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ON A CONJECTURE OF JACQUET ABOUT DISTINGUISHED REPRESENTATIONS OF GL(n)

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Abstract
In this paper we prove a conjecture of Jacquet about supercuspidal representations of $\mathrm{GL}_n(K)$ *distinguished by* $\mathrm{GL}_n(k)$*, or by* $U_n(k)$*, for* K *a quadratic unramified extension of a non-Archimedean local field* k*.*

1. Introduction

Let $\underline{G}$ be a reductive algebraic group over a non-Archimedean local field k. Let K be a quadratic field extension of k. There has recently been much interest in trying to classify representations of $\underline{G}(K)$ which have $\underline{G}(k)$-invariant linear forms. The initial impetus for such a study came from the work of G. Harder, R. Langlands, and M. Rapoport [HLR] for $\underline{G} = \mathrm{GL}_2$ which was done in the global context. H. Jacquet (cf. [JY1], [JY2]) has made the following conjectures for $\underline{G} = \mathrm{GL}_n$ and $\underline{G} = U_n$, where U_n is the unique quasi-split unitary group in n variables over k which is split over K. We also refer to the paper [F] by Y. Flicker.

CONJECTURE 1
Let π *be an irreducible admissible representation of* $\mathrm{GL}_n(K)$*, where* K *is a quadratic extension of a non-Archimedean local field* k*. Assume that the central character of* π *restricted to* k^* *is trivial. Then we have the following.*

(1) *If* n *is odd,* $\pi^\sigma \cong \pi^*$ *if and only if* π *has a* $\mathrm{GL}_n(k)$*-invariant linear form, where* π^σ *denotes the representation of* $\mathrm{GL}_n(K)$ *obtained from* π *by using the automorphism of* $\mathrm{GL}_n(K)$ *coming from the Galois automorphism* σ *of* K *over* k*.*

(2) *If* n *is even,* $\pi^\sigma \cong \pi^*$ *if and only if either* π *has a* $\mathrm{GL}_n(k)$*-invariant linear form or* π *has a linear form* ℓ *with* $\ell(gv) = \omega_{K/k}(\det g)\ell(v)$ *for* $g \in \mathrm{GL}_n(k)$ *and* $v \in \pi$*, where* $\omega_{K/k}$ *is the quadratic character of* k^* *associated to the extension* K *of* k*.*

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CONJECTURE 2
Let π be an irreducible admissible representation of $\mathrm{GL}_n(K)$, where K is a quadratic extension of a non-Archimedean local field k. Then $\pi^\sigma \cong \pi$ if and only if π has a $U_n(k)$-invariant linear form for U_n, the unique quasi-split unitary group in n variables over k which is split over K, and where π^σ denotes the representation of $\mathrm{GL}_n(K)$ obtained from π by using the automorphism of $\mathrm{GL}_n(K)$ coming from the Galois automorphism σ of K over k.

The aim of this paper is to prove these conjectures for supercuspidal representations of $\mathrm{GL}_n(K)$ when K is an unramified quadratic extension of k.

The analogues of these conjectures in the case of finite fields is due to R. Gow [G] (cf. also [P2]).

The proof of this conjecture is accomplished via the methods of our earlier paper [P2], in which we treated a similar question for certain representations of finite groups of Lie type, together with the theorem of C. Bushnell and P. Kutzko that realizes any supercuspidal representation of GL_n by compact induction from a finite-dimensional representation of an open subgroup which is compact modulo the center. Since the methods of compact induction are expected to be true for supercuspidal representations in great generality, it appears that the methods of this paper, which treats representations of compact open subgroups via finite groups of Lie type, though usually not reductive, may have greater applicability.

We note that in an ongoing work of J. Hakim and F. Murnaghan (cf. [HM]), the authors are able to obtain certain results for both the ramified and unramified quadratic extension K of k by an elaborate structure theory of the representations of GL_n, but they are not able to get as complete results as we obtain here for the quadratic unramified case. We refer to the paper of Hakim and Z. Mao [HMa] for some of the earlier results in this direction.

