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Determinants of parabolic bundles on Riemann surfaces*

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Abstract. Let X be a compact Riemann surface and $M_s^p(X)$ the moduli space of stable parabolic vector bundles with fixed rank, degree, rational weights and multiplicities. There is a natural Kähler metric on $M_s^p(X)$. We obtain a natural metrized holomorphic line bundle on $M_s^p(X)$ whose Chern form equals mr times the Kähler form, where m is the common denominator of the weights and r the rank.

Keywords. Parabolic bundles; Riemann surfaces; Kähler metric; Chern forms; π-bundles; Hermitian line bundles; parabolic determinant bundles.

1. Introduction

In a foundational paper [18] in 1985, Quillen studied determinants of $\bar{\partial}$-operators over a compact Riemann surface. Suppose X is a compact Riemann surface whose genus g is at least 2. If E is a C^∞ vector bundle of rank r and degree d on X, the space $\mathscr{A}$ of all holomorphic structures (or $\bar{\partial}$-operators) in E is a complex affine space. For each $D \in \mathscr{A}$, the object

$$\mathscr{L}_D = \lambda(\ker D)^* \otimes \lambda(\operatorname{coker} D),$$

where λ denotes the top exterior power, is a well-defined line and, as D varies in $\mathscr{A}$ these lines form a holomorphic line bundle $\mathscr{L}$ on $\mathscr{A}$, which is called the determinant line bundle. Using the theory of analytic torsion, Quillen defined a smooth Hermitian metric in the determinant bundle and proved that its curvature equals a natural Kähler form on $\mathscr{A}$.

The theorem of Quillen has an immediate application to moduli spaces. The collection $\mathscr{A}_s$ of stable holomorphic structures is an open subset of $\mathscr{A}$ and the complex gauge group $\mathscr{P}\mathscr{G} = \operatorname{Aut}(E)/\mathbb{C}^*$ acts freely on $\mathscr{A}_s$. The orbit space of this action is the moduli space M_s of stable bundles of rank r and degree d on X. When $d = r(g-1)$, the action of $\mathscr{P}\mathscr{G}$ on $\mathscr{A}_s$ lifts to an action on $\mathscr{L}$ and, hence, the determinant $\mathscr{L}$ descends to a holomorphic line bundle L over M_s. The Quillen metric in $\mathscr{L}$ induces a metric in the bundle L on M_s. It can be shown, as a corollary to Quillen's theorem, that the curvature of this metric in L is a naturally defined Kähler form on the moduli space M_s. In particular, the metric in L is positive. In the above argument the

* After this paper was written we came across an announcement of similar results by Witten [22].

restriction $d = r(g-1)$ is purely technical and can be bypassed. Thus a modification of the above construction yields a positive line L on M_s for all arbitrary r and d. When r and d are coprime, M_s is known to be compact, so Kodaira's embedding theorem implies that the positive line bundle L is ample on M_s.

There is another interesting interpretation of the above corollary to Quillen's theorem. A well known result of Narasimhan and Seshadri [16] shows that the moduli space is isomorphic to a certain variety R of irreducible projective unitary representations of the fundamental group $\pi_1(X)$. For each $\rho:\pi_1(X)\to PU(r)$, in R let $\mathrm{ad}\rho:\pi_1(X)\to \mathrm{Aut}(\mathfrak{g})$ denote the composition with ρ of the adjoint representation of $PU(r)$ on $\mathfrak{g} = \mathrm{Lie}\, PU(r)$ and let $W(\mathrm{ad}\rho)$ denote the local system of real vector spaces on X defined by ad ρ. Then the real tangent space to R at ρ is isomorphic to the de Rham group $H^1(X, W(\mathrm{ad}\rho))$.
Define

$$\Omega_\rho: H^1(X, W(\mathrm{ad}\rho)) \times H^1(X, W(\mathrm{ad}\rho)) \to \mathbb{R}$$

by

$$\Omega_\rho(\alpha, \beta) = \int_X \mathrm{tr}(\alpha \wedge \beta).$$

As ρ varies, the family Ω_ρ gives a 2-form Ω on R. This 2-form Ω is symplectic structure on R. One can ask wether Ω represents an integral cohomology class in $H^2(R, \mathbb{R})$. Yes is the answer to this question: under the isomorphism $R \simeq M_s$, Ω corresponds to the Kähler form on M_s which, in turn, equals the Chern form of the metrized line bundle L on M_s. Since all Chern classes are integral cohomology classes, Quillen's theorem implies that the symplectic form Ω on R is integral.

In this paper, we extend these results to the case of stable parabolic bundles on Riemann surfaces. Parabolic bundles are vector bundles over marked Riemann surfaces with weighted flags in the fibres at the marked points. There are two ways to study parabolic bundles: one way is to approach parabolic bundles directly; an alternative approach is through π-bundles. A π-bundle is an equivariant vector bundle over a Riemann surface provided with an action of a finite group π. The above results easily go through for π-bundles and define a metrized π-determinant. The more difficult part is to interpret this π-determinant in terms of the parabolic structure. Reserving the relevant definitions till later, we just state the main results here.

