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ON THE LOCAL HOWE DUALITY CORRESPONDENCE

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1. The Howe duality conjecture, which has recently been proved by Waldspurger in [W1] for all local fields of residue characteristic $\neq 2$ and all dual reductive pairs, gives in particular for the pair $(O(2m), Sp(2n))$ a bijective correspondence between certain representations of $O(2m)$ and $Sp(2n)$. The present work is an attempt to understand the correspondence of the representations arising out of Howe duality in terms of functoriality. In the Archimedean case, Jeffrey Adams made some conjectures in [A] about Langlands parameters of the representations which correspond under the Howe duality correspondence. Later Moeglin [Mo] gave explicit results about the Howe duality correspondence for all orthogonal and symplectic groups, and thereby proved those parts of Adams's conjectures in which it was not necessary to consider Arthur packets. We recall that some of Adams's conjectures were in the cases where the absolute rank of one of the groups was smaller than or equal to half the real rank of the other (the "stable" range). However, it appears that, when the groups have similar absolute rank and we restrict our consideration to only tempered representations, then the Howe duality correspondence respects functoriality (unlike in the general case considered by Adams), and it is in a certain sense the simplest possible for both Archimedean and non-Archimedean fields. In this paper we give precise conjectures about what representations of $O(2m)$ and $Sp(2n)$ appear in the duality correspondence and how their Langlands parameters (and their refinement by Vogan) are related in the cases $2m = 2n$ and $2m = 2n + 2$. In particular, the difference between the roles of $SO(2m)$ and $O(2m)$ in duality correspondence is clearly brought out.

The conjectures we make here are in a similar spirit to the ones made in [GP1] and [GP2], and though there are no epsilon factors in this work, the role of Vogan packets and their parametrization is as crucial here as it was there.

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2. Let V be a quadratic space over a local field k of characteristic $\neq 2$, of dimension $2n$, and with normalised discriminant $D = (-1)^n d$ where d is the usual discriminant of V. Let V_α denote the isomorphism classes of quadratic spaces of dimension $2n$ and normalised discriminant D. Therefore $SO(V_\alpha)$ are the various *pure*

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inner forms of the group $SO(V)$ in the sense of Vogan. Let W be a fixed symplectic space of dimension $2n$. Since the first Galois cohomology of the symplectic group is zero, the symplectic group does not have any other pure inner form, and its Vogan L-packet coincides with the Langlands L-packet. The space $V_\alpha \otimes W$ has a natural symplectic structure, and we let $\widetilde{Sp}(V_\alpha \otimes W)$ denote the two-sheeted metaplectic cover of the symplectic group $Sp(V_\alpha \otimes W)$.

Since the Weil representation of the metaplectic group depends on the choice of an additive character ψ of k, we will fix this character in all that follows. We would like to emphasize here that the Weil representation of $\widetilde{Sp}(W)$ depends also on the symplectic form on W.

We follow the convention of [K] for the Howe duality correspondence between representations of $O(V_\alpha)$ and $Sp(W)$. We simply note here that the metaplectic covering of $Sp(V_\alpha \otimes W)$ need not split over $O(V_\alpha)$ if $4 \nmid \dim W$, but Kudla [K, 1.0.1] constructs a character χ of the possibly nontrivial covering of $O(V_\alpha)$ by $\mu_2 = \{\pm 1\}$ such that multiplication by χ induces a bijection between representations of the covering nontrivial on μ_2 with representations of $O(V_\alpha)$.