2. Recollection of earlier results

We begin by recalling the theorem proved in our earlier paper [P2].

Let $\underline{G}(\mathbb{F})$ be the $\mathbb{F}$ rational points of a connected algebraic group $\underline{G}$ over a finite field $\mathbb{F}$. Let $\mathbb{E}$ be the quadratic field extension of $\mathbb{F}$.

We recall that in [P2] we called a representation of $G = \underline{G}(\mathbb{E})$ *stable* if its character takes the same value on any two elements in $\underline{G}(\mathbb{E})$ which become conjugate when we extend the field $\mathbb{E}$ to its algebraic closure.

Let σ denote the automorphism of $\underline{G}(\mathbb{E})$ obtained from the Galois automorphism of $\mathbb{E}$ over $\mathbb{F}$, and let π^σ denote the representation of $\underline{G}(\mathbb{E})$ obtained from a representation π of $\underline{G}(\mathbb{E})$ by using the automorphism σ of the group $\underline{G}(\mathbb{E})$.

THEOREM 1
For a connected algebraic group $\underline{G}$ over $\mathbb{F}$, an irreducible stable representation π of $\underline{G}(\mathbb{E})$ has a fixed vector for $\underline{G}(\mathbb{F})$ if and only if $\pi^\sigma \cong \pi^*$.

Remark. For a Hermitian matrix J in $\mathrm{GL}_n(\mathbb{E})$, the unitary group $U_n(J)$ can be defined to be the set of matrices $g \in \mathrm{GL}_n(\mathbb{E})$ such that

$$gJ\sigma({}^t g) = J,$$

or $g = J\sigma({}^t g^{-1})J^{-1}$. Thus the unitary group $U_n(J)$ can be considered as the fixed points of the involution $g \to J\sigma({}^t g^{-1})J^{-1}$ on $\mathrm{GL}_n(\mathbb{E})$, which is to be thought of as the new Frobenius action on $\mathrm{GL}_n(\mathbb{E})$ whose fixed point subgroup is $U_n(J)$. Under this Frobenius action the transform of π, to be denoted by π^{Fr}, becomes $(\pi^\sigma)^*$; hence the condition $\pi^{\mathrm{Fr}} \cong \pi^*$ becomes, for unitary groups, $(\pi^\sigma)^* \cong \pi^*$ or $\pi^\sigma \cong \pi$. This explains the difference in the condition on a representation of $\mathrm{GL}_n(K)$—to have a $\mathrm{GL}_n(k)$-invariant linear form, or to have a $U_n(k)$-invariant linear form in Conjectures 1 and 2 due to Jacquet.

We apply this theorem to prove the following theorem.

THEOREM 2
Let K be a quadratic unramified extension of a non-Archimedean local field k. Suppose that $\mathscr{O}_K$ and $\mathscr{O}_k$ are the ring of integers in the two fields. Let $\underline{G}$ be either GL_n over $\mathscr{O}_k$ or the unitary group over $\mathscr{O}_k$ defined in terms of a nondegenerate Hermitian form over $\mathscr{O}_K$. (A nondegenerate Hermitian form over $\mathscr{O}_K$ means, in concrete terms, that the matrix of the Hermitian form has entries in $\mathscr{O}_K$ and that its determinant is a unit, i.e., an element of $\mathscr{O}_K^*$.) Then an irreducible representation π of $\mathrm{GL}_n(\mathscr{O}_K)$ has a fixed vector for $\underline{G}(\mathscr{O}_k)$ if and only if $\pi^\sigma \cong \pi^*$. Here the action of σ on representations of $\mathrm{GL}_n(K)$ is the standard Galois action for $\underline{G} = \mathrm{GL}_n$ and is the standard Galois action composed with the dual for $\underline{G}$, the unitary group.