Theorem 1.1. *Let Y be a compact Riemann surface and π a finite group acting effectively on Y. Denote by $M_s^\pi(Y)$ the moduli space of stable π-bundles of rank r and fixed local type on Y. Then there exists on $M_s^\pi(Y)$ a metrized line bundle L^π whose Chern form equals r times the natural Kähler form Θ^π.*

Using a formalism of Deligne, Beilinson and Manin [3, 5] we interpret the above theorem in terms of parabolic structures as follows.

Theorem 1.2. *Let X be a compact Riemann surface and J a finite subset of X. Denote by $M_s^P(X)$ the moduli space of stable parabolic bundles of rank r and degree d with parabolic structure concentrated on J and having fixed rational weights, fixed multiplicities and fixed top exterior power. Then there exists on $M_s^P(X)$ a metrized line bundle L^P whose Chern form equals mr times the natural Kähler form, where m is the common denominator of all the weights.*

Now we sketch the outline of the paper. In §2 we recall the basic notions about parabolic bundles and π-bundles, and display the relation between them. The determinants of π-bundles form the topic of §4 and here Theorem 1.1 is proved. The π-determinant is expressed in terms of parabolic structures in §4. The proof of Theorem 1.2 is contained in §5 and, finally, in §6 we give an application of our results.

2. Parabolic bundles and π-bundles

In this section we define parabolic bundles and π-bundles and describe how the two notions are related to each other. The two basic references for this section are Seshadri [20] and, Mehta and Seshadri [13].

Let X be a compact Riemann surface and E a holomorphic vector bundle over X.

DEFINITION 2.1

A quasi-parabolic structure on E at a point $x \in X$ is a strictly decreasing flag

$$E_x = F^1 E_x \supseteq F^2 E_x \ldots\ldots \supseteq F^k E_x \supseteq F_x^{k+1} E = 0$$

of linear sub-spaces in E_x. We define

$$r_j = \dim F^j E_x - \dim F^{j+1} E_x.$$

The integer k is called the length of the flag and the sequence $(r_1, \ldots, r_k)$ is called the type of the flag. A parabolic structure in E at x is, by definition, a quasi-parabolic structure at x as above, together with a sequence of real numbers $0 \leqslant \alpha_1 < \ldots < \alpha_k < 1$. The α_j are called the weights and we set

$$d_x(E) = \sum_{j=1}^{k} r_j \alpha_j.$$

DEFINITION 2.2

Fix a finite set I of points in X. We say that E is a parabolic bundle with parabolic structure on I if we are given a parabolic structure in E at each point $x \in I$. The parabolic degree, $p\deg(E)$, is defined by

$$p\deg(E) = \deg(E) + \sum_{x \in I} d_x(E),$$

where $\deg(E)$ denotes the topological degree of E, and we put

$$p\mu(E) = p\deg(E)/\mathrm{rank}(E).$$

The points in I are called the parabolic points.

DEFINITION 2.3

Let E and F be parabolic bundles of the same type over X (i.e. E and F have the same parabolic points, the same types of flags, same weights etc.). A morphism from

E to F is a homomorphism of vector bundles $f: E \to F$, which preserves the flags at the parabolic points.

DEFINITION 2.4

Let E be parabolic bundle with parabolic locus I. A parabolic subbundle of E is a parabolic bundle F with parabolic locus I, such that:
(a) F is a subbundle of E; and
(b) for each point $x \in I$ and for every $j \in \{1, \ldots, k_x\}$, we have $\alpha_{x,j}(F) = \alpha_{x,i}(E)$, where i is the largest integer such that $F^j F_x \subseteq F^i E_x$ and k_x is the length of the flag in F_x.

DEFINITION 2.5

A parabolic bundle E is said to be semistable if for every proper subbundle F of E, we have

$$p\mu(F) \leqslant p\mu(E). \tag{2.1}$$

In case strict inequality holds in (2.1), for all F, we say that E is stable.

In [13], Mehta and Seshadri construct the moduli space $M^p(X)$ of semistable parabolic bundles over X with fixed rank, degree and parabolic type, and show that $M^p(X)$ is a projective variety. They also prove that the set $M^p_s(X)$ of stable parabolic bundles is an open smooth subset of $M^p(X)$.

We now proceed to define π-bundles. Consider a compact Riemann surface Y and let π be a finite group acting holomorphically and effectively on the left of Y. Then $X = \pi \backslash Y$ has a natural structure of a Riemann surface such that the canonical projection $p: Y \to X$ is a ramified covering. The ramification points of p in Y are exactly those points y where the isotropy subgroup π_y of y in π is non-trivial. Note that π_y is a cyclic subgroup of π for every $y \in Y$.

DEFINITION 2.6

A π-bundle over Y is a holomorphic vector bundle E over Y such that:
(a) the bundle projection $p: E \to Y$ is π-equivariant;
(b) if $y \in Y$ and $\gamma \in \pi$, the map $E_y \to E_{\gamma \cdot y}$ given by $E \mapsto \gamma \cdot \xi$, is a linear isomorphism.

DEFINITION 2.7

A π-subbundle of a π-bundle E is a subbundle F of E which is invariant under the action of π on E.

