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EIGENVALUES OF MATRIX-VALUED ANALYTIC MAPS

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Abstract

An elementary and self-contained account of analytic Jordan decomposition of matrix-valued analytic functions is given. An integral representation for their eigenvalues is obtained. This leads to estimates of the differences in eigenvalues and the number of points of degeneracy.

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1. Introduction

The aim of the present exposition is to present a self-contained and elementary account of analytic Jordan decomposition of matrix-valued analytic functions and obtain an integral representation for the eigenvalues. When the entries of the matrix are polynomials we obtain estimates for the number of points of degeneracy and the order of the poles of the semi-simple parts. An alternative approach to the theory of analytic Jordan decomposition may be found in the first chapter of T. Kato's book.

2. Some elementary lemmas on symmetric polynomials

We recall some of the well-known results from the theory of equations and present them in a form which will be used subsequently. For any n complex variables $z_1, z_2, \ldots, z_n$ we write

$$s_j(z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} z_i^j, \tag{2.1}$$

$$(-1)^j p_j(z_1, z_2, \ldots, z_n) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_j \leqslant n} z_{i_1} z_{i_2} \ldots z_i \tag{2.2}$$

for $j = 1, 2, \ldots, n$. Then the polynomials s_j and p_j satisfy the well-known (see Uspensky, 1948, p. 261) Newton's identities:

$$\begin{aligned} s_1 + p_1 &= 0, \\ s_2 + p_1 s_1 + 2p_2 &= 0, \\ \ldots \ldots \ldots \ldots & \\ s_n + p_1 s_{n-1} + p_2 s_{n-2} + \ldots + p_{n-1} s_1 + np_n &= 0. \end{aligned} \tag{2.3}$$

Lemma 2.1. *Let $(a_1, a_2, \ldots, a_n)$, $(b_1, b_2, \ldots, b_n)$ be two ordered n-tuples of complex numbers such that*

$$s_j(a_1, a_2, \ldots, a_n) = s_j(b_1, b_2, \ldots, b_n)$$

for all $1 \leqslant j \leqslant n$. Then $(a_1, a_2, \ldots, a_n)$ is a permutation of $(b_1, b_2, \ldots, b_n)$. In particular, if $s_j(a_1, a_2, \ldots, a_n) = 0$ for all $1 \leqslant j \leqslant n$, then $a_j = 0$ for all $1 \leqslant j \leqslant n$.

Proof. The conditions of the lemma and (2.3) imply that

$$p_j(a_1, a_2, \ldots, a_n) = p_j(b_1, b_2, \ldots, b_n) = \alpha_j,$$

say for all $1 \leqslant j \leqslant n$. Hence $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are the roots of one and the same nth degree polynomial $z^n + \alpha_1 z^{n-1} + \ldots + \alpha_n$. This implies the required result and completes the proof.

Corollary 2.2. *Let A_k, $k = 1, 2, \ldots$, be a sequence of $n \times n$ complex matrices such that $\lim_{k\to\infty} A_k = A$. Let $(\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn})$ and $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the eigenvalues of A_k and A respectively. Then $\sup_{k,j} |\lambda_{kj}| < \infty$ and any limit point of the sequence $(\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kn})$ in the n-dimensional complex Euclidean space $\mathbf{C}^n$ is a permutation of $(\lambda_1, \lambda_2, \ldots, \lambda_n)$.*

Proof. This is immediate from the fact that

$$\sum_{j=1}^{n} \lambda_j^r = \operatorname{tr} A^r = \lim_{k\to\infty} \operatorname{tr} A_k^r = \lim_{k\to\infty} \lambda_{k1}^r + \lambda_{k2}^r + \ldots + \lambda_{kn}^r,$$

where tr denotes trace.

Lemma 2.3. *Let $P(z_1, z_2, \ldots, z_n)$ be any homogeneous symmetric polynomial of degree m in n complex variables $z_1, z_2, \ldots, z_n$. Then P can be expressed as*

$$P(z_1, z_2, \ldots, z_n) = \sum_{j_1 + 2j_2 + \ldots + nj_n = m} a_{j_1, j_2, \ldots, j_n} s_1^{j_1} \ldots s_n^{j_n},$$

where $s_j = s_j(z_1, z_2, \ldots, z_n)$ are the polynomials defined by (2.1), $a_{j_1, j_2, \ldots, j_n}$ are complex numbers and $j_1, j_2, \ldots, j_n$ are non-negative integers.

PROOF. This is just a restatement of the well-known result on symmetric polynomials which is usually described in terms of the elementary symmetric functions p_j defined by (2.2). The p_j's can be replaced by the s_j's according to (2.3). For a proof we refer to Uspensky (1948, pp. 264–266.)

LEMMA 2.4. *Let $\mathbf{C}^k$ be the k-dimensional complex Euclidean space and let $m_1, m_2, \ldots, m_k$ be k positive integers. Let φ: $\mathbf{C}^k \to \mathbf{C}^k$ be the map defined by*

$$\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_k), \quad \varphi_j(z_1, z_2, \ldots, z_k) = \sum_{i=1}^{k} m_i z_i^j.$$

Then

$$\det\left(\left(\frac{\partial \varphi_i}{\partial z_j}\right)\right) = k!\, m_1 m_2 \ldots m_k \prod_{1 \leqslant i < j \leqslant k} (z_i - z_j).$$

PROOF. It is left to the reader.

LEMMA 2.5. *Let*

$$P_{r,k}(z_1, z_2, \ldots, z_n) = \sum_{r=1}^{\binom{n}{k}} \prod_{\substack{i<j \\ i,j \in J_m}} (z_i - z_j)^{2r}, \tag{2.4}$$

for $r = 1, 2, \ldots$, where $J_1, J_2, \ldots, J_{\binom{n}{k}}$ is an enumeration of all subsets of cardinality k from the set $\{1, 2, \ldots, n\}$. Let $(a_1, a_2, \ldots, a_n)$ be any ordered n-tuple of complex numbers. In order that the number of distinct elements among $a_1, a_2, \ldots, a_n$ is less than or equal to $k-1$ it is necessary and sufficient that

$$P_{r,k}(a_1, a_2, \ldots, a_n) = 0$$

for $r = 1, 2, \ldots, \binom{n}{k}$. If there are exactly k distinct numbers $b_1, b_2, \ldots, b_k$ occurring among $a_1, a_2, \ldots, a_n$ with multiplicities $m_1, m_2, \ldots, m_k$ respectively then

$$P_{r,k}(a_1, a_2, \ldots, a_n) = m_1 m_2 \ldots m_k \prod_{1 \leqslant i < j \leqslant k} (b_i - b_j)^{2r}$$

for all $r = 1, 2, \ldots$.