Proof
An irreducible representation of $\mathrm{GL}_n(\mathscr{O}_K)$ is actually a representation of $\mathrm{GL}_n(\mathscr{O}_K/\pi_k^m)$ for some integer $m \geq 1$, where π_k denotes a uniformizing parameter of $\mathscr{O}_k$ and hence also of $\mathscr{O}_K$. It is a consequence of a theorem of M. Greenberg (cf. [G1], [G2]), generalizing the notion of *Witt group schemes,* that the group $\underline{G}(\mathscr{O}_k/\pi_k^m)$ is representable by a connected algebraic group over the finite field $\mathscr{O}_k/\pi_k$ in the sense that there is a connected algebraic group $\underline{G}_{n,m}$ over the finite field $\mathscr{O}_k/\pi_k$ such that for any finite field extension $\mathbb{E}$ of $\mathscr{O}_k/\pi_k$, $G_{n,m}(\mathbb{E}) = \underline{G}(\mathscr{O}_L/\pi_k^m)$, where L is the unique unramified extension of k which corresponds to the extension $\mathbb{E}$ of the residue field $\mathscr{O}_k/\pi_k$ of k.

The proof of this theorem will therefore follow from Theorem 1 if we can check that all representations of $\mathrm{GL}_n(\mathscr{O}_K/\pi_k^m)$ are stable. This is because in fact in $G_{n,m}$ there is no difference between conjugacy and stable conjugacy. This follows, for instance, by an application of Lang's theorem, as the centralizer of any element in $G_{n,m}$ is connected. To substantiate our claim about the connectedness of the centralizer, we only point out that the invertible elements in any $\mathscr{O}_K$ subalgebra (not necessarily free over $\mathscr{O}_K/\pi_k{}^n$) of the matrix algebra $M_n(\mathscr{O}_K/\pi_k{}^n)$ define a connected group. □

3. The theorem of Bushnell and Kutzko

The following theorem is due to Bushnell and Kutzko [BK, Chap. 6].

THEOREM 3

Given a supercuspidal representation π of $\mathrm{GL}_n(K)$, there exists an irreducible representation Π of $K^*\mathrm{GL}_n(\mathscr{O}_K)$ such that

$$\mathrm{Ind}_{K^*\mathrm{GL}_n(\mathscr{O}_K)}^{\mathrm{GL}_n(K)} \Pi \cong \sum_{\mu} \pi \otimes \mu,$$

where the characters μ of K^* are certain distinct unramified characters of K^* with $\mu^n = 1$ which form a group under multiplication; the representations $\pi \otimes \mu$ are distinct for distinct characters μ.

Proof

Since this is not the usual form of the theorem of Bushnell and Kutzko, we give a detailed proof. We recall that Bushnell and Kutzko realized any supercuspidal representation of $\mathrm{GL}_n(K)$ as an induced representation from a maximal compact modulo center subgroup of $\mathrm{GL}_n(K)$,

$$\pi \cong \mathrm{ind}_{\mathscr{K}}^{\mathrm{GL}_n(K)} \Lambda,$$

for a certain maximal compact modulo center subgroup $\mathscr{K}$ of $\mathrm{GL}_n(K)$ which can be written as

$$\mathscr{K} = \mathscr{K}_0 \cdot E^*$$

with $\mathscr{K}_0 \subset \mathrm{GL}_n(\mathscr{O}_K)$, a normal subgroup of $\mathscr{K}$, and E a field extension of K of degree n. The mapping val $\circ$ det on $\mathscr{K}$ induces an isomorphism

$$\mathscr{K}/(\mathscr{K}_0 \cdot K^*) \cong f\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/e\mathbb{Z},$$

where $f\mathbb{Z}$ is the image of val $\circ$ det on E^* and e is the ramification index of E. It follows that