We now fix base points for Vogan L-packets for both $SO(V_\alpha)$ and $Sp(W)$. For this we need to fix a quasi-split inner form of the group together with a nondegenerate character on the unipotent radical of a Borel subgroup. We first take up the case of the symplectic group. Let us fix a symplectic basis $\{e_1, \ldots, e_n; f_1, \ldots, f_n\}$ with $\langle e_i, f_j \rangle = \delta_{ij}$, and $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$ for all $1 \leqslant i, j \leqslant n$. Let B be the Borel subgroup stabilising the isotropic flag $(e_1) \subset (e_1, e_2) \subset \cdots \subset (e_1, \ldots, e_n)$, U its unipotent radical, and T its standard maximal torus. Since the simple roots in the standard notation are $\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$, it is easy to see that the T orbit of nondegenerate characters of U are in one-to-one correspondence to the T orbit of its restriction to the long simple root corresponding to $2\varepsilon_n$. We identify this root space to k via the map taking $x \in k$ to the transformation taking e_n to e_n and f_n to $f_n + xe_n$, and taking all the other basis vectors to themselves. Using the character ψ, we therefore get a T-conjugacy class of a nondegenerate character on U. Let us emphasize here that the identification of the long simple root space to k depends on the symplectic from on W, and therefore the character on U also depends on the symplectic from on W (besides depending on ψ).

For fixing the base point in the Vogan L-packet of $SO(V)$, we first fix a quasi-split quadratic space V_α (i.e., a quadratic space of the same discriminant as V and having an isotropic subspace of as large a dimension as possible; this fixes the quadratic space if the normalised discriminant is 1, otherwise there are two possibilities). If the dimension of V is 2, then the unipotent radical of a Borel is trivial, and we do not have to worry about fixing a character on it. So assume that the dimension of V is $\geqslant 4$. In this case the quadratic form on V_α represents 1, so choose a vector v of length 1 in V_α. Under this circumstance, we have constructed a unique $SO(V_\alpha)$ conjugacy class of nondegenerate homomorphisms of a maximal unipotent subgroup to k in [GP2, Proposition 4.3]. We use this nondegenerate homomorphism composed with ψ on $SO(V_\alpha)$ to act as the base point of the Vogan L-packet of $SO(V)$.

We let $E = k[x]/(x^2 - D)$ denote the quadratic discriminant algebra associated to the quadratic space V, and ω_D the character of k^* of order $\leqslant 2$ associated by the local classfield theory to E (so $\omega_D \equiv 1$ if and only if $E \cong k \oplus k$).

3. We observe that a representation of $O(V_\alpha)$ decomposes in at most two representations of $SO(V_\alpha)$ which are permuted by any reflection in $O(V_\alpha)$. For a representation μ of $SO(V_\alpha)$, we let μ^ε denote the representation of $SO(V_\alpha)$ obtained by conjugating the representation μ by any reflection in $O(V_\alpha)$. The representation μ of $SO(V_\alpha)$ extends to a representation of $O(V_\alpha)$ if and only if $\mu \cong \mu^\varepsilon$, and a representation of $O(V_\alpha)$ does not remain irreducible when restricted to $SO(V_\alpha)$ if and only if $\pi \cong \pi \otimes \det$.

As is standard, we let $W(k)$ denote the Weil group of the local field k, and $W'(k)$ the Weil-Deligne group. Several of our conjectures depend on the conjectural notion of L-packets of representations of a group, their parametrization in terms of certain mappings (called the Langlands parameter) of the Weil-Deligne group into the L-group, and the refinement of these conjectures of Langlands by Vogan [Vo] which identifies the individual representations in the L-packet in terms of the component group of the centraliser of the Langlands parameter. These conjectures of Langlands and Vogan are known only for the Archimedean fields but we assume them in all of this paper. We recall that a parameter for an irreducible representation of (a pure inner form of) $SO(V)$ is a homomorphism

$$\phi\colon W'(k) \to O(M), \qquad \text{with } \det(\phi) = \omega_D$$

up to a conjugation by $SO(M)$ where M is a complex orthogonal space with $\dim(M) = \dim(V)$, as well as a character

$$\chi\colon A_\phi \to \{\pm 1\}$$

where A_ϕ is the component group of the centraliser of the image of ϕ in $SO(M)$. We expect that a parameter for an irreducible representation of (a pure inner form of) $O(V)$ is a homomorphism

$$\phi\colon W'(k) \to O(M), \qquad \text{with } \det(\phi) = \omega_D$$

up to conjugation by $O(M)$ where M is a complex orthogonal space with $\dim(M) = \dim(V)$, as well as a character

$$\chi\colon B_\phi \to \{\pm 1\}$$

where B_ϕ is the component group of the centraliser of the image of ϕ in $O(M)$.