PROOF. The first part is an immediate consequence of the second part of Lemma 2.1 if we replace n by $\binom{n}{k}$ and put

$$a_m = \prod_{\substack{i<j \\ i,j \in J_m}} (z_i - z_j)^2, \quad m = 1, 2, \ldots, \binom{n}{k}.$$

The second part of the lemma is obvious.

We shall state a corollary to the above lemma after introducing a notation. We observe that $P_{r,k}$ defined by (2.4) is a homogeneous symmetric polynomial of degree $rk(k-1)$. By Lemma 2.3 there exist polynomials

$$p_{r,k}(s_1, s_2, \ldots, s_n) = \sum_{j_1+2j_2+\ldots+nj_n=rk(k-1)} a^{r,k}_{j_1,j_2,\ldots,j_n} s_1^{j_1} s_2^{j_2} \ldots s_n^{j_n} \tag{2.5}$$

with the property that when we substitute the values s_j defined by (2.1) in (2.5) we obtain the polynomials $P_{r,k}(z_1, z_2, \ldots, z_n)$. With this definition we have the following corollary.

COROLLARY 2.6. *Let the polynomials $p_{r,k}$ be defined by* (2.5) *for* $r = 1, 2, \ldots, \binom{n}{k}$ *Then the number of distinct eigenvalues of any $n \times n$ complex matrix is less than or equal to $k-1$ if and only if*

$$p_{r,k}(\operatorname{tr} A, \operatorname{tr} A^2, \ldots, \operatorname{tr} A^n) = 0 \tag{2.6}$$

or every $r = 1, 2, \ldots, \binom{n}{k}$

PROOF. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A with multiplicity included. Then $\operatorname{tr} A^j = \lambda_1^j + \lambda_2^j + \ldots + \lambda_n^j = s_j(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Hence the left-hand side of (2.6) is equal to $P_{r,k}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Now an application of Lemma 2.5 completes the proof.

3. Properties of the Jordan decomposition of a matrix

We shall consider a fixed complex vector space V of dimension n and denote by $\mathscr{E}(V)$ the vector space of all endomorphisms of V. Any element of $\mathscr{E}(V)$ is called an *operator*. The identity operator will be denoted by I. An operator P is called a *projection* if $P^2 = P$. If P is a projection so is $I-P$. For any projection the dimension of the subspace $\{v: Pv = v\}$ is called the *dimension* of P and denoted by $\dim P$. If $P_1, P_2, \ldots, P_k$ are projections such that $P_i P_j = 0$ for $i \neq j$ and $\sum_j P_j = I$, then we can decompose V into a direct sum of subspaces M_j, $j = 1, 2, \ldots, k$, satisfying the following:

(i) $V = \oplus_{j=1}^k M_j$;
(ii) $P_j v = v$ if $v \in M_j$;
(iii) $P_j v = 0$ if $v \in M_i$ and $i \neq j$;
(iv) $\operatorname{tr} P_j = \dim P_j$, $j = 1, 2, \ldots, k$.

An operator A is said to be *semisimple* if V has a basis in which the matrix of A is diagonal. A is said to be *nilpotent* if $A^r = 0$ for some positive integer r.

We recall briefly the Jordan canonical decomposition theorem in the co-ordinate-free form. For a proof the reader may refer to Kato (1976, Chapter 1).

THEOREM 3.1 (Jordan). *Let A be an operator whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$ respectively. Then there exist* *projections $P_1, P_2, \ldots, P_k$ and nilpotent operators $D_1, D_2, \ldots, D_k$ such that the following properties hold:*

(i) $A = \sum_{j=1}^{k}(\lambda_j P_j + D_j)$;
(ii) $AP_j = \lambda_j P_j + D_j$;
(iii) $P_i P_j = P_i D_j = D_i P_i = D_j^{m_j} = 0$ for all $i \neq j$;
(iv) $\sum_{j=1}^{k} P_j = I$; $P_i D_i = D_i P_i = D_i$;
(v) $\dim P_j = m_j$.

The decomposition (i) *with properties* (ii)–(v) *is unique. The operator $S(A) = \sum_{j=1}^{k} \lambda_j P_j$ is semisimple. The operator $D(A) = \sum_{j=1}^{k} D_j$ is nilpotent. If $A = S_1 + N_1$ where S_1 is semisimple, N_1 is nilpotent and $S_1 N_1 = N_1 S_1$ then $S_1 = S(A)$ and $D_1 = D(A)$.*

DEFINITION 3.2. In the above theorem equation (i) is called the *canonical decomposition* of A. P_j is called the *canonical projection* of A *corresponding to the eigenvalue* λ_j. D_j is called the *nilpotent part* of A *corresponding to the eigenvalue* λ_j. The operators $S(A)$ and $D(A)$ are called the *semisimple* and *nilpotent parts* of A. (In particular, the maps $A \to S(A)$ and $A \to D(A)$ are well defined on $\mathscr{E}(V)$.)

We shall now express the canonical projections in terms of polynomials in A whose co-efficients are rational functions of the eigenvalues. For any polynomial $p(z) = a_0 + a_1 z + \ldots + a_r z^r$ we shall write $p(A) = a_0 + a_1 A + \ldots + a_r A^r$ for every $A \in \mathscr{E}(V)$.

THEOREM 3.3. *Let A be an operator whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$ with multiplicities $m_1, m_2, \ldots, m_k$ respectively. For any eigenvalue λ, let $P(\lambda, A)$ denote the corresponding canonical projection of A. Suppose*

$$H_j(z) = \prod_{\substack{1 \leqslant i \leqslant k \\ i \neq j}} (z - \lambda_i)^{m_i}, \tag{3.1}$$

$$p_j(z) = H_j(z) \sum_{r=0}^{m_j - 1} \frac{(z - \lambda_j)^r}{r!} (H_j^{-1})^{(r)}(\lambda_j), \tag{3.2}$$

where the superscript (r) *indicates the* rth *derivative of the function $H_j(z)^{-1}$. Then*

$$P(\lambda_j, A) = p_j(A) \quad \text{for } 1 \leqslant j \leqslant k, \tag{3.3}$$

$$S(A) = \sum_{j=1}^{k} \lambda_j p_j(A). \tag{3.4}$$

PROOF. If $p(z) = a_0 + a_1 z + \ldots + a_r z^r$ is any polynomial then it follows from Theorem 3.1 that

$$p(A) = \sum_{j=1}^{k} \left(p(\lambda_j) P_j + \sum_{s=1}^{m_j - 1} \frac{p^{(s)}(\lambda_j)}{s!} D_j^s \right), \tag{3.5}$$

where λ_j, P_j, D_j are as in Theorem 3.1. It is routine to check that the polynomial p_j defined by (3.2) has the property

$$p_j(\lambda_i) = 0 \quad \text{if } i \neq j,$$
$$p_j(\lambda_j) = 1, \quad p_j^{(r)}(\lambda_j) = 0 \quad \text{for } 1 \leqslant r \leqslant m_j - 1.$$

Hence (3.5) implies that

$$p_j(A) = P_j = P(\lambda_j, A).$$

This proves (3.3). Equation (3.4) follows from the definition of $S(A)$. The proof is complete.

