has an abstract Whittaker model with respect to a character $\psi : \mathbb{A}_F/F \longrightarrow \mathbb{C}^*$.*
(2) *π has a nonzero Fourier coefficient with respect to $\psi : \mathbb{A}_F/F \longrightarrow \mathbb{C}^*$.*
(3) *π is automorphic.*

Lemma 2.4. *Let k be a local field, $\tilde{\pi}$ an irreducible admissible representation of $GL_n(k)$, and π an irreducible subrepresentation for $SL_n(k)$ of $\tilde{\pi}$. Let*

$$X_{\tilde{\pi}} = \{\chi : k^* \to \mathbb{C}^* \mid \tilde{\pi} \otimes \chi \cong \tilde{\pi}\}$$

and

$$G_\pi = \{g \in GL_n(k) \mid \pi^g \cong \pi\}.$$

Then $G_\pi = \bigcap_{\chi \in X_{\tilde{\pi}}} \ker(\chi)$ where for a character χ of k^*,

$$\ker(\chi) = \{g \in GL_n(k) \mid \chi(\det g) = 1\}.$$

Proof. See, for example, Theorem 4.2 of [G-K] for a proof of this well-known lemma. □

Lemma 2.5. *If $\pi = \otimes \pi_v$ is an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_F)$ which is not a monomial automorphic form, then any $\pi' = \otimes \pi'_v$ with π'_v in the L-packet containing π_v and equal to π_v at almost all places v of F is automorphic.*

Proof. It suffices to prove that $\pi' \cong {}^g\pi$ for $g \in \mathrm{GL}_2(F)$. Define

$$G_\pi = \{g \in \mathrm{GL}_2(\mathbb{A}_F) \mid {}^g\pi \cong \pi\}.$$

Clearly G_π contains $\mathrm{SL}_2(\mathbb{A}_F)$ as well as $\mathbb{A}_F^*$ embedded diagonally in $\mathrm{GL}_2(\mathbb{A}_F)$. To prove the lemma, it suffices to prove that the double coset

$$\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/G_\pi,$$

consists of a single element. This is clearly an abelian group which is a quotient of $\mathbb{A}_F^*/F^*$.

Assume that $\tilde{\pi} = \otimes\tilde{\pi}_v$ is an irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ containing the automorphic representation π of $\mathrm{SL}_2(\mathbb{A}_F)$. From Lemma 2.4, we find that

$$G_\pi = \{g \in \mathrm{GL}_2(\mathbb{A}_F) \mid \chi(\det g) = 1 \ \forall \chi : \mathbb{A}_F^* \to \mathbb{C}^* \text{such that } \pi \otimes \chi \cong \pi\}.$$

Therefore the characters of $\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)/G_\pi$ are Grössencharacters χ such that $\tilde{\pi} \otimes \chi \cong \tilde{\pi}$. However as $\tilde{\pi}$ is non-monomial, there are no such characters, proving the lemma. □

Remark 2.6. *The same proof yields that in the monomial case there are either two or four orbits of the* $\mathrm{GL}_2(F)$ *action on representations of* $\mathrm{SL}_2(\mathbb{A}_F)$ *(not necessarily automorphic) belonging to one* L*-packet. Exactly one orbit consists of automorphic representations, and the other (one or three) orbits do not have any automorphic representation.*

Remark 2.7. *Our proof works more generally for* SL_n *to prove stability of primitive representations of* GL_n*, i.e., those automorphic representations* $\tilde{\pi}$ *of* GL_n *for which there are no nontrivial characters* χ *with* $\tilde{\pi} \otimes \chi \cong \tilde{\pi}$*.*

2.1. **Size of L-packets.** Lemma 2.5 says that for a global automorphic L-packet on SL_2 which by definition is made up of local packets, one can change any local component in its L-packet in the non-monomial case. This brings us to the interesting question whether the size of a non-monomial global L-packets is infinite or finite. This does not seem to have been studied in the literature, either for SL_2, or for other groups. We take this opportunity to make a remark about it.

Observe that since most local components of an automorphic form on GL_2 are unramified principal series, therefore given by a pair of complex numbers (α_v, β_v), v running over all but finitely many places of F, the question amounts to whether for infinitely many places v of F, the corresponding principal series representation of $\mathrm{GL}_2(F_v)$ reduces into more than one component when restricted to $\mathrm{SL}_2(F_v)$. This is the case if and only if $\alpha_v = -\beta_v$, i.e., $\alpha_v + \beta_v = 0$. Thus for modular forms for $\Gamma_1(N) \subset \mathrm{SL}_2(\mathbb{Z})$, given by classical Fourier expansion

$$f(z) = \sum_n a_n e^{2\pi i n z},$$

the question amounts to whether $a_p = 0$, for infinitely many primes p. As is well known, N. Elkies proved the existence of infinitely many such primes, called *supersingular* primes, when the modular form comes from an Elliptic curve. However, existence of infinitely many such primes is perhaps a special feature of modular forms of weight 2 with values in $\mathbb{Z}$, and is not expected for higher weights, or when the values belong to a larger number field (always of course in the non-monomial case). We refer to the article of Kumar Murty, which establishes upper bounds for such primes in [M].

3. Global distinguishedness of an L-packet for SL_2

We introduce some notation. For any number field F, let

$$\mathbb{A}_F^1 = \{x = (x_v) \in \mathbb{A}_F^* \mid |x| = \prod_v |x_v|_v = 1\}.$$

By the product formula, $F^* \subseteq \mathbb{A}_F^1$, and it is well-known that $F^*\backslash\mathbb{A}_F^1$ is a compact group.

Similarly, let

$$\mathrm{GL}_2^1(\mathbb{A}_F) = \{g \in \mathrm{GL}_2(\mathbb{A}_F) \mid \det g \in \mathbb{A}_F^1\}.$$

Lemma 3.1. *Let E be a quadratic extension of a number field F. Let ϕ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_E)$ whose central character restricted to $\mathbb{A}_F^*$ is trivial. Then*

$$\mathrm{vol}(F^*\backslash\mathbb{A}_F^1) \cdot \int_{\mathbb{A}_F^*\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)} \phi(g)dg = \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2^1(\mathbb{A}_F)} \phi(g)dg.$$

Proof. The absolute convergence of the two integrals above is a well-known consequence of the decay properties of cusp forms; we omit the details. The equality of the integrals is clear as the natural group homomorphism from $\mathrm{GL}_2^1(\mathbb{A}_F)$ to $\mathrm{PGL}_2(\mathbb{A}_F)$ is surjective with kernel consisting of $x \in \mathbb{A}_F^*$ with $|x|^2 = 1$ which is nothing but $\mathbb{A}_F^1$. □

Proposition 3.2. *Let E be a quadratic extension of a number field F. Let ϕ be a cusp form on $\mathrm{GL}_2(\mathbb{A}_E)$. Then*

$$\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg = \frac{1}{\mathrm{vol}(F^*\backslash\mathbb{A}_F^1)} \sum_\eta \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2^1(\mathbb{A}_F)} \phi(g)\eta(\det g)dg$$

where the sum on the right hand side of the equality sign is over all characters η of the compact abelian group $F^*\backslash\mathbb{A}_F^1$.

Proof. We note that for a locally compact topological group G with closed subgroups $H_1 \subset H_2$, which are all assumed to be unimodular, there exists a choice of invariant measures on $H_1\backslash G$, $H_2\backslash G$, $H_1\backslash H_2$, denoted by d_1g, d_2g, dh, such that for a function $f \in L^1(H_1\backslash G)$,

$$\int_{H_1\backslash G} f(g)d_1g = \int_{H_2\backslash G}\left(\int_{H_1\backslash H_2} f(hg)dh\right)d_2g.$$

Applying this general result to $\mathrm{GL}_2(F) \subset \mathrm{GL}_2(F)\mathrm{SL}_2(\mathbb{A}_F) \subset \mathrm{GL}_2^1(\mathbb{A}_F)$, we have,

$$\int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2^1(\mathbb{A}_F)} \phi(g)dg = \int_{F^*\backslash\mathbb{A}_F^1}\left(\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(gx)dg\right)dx. \tag{1}$$

Define

$$F(x) = \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(gx)dg$$

for $x \in \mathbb{A}_F^1$ embedded inside $\mathrm{GL}_2^1(\mathbb{A}_F)$ as $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$. Clearly $F(x)$ is a function on $F^*\backslash\mathbb{A}_F^1$. By the Fourier inversion theorem

$$F(1) = \frac{1}{\mathrm{vol}(F^*\backslash\mathbb{A}_F^1)}\sum_\eta \int_{F^*\backslash\mathbb{A}_F^1} F(x)\eta(x)dx,$$

where the sum on the right hand side of the equality sign is over all characters η of the compact abelian group $F^*\backslash\mathbb{A}_F^1$. Thus by (1), the proof of the proposition is completed. □

We next note the following lemma.