The set of all representations of (pure inner forms of) $SO(V)$ or $O(V)$ with a given ϕ is called a Vogan L-packet. A given $\phi\colon W'(k) \to O(M)$ with $\det \phi = \omega_D$ defines Vogan L-packets for both $SO(V)$ and $O(V)$. The set of representations in the Vogan

L-packet of $O(V)$ determined by ϕ is precisely those irreducible representations of $O(V_\alpha)$ whose restriction to $SO(V_\alpha)$ contains an irreducible representation from the Vogan L-packet determined by the same ϕ. In particular, for a representation π of $O(V)$, both π and $\pi \otimes \det$ belong to the same Vogan L-packet.

4. Let M be a $2n$-dimensional quadratic space over $\mathbb{C}$, and let ϕ: $W'(k) \to O(M)$ such that $\det \phi = \omega_D$. Let $M = \oplus M(i)$ be the isotypical decomposition of M as a module of $W'(k) = W(k) \times SL(2, \mathbb{C})$ under ϕ. Write $M(i) = e_i N_i$ with N_i irreducible.

We record the following proposition from [GP1] for the convenience of the reader.

PROPOSITION 1. (1) *If $M(i)^\vee \neq M(i)$, then the centraliser of ϕ in $M(i) \oplus M(i)^\vee$ is isomorphic to $GL_{e_i}(\mathbb{C})$.*

(2) *If $M(i)^\vee = M(i)$ and N_i is a symplectic irreducible representation, then $e_i = 2d_i$ is even and the centraliser of ϕ in $M(i)$ is isomorphic to $Sp_{2d_i}(\mathbb{C})$.*

(3) *If $M(i)^\vee = M(i)$ and N_i is an orthogonal irreducible representation, then the centraliser of ϕ in $M(i)$ is isomorphic to $O_{e_i}(\mathbb{C})$.*

The following corollary is clear.

COROLLARY 1. (1) *For an irreducible representation μ of $SO(V)$ with parameter ϕ: $W'(k) \to O(M)$, $\mu \cong \mu^\varepsilon$ if and only if the decomposition of M for the representation ϕ contains an odd-dimensional irreducible orthogonal subspace.*

(2) *Given a parameter ϕ: $W'(k) \to SO(2n-1, \mathbb{C})$, and $\phi' = \phi \oplus \omega_D$: $W'(k) \to O(2n, \mathbb{C})$, there is an induced map of the component group of the centralisers: $A_\phi \to A_{\phi'}$. This mapping of the component groups is an injection with a cokernel of order 2 except when ϕ contains the character ω_D, in which case the mapping of the component groups is an isomorphism.*

(3) *Given a parameter ϕ: $W'(k) \to O(2n, \mathbb{C})$, and $\phi' = \phi \oplus \omega_D$: $W'(k) \to SO(2n + 1, \mathbb{C})$, there is an induced map of the component group of the centralisers: $B_\phi \to A_{\phi'}$. This mapping of the component groups is an isomorphism except when ϕ contains the character ω_D, in which case the mapping of the component groups is a surjection with a kernel of order 2.*

5. In this section we make our conjectures for the duality correspoondence between $O(2n)$ and $Sp(2n-2)$ and also between $O(2n)$ and $Sp(2n)$. Before we do that, we would like to note that, according to a well-known result (cf. [Ra, appendix]), the determinant representation of $O(V)$ appears in the duality correspondence with $Sp(2r)$ if and only if $\dim(V) \leqslant r$. This implies that, if $\dim(V) = 2n$, then for a representation π of $O(V)$ at most one of the representations π or $\pi \otimes \det$ appears in the duality correspondence with $Sp(2n-2)$. In particular, if a representation π of $O(V)$ appears in the duality correspondence with $Sp(2n-2)$, then $\pi \ncong \pi \otimes \det$, and therefore π remains irreducible when restricted to $SO(V)$.

Here are the conjectures for the duality correspondence, first for the case of $O(2n)$ and $Sp(2n-2)$, and then for $O(2n)$ and $Sp(2n)$.

We consider only tempered representations in all the conjectures below.