$$\mathrm{Ind}_{K^*\mathscr{K}_0}^{\mathscr{K}} \left(\Lambda|_{K^*\mathscr{K}_0}\right) = \sum \Lambda \otimes \mu$$

for unramified characters μ of K^* coming from the characters of $\mathbb{Z}/e\mathbb{Z}$. Hence

$$\begin{aligned}\sum \pi \otimes \mu &= \mathrm{ind}_{K^*\mathscr{K}_0}^{\mathrm{GL}_n(K)}\big(\Lambda|_{K^*\mathscr{K}_0}\big)\\ &= \mathrm{ind}_{K^*\mathrm{GL}_n(\mathscr{O}_K)}^{\mathrm{GL}_n(K)}\Big[\mathrm{ind}_{K^*\mathscr{K}_0}^{K^*\mathrm{GL}_n(\mathscr{O}_K)}\big(\Lambda|_{K^*\mathscr{K}_0}\big)\Big]\\ &= \mathrm{ind}_{K^*\mathrm{GL}_n(\mathscr{O}_K)}^{\mathrm{GL}_n(K)}\Pi\end{aligned}$$

with

$$\Pi = \mathrm{ind}_{K^*\mathscr{K}_0}^{K^*\mathrm{GL}_n(\mathscr{O}_K)}\big(\Lambda|_{K^*\mathscr{K}_0}\big).$$

That the representations $\pi \otimes \mu$ are distinct follows from the uniqueness of the representation Λ of $\mathscr{K}$ with $\pi \cong \mathrm{ind}_{\mathscr{K}}^{\mathrm{GL}_n(K)}\Lambda$ together with the property of Λ that it is irreducible when restricted to $\mathscr{K}_0$. □

4. Some known results

In this section we recall the following lemma due to Flicker (cf. [F]) about the double coset decomposition of $\mathrm{GL}_n(K)$ by $\mathrm{GL}_n(k)$, whose simple proof we supply for completeness. Here K is a separable quadratic extension of k.

LEMMA 1

For any g in $\mathrm{GL}_n(K)$, $\sigma(g^{-1}) = g_1 g g_2$ for matrices $g_1, g_2 \in \mathrm{GL}_n(k)$.

Proof

It suffices to prove that, given any g in $\mathrm{GL}_n(K)$, there is g_1 in $\mathrm{GL}_n(k)$ such that $\sigma(g)g_1g$ belongs to $\mathrm{GL}_n(k)$. For this it suffices to prove that the equation

$$\sigma(g)Xg = gX\sigma(g)$$

has a solution for X in $\mathrm{GL}_n(k)$. It is clear that the set of solutions in the matrix algebra $M_n(K)$ forms a vector space V that is stable under the Galois involution, hence defined over k, and is nonzero (as it contains, for instance, g^{-1}). Since the determinant takes nonzero values on V, it does so over the k structure V_k of V too. (A nonzero polynomial cannot vanish on an affine space over an infinite field!) □

The following two corollaries were also obtained by Flicker via standard techniques.

COROLLARY 1

The space of $\mathrm{GL}_n(k)$-invariant linear forms on any irreducible admissible representation of $\mathrm{GL}_n(K)$ is at most one-dimensional.

COROLLARY 2
If an irreducible representation π of $\mathrm{GL}_n(K)$ has a $\mathrm{GL}_n(k)$-invariant linear form, then one has $\pi^\sigma \cong \pi^*$.

5. Main theorem

We now give the proof of Jacquet's conjecture when K is an unramified quadratic extension of k.

THEOREM 4
Let π be an irreducible admissible supercuspidal representation of $\mathrm{GL}_n(K)$, where K is an unramified quadratic extension of a non-Archimedean local field k. Assume that the central character of π restricted to k^* is trivial. Then we have the following.