Notation 2.8. If E is a vector bundle over Y (which need not be a π-bundle), we denote

$$\mu(V) = \frac{\text{degree}(V)}{\text{rank}(V)}.$$

DEFINITION 2.9

A π-bundle E is said to be semistable if for every proper subbundle F of E (note that we do not assume that F is a π-subbundle of E), we have

$$\mu(F) \leqslant \mu(E). \tag{2.2}$$

We say that a π-bundle E is stable if for every proper π-subbundle F of E, we have a strict inequality

$$\mu(F) < \mu(E). \tag{2.3}$$

Remark 2.10 Given a π-bundle E of rank r over Y and a point $y \in Y$, there exists a π_y-invariant neighbourhood U of y in Y such that $E|_U$ is defined by a representation $\rho: \pi_y \to GL(r)$, and ρ is unique up to equivalence (cf. Seshadri [20], Proposition 2 of § 2). The equivalence class of the representation ρ is called the local type of E at y and is completely determined by: (a) an integer k such that $1 \leqslant k \leqslant r$; (b) a sequence of integers $0 \leqslant l_1 \leqslant l_2 \leqslant \ldots \leqslant l_k \leqslant n-1$, where $n = |\pi_y|$; and, (c) positive integers $r_1, r_2, \ldots, r_k$ such that $r_1 + \ldots, + r_k = r$. In fact, given the above data, we can define ρ as the block matrix

$$\rho(\zeta) = \begin{bmatrix} \exp(2\pi i \alpha_1)\cdot I_1 & & 0 \\ & \ddots & \\ 0 & & \exp(2\pi i \alpha_k)\cdot I_k \end{bmatrix}$$

where ζ is a generator of π_y, $i^2 = -1$, $\alpha_j = I_j/n$ and I_j is the identity matrix of order r_j.

In [20] Seshadri constructs the moduli space $M^\pi(Y)$ of semistable π-bundles of fixed rank, degree and local type over Y and shows that $M^\pi(Y)$ is a projective variety. It is proved by him that the set $M^\pi_s(Y)$ of stable π-bundles is a smooth open subset of $M^\pi(Y)$.

We now describe the connection between π-bundles on Y and parabolic bundles on $X = \pi \backslash Y$. Let E be a π-bundle over Y. Consider the invariant direct image $p^\pi_* \mathcal{O}(E)$ of $\mathcal{O}(E)$. This is the $\mathcal{O}_X$-module defined on each open set $U \subseteq X$ by

$$p^\pi_* \mathcal{O}(E) = \{\pi\text{-invariant sections of } E \text{ on } p^{-1}(U)\}.$$

Since p is a proper map, $p^\pi_* \mathcal{O}(E)$ is a coherent $\mathcal{O}_X$-module and is clearly torsion-free. Since $\dim(X) = 1$, it follows that $p^\pi_* \mathcal{O}(E)$ is a locally free $\mathcal{O}_X$-module of the same rank as E, and we denote the corresponding vector bundle on X by E^π.

Denote by J the ramification set of p in X. For each $x \in J$ choose a point $y(x) \in p^{-1}(x)$ and let $n_x = |\pi_{y(x)}|$. Let E be a π-bundle of rank r over Y. Suppose that the local type (cf. Remark 2.10) of Y is given by the integers $0 \leqslant l_{x,1} \leqslant l_{x,2} \leqslant \ldots \leqslant l_{x,k_x} \leqslant n_x - 1$ and $(r_{x,1}, \ldots, r_{x,k_x})$. We then have the following result for which we refer to Bhosle [4] and Mehta–Seshadri [13].

PROPOSITION 2.11

There exists a natural parabolic structure in the bundle E^π on X with parabolic set $I = p(J)$. For each $x \in I$, the type of the flag in E^π_x is $(r_{x,1}, \ldots, r_{x,k_x})$ and the weights at x are $(\alpha_{x,1}, \ldots, \alpha_{x,k_x})$, where $\alpha_{x,j} = l_{x,j}/n_x$. Conversely, given a parabolic bundle F on X of the above parabolic type, there exists on Y a π-bundle E, unique up to isomorphism such that $E^\pi \simeq F$ as parabolic bundles. The correspondence $E \mapsto E^\pi$ takes stable π-bundles on Y to stable parabolic bundles on X and semistable π-bundles to semistable parabolic bundles. The resulting bijection between the semistable moduli spaces $M^\pi(Y)$ and $M^P(X)$ is an isomorphism of varieties.

3. The determinant bundle on $M_s^\pi(Y)$

Consider a compact Riemann surface Y and let π be a finite group acting holomorphically and effectively on Y. Let μ be a Hermitian metric on Y which is invariant under the action of π. Denote by $A^{p,q}$ the space of all C^∞ (p,q)-forms on Y and set $A^r = \Sigma_{p+q=r} A^{p,q}$. Let $\omega \in A^{1,1}$ be the fundamental $(1,1)$-form of the metric μ and assume that μ is normalized so that $\int_Y \omega = 1$.