CONJECTURE 1. (1.1) *All the representations of* $Sp(2n-2)$ *occur in the duality correspondence with some* $O(V_\alpha)$.

(1.2) *If the Howe lift of a representation* π *of* $Sp(2n-2)$ *with Langlands parameter* $\phi\colon W'(k) \to SO(2n-1, \mathbb{C})$ *to* $O(V_\alpha)$ *is* π' *with Langlands parameter* $\phi'\colon W'(k) \to O(2n, \mathbb{C})$, *then* $\phi' = \phi \oplus \omega_D$. *In particular, the duality correspondence takes tempered representations to tempered representations.*

(1.3) *Given a parameter* $\phi\colon W'(k) \to SO(2n-1, \mathbb{C})$, *and* $\phi' = \phi \oplus \omega_D\colon W'(k) \to O(2n, \mathbb{C})$, *there is an induced map of the component group of the centralisers:* $A_\phi \to A_{\phi'}$. *The Vogan parameters of the restriction to* $SO(V_\alpha)$ *of the Howe lift to* $O(V_\alpha)$ *of a representation* π *of* $Sp(2n-2)$ *with parameter* ϕ, *and character* χ *of the component group* A_ϕ, *is the representation* ϕ' *of* $W'(k)$ *together with all the possible ways of extending* χ *to* $A_{\phi'}$. *Therefore, the Howe duality correspondence takes a Vogan L-packet (= Langlands L-packet) of* $Sp(2n-2)$ *surjectively to a Vogan L-packet of* $SO(V)$ *taking one representation to exactly two representations unless the L-packet of* $Sp(2n-2)$ *came from the duality correspondence with* $O(\tilde{V})$ *for some* $\tilde{V}$ *of dimension* $2n-2$ *and normalised discriminant* D, *in which case it gives a one-to-one mapping.*

Here are the conjectures for the Howe duality correspondence between $O(2n)$ and $Sp(2n)$.

CONJECTURE 2. (2.1) *If* π *is a representation of* $O(V_\alpha)$, *then either the Howe lift of* π *is nonzero or the Howe lift of* $\pi \otimes \det$ *is nonzero. In particular, if* $\pi \cong \pi \otimes \det$, *the Howe lift of* π *is nonzero.*

(2.2) *If a representation* π *of* $O(V_\alpha)$ *does not occur for the duality correspondence with* $Sp(2n)$, *then* $\pi \otimes \det$ *appears for the duality correspondence with* $Sp(2n-2)$.

(2.3) *The Howe lift of an irreducible admissible representation* π' *of* $Sp(W)$ *is zero for all* V_α *if and only if the Langlands parameter* $\phi'\colon W'(k) \to SO(2n+1, \mathbb{C})$ *associated to* π' *does not factor through* $O(2n, \mathbb{C})$ *with determinant* ω_D. *In particular, the property that the Howe lift of* π' *is nonzero for some* V_α *is shared by all elements of the L-packet containing* π'.

(2.4) *Let* $\phi\colon W'(k) \to O(2n, \mathbb{C})$ *be the Langlands parameter of the representation* π *of* $O(V_\alpha)$. *Then the Langlands parameter of the Howe lift* π' *to* $Sp(W)$ *is* $\phi' = \phi \oplus \omega_D$. *In particular, the Howe lift takes tempered representations of* $O(V_\alpha)$ *to tempered representations of* $Sp(2n, \mathbb{C})$.

(2.5) *Given a parameter* $\phi\colon W'(k) \to O(2n, \mathbb{C})$, *and* $\phi' = \phi \oplus \omega_D\colon W'(k) \to SO(2n+1, \mathbb{C})$, *there is an induced map of the component group of the centralisers:* $B_\phi \to A_{\phi'}$. *Therefore a character of* $A_{\phi'}$ *gives rise to a character of* B_ϕ *which (by Vogan parametrisation) determines the group* $O(V_\alpha)$ *and the representation* π *which is associated to* π' *by the Howe correspondence.*