(1) *If n is odd, $\pi^\sigma \cong \pi^*$ if and only if π has a $\mathrm{GL}_n(k)$-invariant linear form, where π^σ denotes the representation of $\mathrm{GL}_n(K)$ obtained from π by using the automorphism of $\mathrm{GL}_n(K)$ coming from the Galois automorphism σ of K over k.*

(2) *If n is even, $\pi^\sigma \cong \pi^*$ if and only if either π has a $\mathrm{GL}_n(k)$-invariant linear form or π has a linear form ℓ with $\ell(gv) = \omega_{K/k}(\det g)\ell(v)$ for $g \in \mathrm{GL}_n(k)$ and $v \in \pi$, where $\omega_{K/k}$ is the quadratic character of k^* associated to the extension K of k.*

Proof
From Corollary 2 we already know that if π has a $\mathrm{GL}_n(k)$-invariant linear form, then $\pi^\sigma \cong \pi^*$. It therefore suffices to prove the converse statement. From the theorem of Bushnell and Kutzko recalled in Section 3, there exists an irreducible representation Π of $K^*\mathrm{GL}_n(\mathcal{O}_K)$ such that

$$\mathrm{Ind}_{K^*\mathrm{GL}_n(\mathcal{O}_K)}^{\mathrm{GL}_n(K)}\Pi \cong \sum_\mu \pi \otimes \mu,$$

where the characters μ of K^* are certain distinct unramified characters of K^*. The isomorphism $\pi^\sigma \cong \pi^*$ implies that the same is true for Π: $\Pi^\sigma \cong \Pi^*$. To prove this, note that Bushnell and Kutzko work with "simple types", say (J, λ), and prove a uniqueness theorem for these up to G conjugacy. The isomorphism of π with $\pi^{\sigma *}$ would, however, only give an element g in $G = \mathrm{GL}_n(K)$ which preserves J under inner conjugation and takes λ to $\lambda^{\sigma *}$. This g has the property that g^2 takes λ to λ and hence belongs to J. It follows that the group $\mathbb{J}$ generated by g and J is compact modulo center and that the induction of λ to $\mathbb{J}$, say Λ, has the property that $\Lambda \cong \Lambda^{\sigma *}$. Induction of Λ to a maximal compact modulo center subgroup will continue to have this property, and hence the same is true for Π by the proof of Theorem 3.

From a simple application of Mackey theory about the restriction of an induced representation to a subgroup, it follows that

$$\mathrm{Res}_{\mathrm{GL}_n(k)}\mathrm{Ind}_{K^*\mathrm{GL}_n(\mathscr{O}_K)}^{\mathrm{GL}_n(K)}\Pi = \mathrm{Ind}_{k^*\mathrm{GL}_n(\mathscr{O}_k)}^{\mathrm{GL}_n(k)}\Pi|_{k^*\mathrm{GL}_n(\mathscr{O}_k)} \oplus \dots,$$

where the terms omitted in the above expression come from the nontrivial double cosets of

$$K^*\mathrm{GL}_n(\mathscr{O}_K)\backslash\mathrm{GL}_n(K)/\mathrm{GL}_n(k).$$

Noting that Π restricted to k^* is trivial, Theorem 2 together with an application of the Frobenius reciprocity implies that one of the twists of π by an unramified character μ of K^* has a $\mathrm{GL}_n(k)$-invariant form. We claim that the only possible μ for which $\pi \otimes \mu$ could possibly have a $\mathrm{GL}_n(k)$-invariant linear form is the unramified character μ of order 2. For this we note by Corollary 2 that if $\pi \otimes \mu$ has a $\mathrm{GL}_n(k)$-invariant linear form, then $(\pi \otimes \mu)^\sigma \cong (\pi \otimes \mu)^*$, or $\pi^\sigma \cong \pi^* \otimes \mu^{-2}$. Since we are already given that $\pi^\sigma \cong \pi^*$, it follows that $\pi \cong \pi \otimes \mu^2$. But the twists that appear in Theorem 3 (due to Bushnell and Kutzko) are all distinct. Hence $\mu^2 = 1$. In particular, if n is odd, μ is trivial, and hence a representation π of $\mathrm{GL}_n(K)$, n odd, with $\pi^\sigma \cong \pi^*$, has a $\mathrm{GL}_n(k)$-invariant linear form. If n is even, then either π or $\pi \otimes \omega_{K/k}$, has a $\mathrm{GL}_n(k)$-invariant form. □