Fix a C^∞ π-vector bundle E of rank r and degree d over Y. Let h be a π-invariant Hermitian metric in E. Denote by $A^{p,q}(E)$ the space of C^∞ (p,q)-forms on Y with values in E and let $A^r(E) = \Sigma_{p+q=r} A^{p,q}(E)$. There is a natural linear action of π on each of the spaces $A^{p,q}$ and $A^{p,q}(E)$. We denote the corresponding subspaces of π-invariant forms by $(A^{p,q})^\pi$ and $(A^{p,q}(E))^\pi$.

DEFINITION 3.1

A holomorphic structure in E is a $\mathbb{C}$-linear map $D: A^0(E) \to A^{0,1}(E)$ which satisfies the Leibniz identity

$$D(fs) = f.D(s) + \bar{\partial} f \otimes s$$

for all $f \in A^0$ and $s \in A^0(E)$.

If D_1 and D_2 are holomorphic structures, the difference

$$\alpha = D_1 - D_2 : A^0(E) \to A^{0,1}(E)$$

is an A^0-linear map, i.e., $\alpha \in A^{0,1}(\text{End } E)$. Conversely, if D_1 is a holomorphic structure and if $\alpha \in A^{0,1}(\text{End } E)$, the sum $D_2 = D_1 + \alpha$ is again a holomorphic structure. Thus the set $\mathscr{A}$ of all holomorphic structures in E is an affine space modelled after the $\mathbb{C}$-vector space $A^{0,1}(\text{End } E)$.

Let $\mathscr{G}$ denote the group of C^∞ automorphisms of the bundle E. Given $D \in \mathscr{A}$ and $g \in \mathscr{G}$, we define a new holomorphic structure $D \cdot g \in \mathscr{A}$ by the prescription

$$D \cdot g = g^{-1} \circ D \circ g.$$

Any holomorphic structure D in E induces a holomorphic structure, also denoted D, in End E as follows

$$D: A^0(\text{End } E) \to A^{0,1}(\text{End } E)$$
$$f \mapsto [D, f] = D \circ f - f \circ D.$$

Having made this definition, we can write $D \cdot g = g^{-1} \circ (Dg) + D$.
Anyway, we now have a map

$$\mathscr{A} \times \mathscr{G} \to \mathscr{A}$$
$$(D, g) \mapsto D \cdot g.$$

This map is clearly an affine linear right action of $\mathscr{G}$ on $\mathscr{A}$.

The multiplicative subgroup $\mathbb{C}^*$ is embedded in $\mathscr{G}$ as the normal subgroup consisting of constant multiples of the identity automorphism of E. This subgroup $\mathbb{C}^*$ acts trivially on $\mathscr{A}$. Hence the above action induces an action of the group

$\mathscr{P}\mathscr{G} = \mathscr{G}/\mathbb{C}^*$ on $\mathscr{A}$. We now state a well-known integrability theorem, whose proof can be found in Donaldson [6].

PROPOSITION 3.2

Given a holomorphic structure $D \in \mathscr{A}$, there exists a unique structure of a holomorphic vector bundle E_D in E such that the $\bar{\partial}$-operator of E_D equals D.

Now we have the trivial

PROPOSITION 3.3

Let D_1 and D_2 be holomorphic structures in E. Then E_{D_1} and E_{D_2} are isomorphic as holomorphic vector bundles if and only if D_1 and D_2 lie in the same $\mathscr{G}$-orbit of $\mathscr{A}$.

Proof. The bundles E_{D_1} and E_{D_2} are isomorphic as holomorphic vector bundles if and only if there is a C^∞ isomorphism $g: E_{D_1} \to E_{D_2}$ which commutes with the $\bar{\partial}$-operators of the bundles, i.e., such that $g \circ D_1 = D_2 \circ g$. This means precisely that $g \in \mathscr{G}$ and $D_1 = D_2 \cdot g$.

DEFINITION 3.4

A holomorphic structure D in E is called a holomorphic π-structure if the map $D: A^0(E) \to A^{0,1}(E)$ is π-equivariant.

The set $\mathscr{A}^\pi$ of all holomorphic π-structures in E is an affine subspace of $\mathscr{A}$ in bijection with the subspace $A^{0,1}(\mathrm{End}\ E)^\pi$ of $A^{0,1}(\mathrm{End}\ E)$. Let $\mathscr{G}^\pi$ denote the subgroup of $\mathscr{G}$ consisting of π-automorphisms of E. It is easily seen that $\mathscr{A}^\pi$ is invariant under the action of $\mathscr{G}^\pi$ on $\mathscr{A}$. Further $\mathbb{C}^*$ is contained in $\mathscr{G}^\pi$, hence we get an action of $\mathscr{P}\mathscr{G}^\pi = \mathscr{G}^\pi_{/\mathbb{C}^*}$ on $\mathscr{A}^\pi$.

Remark 3.5. If V and W are C^∞ π-bundles on Y of the same rank, degree and local type (cf. Remark 2.10), then V and W are isomorphic as C^∞ π-bundles. Thus the set $\mathscr{A}^\pi/\mathscr{P}\mathscr{G}^\pi$ can be thought of as the set of isomorphism classes of holomorphic π-bundles on Y of rank r, degree d and of the local type as E.

DEFINITION 3.6

We say that a holomorphic π-structure D is stable if the corresponding holomorphic π-bundles E_D on Y is stable.