(2.6) *The Howe duality correspondence gives a mapping from a Vogan L-packet of $O(V)$ to a Vogan L-packet (= Langlands L-packet) of $Sp(2n)$ which is a bijection unless the Vogan L-packet of $O(V)$ came from the duality correspondence with $Sp(2n-2)$, in which case the Howe lift of a representation of $O(V)$ is nonzero if and only if its lift to $Sp(2n-2)$ is nonzero, and the Howe lift gives a bijection from such representations of $O(V)$ (whose lift to $Sp(2n-2)$ is nonzero) to representations of $Sp(2n)$ in a given L-packet.*

Remark. The principal example, because of which we restricted ourselves to only tempered representations, is a result of Howe and Piatetski-Shapiro [HP, Theorem 9.4] that there is a representation (in fact exactly one) of $Sp(4)$ which can be lifted from both split and anisotropic quadratic forms in 4 variables, and the associated representations of orthogonal groups are one dimensional.

Remark. The inner conjugation action of the adjoint group of $Sp(2n, k)$ on $Sp(2n, k)$ gives an action of k^*/k^{*2} on the representations of $Sp(2n, k)$ preserving L-packets. If V_α and $\lambda \cdot V_\alpha$ are two quadratic spaces, then the corresponding orthogonal groups are canonically isomorphic, and if the Howe lift of a representation π of $O(V_\alpha)$ is π', then it can be easily seen that the Howe lift of the representation π of $O(\lambda \cdot V_\alpha)$ is π'^{λ}.

Remark. If an irreducible representation of $SO(V)$ extends to $O(V)$ in two distinct ways, then our conjectures predict that the corresponding Howe lifts are in the same L-packet (one of which may be zero). Perhaps the reasons behind this lie deep as one knows that the two representations of $O(2)$ which restrict trivially on $SO(2)$ lift to representations of $Sp(4)$, one of which is tempered (even supercuspidal in the p-adic case) and the other one is not; in particular they do not lie in the same L-packet; however they do lie in the same Arthur packet. (Our conjectures do not cover lifting from $O(2m)$ to $Sp(2n)$ for $m < n$.)

6. We list some examples corroborating the conjectures made above.

1. $(O(2), SL(2))$. This example has been studied for a long time by various people, and it is completely analysed for all local fields and all quadratic forms in two variables. The theory of L-packets is also known for $SL(2)$, and the results lead to proof of all of our conjectures.

2. $(O(4), SL(2))$. This example has been studied by Jacquet and Langlands. If the four-dimensional quadratic space V has a two-dimensional isotropic subspace, then $SO(V)$ is isomorphic to the group $\{(x, y) \in GL(2) \times GL(2): \det(x) = \det(y)\}/\Delta(k^*)$, and an L-packet of representations of $SO(V)$ is the restriction to $SO(V)$ of a representation $\pi_1 \otimes \pi_2$ of $GL(2) \times GL(2)$. It is proved by Jacquet-Langlands that, if π is a representation of $GL(2)$, then the Howe lift of any representation in the restriction of π to $SL(2)$ lies in the L-packet determined by $\pi \otimes \pi^*$. If σ is the Langlands parameter of π (a two-dimensional representation of $W'(k)$) or of any irreducible representation of π restricted to $SL(2)$, the Langlands parameter of $\pi \otimes \pi^*$ is $\sigma \otimes \sigma^* = Ad(\sigma) \oplus 1$, in accord with our conjectures. There is a similar result if the quadratic space V does not represent zeros with D^* for D the unique quaternion division algebra replacing $GL(2)$. However if V has a one-dimensional isotropic subspace but not two, then the discriminant algebra E of V is a field, and $SO(V)$ is the group $\{x \in GL(2, E)\colon \det(x) \in k^*\}/\Delta(k^*)\}$, and the lifting from $SL(2)$ to $SO(V)$ is the base change map.

Now we compute in some detail what our finer conjectures on the Vogan L-packet give in this case. See the forthcoming papers of Manderscheid for some of the proofs.

Case 1: The normalised discriminant of V is 1. In this case a discrete series representation of $SL(2)$ corresponds to two representations in the corresponding Vogan L-packet of $SO(V)$, one each for the split and anisotropic quadratic spaces. So all the discrete series representations of $SL(2)$ lift to $SO(V)$.