Lemma 3.3. *For a character $\chi : \mathbb{A}_F^*/F^* \longrightarrow \mathbb{C}^*$, there exists a character $\eta : \mathbb{A}_F^*/F^* \longrightarrow \mathbb{C}^*$ such that $\chi = \eta^2$ if and only if there are no local obstruction to solving $\chi = \eta^2$, i.e., if $\chi = \prod \chi_v$, then $\chi_v(-1) = 1$ for all places v of F.*

Proof. The proof follows easily by analysing the exact sequence of topological abelian groups

$$0 \to A[2] \to A \to A,$$

with $A = \mathbb{A}_F^*/F^*$, and $A[2] = \{a \in A | a^2 = 1\}$ together with the fact that an element of F^* is a square if and only if it is a square in F_v^* for all places v of F. □

Proposition 3.4. *If $\tilde{\pi}$ is a cusp form on $\mathrm{GL}_2(\mathbb{A}_E)$ which is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$, then there is a Grössencharacter η of $F^*\backslash\mathbb{A}_F^*$ such that $\tilde{\pi}$ is η-distinguished for $\mathrm{GL}_2(\mathbb{A}_F)$. Conversely if $\tilde{\pi}$ is η-distinguished for some Grössencharacter η of $F^*\backslash\mathbb{A}_F^*$, then $\tilde{\pi}$ is $\mathrm{SL}_2(\mathbb{A}_F)$-distinguished. Hence there is a member of the L-packet of automorphic* *representations of* $\mathrm{SL}_2(\mathbb{A}_E)$ *determined by* $\tilde{\pi}$ *which is globally* $\mathrm{SL}_2(\mathbb{A}_F)$*-distinguished.*

Proof. As $\tilde{\pi}$ is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$, it is locally distinguished. Hence the central character $\omega_{\tilde{\pi}}$ of $\tilde{\pi}$ takes the value 1 at -1 locally at all places v of F. Therefore by the previous lemma, we can assume that $\omega_{\tilde{\pi}}$ restricted to $\mathbb{A}_F^*$ is the square of a Grössencharacter on $\mathbb{A}_F^*$ and hence by twisting that the central character of $\tilde{\pi}$ restricted to $\mathbb{A}_F^*$ is trivial. (Actually, by the same argument $\omega_{\tilde{\pi}}$ itself is the square of a Grössencharacter on $\mathbb{A}_E^*$ and hence by twisting we can assume that the central character of $\tilde{\pi}$ is trivial, but this is not relevant for us.)

Now combining Lemma 3.1 and Proposition 3.2, and assuming that $\mathrm{vol}(F^*\backslash\mathbb{A}_F^1) = 1$, we have:

$$\begin{aligned}\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg &= \sum_{\eta: F^*\backslash\mathbb{A}_F^1\to\mathbb{C}^*} \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2^1(\mathbb{A}_F)} \phi(g)\eta(\det g)dg \\ &= \sum_{\substack{\eta\,:\,F^*\backslash\mathbb{A}_F^1\to\mathbb{C}^* \\ \eta^2=1}} \int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2^1(\mathbb{A}_F)} \phi(g)\eta(\det g)dg \\ &= \sum_{\substack{\tilde{\eta}\,:\,F^*\backslash\mathbb{A}_F^*\to\mathbb{C}^* \\ \tilde{\eta}^2=1}} \int_{\mathbb{A}_F^*\mathrm{GL}_2(F)\backslash\mathrm{GL}_2(\mathbb{A}_F)} \phi(g)\tilde{\eta}(\det g)dg\end{aligned}$$

Thus if $\tilde{\pi}$ is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$, then it is η distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$ for some Grössencharacter η of $F^*\backslash\mathbb{A}_F^*$.

Conversely, assume that $\tilde{\pi}$ is η distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$, and $\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg = 0$ for all $\phi \in \tilde{\pi}$. Twisting by a character, we assume that $\eta = 1$. Then, in particular, $\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(gx)dg = 0$ for all $x \in \mathrm{GL}_2(\mathbb{A}_F)$. By the identity (1) in the proof of Proposition 3.2, we get,

$$\int_{\mathrm{GL}_2(F)\backslash\mathrm{GL}_2^1(\mathbb{A}_F)} \phi(g)dg = \int_{F^*\backslash\mathbb{A}_F^1} \left(\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(gx)dg\right) dx = 0$$

which, by an application of Lemma 3.1, is a contradiction to $\tilde{\pi}$ being distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$, completing the proof of Proposition 3.4. $\square$

Remark 3.5. *When we talk of* χ *distinguished representation,* χ *is a character of* F^* *or* $\mathbb{A}_F^*/F^*$ *as the case may be, whereas in many calculations, we have to extend this character to a character of* E^* *or* $\mathbb{A}_E^*/E^*$ *which we often continue to write as* χ*. The end results naturally depend only on* χ *on* F^* *or* $\mathbb{A}_F^*/F^*$*, and not on the extension chosen.*

4. Criterion for global distinguishedness for SL_2

We begin with the following local result which follows from Theorem 1.1 of [A-P].

Lemma 4.1. *Let K be a quadratic extension of a local field k. Let π be an irreducible admissible representation of $\mathrm{SL}_2(K)$ contained in an irreducible admissible representation $\tilde{\pi}$ of $\mathrm{GL}_2(K)$ which is distinguished by $GL_2(k)$. Then π is distinguished by $\mathrm{SL}_2(k)$ if and only if π has a Whittaker model with respect to a character of K which is trivial on k.*

Here is the theorem about global distiguishedness of an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_E)$ which is the global analogue of the local result contained in Lemma 4.1.

Theorem 4.2. *Let π be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_E)$ contained in a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_E)$. Suppose that $\tilde{\pi}$ is distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$. Then π is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$ if and only if it has a Whittaker model with respect to a non-trivial character of $\mathbb{A}_E/E$ trivial on $\mathbb{A}_F/F$.*

The proof of this theorem will use the following lemma.

Lemma 4.3. *Let ϕ be a square integrable function on $\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)$ such that*

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(ng)dn = 0$$

for all $g \in \mathrm{SL}_2(\mathbb{A}_F)$ where N is the group of all upper triangular unipotent matrices in SL_2. Then

$$\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg = 0.$$

Proof. The condition on ϕ implies that it is a cusp form, hence it belongs to the (completion) of the direct sum of cuspidal automorphic representations in $L^2(\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F))$. The integral

$$f \mapsto \int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} f(g)dg$$

is an $\mathrm{SL}_2(\mathbb{A}_F)$-invariant linear form, and hence must be trivial on any irreducible representation which is not trivial, hence on any irreducible cuspidal representation, and therefore on their sum too. It follows that $\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg = 0$. $\square$