We shall now estimate the norms of the canonical projections of A in terms of the eigenvalues of A. To this end we establish a simple inequality.

LEMMA 3.4. *Let $p(z) = \prod_{j=1}^{n}(z-\theta_j)$ be a polynomial with roots $\theta_1, \theta_2, \ldots, \theta_n$ (not necessarily distinct) and let*

$$\alpha(z) = \min_{1 \leqslant j \leqslant n} |z - \theta_j|.$$

Then there exists a positive constant $c(n)$ depending only on n such that

$$|(p(z)^{-1})^{(r)}| \leqslant c(n)\,\alpha(z)^{-(n+r)}, \quad 0 \leqslant r \leqslant n,$$

for all z.

PROOF. Let $\varphi(z) = p(z)^{-1}$. Differentiating the identity $\varphi(z)p(z) = 1$, r times we obtain

$$|\varphi^{(r)}(z)| \leqslant \sum_{j=0}^{r-1} \binom{r}{j} \left| \frac{p^{(r-j)}(z)}{p(z)} \right| |\varphi^{(j)}(z)|. \tag{3.6}$$

It is clear that

$$|\varphi(z)| \leqslant \alpha(z)^{-n}. \tag{3.7}$$

Suppose

$$|\varphi^{(j)}(z)| \leqslant \alpha(z)^{-(n+j)} \quad \text{for } 1 \leqslant j \leqslant r-1, \tag{3.8}$$

for all z, where $a_0 = 1, a_1, a_2, \ldots, a_{r-1}$ are positive absolute constants. We note that $p^{(k)}(z)/p(z)$ is a sum of $n(n-1)\ldots(n-k+1)$ terms of the form

$$[(z-\theta_{i_1})(z-\theta_{i_2})\ldots(z-\theta_{i_k})]^{-1}$$

with $i_1 < i_2 < \ldots < i_k$. Hence

$$\left| \frac{p^{(k)}(z)}{p(z)} \right| \leqslant n(n-1)\ldots(n-k+1)\,\alpha(z)^{-k}. \tag{3.9}$$

Combining (3.6) and (3.9) we obtain

$$|\varphi^{(r)}(z)| \leqslant \sum_{j=0}^{r-1} a_j \binom{r}{j} n(n-1)\dots(n-r+j+1)\,\alpha(z)^{-(n+r)}$$
$$= a_r\,\alpha(z)^{-(n+r)}, \quad \text{say.}$$

If we now define $a_0, a_1, \dots, a_n$ inductively by $a_0 = 1$,

$$a_r = \sum_{j=0}^{r-1} a_j \binom{r}{j} n(n-1)\dots(n-r+j+1)$$

and put $c(n) = \max(a_0, a_1, \dots, a_n)$ we obtain the required inequality. The proof is complete.

THEOREM 3.5. *Let V be an n-dimensional complex Banach space. For any operator A on V let $\sum(A)$ denote the set of its distinct eigenvalues. For any $\lambda \in \Sigma(A)$, let $P(\lambda, A)$ denote the canonical projection corresponding to the eigenvalue λ and let*

$$d(\lambda, A) = \inf_{\mu \in \Sigma(A)\setminus\{\lambda\}} |\lambda - \mu|,$$
$$d(A) = \inf_{\lambda \in \Sigma(A)} d(\lambda, A).$$

Then there exists a constant $c(n)$ depending on only n such that

$$\|P(\lambda, A)\| \leqslant c(n)\left(\frac{\|A\|}{d(\lambda, A)}\right)^{n-1}, \tag{3.10}$$

$$\|S(A)\| \leqslant c(n)\frac{\|A\|^n}{d(A)^{n-1}}, \tag{3.11}$$

where $S(A)$ is the semisimple part of A.

PROOF. Let $\sum(A) = \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $\lambda = \lambda_j$. Then $P(\lambda_j, A)$ is given by equations (3.1)–(3.3). Applying Lemma 3.4 to the polynomial $H_j(z)$ of degree $n - m_j$ defined by (3.1), where m_j is the multiplicity of λ_j we obtain

$$|(H_j^{-1})^{(r)}(\lambda_j)| \leqslant c_0\, d(\lambda_j, A)^{-(n-m_j+r)},$$

where c_0 is an absolute positive constant. Since $\|A - \lambda_i\| \leqslant 2\|A\|$ for all i we conclude from (3.1)–(3.3) that for some absolute constant c_1,

$$\|P(\lambda_j, A)\| \leqslant c_1 \sum_{r=0}^{m_j-1} (2\|A\|)^{n-m_j+r}\, d(\lambda_j, A)^{-(n-m_j+r)}.$$

Since $d(\lambda_j, A) \leqslant 2\|A\|$ we conclude (3.10) by summing up the finite geometric series on the right-hand side of the above inequality. Inequality (3.11) follows from (3.10) and the fact that $|\lambda_j| \leqslant \|A\|$ for all j. This completes the proof of the theorem.

COROLLARY 3.6. *If $A_k \to A$ as $k \to \infty$ in $\mathscr{E}(V)$ and $d(A_k) \geqslant \delta \neq 0$ for all k, where $d(A)$ is defined as in Theorem* 3.5, *then* $\lim_{k\to 0} S(A_k) = S(A)$.

PROOF. This follows immediately from Theorem 3.5 and the uniqueness of the semi-simple and nilpotent parts of any endomorphism on V.

4. Operator-valued analytic maps

We shall call any path connected open subset $\mathscr{D}$ of the complex plane a *domain*. A map $z \to v(z)$ from $\mathscr{D}$ into V is called *analytic* if for every linear functional λ on V, the scalar function $\lambda(v(z))$ is analytic in $\mathscr{D}$. A map $z \to T(z)$ from $\mathscr{D}$ into $\mathscr{E}(V)$ is called *analytic* if for every $v \in V$, the map $z \to T(z)v$ is analytic. If

$$T(z) = A_0 + zA_1 + z^2 A_2 + \dots + z^d A_d$$

for all $z \in \mathbf{C}$, where $A_0, A_1, \dots, A_d$ are operators and $A_d \neq 0$ we shall say that $T(z)$ is a polynomial of degree d in z.

We shall fix a domain $\mathscr{D}$ and study the properties of eigenvalues of a fixed operator valued analytic map T on $\mathscr{D}$. We establish a few elementary lemmas.