Case 2: The normalised discriminant of V is not 1. Let E be the discriminant field. In this case the two pure inner forms of $SO(V)$ can be taken to be $SO(V)$ and $SO(\lambda V)$ for λ an element of $k^* - \mathbb{N}E^*$. Suppose $\{\pi\}$ is a discrete series L-packet on $SL(2)$ associated to a character χ on a quadratic extension K of k. We first consider the case when the restriction of χ to the norm-one subgroup K^1 of K^* is not of order 2. In this case the L-packet $\{\pi\}$ contains exactly two elements which are permuted by $k^*/\mathbb{N}K^*$. Our conjectures imply the Howe lifting to be one-to-one if $K = E$ and to be one-to-two if $K \neq E$ from the L-packet $\{\pi\}$ to a Vogan L-packet of $SO(V)$. Therefore only one representation from $\{\pi\}$ lifts to $SO(V)$ if $K = E$, and both lift to $SO(V)$ if $K \neq E$.

If the restriction of χ to K^1 is of order two and is nontrivial, then the L-packet $\{\pi\}$ has four elements, and the Langlands parameter $\phi\colon W(k) \to SO(3, \mathbb{C})$ associated to $\{\pi\}$ consists of three characters $(\omega_1, \omega_2, \omega_3)$ of k^* of order 2 such that $\omega_1\omega_2\omega_3 = 1$. (In particular, the image of $W(k)$ for this discrete series packet is abelian.) If E is one of the quadratic fields determined by ω_i for some i, then our conjectures imply a bijection of the L-packet $\{\pi\}$ with a Vogan L-packet of $SO(V)$. Therefore two of the representations in $\{\pi\}$ will correspond to representations of $SO(V)$ and the other two to representations of $SO(\lambda V)$. If E is not one of the fields determined by ω_i for any i (a possibility only if k is non-Archimedean and has residue characteristic 2) then our conjectures imply a one-to-two mapping, and therefore all the representations in the L-packet $\{\pi\}$ correspond with representations of $SO(V)$.

3. $(O(4), Sp(4))$. See the extensive work of Przebinda [Pr] in the real case. The p-adic case is studied by Vigneras in [Vi].

4. $(O(6), Sp(4))$. See the work of Waldspurger in [Wa2] where he considers the duality correspondence for the representations of $Sp(2)$ induced from a supercuspidal representation of a Levi component of the parabolic stabilising a line.

5. The conjectures are true in the case when the representations are spherical (and one does not have to restrict to the tempered representations); this is due to the work of Piatetski-Shapiro and Rallis, [PsR]. More generally, the conjectures are true for representations with Iwahori fixed vector, called tamely ramified representations, which were classified by Kazhdan and Lusztig, by the work of Aubert [Au].

7. There are obvious global analogues of the local conjectures made here. It appears that, if we take a cuspidal tempered automorphic representation on one of the dual pairs considered in this paper, then its theta lift is nonzero if and only if there are no local obstructions (i.e., the Howe lift of the local components of the automorphic representation is nonzero for all places). One will also want to understand the structure of L-indistinguishable automorphic representations on one member of the dual pair in terms of the other. We have not seen a complete treatment of these questions even for the simplest pair ($O(2)$, $SL(2)$), though it has been much studied for the similitude groups. For example, one would expect that if an automorphic representation on $SL(2)$ comes from a quadratic field (so by theta lift of an automorphic form on $O(2)$), then all automorphic representations of $SL(2)$ which are L-indistinguishable from it are also obtained by changing the automorphic form on $O(2)$, keeping its restriction to $SO(2)$ the same (and one is of course allowed to scale the quadratic form, which amounts to conjugating the automorphic representation of $SL(2)$ by $GL(2, k)$). As the structure of L-indistinguishable automorphic representations of $SL(2)$ is completely analysed by Labesse and Langlands, it should be possible to answer this question. There is some work of Howe and Piatetski-Shapiro [HP] in the global situation for ($O(4)$, $Sp(4)$) when the orthogonal group is split.

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