Proof of Theorem 4.2. Suppose that π is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. Then $\int_{\mathrm{SL}_2(F)\backslash\mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg \neq 0$, for some $\phi \in \pi$. By the previous lemma, this implies that $\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(ng)dn \neq 0$ for some $g \in \mathrm{SL}_2(\mathbb{A}_F)$. By considering the translate of ϕ by g, one can in fact assume that $\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(n)dn \neq 0$. Now ϕ is a cusp form on $\mathrm{SL}_2(E)\backslash\mathrm{SL}_2(\mathbb{A}_E)$. Considering it as a function on $N(E)\backslash N(\mathbb{A}_E)$, which we henceforth write as $\mathbb{A}_E/E$, and expanding it by Fourier series, we have

$$\phi(n) = \sum_{\psi} \hat{\phi}(\psi)\psi(n)$$

where ψ runs over all characters $\psi : \mathbb{A}_E/E \longrightarrow \mathbb{C}^*$, and

$$\hat{\phi}(\psi) = \int_{\mathbb{A}_E/E} \phi(v)\psi(-v)dv.$$

Since the integral of a non-trivial character on $\mathbb{A}_F/F$ is zero, we find that

$$\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(n)dn = \sum_{\psi} \hat{\phi}(\psi),$$

where ψ runs over all characters $\psi : \mathbb{A}_E/E \longrightarrow \mathbb{C}^*$ which are trivial on $\mathbb{A}_F$. Since $\int_{N(F)\backslash N(\mathbb{A}_F)} \phi(n)dn \neq 0$, there must be a ψ which is trivial on $\mathbb{A}_F/F$, for which $\hat{\phi}(\psi) \neq 0$. By the cuspidality condition, ψ must be non-trivial. This proves the existence of a Whittaker model with respect to a character of $\mathbb{A}_E/E$ trivial on $\mathbb{A}_F/F$.

We now prove the converse statement, i.e., if π has a Whittaker model with respect to a character $\psi : \mathbb{A}_E/E \longrightarrow \mathbb{C}^*$ which is trivial on $\mathbb{A}_F/F$, then π is distinguished. For this observe that by Proposition 3.4, $\tilde{\pi}$ is $\mathrm{SL}_2(\mathbb{A}_F)$-distinguished, and hence some cuspidal representation in the global L-packet of π is $\mathrm{SL}_2(\mathbb{A}_F)$-distinguished. By Lemma 2.1, we can assume that ${}^g\pi$ is distinguished for some $g \in \mathrm{GL}_2(E)$, hence from what has been just proved, ${}^g\pi$ has a Whittaker model by a character $\psi' : \mathbb{A}_E/E \longrightarrow \mathbb{C}^*$ which is trivial on $\mathbb{A}_F/F$. But we are given that π has a Whittaker model by a character $\psi : \mathbb{A}_E/E \longrightarrow \mathbb{C}^*$ which is trivial on $\mathbb{A}_F/F$. Since the set of non-trivial characters of $\mathbb{A}_E/E$ trivial on $\mathbb{A}_F/F$ is a principal homogenous space for F^*, and since clearly π is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$ if and only is ${}^h\pi$ is for any $h \in \mathrm{GL}_2(F)$, we can assume that $\psi = \psi'$, i.e., both π and ${}^g\pi$ have Whittaker models by the same character ψ and ${}^g\pi$ is distinguished. But by the uniqueness of Whittaker model (for GL_2), this implies that $\pi = {}^g\pi$, and hence π is distinguished. $\square$

Remark 4.4. *E. Lapid has pointed out to us that Lemma 4.3 can also be proved as follows. Every cusp form is orthogonal to any pseudo-Eisenstein series, and the pseudo-Eisenstein series contain the constant functions in their closure, thus a cusp form is orthogonal to the constants.*

5. Locally but not globally distinguished I

In this section we use the theorem of the previous section to show that there are cuspidal representations of $\mathrm{SL}_2(\mathbb{A}_E)$ which are not distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$ but for which each of its local component is SL_2-distinguished. To this end, fix a nontrivial character $\psi : \mathbb{A}_E/E \longrightarrow \mathbb{C}^*$ which is trivial on $\mathbb{A}_F$. Let π be a cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$ which occurs in the restriction of a cuspidal representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_E)$ which is distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$.

Our examples will depend on understanding and identifying the distinguished parts of the restriction of $\tilde{\pi}$ to the successive subgroups

$$GL_2(\mathbb{A}_E) \supseteq \mathbb{A}_E^*\mathrm{SL}_2(\mathbb{A}_E)\mathrm{GL}_2(\mathbb{A}_F) \supseteq \mathbb{A}_E^*\mathrm{SL}_2(\mathbb{A}_E)\mathrm{GL}_2(F) \supseteq SL_2(\mathbb{A}_E).$$

We denote by π' the irreducible representation of $G' = \mathbb{A}_E^*\mathrm{SL}_2(\mathbb{A}_E)\mathrm{GL}_2(\mathbb{A}_F)$ that occurs in the restriction of $\tilde{\pi}$ to G', and which is ψ-generic. By Lemma 4.1, π' is the unique irreducible component of the restriction of $\tilde{\pi}$ to G' which is abstractly distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. Further, an irreducible representation of $\mathrm{SL}_2(\mathbb{A}_E)$ occurring in $\tilde{\pi}$ is abstractly distinguished with respect to $\mathrm{SL}_2(\mathbb{A}_F)$ if and only if it occurs in the restriction of π' to $\mathrm{SL}_2(\mathbb{A}_E)$. From Theorem 4.2, it follows that there is exactly one irreducible cuspidal representation of $G'' = \mathbb{A}_E^*\mathrm{SL}_2(\mathbb{A}_E)\mathrm{GL}_2(F)$ occurring in the space of functions in $\tilde{\pi}$ that is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$, say π''. Also, an irreducible cuspidal representation of $\mathrm{SL}_2(\mathbb{A}_E)$ occurring in the space of functions in $\tilde{\pi}$ is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$ if and only if it occurs in the restriction of π'' to $\mathrm{SL}_2(\mathbb{A}_E)$. Now if we choose $\tilde{\pi}$ such that $\tilde{\pi} \otimes \omega \cong \tilde{\pi}$ where ω is a character of $\mathbb{A}_E^*/E^*$ with non-trivial restriction to $\mathbb{A}_F^*$, then $\pi' \otimes \omega \cong \pi'$, since π' is the unique irreducible representation of G' which is (abstractly) ψ-generic. Hence the restriction of π' to G'' is not irreducible. Hence we get cuspidal representations of $\mathrm{SL}_2(\mathbb{A}_E)$ which appear in the restriction of π' but not in the restriction of π''. These representations are abstractly distinguished but not distinguished.

It remains to construct cuspidal representations $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_E)$ which are distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\tilde{\pi} \otimes \omega \cong \tilde{\pi}$ where ω restricts non-trivially to $\mathbb{A}_F^*$. We need the following lemma.

Lemma 5.1. *Let L/F be a quadratic extension of number fields. Given a positive integer n, there exists a Grössencharacter η of $\mathbb{A}_L^*$ of order n such that η has trivial restriction to $\mathbb{A}_F^*$.*

Proof. Let v be a place of F that splits in L, say $v = w_1 w_2$, such that F_v has odd residue characteristic. Let η_{w_1} be a character of order n of $L_{w_1}^*$. Consider the character $(\eta_{w_1}, 1)$ of $L_{w_1}^* \times L_{w_2}^*$. By Grunewald-Wang theorem, we get a Grössencharacter of $\mathbb{A}_L^*$ of order n whose component at the place above v is $(\eta_{w_1}, 1)$. It follows from our construction that η/η^τ is also a Grössencharacter of order n, where τ is the non-trivial element of $Gal(L/F)$. Further η/η^τ restricted to $\mathbb{A}_F^*$ is trivial. □

Now let η be a Grössencharacter of $\mathbb{A}_L^*$ of order 8 such that η has trivial restriction to $\mathbb{A}_F^*$. Let M be the quadratic extension of L such that $\eta^4 = \omega_{M/L}$. Since $\omega_{M/L}$ has trivial restriction to $\mathbb{A}_F^*$, we see that there is a quadratic extension E of F such that $M = EL$ (cf. Corollary 6.7). The conditions on η imply that the representations $\mathrm{Ind}_{W_L}^{W_F}\eta$ and $\mathrm{Ind}_{W_L}^{W_F}\eta^2$ of W_F, the Weil group of F, are irreducible and the restriction of $\mathrm{Ind}_{W_L}^{W_F}\eta^2$ to W_E, the Weil group of E, is a sum of two distinct Grössencharacters, necessarily of the form γ and γ^σ. Now let $\rho_{\tilde\pi}$ be the restriction to W_E of the representation $\mathrm{Ind}_{W_L}^{W_F}\eta$ of W_F, and $\tilde\pi$ the associated automorphic form on $\mathrm{GL}_2(\mathbb{A}_E)$. Since $\eta^4 \neq 1$, $\rho_{\tilde\pi}$ is an irreducible representation.