LEMMA 4.1. *Let $m(T,z)$ be the number of distinct eigenvalues of the operator $T(z)$ and let*

$$k(T) = \max\{m(T,z), z \in \mathscr{D}\}, \tag{4.1}$$

$$\mathscr{S}(T) = \{z\colon z \in \mathscr{D}, m(T,z) < k(T)\}. \tag{4.2}$$

The set $\mathscr{S}(T)$ is discrete in $\mathscr{D}$. If $\mathscr{D} = \mathbf{C}$ and $T(z)$ is a polynomial of degree d in z then the cardinality of $\mathscr{S}(T)$ is at most $k(T)(k(T)-1)d$.

PROOF. Let $k(T) = k$ and let $\lambda_1(z), \lambda_2(z), \dots, \lambda_n(z)$ be an enumeration of all the eigenvalues (with multiplicity) of the operator $T(z)$. Then the functions

$$\xi_j(z) = \lambda_1(z)^j + \lambda_2(z)^j + \dots + \lambda_n(z)^j = \operatorname{tr} T(z)^j$$

are analytic in $\mathscr{D}$ for every $j = 1, 2, \dots$. Consider the polynomials $P_{1,k}$ and $p_{1,k}$ defined by (2.4) and (2.5) for $r = 1$. Then the function

$$\begin{aligned} q(z) &= P_{1,k}(\lambda_1(z), \lambda_2(z), \dots, \lambda_n(z)) \\ &= p_{1,k}(\operatorname{tr} T(z), \operatorname{tr} T(z)^2, \dots, \operatorname{tr} T(z)^n) \end{aligned} \tag{4.3}$$

is analytic in $\mathscr{D}$. If $T(z)$ is a polynomial of degree d then $q(z)$ is a polynomial of degree $k(k-1)d$. Let now z_0 be a point in $\mathscr{D}$ such that $m(T, z_0) = k$. Then there are exactly k distinct elements $\mu_1, \mu_2, \ldots, \mu_k$ among the sequence

$$\lambda_1(z_0), \lambda_2(z_0), \ldots, \lambda_n(z_0).$$

If μ_i occurs with multiplicity m_i then the second part of Lemma 2.5 implies that

$$q(z_0) = m_1 m_2 \ldots m_k \prod_{1 \leqslant i < j \leqslant k} (\mu_i - \mu_j)^2 \neq 0.$$

Thus q is not identically zero. If $q(z') \neq 0$ for some z' it follows from the definition of $P_{1,k}$ that there must be at least k distinct numbers among $\lambda_1(z'), \lambda_2(z'), \ldots, \lambda_n(z')$. By the definition of k it then follows that $z' \notin \mathscr{S}(T)$. Thus $\mathscr{S}(T) \subset \{z : q(z) \neq 0\}$. This implies the required result and completes the proof of the lemma.

DEFINITION 4.2. Let T: $z \to T(z)$ be an operator valued analytic map in a domain $\mathscr{D}$. Then the integer $k(T)$ defined by (4.1) is called the *index* of the map T. The set $\mathscr{S}(T)$ defined by (4.2) is called the *set of degeneracy* of the map T.

LEMMA 4.3. *Let T be an analytic operator valued map on $\mathscr{D}$ with index $k(T)$ and set of degeneracy $\mathscr{S}(T)$. Then there exist positive integers $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k$, $\sum m_j = n$ with the following property: for every $z \in \mathscr{D} \setminus \mathscr{S}(T)$ the distinct eigenvalues of $T(z)$ can be arranged as $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ where $\lambda_j(z)$ has multiplicity m_j for every $j = 1, 2, \ldots, k$.*

PROOF. Choose and fix any point $z_0 \in \mathscr{D} \setminus \mathscr{S}(T)$. From Corollary 2.2 it follows that the eigenvalues of $T(z)$ converge to those of $T(z_0)$ as $z \to z_0$. Hence we can choose a neighbourhood N_0 of z_0 and neighbourhoods $N_1, N_2, \ldots, N_k$ of $\lambda_1(z_0), \lambda_2(z_0), \ldots, \lambda_k(z_0)$ respectively such that

(i) $N_0 \subset N$, $N_i \cap N_j = \emptyset$ for $i \neq j$, $1 \leqslant i, j \leqslant k$;

(ii) for every $z \in N_0$, the distinct eigenvalues of $T(z)$ can be arranged as $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ where $\lambda_j(z) \in N_j$ and $\lambda_j(z)$ has multiplicity d_j for every j.

We can now restate this in the following manner. For any set of positive integers $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$, $\sum_{j=1}^{k} d_j = n$, let

$$\begin{aligned} U(d_1, d_2, \ldots, d_k) = \{z : z \in \mathscr{D} \setminus \mathscr{S}(T), &\text{ the distinct eigenvalues of } T(z) \text{ can be} \\ &\text{arranged as } \lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z) \text{ with multiplicities} \\ &d_1, d_2, \ldots, d_k \text{ respectively}\}. \end{aligned}$$

Then $U(d_1, d_2, \ldots, d_k)$ is open. By Lemma 4.1, $\mathscr{D} \setminus \mathscr{S}(T)$ is path connected and can be expressed as a disjoint union of open sets $U(d_1, d_2, \ldots, d_k)$. This implies that

$$\mathscr{D} \setminus \mathscr{S}(T) = U(m_1, m_2, \ldots, m_k)$$

for some $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k$ such that $\sum_j m_j = n$. The proof is complete.

COROLLARY 4.4. *Let $\mathscr{D}$, T, $k(T)$ and $\mathscr{S}(T)$ be as in Lemma 4.3, and let $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ be the distinct eigenvalues of $T(z)$ with multiplicities $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k$ for $z \in \mathscr{D} \backslash \mathscr{S}(T)$. Let*

$$\psi(z) = \prod_{1 \leqslant i < j \leqslant k} (\lambda_i(z) - \lambda_j(z))^2 \quad \text{if } z \in \mathscr{D} \backslash \mathscr{S}(T),$$
$$= 0 \qquad \text{if } z \in \mathscr{S}(T).$$

Then ψ is analytic in $\mathscr{D}$. If $\mathscr{D} = \mathbf{C}$ and $T(z)$ is a polynomial of degree d in z then ψ is a polynomial of degree $k(k-1)\,d$. Further

$$\mathscr{S}(T) = \{z : \psi(z) = 0\}.$$

PROOF. If we consider the function $q(z)$ defined by (4.3) and use the second part of Lemma 2.5 for $r = 2$ we obtain

$$q(z) = m_1 m_2 \ldots m_k \prod_{1 \leqslant i < j \leqslant k} (\lambda_i(z) - \lambda_j(z))^2$$

for all $z \in \mathscr{D} \backslash \mathscr{S}(T)$. The required result follows from the proof of Lemma 4.1.