Let $\mathscr{A}^\pi_s$ denote the subset of $\mathscr{A}^\pi$ consisting of stable holomorphic π-structures. Then it is obvious that $\mathscr{A}^\pi_s$ is a $\mathscr{G}^\pi$-invariant subset. We know from Seshadri [20] that the isotropy subgroup of $\mathscr{G}^\pi$ at any $D \in \mathscr{A}^\pi_s$ is exactly $\mathbb{C}^*$. Thus the group $\mathscr{P}\mathscr{G}^\pi$ acts freely on $\mathscr{A}^\pi_s$. Let $M^\pi_s(Y) = \mathscr{A}^\pi_{s/\mathscr{P}\mathscr{G}^\pi}$. We state without proof.

PROPOSITION 3.7

The canonical projection $\varphi: \mathscr{A}^\pi_s \to M^\pi_s(Y)$ is a holomorphic principal $\mathscr{P}\mathscr{G}^\pi$-bundle.

Remark 3.8. We have not defined any topology or complex structure on $\mathscr{A}^\pi_s$ and $M^\pi_s(Y)$, so Proposition 3.7, as it stands, is meaningless. But using Sobolev spaces it is possible to give a natural complex manifold structure to $M^\pi_s(Y)$ and to make sense of Proposition 3.7. Since there are many accounts now of this kind of argument using

Sobolev spaces, we omit all the details and make a blanket reference to Narasimhan and Ramadas [23], Donaldson [6], Lübke and Okonek [12], Kim [10] and, Atiyah and Bott [1].

PROPOSITION 3.9

If $\varphi:\mathscr{A}^\pi_s \to M^\pi_s(Y)$ denotes the quotient projection and if $D \in \mathscr{A}^\pi_s$, then the tangent space to $M^\pi_s(Y)$ at the point $\varphi(D)$ is naturally isomorphic to $H^1(Y, \pi, \text{End } E_D)$.

Remark 3.10. If V is a π-sheaf on Y, define $H^0(Y, \pi, V)$ to be the set of all π-invariant sections of V on Y. The correspondence

$$V \mapsto H^0(Y, \pi, V)$$

is then a left exact additive covariant functor from the abelian category of π-sheaves on Y to the category of abelian groups. For $i \geqslant 1$, the cohomology group $H^i(Y, \pi, V)$ is, by definition, the ith right derived functor of the above functor. It is known (cf Seshadri [20]) that $H^i(Y, \pi, V)$ is naturally isomorphic to $H^i(X, p^\pi_* \mathcal{O}(V))$, where $X = \pi \backslash Y$ and $p: Y \to X$ is the canonical projection.

Proof of Proposition 3.9. Fix $D \in \mathscr{A}^\pi_s$ and define

$$\sigma: \mathscr{P}\mathscr{G}^\pi \to \mathscr{A}^\pi_s.$$
$$g \mapsto D.g.$$

The Lie algebra of $\mathscr{P}\mathscr{G}^\pi$ is $\mathfrak{P}\mathfrak{G}^\pi = A^0(\text{End } E)^\pi$. The differential of σ at the identity element of $\mathscr{P}\mathscr{G}^\pi$ is given by

$$D^\pi: A^0(\text{End } E)^\pi \to A^{0,1}(\text{End } E)^\pi$$
$$f \mapsto Df.$$

Thus the tangent space along the fibre of φ at D is precisely $\text{Im}(D^\pi)$. The Hodge theory implies that

$$A^{0,1}(\text{End } E)^\pi = \text{Im}(D^\pi) \oplus \ker(D^{*,\pi}),$$

where $D^{*,\pi}$ is the Hermitian adjoint of D^π. Hence the tangent space to $M^\pi_s(Y)$ at $\varphi(D)$ is given by

$$\frac{\text{Tan. space to } \mathscr{A}^\pi_s \text{ at } D}{\text{Vertical tan. space}} = \frac{A^{0,1}(\text{End } E)^\pi}{\text{Im}(D^\pi)} \simeq \ker(D^{*,\pi}).$$

As usual, let $X = \pi \backslash Y$ and $p: Y \to X$ the canonical projection. Let $\alpha^{p,q}(\text{End } E)$ denote the sheaf of $C^\infty(p,q)$-forms on Y with values in End E. Consider the sequence

$$0 \to p^\pi_* \mathcal{O}(\text{End } E_D) \to p^\pi_* \alpha^0(\text{End } E) \xrightarrow{D^\pi} p^\pi_* \alpha^{0,1}(\text{End } E) \to 0.$$

By averaging over π, this sequence is easily seen to be a fine resolution of $p^\pi_* \mathcal{O}(\text{End } E_D)$

on X. Hence the sheaf theory implies that

$$\begin{aligned} H^1(X, p_*^\pi \mathcal{O}(\text{End } E_D)) &= \text{coker}(D^\pi : A^0(\text{End } E)^\pi \to A^{0,1}(\text{End } E)^\pi) \\ &= \ker D^{*,\pi} \\ &= T_{\varphi(D)}(M_s^\pi(Y)). \end{aligned}$$

On the other hand, we know (cf. Remark 3.10) that

$$H^1(X, p_*^\pi \mathcal{O}(\text{End } E_D)) \simeq H^1(Y, \pi, \text{End } E_D).$$