Let $r(\rho_{\tilde\pi})$ be the 4 dimensional representation of W_F obtained from $\rho_{\tilde\pi}$ of W_E by the process of twisted tensor induction. It is a general and simple fact that if H is a subgroup of a group G of index two and V a representation of G, then

$$r(V|_H) \cong \mathrm{Sym}^2(V) \oplus \wedge^2(V) \cdot \omega_{G/H}$$

where $\omega_{G/H}$ is the nontrivial character of G trivial on H. Applying this to our situation, we have:

$$\begin{aligned} r(\rho_{\tilde\pi}) &\cong \mathrm{Sym}^2(\mathrm{Ind}_{W_L}^{W_F}\eta) \oplus \wedge^2(\mathrm{Ind}_{W_L}^{W_F}\eta) \cdot \omega_{E/F} \\ &\cong \mathrm{Ind}_{W_L}^{W_F}\eta^2 \oplus 1 \oplus \omega_{E/F}\omega_{L/F}. \end{aligned}$$

It follows that $\tilde\pi$ is distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$. Also:

$$\rho_{\tilde\pi} \otimes \rho_{\tilde\pi^\sigma} \cong \rho_{\tilde\pi} \otimes \rho_{\tilde\pi}^\vee = 1 \oplus \omega_{L/F} \circ Nm_{E/F} \oplus \gamma \oplus \gamma^\sigma.$$

Therefore γ is a self-twist for $\tilde\pi$. Observe that γ has non-trivial restriction to $\mathbb{A}_F^*$. We have thus proved the following theorem.

Theorem 5.2. *There is a cuspidal representation of* $\mathrm{SL}_2(\mathbb{A}_E)$ *which is not distinguished by* $\mathrm{SL}_2(\mathbb{A}_F)$ *but for which each of its local component is* SL_2*-distinguished.*

The above analysis also gives the following proposition.

Proposition 5.3. *Let* $\tilde{\pi}$ *be a non-monomial cuspidal representation of* $\mathrm{GL}_2(\mathbb{A}_E)$ *that is distinguished by* $\mathrm{GL}_2(\mathbb{A}_F)$*. Then any irreducible cuspidal representation of* $\mathrm{SL}_2(\mathbb{A}_E)$ *in the L-packet associated to* $\tilde{\pi}$ *that is abstractly distinguished with respect to* $\mathrm{SL}_2(\mathbb{A}_F)$ *is in fact distinguished by* $\mathrm{SL}_2(\mathbb{A}_F)$*.*

Proof. Note that since $\tilde{\pi}$ is non-monomial, it cannot be χ-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ for any non-trivial Grössencharacter χ of $\mathbb{A}_F^*$ (see for example Corollary 6.4 below). Suppose that μ is a character of $\mathbb{A}_E^*$ (not necessarily a Grössencharacter) such that $\tilde{\pi} \otimes \mu \cong \tilde{\pi}$ and such that μ restricted to F^* is trivial. Since $r(\tilde{\pi} \otimes \mu) = r(\tilde{\pi}) \otimes \mu|_{\mathbb{A}_F^*}$, it follows that $\tilde{\pi}$ is distinguished with respect to the Grössencharacter $\mu|_{\mathbb{A}_F^*}$. This forces $\mu|_{\mathbb{A}_F^*} = 1$. In other words, any irreducible representation of G' that occurs in the restriction of $\tilde{\pi}$ to G' restricts irreducibly to G''. This proves the proposition. □

6. Factorisation

In this section we analyse whether the period integral on SL_2 is factorisable or not. We begin by making a precise definition for factorisation of a linear form ℓ on $\otimes_v \pi_v$, a restricted direct product of vector spaces π_v with respect to vectors $w_v^0 \in \pi_v$ where v runs over any infinite set, say X, such as the set of places of a number field.

We say that ℓ is factorisable, if there are linear forms ℓ_v for each $v \in X$ such that $\ell_v(w_v^0) = 1$ outside a finite subset T of X, and such that for any finite subset S of X containing T,

$$\ell(w_S \otimes w^S) = (\otimes_{v \in S} \ell_v)(w_S),$$

where $w_S \otimes w^S$ is a vector in $\otimes_{v \in X} \pi_v$ with $w_S \in \otimes_{v \in S} \pi_v$, and $w^S = \otimes_{v \notin S} w_v^0$.

We state the following two elementary lemmas without proof.

Lemma 6.1. *Suppose that* π_v' *is a subspace of* π_v *(containing the vector* w_v *for almost all* v*), and* ℓ *is a factorisable linear form on* $\pi = \otimes_v \pi_v$*, then the restriction of* ℓ *to* $\pi' = \otimes_v \pi_v'$ *is also factorisable.*

Lemma 6.2. *Suppose that* ℓ_i *are finitely many factorisable linear forms* $\ell_i = \otimes_v \ell_{i,v}$ *on* $\pi = \otimes_v \pi_v$*, then* $\ell = \sum_i \ell_i$ *is not factorisable if there is* *an infinite subset $Y \subset X$ such that the subspace of linear forms on π_v generated by $\ell_{i,v}$ has dimension > 1 for $v \in Y$.*

Before we state the main theorem, we prove the following proposition. This is the global analogue of Proposition 4.2 of [A-P].

Proposition 6.3. *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A}_E)$ which is (globally) distinguished with respect to $\mathrm{SL}_2(\mathbb{A}_F)$. Then the sets*

$$X = \left\{ \chi \in \widehat{\mathbb{A}_F^*/F^*} \mid \begin{array}{c} \pi \text{ is } \chi\text{-distinguished} \\ \text{with respect to } \mathrm{GL}_2(\mathbb{A}_F) \end{array} \right\}$$

and

$$Y = \{\mu \in \widehat{\mathbb{A}_E^*/E^*} \mid \pi \otimes \mu \cong \pi; \mu|_{\mathbb{A}_F^*} = 1\}$$

have the same cardinality; in fact $\chi \mapsto \chi \circ N_{E/F}$ induces an isomorphism of X onto Y if π is $\mathrm{GL}_2(\mathbb{A}_F)$-distinguished.

Proof. We assume without loss of generality that π is (globally) distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$. Then we give explicit maps from X to Y and from Y to X.

For $\chi \in X$, let $\tilde{\chi}$ be a character of $\mathbb{A}_E^*/E^*$ restricting to χ on $\mathbb{A}_F^*$. Then we have $\pi^\vee \cong \pi^\sigma$ and $(\pi \otimes \tilde{\chi}^{-1})^\vee \cong (\pi \otimes \tilde{\chi}^{-1})^\sigma$, and therefore we get

$$\pi \cong \pi \otimes \chi \circ N_{E/F}.$$

Note that since π is both distinguished and χ-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$, consideration of the central character implies that $\omega_\pi|_{\mathbb{A}_F^*} = \omega_\pi|_{\mathbb{A}_F^*}\chi^{-2} = 1$. Therefore $\chi^2 = 1$, thus $\chi \circ N_{E/F} \in Y$. This allows us to define a map from X to Y by sending χ to $\chi \circ N_{E/F}$.