LEMMA 4.5. *Let $\mathscr{D}$, T, $k(T)$, $\mathscr{S}(T)$ and $m_1, m_2, \ldots, m_k$ be as in Lemma 4.3. Then for any $z_0 \in \mathscr{D} \backslash \mathscr{S}(T)$ there exists a neighbourhood $N_0 \subset \mathscr{D} \backslash \mathscr{S}(T)$ of z_0 with the following property: the distinct eigenvalues of $T(z)$, $z \in N_0$ can be arranged as $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ where $\lambda_j(z)$ has multiplicity m_j and $\lambda_j(.)$ is an analytic function in N_0 for every j.*

PROOF. Let $z_0 \in \mathscr{D} \backslash \mathscr{S}(T)$. For every $z \in \mathscr{D} \backslash \mathscr{S}(T)$ arrange the distinct eigenvalues of $T(z)$ as $\lambda_1(z), \ldots, \lambda_k(z)$ so that the properties of Lemma 4.3 are fulfilled and

$$\lim_{z \to z_0} \lambda_j(z) = \lambda_j \quad \text{for } 1 \leqslant j \leqslant k.$$

Choose a neighbourhood N_0 of z_0 and neighbourhoods N_j of $\lambda_j(z_0)$ such that

(i) $N_i \cap N_j = \emptyset$ if $i \neq j$, $1 \leqslant i < j \leqslant k$;

(ii) if $z \in N_0$, then $\lambda_j(z) \in N_j$ for all $1 \leqslant j \leqslant k$;

(iii) the map φ defined in Lemma 2.4 is an analytic diffeomorphism on $N_1 \times N_2 \times \ldots \times N_k$.

Property (iii) can be achieved because Lemma 2.4 implies that the map φ has a non-vanishing Jacobian at the point $(\lambda_1(z_0), \lambda_2(z_0), \ldots, \lambda_k(z_0))$ in $\mathbf{C}^k$. Now we observe that the functions

$$\xi_j(z) = m_1 \lambda_1(z)^j + m_2 \lambda_2(z)^j + \ldots + m_k \lambda_k(z)^j$$
$$= \operatorname{tr} T(z)^j, \quad 1 \leqslant j \leqslant k,$$

are analytic in $\mathscr{D}$. Property (ii) implies that

$$(\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)) = \varphi^{-1}(\xi_1(z), \xi_2(z), \ldots, \xi_k(z))$$

for all $z \in N_0$, where φ^{-1} is the analytic inverse of φ on the set $\varphi(N_1 \times N_2 \times \ldots \times N_k)$. This shows that each $\lambda_j(z)$ is analytic in $z \in N_0$ and completes the proof.

LEMMA 4.6. *Let $\mathscr{G} \subset \mathscr{D} \setminus \mathscr{S}(T)$ be any simply connected domain and let*

$$m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k$$

be as in the preceding lemma. For every $z \in \mathscr{G}$, the eigenvalues of $T(z)$ can be arranged as $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ so that each $\lambda_j(.)$ is analytic in $\mathscr{G}$ and $\lambda_j(z)$ has multiplicity m_j.

PROOF. By Lemma 4.5 we know that for any $z_0 \in \mathscr{G}$ we can find a neighbourhood N_{z_0} of z_0 such that $N_{z_0} \subset \mathscr{G}$ and the distinct eigenvalues of $T(z)$, $z \in N_{z_0}$, can be arranged as $\lambda_1(z_0, z), \lambda_2(z_0, z), \ldots, \lambda_k(z_0, z)$ so that $\lambda_j(z_0, z)$ is analytic in $z \in N_{z_0}$. Let $z_0, z_1 \in \mathscr{G}$ be such that $N_{z_0} \cap N_{z_1} \neq \emptyset$. Then for each $z \in N_{z_0} \cap N_{z_1}$ we can find a permutation $s(z)$ (of k objects) such that

$$s(z)(\lambda_1(z_0, z), \lambda_2(z_0, z), \ldots, \lambda_k(z_0, z)) = (\lambda_1(z_1, z), \lambda_2(z_1, z), \ldots, \lambda_k(z_1, z)).$$

For any permutation s of k objects, let

$$M_s = \{z \colon z \in N_{z_0} \cap N_{z_1}, s(z) = s\}.$$

Since the union of all M_s is $N_{z_0} \cap N_{z_1}$ it follows that one of the M_s is uncountable, that is, there exists a permutation s_0 such that for uncountably many z in $N_{z_0} \cap N_{z_1}$

$$s_0(\lambda_1(z_0, z), \lambda_2(z_0, z), \ldots, \lambda_k(z_0, z)) = (\lambda_1(z_1, z), \lambda_2(z_1, z), \ldots, \lambda_k(z_1, z)).$$

Since both sides are analytic it follows that the above equality holds for all $z \in N_{z_0} \cap N_{z_1}$. Thus for each Jordan arc Γ in $\mathscr{G}$ the eigenvalues $\lambda_j(z)$, $j = 1, 2, \ldots, k$, can be analytically continued as eigenvalues of $T(z)$ along Γ with multiplicities m_j, $j = 1, 2, \ldots, k$. Hence the required result follows from the monodromy principle. (See Knopp, 1945, p. 105.) This completes the proof.

LEMMA 4.7. *Let T, $\mathscr{D}$, $\mathscr{G}$, $\lambda_j(z)$, $j = 1, 2, \ldots, k$, be as in Lemma 4.6. Let $P_j(z)$ be the canonical projection of $T(z)$ corresponding to the eigenvalue $\lambda_j(z)$ for every $z \in \mathscr{G}$. Then the map $z \to P_j(z)$ is analytic in $\mathscr{G}$ for every j.*

PROOF. This is immediate from the formula for $P_j(z) = P(\lambda_j(z), T(z))$, given by Theorem 3.3.

We can now summarize the results obtained so far in the present section and conclude the following theorem by using Theorem 3.1.

THEOREM 4.8. *Let $\mathscr{D} \subset \mathbf{C}$ be a domain in the complex plane. Let $T: z \to T(z)$ be an analytic map from $\mathscr{D}$ into the space of endomorphisms of a complex vector space V of dimension n. Then there exists a set $\mathscr{S}(T) \subset \mathscr{D}$ and positive integers k, $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k$, such that $m_1 + m_2 + \ldots + m_k = n$ and the following properties hold:*

(i) *$\mathscr{S}(T)$ is discrete in $\mathscr{D}$;*

(ii) *for $z \notin \mathscr{S}(T)$, $T(z)$ has exactly k distinct eigenvalues $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$ with multiplicities $m_1, m_2, \ldots, m_k$ respectively;*

(iii) *for every $z \in \mathscr{S}(T)$, the number of distinct eigenvalues of $T(z)$ is strictly less than k;*