Thus we conclude that

$$T_{\varphi(D)}(M_s^\pi(Y)) = H^1(Y, \pi, \text{End } E_D).$$

■

We recall now the definition of the determinant bundle on $\mathscr{A}$ as in Quillen [18]. For each $D \in \mathscr{A}$, define a complex line $\mathscr{L}_D$ by

$$\mathscr{L}_D = \lambda(\ker D)^* \otimes \lambda(\text{coker } D),$$

where λ denotes the top exterior power. Since each D is an elliptic operator, the kernel and cokernel of D are finite dimensional, so $\mathscr{L}_D$ is well-defined. We thus have a family

$$\mathscr{L} = \bigcup_{D \in \mathscr{A}} \mathscr{L}_D \tag{3.1}$$

of lines parametrized by $\mathscr{A}$. It is a fact, proved by Quillen [18], that $\mathscr{L}$ is actually a holomorphic line bundle over $\mathscr{A}$. There is a right $\mathscr{G}$-action on $\mathscr{L}$ given by

$$\begin{aligned} &(D, (s_1^* \wedge \ldots \wedge s_k^*) \otimes (t_1 \wedge \ldots \wedge t_1)) \times g \\ &\mapsto (D \cdot g, (g^{-1}(s_1^*) \wedge \ldots \wedge g^{-1}(s_k^*)) \otimes (g(t_1) \wedge \ldots \wedge g(t_1)) \end{aligned}$$

where $(s_1^*, \ldots, s_k^*)$ is a basis of $(\ker D)^*$ and $(t_1, \ldots, t_l)$ is a basis of coker D. If $g = \lambda \cdot id_E$ where $\lambda \in \mathbb{C}^*$, then $D \cdot g = D$ and g acts on $\mathscr{L}_D$ as

$$\begin{aligned} \xi &= (s_1^* \wedge \ldots \wedge s_k^*) \otimes (t_1 \wedge \ldots \wedge t_l) \\ &\mapsto ((\lambda^{-1} s_1^*) \wedge \ldots \wedge (\lambda^{-1} s_k^*)) \otimes ((\lambda t_1) \wedge \ldots \wedge (\lambda t_l)) \\ &= \lambda^{l-k} \cdot \xi = \lambda^{-\chi(E)} \cdot \xi \end{aligned}$$

where $\chi(E) = \text{index}(D)$ which by Riemann–Roch, equals $d + r(1-g)$, d being the degree of E, r the rank of E and g the genus of Y.

We would like to modify the line bundle $\mathscr{L}$ on $\mathscr{A}$ in order to obtain a line bundle $\tilde{\mathscr{L}}$ on which the action of $\mathbb{C}^*$ is trivial. To do this, fix a point $y \in Y$ such that the isotropy subgroup π_y of y in π is trivial. Define

$$\tilde{\mathscr{L}} = \mathscr{L}^r \otimes \lambda(\mathscr{A} \times E_y)^{\chi(E)}, \tag{3.2}$$

where $\mathscr{L}^r$ denotes the rth tensor power of $\mathscr{L}$ and $\mathscr{A} \times E_y$ is the trivial vector bundle on $\mathscr{A}$ with fibre E_λ. The computation of the previous paragraph shows that $\mathbb{C}^*$ acts trivially on $\tilde{\mathscr{L}}$ and, hence, $\tilde{\mathscr{L}}$ becomes a $\mathscr{PG}$-line bundle on $\mathscr{A}$. We denote the restriction of $\tilde{\mathscr{L}}$ to $\mathscr{A}_s^\pi$ by the symbol $\tilde{\mathscr{L}}$ itself. Since the natural projection

$$\varphi:\mathscr{A}_s^\pi \to M_s^\pi(Y)$$

is a holomorphic principal $\mathscr{PG}^\pi$-bundle, the $\mathscr{PG}^\pi$-line bundle $\tilde{\mathscr{L}}$ on $\mathscr{A}_s^\pi$ descends to a holomorphic line bundle L^π over $M_s^\pi(Y)$.

DEFINITION 3.11

The line bundle L^π is called the determinant bundle on the moduli space $M_s^\pi(Y)$.

Remark 3.12. If $\chi(E)=0$, then the above argument shows that $\mathbb{C}^*$ acts trivially on the initially unmodified line bundle $\mathscr{L}$ (cf. equation (3.1)). Thus $\mathscr{L}$ itself descends to $M_s^\pi(Y)$ when the index of E is zero.

In the earlier part of this section we have constructed certain holomorphic objects. We would now like to study their metric properties. To do this we need the space of unitary connections and the unitary gauge group. Recall that we have fixed a π-invariant Hermitian metric h in the C^∞ vector bundle E.

DEFINITION 3.13

A unitary connection in E is a connection

$$\nabla: A^0(E) \to A^1(E)$$

such that for all s, $t \in A^0(E)$, we have

$$d(h(s,t)) = h(\nabla s, t) + h(s, \nabla t).$$

Let $\mathrm{End}(E,h)$ denote the C^∞ $\mathbb{R}$-vector bundle of skew-Hermitian endomorphisms of (E,h). It is easily checked that the set $\mathscr{C}$ of all unitary connections in E is an affine space modelled after the $\mathbb{R}$-vector space $A^1(\mathrm{End}(E,h))$. If ∇ is a unitary connection in E, the $(0,1)$-part D_∇ of ∇ is a holomorphic structure in E.