If $\mu \in Y$, then, since $\mu|_{\mathbb{A}_F^*} = 1$, and $\mu^2 = 1$, we have that μ factors through the norm map $N_{E/F}$. Let $\mu = \eta\eta^\sigma$ for a Grössencharacter η of $\mathbb{A}_E^*$. Now consider the representation $\pi \otimes \eta$. Observe that $(\pi \otimes \eta)^\vee \cong (\pi \otimes \eta)^\sigma$, and that $\omega_{\pi\otimes\eta}|_{F^*} = 1$. Therefore $\pi \otimes \eta$ is either distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ or $\omega_{E/F}$-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$. We map μ to $\eta|_{\mathbb{A}_F^*}$ or $\eta|_{\mathbb{A}_F^*}\omega_{E/F}$ accordingly. Clearly the above two maps are inverses of each other and hence X and Y have the same cardinality, completing the proof of the proposition. $\square$

Corollary 6.4. *A non-monomial automorphic representation is χ-distinguished for at most one Grössencharacter. A distinguished monomial automorphic representation is χ-distinguished for at least two (and at most four) Grössencharacters of $\mathbb{A}_F^*$.*

Proof. We need to supply a proof only for monomial representations. Let $\tilde{\pi}$ be a distinguished monomial automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$.

We need to show that there exists a non-trivial Grössencharacter μ of $\mathbb{A}_E^*/E^*$ with $\mu|_{\mathbb{A}_F^*} = 1$, and $\tilde{\pi}\otimes\mu \cong \tilde{\pi}$. Since $\tilde{\pi}$ is distinguished, $\tilde{\pi}^\vee \cong \tilde{\pi}^\sigma$ from which it follows that if $\tilde{\pi} \otimes \mu \cong \tilde{\pi}$, then $\tilde{\pi} \otimes \mu^\sigma \cong \tilde{\pi}$ also, and hence $\tilde{\pi} \otimes (\mu\mu^\sigma) \cong \tilde{\pi}$. Since $\tilde{\pi}$ is monomial, it has a non-trivial self-twist μ, hence we are done unless this self-twist μ restricted to $\mathbb{A}_F^*$ equals $\omega_{E/F}$. But this would mean that $\tilde{\pi}$ is both distinguished, and $\omega_{E/F}$-distinguished, which is not possible. □

From the proof of the previous corollary, we isolate the following fact which we will have occasion to use in the next theorem about factorisation.

Lemma 6.5. *Let $\tilde{\pi}$ be a distinguished cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$. Then if $\tilde{\pi} \otimes \mu \cong \tilde{\pi}$, μ restricted to $\mathbb{A}_F^*$ cannot be equal to $\omega_{E/F}$. Thus, if $\tilde{\pi}$ is a monomial representation coming from a quadratic extension M of E, M cannot be a cyclic quartic extension of F.*

Proof. The last conclusion is part of classfield theory, see corollary 6.7 below. □

Before we proceed further, we recall the following lemma from class field theory.

Lemma 6.6. *Let E be a finite extension of a number field or a local field F. Let $\chi : \mathbb{A}_F^*/F^* \longrightarrow \mathbb{C}^*$ (or $\chi : F^* \longrightarrow \mathbb{C}$ if F is local) be a character of finite order cutting out a finite cyclic extension L of F. Then the character $\chi \circ Nm :\ \mathbb{A}_E^*/E^* \xrightarrow{Nm} \mathbb{A}_F^*/F^* \xrightarrow{\chi} \mathbb{C}^*$ defines the cyclic extension LE of E.*

Corollary 6.7. *If E is a quadratic extension of F, and $\omega : \mathbb{A}_E^*/E^* \longrightarrow \mathbb{C}^*$ a quadratic character defining an extension M of E, then*

(1) *M is biquadratic over F if and only if ω restricted to $\mathbb{A}_F^*/F^*$ is trivial.*
(2) *M is cyclic quartic over F if and only if ω restricted to $\mathbb{A}_F^*/F^*$ is $\omega_{E/F}$ where $\omega_{E/F}$ is the quadratic character on $\mathbb{A}_F^*/F^*$ defined by the quadratic extension E of F.*
(3) *M is non-Galois over F if and only if $\omega/\omega^\sigma \neq 1$, and this is so if and only if ω restricted to F^* is not 1 or $\omega_{E/F}$; the restriction of ω to F^* defines a quadratic extension, say L' of F such that EL' is the quadratic extension of E defined by $\omega\omega^\sigma$.*

Proof of corollary. One only needs to observe that M is Galois over F if and only if ω is invariant under $Gal(E/F)$. □

Here is the main theorem regarding factorisation of period integrals on SL_2.

Theorem 6.8. *Let π be an automorphic representation of $\mathrm{SL}_2(\mathbb{A}_E)$ contained in a cuspidal automorphic representation $\tilde{\pi}$ of $\mathrm{GL}_2(\mathbb{A}_E)$. Suppose that $\tilde{\pi}$ is distinguished by $\mathrm{GL}_2(\mathbb{A}_F)$. Then the period integral on π is factorisable if $\widetilde{\pi}$ is non-monomial, or if $\widetilde{\pi}$ is monomial, and comes from three quadratic extensions of E of which exactly one is Galois over F. If $\widetilde{\pi}$ is monomial, and comes from a unique quadratic extension, say M, of E, or comes from three quadratic extensions of E which are all Galois over F, then the period integral is not factorisable.*

Proof. We recall an identity established earlier:

$$\int_{\mathrm{SL}_2(F)\backslash \mathrm{SL}_2(\mathbb{A}_F)} \phi(g)dg$$

$$= \sum_{\substack{\tilde{\eta}: F^*\backslash \mathbb{A}_F^* \to \mathbb{C}^* \\ \tilde{\eta}^2 = 1}} \int_{\mathbb{A}_F^* \mathrm{GL}_2(F)\backslash \mathrm{GL}_2(\mathbb{A}_F)} \phi(g)\tilde{\eta}(\det g)dg.$$

In the non-monomial case, the above sum of integrals reduces to a single term by Corollary 6.4, hence is factorisable for the GL_2 automorphic representation by multiplicity 1 theorem for GL_2, and hence also for SL_2 automorphic subrepresentations by Lemma 6.1.

If $\tilde{\pi}$ is monomial, and comes from three quadratic extensions of E of which exactly one is Galois over F, then we have an isomorphism $\tilde{\pi} \cong \tilde{\pi} \otimes \omega$, where ω is a Grössencharacter of $\mathbb{A}_E^*$ which does not restrict trivially to $\mathbb{A}_F^*$ (cf. Corollary 6.7). Also in this case, $\tilde{\pi}$ is distinguished for exactly two Grössencharacters of $\mathbb{A}_F^*$, namely 1 and $\chi = \omega|_{\mathbb{A}_F^*}$ (cf. Proposition 6.3). An isomorphism of $\mathrm{GL}_2(\mathbb{A}_E)$-modules between $\tilde{\pi}$ and $\tilde{\pi} \otimes \omega$ can be interpreted as an isomorphism, say T, between $\tilde{\pi}$ and itself such that $T(gv) = \omega(\det g)gT(v)$ for all $g \in \mathrm{GL}_2(\mathbb{A}_E)$, and $v \in \tilde{\pi}$. Upon modifying T by a scalar, we can assume that T has order 2, and splits $\tilde{\pi}$ into $\tilde{\pi}^+ \oplus \tilde{\pi}^-$ on which T operates by $+1$ and -1 respectively. Since the period integral is the unique abstract $\mathrm{GL}_2(\mathbb{A}_F)$-invariant linear form up to scalar, the χ-period integral is the composite of the period integral with T. The key fact is that π, being an irreducible representation of $\mathrm{SL}_2(\mathbb{A}_E)$, sits either inside $\tilde{\pi}^+$ or inside $\tilde{\pi}^-$. Therefore, the χ-period integral on π is a scalar multiple of the period integral restricted to π. Hence the SL_2-period integral is factorisable by the above identity and Lemma 6.1.

In the other cases, the period integral is a sum of more than one linear form by Corollary 6.4, each of which is factorisable. We argue below using Lemma 6.2 that the sum is not factorisable.