(iv) *for any simply connected domain $\mathscr{G} \subset \mathscr{D} \setminus \mathscr{S}(T)$ the distinct eigenvalues of $T(z)$, $z \in \mathscr{G}$, can be arranged as $\lambda_1(z), \lambda_2(z), \ldots, \lambda_k(z)$, where each $\lambda_j(z)$ is analytic in $\mathscr{G}$ and $\lambda_j(z)$ has multiplicity m_j; if*

$$T(z) = \sum_{j=1}^{k} (\lambda_j(z) P_j(z) + D_j(z))$$

is the corresponding canonical decomposition of $T(z)$ for $z \in \mathscr{G}$ then *$P_j(\cdot)$ and $D_j(\cdot)$ are analytic in $\mathscr{G}$;*

(v) *the semisimple part $S(z)$ of $T(z)$ is analytic in $\mathscr{D} \setminus \mathscr{S}(T)$;*

(vi) *if $T(z)$ is a polynomial of degree d in z and $\mathscr{D} = \mathbf{C}$ then $\mathscr{S}(T)$ has at most $k(k-1)d$ points. Further the singularities of the semisimple part $S(z)$ can be only poles of order $\leqslant \frac{1}{2}(n-1)k(k-1)d$.*

PROOF. Only the second part of the property (vi) remains to be proved. Let us assume that $T(z)$ is a polynomial of degree d_0. Consider the function $d(T(z))$ defined by the notations of Theorem 3.5 and the polynomial $\psi(z)$ defined by Corollary 4.4. It is clear that the function $\psi(z)\, d(T(z))^{-2}$ is bounded in every compact subset of $\mathbf{C}$. By Theorem 3.5 it now follows that the function

$$f(z) = |\psi(z)|^{\frac{1}{2}(n-1)} \| S(z) \|$$

is bounded in every compact set. Since $\psi(z)$ is a polynomial of degree $k(k-1)d_0$ and $S(z)$ can be singular only at the zero's of $\psi(z)$ the required result follows and the proof is complete.

COROLLARY 4.9. *Let T be an analytic operator-valued map in a domain $\mathscr{D}$ and let*

$$\Delta(T) = \{z \colon T(z) \text{ is semisimple}\}.$$

Then either $\Delta(T)$ is countable or the complement of $\Delta(T)$ is countable.

PROOF. Let $N(z)$ denote the nilpotent part of $T(z)$. Then $z \in \Delta(T)$ if and only if $N(z) = 0$. If $\Delta(T)$ is uncountable then $\Delta(T) \cap (\mathscr{D} \setminus \mathscr{S}(T))$ is uncountable. Property (v) of Theorem 4.8 implies that $N(z) = 0$ for all $z \notin \mathscr{S}(T)$. Hence $\mathscr{D} \setminus \Delta(T) \subset \mathscr{S}(T)$. This completes the proof.

COROLLARY 4.10. *If A, B are complex Hermitian matrices of order n then the set $\{z\colon A+zB$ is not semisimple$\}$ has at most $k(k-1)$ points where k is the index of the map $T(z) = A+zB$.*

PROOF. This is immediate from Theorem 4.8, the proof of Corollary 4.9 and the fact that $A+tB$ is semisimple for all real t.

Our next result is taken from Kato (1976, p. 99) but the proof given here makes use of the finite dimensionality of V. This simple proof was suggested to me by Ray Vanstone.

THEOREM 4.11. *Let $P\colon z\to P(z)$ be an analytic projection-valued map in a simply connected domain $\mathscr{G}$ and let $z_0 \in \mathscr{G}$ be any fixed point. Then there exists an analytic operator-valued map $U\colon z\to U(z)$ in $\mathscr{G}$ such that* det $U(z) = 1$ *for all z and*

$$U(z)\,P(z_0)\,U(z)^{-1} = P(z)$$

for all $z \in \mathscr{G}$.

PROOF. Since $P(z)$ is a projection we differentiate the identity $P^2 = P$ and get

$$P' = PP' + P'P. \tag{4.4}$$

Multiplying by P on the left and the right we now obtain

$$PP'P = 0. \tag{4.5}$$

Let U be the solution of the ordinary operator differential equation

$$U' = [P', P]\,U \tag{4.6}$$

with the initial condition

$$U(z_0) = I,$$

where $[P', P] = P'P - PP'$. If $u(z) = \det U(z)$ then (4.6) implies that

$$u' = (\operatorname{tr}[P', P])\,u = 0.$$

Hence $u(z) = u(z_0) = 1$ for all z. In particular $U(z)$ is nonsingular everywhere. Define

$$Q(z) = U(z)^{-1}P(z)\,U(z).$$

Then

$$Q' = -U^{-1}U'U^{-1}PU + U^{-1}P'U + U^{-1}PU'.$$

Substituting for U' in the right-hand side of (4.6) and using (4.4) and (4.5) we have

$$Q' = U^{-1}(-[P', P]P + P' + P[P', P])\,U = 0.$$

Hence $Q(z) = Q(z_0) = P(z_0)$ for all z. This completes the proof.

COROLLARY 4.12. *Let $P\colon z \to P(z)$ be an analytic projection valued map in a simple connected domain $\mathscr{G}$. Let*

$$M(z) = \{v : P(z)v = v\}.$$

Then $\dim M(z) = m$ *is a constant. We can choose a basis* $\{v_1(z), v_2(z), \ldots, v_m(z)\}$ *in* $M(z)$ *such that the maps* $z \to v_j(z)$ *are analytic in* $\mathscr{G}$ *for* $1 \leqslant j \leqslant m$.

PROOF. Define the map $U\colon z \to U(z)$ satisfying the properties of Theorem 4.11 and select a basis $v_1, v_2, \ldots, v_m$ for $M(z_0)$. Put $v_j(z) = U(z)v_j$ for all $z \in \mathscr{G}$. Then $\{v_1(z), v_2(z), \ldots, v_m(z)\}$ is a basis of $M(z)$ with the required property. The proof is complete.

THEOREM 4.13. *Let $T\colon z \to T(z)$ be an analytic operator valued map in a domain $\mathscr{D}$ with index k and set of degeneracy $\mathscr{S}(T)$. Let $\mathscr{G} \subset \mathscr{D} \backslash \mathscr{S}(T)$ be a simply connected domain and let m_j, $\lambda_j(z)$, $P_j(z)$, $z \in \mathscr{G}$, $j = 1, 2, \ldots, k$, be as in Theorem* 4.8, *so that properties* (i)–(iv) *are fulfilled. Then for any two points $z_1, z_2 \in \mathscr{G}$ and any Jordan arc Γ joining z_1 and z_2 and completely contained in $\mathscr{G}$,*

$$\lambda_j(z_2) - \lambda_j(z_1) = m_j^{-1} \int_\Gamma \operatorname{tr} T'(z)\, P_j(z)\, dz$$

for all $1 \leqslant j \leqslant k$.