THEOREM 5

Let π be an irreducible admissible supercuspidal representation of $\mathrm{GL}_n(K)$, where K is a quadratic unramified extension of a non-Archimedean local field k. Then $\pi^\sigma \cong \pi$ if and only if π has a $U_n(k)$-invariant linear form for U_n, the unique quasi-split unitary group in n variables over k which is split over K, and where π^σ denotes the representation of $\mathrm{GL}_n(K)$ obtained from π by using the automorphism of $\mathrm{GL}_n(K)$ coming from the Galois automorphism σ of K over k.

Proof

If a representation π of $\mathrm{GL}_n(K)$ has a $U_n(k)$-invariant linear form, then one has $\pi^\sigma \cong \pi$. This is proved via global methods by embedding a representation of $\mathrm{GL}_n(K)$ which has a nontrivial $U_n(k)$-invariant linear form into a global automorphic representation with nonzero period on the unitary group and then appealing to a global theorem. We refer to [F], [HF], and [H] for various contexts in which such a result has been proved and to the most recent and most complete work by Jacquet in [J].

It therefore suffices to prove that supercuspidal representations π of $\mathrm{GL}_n(K)$ with $\pi^\sigma \cong \pi$ carry a $U_n(k)$-invariant linear form. The proof of the previous theorem constructs in this case a $U_n(k)$-invariant linear form on some twist $\pi \otimes \mu$ of π by an unramified character of K^*. Notice that if $\pi \otimes \mu$ has a $U_n(k)$-invariant linear form, then π itself carries a $U_n(k)$ linear form. This follows as the determinant map on

$\mathrm{GL}_n(K)$ which, when restricted to $U_n(k)$, takes values in $U_1 = \{z \in K^* | z\sigma(z) = 1\}$ on which an unramified character such as μ must be trivial. □

Remark. It will be nice to be able to carry out a generalization of the method used here to the case of the ramified quadratic extensions. One of the difficulties in this case, which was also encountered in [P1] but taken care of there by explicit character formulae, is that the unique invariant linear form that one wants to construct does not arise from the *trivial* double coset used in the arguments of the above theorems. Thus although Theorem 2 is not true for ramified field extensions, as one can easily see, Conjectures 1 and 2 are expected to be true.

Remark. It is expected that if π is a supercuspidal representation of $\mathrm{GL}_n(K)$ with $\pi^\sigma \cong \pi^*$, then either π or $\pi \otimes \omega_{K/k}$ has a $\mathrm{GL}_n(k)$-invariant linear form, where $\omega_{K/k}$ is the quadratic character of k^* associated to the extension K of k, but that the two possibilities do not hold simultaneously. Also, it is expected that if π is a supercuspidal representation of $\mathrm{GL}_n(K)$ with $\pi^\sigma \cong \pi$, then π has a $U_n(k)$-invariant form which is unique up to scalars. Both of these expectations are false for principal series representations, as can be easily seen. Hence methods of Gelfand pairs are inadequate to prove these multiplicity-1 expectations. Having constructed the desired linear forms, what needs to be proved is that *nontrivial* double cosets do not contribute to invariant linear forms. This can be done by the property of inducing data in many cases (cf. [HM], [HMa]).