Suppose that $\tilde{\pi}$ is 'induced' from a Grössencharacter of a quadratic extension M of E. Notice that because of Lemma 6.5 we can assume that M is Galois over F with Galois group $(\mathbb{Z}/2)^2$. However, because of $\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee$, if $\tilde{\pi}$ arises from a quadratic extension M of E, it also arises from M^σ. It is well known that $\tilde{\pi}$ arises either from 1 quadratic extension, or 3 quadratic extensions. It follows that one of these quadratic extensions, say M, of E from which $\tilde{\pi}$ arises must be Galois over F. In this case, the Grössencharacter $\omega_{M/E}$ of $\mathbb{A}_E^*$ is σ invariant, hence there is a character χ of $\mathbb{A}_E^*/E^*$ such that $\chi\chi^\sigma = \omega_{M/E}$. Clearly χ restricted to $\mathbb{A}_F^*$ cannot be 1 or $\omega_{E/F}$. Since $\tilde{\pi}$ is distinguished,

$$\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee \cong \tilde{\pi}^\vee \otimes \omega_{M/E} = \tilde{\pi}^\vee \otimes (\chi\chi^\sigma).$$

It follows that $(\tilde{\pi} \otimes \chi^{-1})^\sigma \cong (\tilde{\pi} \otimes \chi^{-1})^\vee$, and therefore $\tilde{\pi}$ is χ or $\chi\omega_{E/F}$-distinguished, which after perhaps changing the choice of χ with $\chi\chi^\sigma = \omega_{M/E}$, we can assume that $\tilde{\pi}$ is χ distinguished (besides being distinguished). We note that this implies, in particular, that χ restricted to $\mathbb{A}_F^*$ is of order 2 (we have already noted earlier that χ restricted to $\mathbb{A}_F^*$ is not trivial).

Observe now that the quadratic character χ restricted to $\mathbb{A}_F^*$ defines a quadratic extension L of F, and from the equality $\chi\chi^\sigma = \omega_{M/E}$, M is a biquadratic extension, $M = LE$ of F. Assume that τ is the non-trivial automorphism of M over E, and abusing notation, let σ be the non-trivial automorphism of M over L.

$$\begin{tikzcd} & M \arrow[dl, dash, "\tau"'] \arrow[dr, dash, "\sigma"] & \\ E \arrow[dr, dash, "\sigma"'] & & L \arrow[dl, dash, "\tau"] \\ & F & \end{tikzcd}$$

Suppose that $\tilde{\pi}$ arises from a Grössencharacter μ of $\mathbb{A}_M^*/M^*$, and is distinguished. Therefore,

$$\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee,$$

which assuming $\tilde{\pi}$ arises from μ implies that

$$\mathrm{Ind}_{W_M}^{W_E}(\mu^\sigma) = \mathrm{Ind}_{W_M}^{W_E}(\mu^{-1}).$$

This implies that either,

$$\mu^\sigma = \mu^{-1}, \tag{2}$$

or,

$$\mu^{\sigma\tau} \;=\; \mu^{-1}. \tag{3}$$

Defining L_1 to be the field fixed by $\sigma\tau$, we note that in case (3), the character μ/μ^τ is trivial on $\mathbb{A}_{L_1}^*$, because in this case $\mu\mu^{\sigma\tau} = 1$, therefore μ restricted to $\mathbb{A}_{L_1}^*$ is either 1 or ω_{M/L_1}. Therefore μ and μ^τ have the same restriction to $\mathbb{A}_{L_1}^*$, proving our claim.

In case (2), we claim that μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ cannot be ω_{M/L_1}. If μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ was ω_{M/L_1}, then in particular $(\mu/\mu^\tau)(\mu/\mu^\tau)^{\sigma\tau} = 1$. Since we are in case (2), $\mu^\sigma = \mu^{-1}$. Therefore, the condition $(\mu/\mu^\tau)(\mu/\mu^\tau)^{\sigma\tau} = 1$ becomes $(\mu/\mu^\tau)^2 = 1$. If μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ is ω_{M/L_1}, then by Corollary 6.7, the quadratic extension of M defined by μ/μ^τ, call it M_1, is a cyclic quartic extension of L_1. Now we note the following elementary lemma whose proof is omitted.

Lemma 6.9. *Let N be an abelian normal subgroup of a group G with G/N cyclic. Assume that the action of G/N on N via inner conjugation is trivial. Then G is abelian.*

We apply the above lemma to $G = Gal(M_1/F)$ which contains $N = Gal(M_1/E) = (\mathbb{Z}/2)^2$ as an abelian normal subgroup on which $Gal(E/F)$ acts trivially since we are in the situation in which all the three quadratic extensions of E from which $\tilde{\pi}$ arises are Galois over F. Thus we find that $G = Gal(M_1/F)$ is abelian. Because of Lemma 6.5, G does not contain $\mathbb{Z}/4$ as a quotient and hence neither as a subgroup. This implies that the Galois group of M_1 over L_1 cannot be $\mathbb{Z}/4$, proving our claim that μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ cannot be ω_{M/L_1}.

Before proceeding further, we note the following lemma which is at the basis of our proof of non-factorisation of period integral. This is part of case 3 of Theorem 1.3 of our paper [A-P]. It can be easily proved by a direct analysis of the $\mathrm{GL}_2(F_v)$ action on $\mathbb{P}^1(E_v)$.

Lemma 6.10. *Let E_v be a quadratic extension of a local field F_v, and $\pi = Ps(\chi_1, \chi_2)$ a principal series representation of $\mathrm{GL}_2(E_v)$. Then if $\chi_1 = \chi_2$, π remains irreducible when restricted to $\mathrm{SL}_2(E_v)$, and is ν distinguished for two characters ν of F_v^*.*

In what follows, we will be doing some local analysis for which we assume that all our places in consideration in L_1 or M are unramified over the corresponding place in F, and the character μ is unramified too at these places.

We note that there are infinitely many primes in L_1 which are inert in M. The prime in F below such a prime in L_1 has the property that it is inert in both L and E, and split in L_1. We abuse notation to denote the pair of places in M as well as in L_1 as (v_1, v_2). Since we are going to use only unramified characters, this should not cause any confusion.

If the local components of μ at (v_1, v_2) is (μ_1, μ_2), μ/μ^τ looks like $(\mu_1/\mu_2, \mu_2/\mu_1)$ at this pair of places.

In case (3), since the character μ/μ^τ is trivial on $\mathbb{A}_{L_1}^*$, in particular the pair of characters $(\mu_1/\mu_2, \mu_2/\mu_1)$ is trivial, hence $\mu_1 = \mu_2$.

In case (2), we know that μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ is a certain quadratic character which is not ω_{M/L_1}. Either, μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ is the trivial character, in which case places of L_1 which are inert in M automatically give $\mu_1 = \mu_2$, or the quadratic extension of L_1 defined by μ/μ^τ restricted to $\mathbb{A}_{L_1}^*$ is distinct from M, and together with M gives a Galois extension of F with Galois group $(\mathbb{Z}/2)^3$. Applying Cebotaraev density theorem, we once again find that there are infinitely many primes of L_1 which are inert in M where the restriction of μ/μ^τ is trivial. Hence, once again $\mu_1 = \mu_2$.

Thus Lemma 6.10 applies, and which in conjunction with Lemma 6.2 implies that the period integral is not factorisable, completing the proof of Theorem 6.8. $\square$

Remark 6.11. *Observe that the above theorem can be viewed as an analogue of Theorem 1.2 of* [A-P]. *The cases where the period integral is Eulerian are exactly the global analogues of the cases in Theorem 1.2 of* [A-P] *where the space of local invariant forms has multiplicity one. Note that this analogy holds in the context of Jacquet's result too* [J]. *There the symmetric space* $(\mathrm{Res}_{E/F}\mathrm{GL}(3), \mathrm{U}(3))$ *has the property that locally over a p-adic field, the space of* $\mathrm{U}(3)$*-invariant linear forms on a supercuspidal representation of* $\mathrm{GL}_3(E)$ *has multiplicity at most one, and over global fields, the period integral is factorisable for cuspidal representations.*