PROOF. By using Theorem 4.8 and Corollary 4.12 we can find an analytic map $U\colon z \to U(z)$ in $\mathscr{G}$ such that the map $\tilde{T}(z) = U(z)\,T(z)\,U(z)^{-1}$ admits the canonical decomposition

$$\tilde{T}(z) = \sum_{j=1}^{k} (\lambda_j(z)\, Q_j + \tilde{D}_j(z)), \quad z \in \mathscr{G}, \tag{4.7}$$

where the projection

$$Q_j = U(z)\, P_j(z)\, U(z)^{-1} \tag{4.8}$$

is independent of z. In particular,

$$\tilde{T}(z)\, Q_j = \lambda_j(z)\, Q_j + \tilde{D}_j(z).$$

Differentiating and taking trace we get

$$\operatorname{tr} \tilde{T}'(z)\, Q_j = m_j\, \lambda_j'(z) + \operatorname{tr} \tilde{D}_j'(z).$$

Since $\tilde{D}_j(z)$ is nilpotent and $\operatorname{tr} \tilde{D}_j'(z) = (\operatorname{tr} \tilde{D}_j(z))' = 0$ we have

$$m_j\, \lambda_j'(z) = \operatorname{tr} \tilde{T}'(z)\, Q_j \quad \text{for all } z \in \mathscr{G}. \tag{4.9}$$

Since $(U^{-1})' = -U^{-1}U'U^{-1}$ we have

$$\tilde{T}' = [U'U^{-1}, \tilde{T}] + UT'U^{-1}. \tag{4.10}$$

Since $\operatorname{tr} AB = \operatorname{tr} BA$ for any two operators A, B and $\tilde{T}(z)$ commutes with its canonical projections Q_j we have

$$\operatorname{tr}[U' U^{-1}, \tilde{T}] Q_j = 0.$$

Now (4.8)–(4.10) imply

$$\begin{aligned} m_j \lambda_j'(z) &= \operatorname{tr} U(z) T'(z) U(z)^{-1} Q_j \\ &= \operatorname{tr} T'(z) U(z)^{-1} Q_j U(z) \\ &= \operatorname{tr} T'(z) P_j(z) \end{aligned}$$

for all $z \in \mathscr{G}$. Since λ_j, T', P_j are all analytic in $\mathscr{G}$ an application of Cauchy's theorem completes the proof.

5. Some applications

We shall now indicate a few applications of Theorem 4.8 and Theorem 4.13 in the study of variation of spectra of finite dimensional operators.

THEOREM 5.1. *Let U, V be any two unitary operators in an n-dimensional Hilbert space. Then the eigenvalues of U and V can be arranged as $\lambda_1(U), \lambda_2(U), \ldots, \lambda_n(U)$ and $\lambda_1(V), \lambda_2(V), \ldots, \lambda_n(V)$ respectively such that*

$$\max_j |\lambda_j(U) - \lambda_j(V)| \leqslant \|K\|,$$

where K is any Hermitian operator such that

$$VU^{-1} = \exp(iK).$$

PROOF. Put $T(z) = \exp(izK)\, U$ for all $z \in \mathbf{C}$, where $VU^{-1} = \exp(iK)$ and K is Hermitian. Then the map T is analytic and $T(0) = U$, $T(1) = V$. Further $T(t)$ is unitary for all real t. Let $\mathscr{S}(T)$ be the set of degeneracy and let k be the index of the map T. The set $[0,1] \cap \mathscr{S}(T)$ is finite. We denote the points of this set by $t_1 < t_2 < \ldots < t_r$ and put $t_0 = 0$, $t_{r+1} = 1$. Since there are no points of degeneracy in (t_{i-1}, t_i) we can find a simply connected domain $\mathscr{G}_i$ such that $(t_{i-1}, t_i) \subset \mathscr{G}_i \subset \mathbf{C} \setminus \mathscr{S}(T)$ for all $i = 1, 2, \ldots, r+1$. By Theorem 4.13 we can arrange the distinct eigenvalues of $T(t), t \in (t_{i-1}, t_i)$, as $\lambda_{i1}(t), \lambda_{i2}(t), \ldots, \lambda_{ik}(t)$ so that

$$\lambda_{ij}(b) - \lambda_{ij}(a) = m_j^{-1} \int_a^b \operatorname{tr} T'(t) P_{ij}(t)\, dt \tag{5.1}$$

for all $[a, b] \subset (t_{i-1}, t_i)$, where $m_1 \geqslant m_2 \geqslant \ldots \geqslant m_k$ are defined by Theorem 4.8 and $P_{ij}(t)$ is the canonical projection of $T(t)$ corresponding to the eigenvalue $\lambda_{ij}(t)$.

Since $T(t)$ is unitary, $P_{ij}(t)$ are orthogonal projections of dimensions m_j and

$$\begin{aligned}|m_j^{-1}\operatorname{tr} T'(t)\,P_{ij}(t)| &\leqslant \|T'(t)\|\\ &= \|iK\exp(itK)\,U\|\\ &\leqslant \|K\|.\end{aligned}$$

Hence (5.1) implies

$$|\lambda_{ij}(b)-\lambda_{ij}(a)|\leqslant\|K\|(b-a), \tag{5.2}$$

for all $[a,b]\subset(t_{i-1},t_i)$. Letting $b\uparrow t_i$ and $a\downarrow t_{i-1}$ and using Corollary 2.2 we now conclude the following: for every arrangement of all the eigenvalues of $T(t_{i-1})$ as $\lambda_1(t_{i-1}),\lambda_2(t_{i-1}),\ldots,\lambda_n(t_{i-1})$ we can find an arrangement of all the eigenvalues of $T(t_{i-1})$ as $\lambda_1(t_i),\lambda_2(t_i),\ldots,\lambda_n(t_i)$ such that

$$|\lambda_j(t_i)-\lambda_j(t_{i-1})|\leqslant\|K\|(t_i-t_{i-1}) \tag{5.3}$$

for all $1\leqslant j\leqslant n$. Permuting the eigenvalues of $T(t_i)$ successively for $i=1,2,\ldots,r+1$ in a suitable manner we can ensure (5.3) for all i. Then

$$\begin{aligned}|\lambda_j(0)-\lambda_j(1)| &\leqslant \sum_{i=1}|\lambda_j(t_i)-\lambda_j(t_{i-1})|\\ &\leqslant\|K\|\end{aligned}$$

for all $1\leqslant j\leqslant n$. Since $T(0)=U$ and $T(1)=V$ the proof is complete.

Our next result is a generalization of Lidskii's theorem (see Kato, 1976, pp. 124–126).