Let $\mathscr{U}$ denote the subgroup of $\mathscr{G}$ consisting of unitary automorphisms of (E,h). Given $\nabla \in \mathscr{C}$ and $g \in \mathscr{U}$, we define $\nabla \cdot g \in \mathscr{C}$ by

$$\nabla \cdot g = g^{-1} \circ \nabla \circ g.$$

This gives a right action of $\mathscr{U}$ on $\mathscr{C}$. There is a natural embedding $\lambda \mapsto \lambda \cdot id_E$ of $U(1)$ in $\mathscr{U}$. This subgroup $U(1)$ of $\mathscr{U}$ acts trivially on $\mathscr{C}$ and hence we get an action of $\mathscr{PU} = \mathscr{U}_{/U(1)}$ on $\mathscr{C}$. The proof of the following simple proposition can be found in Kobayashi [11].

PROPOSITION 3.14

The $(0,1)$-*part map*

$$\mathscr{C} \to \mathscr{A}$$

$$\nabla \mapsto D_\nabla = (0,1)\text{-part of } \nabla$$

is a $\mathscr{PU}$-*equivariant affine* $\mathbb{R}$-*linear isomorphism.*

DEFINITION 3.15

A unitary connection ∇ in E is called a unitary π-connection if the map $\nabla: A^0(E) \to A^1(E)$ is π-equivariant.

The set $\mathscr{C}^\pi$ of all unitary π-connections in E is an affine subspace of $\mathscr{C}$ isomorphic to the real subspace $A^1(\mathrm{End}\,(E,h))^\pi$ of $A^1(\mathrm{End}\,(E,h))$. Let $\mathscr{U}^\pi$ denote the subgroup of $\mathscr{U}$ consisting of unitary π-automorphisms of (E,h). It is easily seen that $\mathscr{C}^\pi$ is invariant under the action of $\mathscr{U}^\pi$ on $\mathscr{C}$. Further $U(1)$ is contained in $\mathscr{U}^\pi$, hence we get an action of $\mathscr{P}\mathscr{U}^\pi = \mathscr{U}^\pi/_{U(1)}$ on $\mathscr{C}^\pi$.

DEFINITION 3.16

A unitary π-connection ∇ in E is said to be reducible if there exists proper C^∞ π-subbundles E_1 and E_2 of E together with unitary π-connections ∇_1 and ∇_2 such that $E = E_1 \oplus E_2$ and $\nabla = \nabla_1 \oplus \nabla_2$. If ∇ is not reducible we say that ∇ is irreducible.

DEFINITION 3.17

A unitary π-connection ∇ is said to be Einstein–Hermitian if the curvature $R(\nabla)$ of ∇ satisfies the identity

$$R(\nabla) = \frac{2\pi \cdot \mu(E)}{i \cdot |\pi|} \cdot \omega \otimes id_E$$

where ω is the Kähler form on (Y, μ), $|\pi|$ is the order of the group π and

$$\mu(E) = \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}.$$

DEFINITION 3.18

A holomorphic π-structure $D \in \mathscr{A}^\pi$ is said to be Einstein–Hermitian if the corresponding connection is so. A holomorphic π-structure is called indecomposable if there are no proper holomorphic π-subbundles E_1 and E_2 of E_D such that $E_D = E_1 \oplus E_2$.

The following theorem is a special case of an important theorem of Simpson [21]. This also follows from Seshadri [20].

Theorem 3.19. *An indecomposable holomorphic π-structure $D \in \mathscr{A}^\pi$ is stable if and only if there exists an Einstein–Hermitian holomorphic π-structure in the $\mathscr{P}\mathscr{G}^\pi$-orbit of D. Further if D_1 and D_2 are two Einstein–Hermitian holomorphic π-structures in the $\mathscr{P}\mathscr{G}^\pi$-orbit of D, then D_1 and D_2 lie in the same $\mathscr{P}\mathscr{U}^\pi$-orbit.*

Let $\mathscr{C}_s^\pi$ denote the subset of $\mathscr{C}^\pi$ consisting of irreducible Einstein–Hermitian π-connections. Then it is obvious that $\mathscr{C}_s^\pi$ *is a* $\mathscr{U}^\pi$-invariant subset. It can be easily checked (cf. Kobayashi [11]) that the stabilizer in $\mathscr{U}^\pi$ at any $\nabla \in \mathscr{C}_s^\pi$ is exactly $U(1)$. Thus the group $\mathscr{P}\mathscr{U}^\pi$ acts freely on $\mathscr{C}_s^\pi$. Let $N_s^\pi = \mathscr{C}_s^\pi / \mathscr{P}\mathscr{U}^\pi$. Then we have

PROPOSITION 3.20

The canonical projection $\psi: \mathscr{C}_s^\pi \to N_s^\pi$ is a C^∞ principal $\mathscr{P}\mathscr{U}^\pi$-bundle.