Remark 6.12. *For a reductive algebraic group G over a local field k, K a separable quadratic extension of k, and π an irreducible admissible representation of $G(K)$, it makes sense to study the dimension of the space of $G(k)$-invariant forms $\ell : \pi \to \mathbb{C}$. It is reasonable to expect that this dimension is always finite. In the global study, since the linear form is fixed to be the period integral, there is no obvious global analogue of the concept of the dimension of $G(k)$-invariant forms. However Theorem 6.8, together with Theorem 1.2 of* [A-P], *suggests a reasonable global analogue to be the smallest positive integer d such that the period integral can be written as a sum of d factorisable linear forms. With this notion, we can go a step further in Theorem 6.8 to say that in the* *cases in which the period integral is not factorisable, it is a sum of two or four factorisable linear forms depending on whether the representation comes from a unique quadratic extension of E or three quadratic extensions of E which are all Galois over F. We omit the details of this calculation. It is curious to note that not only is d finite for SL_2, it has a very similar structure to the dimension of the space of local invariant forms. Understanding these local and global dimensions in general seems a very interesting problem. In this connection, we mention the work of Lapid and Rogawski* [L-R2] *which computes the period of an Eisenstein series on GL_3 as a sum of factorizable functionals, its recent generalization by Omer Offen which is to appear in the Duke Math J., as well as the earlier work of Jacquet* [J] *for GL_3, and its recent generalisation to GL_n.*

7. Psuedo-Distinguishedness

If an automorphic representation $\pi = \otimes \pi_v$ of $\mathrm{GL}_2(\mathbb{A}_E)$ has the property that π_v is distinguished by $\mathrm{SL}_2(F_v)$ at all places v of E, then there are characters χ_v of E_v^* such that

$$(\pi_v \otimes \chi_v)^\sigma \cong (\pi_v \otimes \chi_v)^\vee.$$

Thus at all places v of E, π_v^σ and $\pi_v^\vee$ differ by a character of E_v^*. By the multiplicity one theorem of Ramakrishnan, cf. [R], this implies that

$$\pi^\sigma \cong \pi^\vee \otimes \chi$$

for a character χ of $\mathbb{A}_E^*/E^*$.

The aim of this section is to classify representations π of $\mathrm{GL}_2(\mathbb{A}_E)$ such that

$$\pi^\sigma \cong \pi^\vee \otimes \chi, \tag{4}$$

for a character χ of $\mathbb{A}_E^*/E^*$ which we assume fixed in this section, and which is not Galois invariant. We call such representations *pseudo-distinguished*. Although, we write the arguments below for π an irreducible admissible representation of $\mathrm{GL}_2(E)$, E a local field, exactly the same argument works in the case of automorphic forms over global fields. We note that Lapid and Rogawski, cf. [L-R1], have also done an analogous study, of classifying π with $\pi^\sigma \cong \pi \otimes \chi$, via the methods of trace formula.

We note that if χ is Galois invariant, then we can write χ as $\chi = \alpha \cdot \alpha^\sigma$ for a character α of E^*, and therefore equation (4) reduces after twisting by a character to $\pi^\sigma \cong \pi^\vee$, studied in the theory of distinguished representations.

By applying σ to (4), and rewriting, we find,

$$\pi^\sigma \cong \pi^\vee \otimes \chi^\sigma. \tag{5}$$

Therefore from (4) and (5), if $\chi^\sigma \neq \chi$, then χ^σ/χ is a quadratic character, say ω, of E^*, and π has a self-twist by ω, implying that π is a monomial representation arising from a character μ of the quadratic extension M of E defined by by ω: $\pi = \mathrm{Ind}_{W_M}^{W_E}\mu$, where W_M and W_E are respectively the Weil groups of M and E. Since $\chi(x/x^\sigma) = \omega(x)$, ω restricted to $\mathbb{A}_F^*$ is trivial. Therefore by Corollary 6.7, M is a biquadratic extension of F, say $M = EL$ with L a quadratic extension of F. Assume that τ is the non-trivial automorphism of M over E, and abusing notation, let σ be the non-trivial automorphism of M over L.

$$\begin{tikzcd} & M \arrow[dl, dash, "\tau"'] \arrow[dr, dash, "\sigma"] & \\ E \arrow[dr, dash, "\sigma"'] & & L \arrow[dl, dash, "\tau"] \\ & F & \end{tikzcd}$$

Once again, condition (4),

$$\mathrm{Ind}_{W_M}^{W_E}(\mu^\sigma) = \mathrm{Ind}_{W_M}^{W_E}(\mu^{-1}) \otimes \chi,$$

implies that either,

$$\mu^\sigma = \mu^{-1}\chi\chi^\tau \tag{6}$$

or,

$$\mu^{\sigma\tau} = \mu^{-1}\chi\chi^\tau. \tag{7}$$

Let us consider the first case, as the other case is similar. In this case,

$$\mu\mu^\sigma = \chi\chi^\tau. \tag{8}$$

Note that since χ/χ^σ is of order 2 defining M over E, the character $\chi\chi^\tau$ of $\mathbb{A}_M^*/M^*$ is σ-invariant. Therefore there exists a character $\tilde{\mu}$ of $\mathbb{A}_L^*/L^*$ such that

$$\tilde{\mu}(Nmx) = (\chi\chi^\tau)(x),$$

for all $x \in \mathbb{A}_M^*/M^*$ where the norm is taken from $\mathbb{A}_M^*/M^*$ to $\mathbb{A}_L^*/L^*$. Any extension μ of $\tilde{\mu}$ from $\mathbb{A}_L^*/L^*$ to $\mathbb{A}_M^*/M^*$ satisfies condition (6). Further, it can be seen that $\tilde{\mu}$ is unique up to the automorphism of L over F (since the automorphism takes $\tilde{\mu}$ to $\tilde{\mu}\omega_{M/E}$), proving the following proposition.

Proposition 7.1. *Given a character χ of $\mathbb{A}_E^*/E^*$ such that χ/χ^σ is a character of order 2 of $\mathbb{A}_E^*/E^*$, defining a quadratic extension M of E which is of the form $M = LE$ where L is a quadratic extension of F, there exists a character $\tilde{\mu}$ of $\mathbb{A}_L^*/L^*$ such that the characters χ and $\tilde{\mu}$ restricted to $\mathbb{A}_M^*/M^*$ (via the norm maps to $\mathbb{A}_E^*/E^*$ and $\mathbb{A}_L^*/L^*$) are the same, i.e., $\chi(x \cdot x^\tau) = \tilde{\mu}(x \cdot x^\sigma)$ for all $x \in \mathbb{A}_M^*/M^*$. Such a character $\tilde{\mu}$ is unique up to the Galois automorphism of L over F (since the automorphism takes $\tilde{\mu}$ to $\tilde{\mu}\omega_{M/E}$), and any extension μ of $\tilde{\mu}$ from $\mathbb{A}_L^*/L^*$ to $\mathbb{A}_M^*/M^*$ gives rise to an isomorphism*

$$\pi^\sigma \cong \pi^\vee \otimes \chi,$$

for $\pi = \mathrm{Ind}_{W_M}^{W_E}(\mu)$.

Remark 7.2. *Since characters of $\mathbb{A}_M^*/M^*$ extending a given character of $\mathbb{A}_L^*/L^*$ are –after fixing one such character– in bijective correspondence with characters of $\mathbb{A}_M^*/M^*$ trivial on $\mathbb{A}_L^*/L^*$, one can state the proposition in a more suggestive way as follows: representations π of $\mathrm{GL}_2(E)$ with $\pi^\sigma \cong \pi^\vee \otimes \chi$, with χ/χ^σ cutting out a quadratic extension $M = LE$ of E are in bijective correspondence with representations of $\mathbb{A}_M^*/M^*$ distinguished by $\mathbb{A}_L^*/L^*$.*

8. Locally but not globally distinguished II

In this section, we construct an automorphic representation $\pi = \otimes \pi_v$ of $\mathrm{SL}_2(\mathbb{A}_E)$ which is abstractly $\mathrm{SL}_2(\mathbb{A}_F)$ distinguished but none of the elements in the global L-packet determined by π is distinguished by $\mathrm{SL}_2(\mathbb{A}_F)$. We achieve this by the following steps.