THEOREM 5.2. *Let $T(z)=A_0+zA_1+z^2A_2+\ldots$ be a power series such that $\sum_{r=0}^{\infty}|z|^r\|A_r\|$ converges for all $|z|<\rho$, where $A_0,A_1,A_2,\ldots$ are Hermitian matrices of order n. Let $\lambda_{j1}\geqslant\lambda_{j2}\geqslant\ldots\geqslant\lambda_{jn}$ be the eigenvalues of A_j and let*

$$\boldsymbol{\lambda}^{(j)}=\begin{pmatrix}\lambda_{j1}\\ \lambda_{j2}\\ \vdots\\ \lambda_{jn}\end{pmatrix},\quad j=1,2,\ldots.$$

Then the eigenvalues of $T(t)$ can be arranged as $\lambda_1(t),\lambda_2(t),\ldots,\lambda_n(t)$ in $t\in(-\rho,\rho)$ such that

$$\boldsymbol{\lambda}(t)=\sum_{j=0}^{\infty}t^j\,Q^{(j)}(t)\,\boldsymbol{\lambda}^{(j)} \tag{5.4}$$

where

$$\boldsymbol{\lambda}(t)=\begin{pmatrix}\lambda_1(t)\\ \lambda_2(t)\\ \vdots\\ \lambda_n(t)\end{pmatrix}$$

and $Q^{(j)}(t)$ is a continuous doubly stochastic matrix-valued function in $(-\rho, \rho)$ for each j. Outside a fixed countable set of points all the $Q^{(j)}$'s can be chosen to be analytic in $(-\rho, \rho)$.

PROOF. Let $\mathscr{D} = \{z : |z| < \rho\}$ and let $\mathscr{S}(t)$ be the set of degeneracy of the analytic map $z \to T(z)$ in $\mathscr{D}$. Let k be the index of this map. By Theorem 4.8, the set $C_0 = (-\rho, \rho) \cap \mathscr{S}(T)$ is a countable discrete set. Let

$$C_0 = \{t_r, r = 0, \pm 1, \pm 2, \ldots\},$$

where $0 \leqslant t_0 < t_1 < \ldots$ and $0 > t_{-1} > t_{-2} > \ldots$. By the argument employed in the proof of Theorem 5.1 we can arrange the distinct eigenvalues of $T(t)$ in (t_{i-1}, t_i) as $\tilde{\lambda}_{i1}(t), \tilde{\lambda}_{i2}(t), \ldots, \tilde{\lambda}_{ik}(t)$ such that

$$\tilde{\lambda}_{ij}(b) - \tilde{\lambda}_{ij}(a) = m_j^{-1} \int_a^b \operatorname{tr} T'(t) P_{ij}(t)\, dt \tag{5.5}$$

for all $[a, b] \subset (t_{i-1}, t_i)$, where m_j is the multiplicity of $\tilde{\lambda}_{ij}(t)$ and $P_{ij}(t)$ is the corresponding canonical projection. We have

$$T'(t) = \sum_{r=1}^{\infty} r t^{r-1} A_r, \quad t \in (-\rho, \rho).$$

Let $v_{r1}, v_{r2}, \ldots, v_{rn}$ be unit eigenvectors of A_r corresponding to the eigenvalues $\lambda_{r1}, \lambda_{r2}, \ldots, \lambda_{rn}$ of A_r. Then (5.5) can be written as

$$\tilde{\lambda}_{ij}(b) - \tilde{\lambda}_{ij}(a) = \int_a^b \sum_{r=1}^{\infty} \sum_{s=1}^{n} r t^{r-1} \frac{\langle P_{ij}(t) v_{rs}, v_{rs} \rangle}{m_j}\, dt. \tag{5.6}$$

Since $\langle P_{ij}(t) v_{rs}, v_{rs} \rangle$ is a non-negative continuous function in (t_{i-1}, t_i) (indeed, analytic) and $0 \leqslant \langle P_{ij}(t) v_{rs}, v_{rs} \rangle \leqslant 1$ it follows that the limits

$$\lim_{b \uparrow t_i} \tilde{\lambda}_{ij}(b) \quad \text{and} \quad \lim_{a \downarrow t_{i-1}} \tilde{\lambda}_{ij}(a)$$

exist. Using Corollary 2.2 we now conclude the following: permuting the functions $\tilde{\lambda}_{i1}(t), \tilde{\lambda}_{i2}(t), \ldots, \tilde{\lambda}_{ik}(t)$ in each interval (t_{i-1}, t_i) suitably we can write all the eigenvalues of $T(t)$ as $\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t)$ so that

(i) each $\lambda_i(t)$ is continuous in $t \in (-\rho, \rho)$;
(ii) $\lambda_i(t)$ is analytic in $(-\rho, \rho) \setminus C_0$;
(iii) there exists an orthogonal projection-valued function $P_i(t)$ such that

$$\lambda_i(b) - \lambda_i(a) = \int_a^b \operatorname{tr} \frac{T'(t) P_i(t)}{\dim P_i(t)}\, dt$$

for all $[a, b] \subset (-\rho, \rho)$;

(iv) for $t \notin C_0$, $P_i(t)$ is the canonical projection of $T(t)$ corresponding to the eigenvalue $\lambda_i(t)$. In particular,

$$\lambda_i(t) - \lambda_i(0) = \int_0^t \sum_{r=1}^{\infty} r\tau^{r-1} \frac{\operatorname{tr} A_r P_i(\tau)}{\dim P_i(\tau)} d\tau. \tag{5.7}$$

Put

$$Q^{(r)}(\tau) = ((q_{ij}^{(r)}(\tau))),$$

where

$$q_{ij}^{(r)}(t) = t^{-r} \int_0^t r\tau^{r-1} \frac{\langle P_i(\tau) v_{rj}, v_{rj} \rangle}{\dim P_i(\tau)} d\tau,$$

for all $t \in (-\rho, \rho)$. Since $\operatorname{tr} P_i(\tau) = \dim P_i(\tau)$ we have

$$\sum_{j=1}^{n} q_{ij}^{(r)}(t) = t^{-r} \int_0^t r\tau^{r-1} d\tau = 1.$$

Since $\lambda_i(t)$ has multiplicity $\dim P_i(t)$ and the sum of $P_i(t)$ over all i such that $\lambda_i(t)$ runs through all the distinct eigenvalues of $T(t)$ is the identity operator it follows that

$$\sum_{i=1}^{n} q_{ij}^{(r)}(t) = 1.$$

Since $P_i(\tau)$ is an orthogonal projection, $q_{ij}^{(r)}(t) \geqslant 0$. In other words $Q^{(r)}$ is a doubly stochastic matrix-valued function in $(-\rho, \rho)$. Now (5.7) implies that

$$\lambda_i(t) = \lambda_i(0) + \sum_{r=1}^{\infty} t^r (Q^{(r)}(t) \boldsymbol{\lambda}^{(r)})_i,$$

for all $1 \leqslant i \leqslant n$. Without loss of generality we could have assumed

$$\lambda_1(0) \geqslant \lambda_2(0) \geqslant \ldots \geqslant \lambda_n(0).$$

Since $P_i(t)$'s are analytic in $t \notin C_0$ it follows that the $Q^{(r)}$'s are analytic in $t \notin C_0$. This completes the proof of the theorem.

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