6. Question of central characters

In Conjecture 1 and Theorem 4, we restricted ourselves to representations of $\mathrm{GL}_n(K)$ whose central character restricted to k^* is trivial. One can in fact treat the more general situation that might arise from the condition $\pi^\sigma \cong \pi^*$. Observe that if $\pi^\sigma \cong \pi^*$, then the restriction of the central character of π to k^* is either trivial or is $\omega_{K/k}$. Fix a character χ of K^* whose restriction to k^* is $\omega_{K/k}$. It is easy to see that twisting by the character χ preserves the condition $\pi^\sigma \cong \pi^*$, and if n is odd, it takes representations π whose central character restricted to k^* is trivial to representations $\pi \otimes \chi$ whose central character restricted to k^* is nontrivial, and vice-versa. It is clear that if π has a $\mathrm{GL}_n(k)$-invariant linear form, then for n odd, $\pi \otimes \chi$ has a linear form on which $\mathrm{GL}_n(k)$ operates by $\omega_{K/k}$. Hence Theorem 4 implies the following slightly more general theorem

THEOREM 6

A representation π of $\mathrm{GL}_n(K)$ for K a quadratic unramified extension of k and n odd, with $\pi^\sigma \cong \pi^*$, has a $\mathrm{GL}_n(k)$-invariant linear form if and only if its central character *restricted to* k^* *is trivial. If the central character of* π *restricted to* k^* *is* $\omega_{K/k}$*, then* π *has a linear form* $\ell : \pi \to \mathbb{C}$ *with* $\ell(gv) = \omega_{K/k}(\det g)\ell(v)$ *for* $v \in \pi$ *and for* g *in* $\mathrm{GL}_n(k)$.

For n even, we have the following theorem.

THEOREM 7
For K *a quadratic extension of* k *and* n *even, a supercuspidal representation* π *of* $\mathrm{GL}_n(K)$*, with* $\pi^\sigma \cong \pi^*$ *has trivial central character when restricted to* k^*.

Proof
The analogous result for irreducible representations of the Weil group W_K is a simple group-theoretic fact proved in J. Rogawski's book (cf. [R, Lemma 15.1.2(b)]). The result then follows from the local Langlands conjecture proved by Harris, Taylor, and Henniart. □

7. A conjecture
The method of this paper, which tries to retrieve information on representations of a p-adic group via its restriction to compact open subgroups, does not apply to representations other than supercuspidals, most notably to discrete series representations that are not supercuspidal. Based on what is expected for GL_n, it is tempting to speculate about at least one general class of representations as to what may be expected in general. In this section we make a conjecture about when the Steinberg representation of $G(K)$ has a $G(k)$-invariant linear form in the case when G is a quasi-split reductive group over a non-Archimedean local field k. A particular case of the conjecture below is that, for a simply connected semisimple quasi-split group G over a local field k, the Steinberg representation of $G(K)$ carries a unique $G(k)$-invariant linear form. This is not the case for general quasi-split reductive groups, and we make a precise conjecture below.

Observe that if there is an exact sequence of algebraic groups

$$1 \to A \to G \to G' \to 1$$

with A a central subgroup in a reductive algebraic group G whose derived subgroup is quasi-split over k, then the k-rational points of a flag variety G/P of G can be identified to the k-rational points of a flag variety G'/P' of G'. It follows that the Steinberg representation of $G(k)$ is the restriction to $G(k)$ of the Steinberg representation of $G^{\mathrm{ad}}(k)$, where G^{ad} is the group G divided by its center $Z(G)$. This actually gives an extra structure to the Steinberg representation of $G(k)$ since $G^{\mathrm{ad}}(k)$ is in general larger than the image of $G(k)$ in $G^{\mathrm{ad}}(k)$.

We now construct a natural character χ_K on $G(k)$ with values in $\mathbb{Z}/2$ associated to any quadratic extension K of k, where G is any reductive group over the local field k.