(1) We construct a pseudo-distinguished representation $\tilde{\pi} = \otimes \tilde{\pi}_v$ of $\mathrm{GL}_2(\mathbb{A}_E)$ with

$$\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee \otimes \chi,$$

for a Grössencharacter χ with $\chi^\sigma \neq \chi$. Our representation $\tilde{\pi}$ will be monomial arising from exactly one quadratic extension of E, and hence there is exactly one non-trivial quadratic character ω such that

$$\tilde{\pi} \cong \tilde{\pi} \otimes \omega.$$

This implies that the only Grössencharacters α with

$$\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee \otimes \alpha,$$

are χ and χ^σ, and in particular, there are none with $\alpha^\sigma = \alpha$.

(2) Step (1) implies that such an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ is not ν-distinguished with respect to $\mathrm{GL}_2(\mathbb{A}_F)$ for any Grössencharacter ν of $\mathbb{A}_F^*$, and hence by Proposition 3.4, none of

the members of the L-packet of automorphic forms determined by $\tilde{\pi}$ is $\mathrm{SL}_2(\mathbb{A}_F)$ distinguished.

(3) We next ensure that $\tilde{\pi}$ is locally distinguished (with respect to some character of F_v^*) at all the places v of F, and hence is abstractly distinguished by $SL_2(\mathbb{A}_F)$.

(4) At a place v of F which splits as (v_1, v_2) in E, the condition

$$\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee \otimes \chi,$$

amounts to $(\tilde{\pi}_{v_1}, \tilde{\pi}_{v_2})^\sigma \cong (\tilde{\pi}_{v_1}, \tilde{\pi}_{v_2})^\vee \otimes (\chi_1, \chi_2)$. This is equivalent to

$$\begin{aligned} \tilde{\pi}_{v_2} &\cong \tilde{\pi}_{v_1}^\vee \cdot \chi_1 \\ \tilde{\pi}_{v_1} &\cong \tilde{\pi}_{v_2}^\vee \cdot \chi_2. \end{aligned}$$

This implies in particular that $\tilde{\pi}_{v_1} \otimes \tilde{\pi}_{v_2}$ has χ_2-invariant linear form. Therefore in the L-packets determined by $\tilde{\pi}_{v_1}$ and $\tilde{\pi}_{v_2}$, there are representations π_{v_1} and π_{v_2} of $\mathrm{SL}_2(F_v)$ such that $\pi_{v_1} \cong \pi_{v_2}^\vee$. Thus for places of F which are split in E, local distinguishedness is automatic from the pseudo-distinguishedness condition $\tilde{\pi}^\sigma \cong \tilde{\pi}^\vee \otimes \chi$.

(5) For a place v of E which is inert over F, we will ensure that the local representation $\tilde{\pi}_v$ is either unramified, or comes from an unramified character of a quadratic extension, say M_v of E_v which is Galois over F_v. In the latter case, $\tilde{\pi}_v$ is the principal series representation of the form $Ps(\chi, \chi\omega)$ where χ is an unramified character of E_v, and ω is the quadratic character defining the quadratic extension M_v of E_v, and is therefore invariant under the automorphism of E_v over F_v. From lemma 8.1 below, $\tilde{\pi}_v$ is $\mathrm{SL}_2(F_v)$ distinguished.

(6) At places v of E at which $\tilde{\pi}_v$ is unramified, and v itself is unramified over F, all the members of the L-packet determined by $\tilde{\pi}_v$ are distinguished by $\mathrm{SL}_2(F_v)$. This easily follows as under these conditions $\mathrm{GL}_2(F_v)$ operates transitively on the L-packet of $\mathrm{SL}_2(E_v)$ determined by $\tilde{\pi}_v$.

(7) By steps (4),(5),(6), there are $\mathrm{SL}_2(\mathbb{A}_E)$ components of $\tilde{\pi} = \otimes \tilde{\pi}_v$ which are $\mathrm{SL}_2(\mathbb{A}_F)$ distinguished. Because of the flexibility offered by step 6, we can assume that these are even automorphic.

The proof of the following elementary lemma follows from Proposition 2.3 of [A-P].

Lemma 8.1. *For a separable quadratic extension K of a non-archimedean local field k with the nontrivial Galois automorphism σ of K over k, a* *principal series representation $Ps(\chi_1, \chi_2)$ of $GL_2(K)$ is distinguished by $SL_2(k)$ if and only if either $(\chi_1\chi_2^{-1})|_{k^*} = 1$, or $(\chi_1\chi_2^{-1}) = (\chi_1\chi_2^{-1})^{\sigma}$.*

Here is the main theorem of this section.

Theorem 8.2. *There exists a cuspidal automorphic representation π of $\mathrm{SL}_2(\mathbb{A}_E)$ for $E = \mathbb{Q}(\sqrt{-1})$ which is locally distinguished with respect to $SL_2(\mathbb{A}_{\mathbb{Q}})$ at all the places of $\mathbb{Q}$, but for which none of the members of its L-packet is globally distinguished.*

Proof. We will construct a cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_E)$ which is pseudo-distinguished by a character χ of $\mathbb{A}_E^*/E^*$ with $\chi \neq \chi^{\sigma}$, and unramified at all places of E. Such a representation π is locally SL_2-distinguished at all places of $\mathbb{Q}$. The representation π will be a monomial representation, coming from exactly one monomial quadratic extension of E. As we have seen above, the L-packet of automorphic representations of $\mathrm{SL}_2(\mathbb{A}_E)$ determined by π has no globally $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}})$-distinguished member. We now construct the specific example.

Let $L = \mathbb{Q}(\sqrt{-257})$, and $L' = \mathbb{Q}(\sqrt{257})$. From the tables in [B-S], the class group C_L of L is $\mathbb{Z}/16$, and the class group $C_{L'}$ of L' is $\mathbb{Z}/3$. Let $M = \mathbb{Q}(\sqrt{257}, \sqrt{-257})$ be the unique quadratic unramified extension of L. The natural map from the class group C_L to the class group C_M has $\mathbb{Z}/2$ as its kernel, and $\mathbb{Z}/8$ as its image. Further, the image of the natural map from $C_{L'}$ to C_M is $\mathbb{Z}/3$.

Let τ be the automorphism of M over $\mathbb{Q}$ which is nontrivial on both L and L', and we abuse notation to denote its restriction to L or L' also by τ. Further we let σ be the nontrivial automorphism of M over $\mathbb{Q}$ which is trivial on L.

We will be constructing an unramified character μ' of $\mathbb{A}_L^*/L^*$, which is the same as a character of C_L, such that μ'/μ'^{τ} is a quadratic character, and hence defines the quadratic unramified extension M of L. For any extension μ of μ' to $\mathbb{A}_M^*/M^*$, it is easy to see that $\mu\mu^{\sigma}$ is a τ-invariant character on $\mathbb{A}_M^*/M^*$, and hence there is a character $\tilde{\chi}$ of $\mathbb{A}_M^*/M^*$ such that

$$\mu\mu^{\sigma} = \tilde{\chi}\tilde{\chi}^{\tau}.$$

Denote the restriction of $\tilde{\chi}$ to $\mathbb{A}_E^*/E^*$ by χ. It can be checked that χ/χ^{σ} is the quadratic character of E defining the quadratic extension M of E.

It is easy to see that the action of τ on C_L and also on $C_{L'}$ is $x \to -x$, and hence also on the image of these groups in C_M.

Let μ' be a character of C_L of order 4. Such a character μ' is trivial on the kernel of the map from C_L to C_M. Since τ acts by $x \to -x$ on C_L and μ' is of order 4, μ'/μ'^{τ} is a character of order 2, hence defines the unique quadratic unramified extension of L which we are denoting by M. The character μ' being trivial on the kernel of the map from C_L to C_M extends to a character μ of C_M which we take to be nontrivial on the $\mathbb{Z}/3$ coming from $L' = \mathbb{Q}(\sqrt{257})$. Therefore μ/μ^τ is not of order 2. Let π be the cuspidal representation on $\mathrm{GL}_2(\mathbb{A}_E)$ obtained by 'inducing' the character μ of $\mathbb{A}_M^*/M^*$, which by the condition $\mu\mu^\sigma = \tilde{\chi}\tilde{\chi}^\tau$, will be pseudo-distinguished for the character χ. This π is the representation sought after in the statement of the theorem.

□

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