We denote the simply connected cover of G^{ad} by G^{sc}, and we denote the center of G^{sc} by Z. By a theorem due to Kneser and Bruhat-Tits, the first Galois cohomology of G^{sc} vanishes. This gives rise to the following exact sequence of groups:

$$1 \to Z(k) \to G^{\mathrm{sc}}(k) \to G^{\mathrm{ad}}(k) \to H^1(k, Z) \to 1.$$

It is known that $G^{\mathrm{sc}}(k)/Z(k)$ is its own derived subgroup if G^{sc} is not anisotropic; this is a consequence of the so-called Kneser-Tits problem, known to be true for all p-adic fields due to Platonov. Hence, from the exact sequence above, the character group of $G^{\mathrm{ad}}(k)$ can be identified to the character group of $H^1(k, Z)$. By the Tate-Nakayama duality, the character group of $H^1(k, Z)$ can be identified to $H^1(k, Z^\vee)$, where $Z^\vee$ is the Cartier dual of Z.

Let $G^\vee$ be the dual group of G^{ad}. So $G^\vee$ is a complex semisimple simply connected group whose center is isomorphic to $Z^\vee(\mathbb{C})$. The group $G^\vee$ comes equipped with the action of the Galois group of k via algebraic automorphisms on the complex group $G^\vee(\mathbb{C})$, and hence the center of $G^\vee(\mathbb{C})$, which as we have pointed out is $Z^\vee(\mathbb{C})$, gets a Galois action that is the same as it gets as the Cartier dual of Z. (In particular, $Z^\vee$ is a constant group scheme over k for a semisimple split group G^s.) It follows from the Jacobson-Morozov theorem that there is a homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to $G^\vee(\mathbb{C})$ which takes a nontrivial unipotent of SL_2 to a regular unipotent in $G^\vee(\mathbb{C})$. Since the action of the Galois group of k preserves a based root datum in $G^\vee$, there is a regular unipotent in $G^\vee(\mathbb{C})$ on which the Galois action is trivial. Hence the homomorphism from $\mathrm{SL}_2(\mathbb{C})$ to $G^\vee(\mathbb{C})$ can be assumed to be invariant under the Galois action. Under this homomorphism the center of SL_2, consisting of ± 1, goes to the center of $G^\vee$ and thus canonically gives a Galois invariant element in the center of $G^\vee$ which is either trivial or is of order 2. (This is the element that decides whether an algebraic self-dual representation of $G^\vee$ is orthogonal or symplectic (cf. [P3]).) The associated mapping from $\mathbb{Z}/2$ to $Z^\vee$ gives rise to a homomorphism from $H^1(k, \mathbb{Z}/2)$ to $H^1(k, Z^\vee)$. We now define an element in $H^1(k, Z^\vee)$ to be the image of the element in $H^1(k, \mathbb{Z}/2)$ which defines the quadratic extension K of k. This, as we saw earlier, defines a character, say χ_K, which is either trivial or of order 2 on the group $G^{\mathrm{ad}}(k)$ with values in $\mathbb{Z}/2$ associated to any quadratic extension K of a local field k. If G is any reductive group over k, the natural map from G to G^{ad}, when composed with the character χ_K defined here for G^{ad}, thus defines a character on $G(k)$ for any reductive group G.

We are now ready to make our conjecture.

CONJECTURE 3
For a reductive algebraic group G over a local field k whose derived subgroup is quasi-split, the Steinberg representation of $G(K)$, K a quadratic extension of k, carries a unique linear form ℓ such that

$$\ell(gv) = \chi_K(g)\ell(v)$$

for all $g \in G^{\mathrm{ad}}(k)$, and v a vector in the Steinberg representation of $G(K)$. The Steinberg representation of $G(K)$ does not carry a χ-invariant linear form for the action of the group $G^{\mathrm{ad}}(k)$ on the Steinberg representation of $G(K)$ for any other character χ of $G^{\mathrm{ad}}(k)$.

Remark. For $G = \mathrm{GL}_2(K)$, this conjecture follows from the results in [P1].

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