Remark 3.21. The comments made in Remark 3.8 apply to the above Proposition too. We will assume Proposition 3.20 as proved and quote the same references as in Remark 3.8.

Let ∇ be an Einstein–Hermitian holomorphic π-connection in E. Then the curvature $R(\tilde{\nabla})$: $A^0(\mathrm{End}(E,h)) \to A^2(\mathrm{End}(E,h))$ of the induced connection $\tilde{\nabla}$ in $\mathrm{End}(E,h)$ is given by

$$R(\tilde{\nabla})(f) = [R(\nabla), f] = [\alpha \otimes id_E, f], \quad \alpha \in A^{1,1}$$
$$= \alpha \otimes [id_E, f] = 0.$$

Thus $\tilde{\nabla}$ is a flat π-connection in the real vector bundle $\mathrm{End}(E,h)$. We denote $\tilde{\nabla}$ also by ∇. The vector bundle $\mathrm{End}(E,h)$ together with the flat connection $\tilde{\nabla} = \nabla$ becomes a local system of real vector spaces on Y which we shall denote by $\mathrm{End}\, \mathscr{E}_\nabla$. In Seshadri [19], the de Rham and Hodge theories are developed for C^∞ forms with values in local systems. We denote the de Rham cohomology groups of Y with values in $\mathrm{End}\, \mathscr{E}_\nabla$ by $H^i_d(Y, \mathrm{End}\, \mathscr{E}_\nabla)$. These cohomology groups are finite dimensional vector spaces. The π-invariants in $H^i_d(Y, \mathrm{End}\, \mathscr{E}_\nabla)$ are denoted by $H^i_d(Y, \pi, \mathrm{End}\, \mathscr{E}_\nabla)$.

We now compute the tangent space of the space N^π_s.

Lemma 3.22: *The tangent space to $\mathscr{C}^\pi_s$ at a point $\nabla \in \mathscr{C}^\pi_s$ equals the group of cocycles*

$$Z^i_d(Y, \pi, \mathrm{End}\, \mathscr{E}_\nabla) = \mathrm{Ker}\{\nabla^\pi : A^1(\mathrm{End}(E,h))^\pi \to A^2(\mathrm{End}(E,h))^\pi\}$$

Proof. Let $\mathscr{C}^\pi_0$ denote the set of irreducible π-connections. $\mathscr{C}^\pi_0$ is an open set in $\mathscr{C}^\pi$. Let V be the $\mathbb{R}$-subspace of $A^2(\mathrm{End}(E,h))$ given by

$$V = \{\alpha \otimes id_E / i\alpha \in A^{1,1}_{\mathbb{R}}\},$$

where $A^{1,1}_{\mathbb{R}}$ denotes the space of real (1, 1)-forms on Y. Let

$$B : A^2(\mathrm{End}(E,h))^\pi \to \frac{A^2(\mathrm{End}(E,h))^\pi}{V}$$

denote the natural projection and define

$$\theta : \mathscr{C}^\pi_0 \to \frac{A^2(\mathrm{End}(E,h))^\pi}{V}$$
$$\nabla_0 \mapsto B(R(\nabla_0)).$$

We now have the Hodge decomposition

$$A^2(\mathrm{End}(E,h))^\pi = \mathrm{Im}(\nabla^\pi_0) \oplus \ker(\nabla^{*,\pi}_0).$$

We claim that $\ker(\nabla^{*,\pi}_0) = V$. To see this, suppose $\nabla^{*,\pi}_0(\alpha) = 0$ for some $\alpha \in A^2(\mathrm{End}(E,h))^\pi$. Then $*^{-1} \circ \nabla_0 \circ *(\alpha) = 0$. But the irreducibility of ∇_0 implies that the parallel sections of ∇_0 in $\mathrm{End}(E,h)$ are exactly the scalars. Therefore $*(\alpha) = \lambda \cdot id_E$ for some $\lambda \in \mathbb{C}$. Hence $*^2(\alpha) = \lambda \cdot \omega \otimes id_E$ i.e., $\alpha = -\lambda \cdot \omega \otimes id_E$. Since α is skew-Hermitian, $i\lambda \in \mathbb{R}$. This proves that $\ker(\nabla^{*,\pi}_0) = V$. Therefore, the Hodge decomposition above implies that

$$\frac{A^2(\mathrm{End}\,(E,h))}{V} \simeq \mathrm{Im}(\nabla_0^\pi).$$

We conclude from this that θ is a submersion on $\mathscr{C}_0^\pi$. Clearly $\mathscr{C}_s^\pi = \theta^{-1}(0)$. Thus the tangent space to $\mathscr{C}_0^\pi$ at any point ∇ is given by the kernel of $d\theta$ at ∇, i.e., equals $\ker(\nabla^\pi)$. This is precisely $Z_d^1(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla)$. ∎

Recall that $\psi: \mathscr{C}_s^\pi \to N_s^\pi$ is the canonical quotient projection.

PROPOSITION 3.23

The tangent space to N_s^π at a point $\psi(\nabla)$, $\nabla \in \mathscr{C}_s^\pi$, is given by $H_d^1(Y, \pi, End\,\mathscr{E}_\nabla)$.

Proof. As in the proof of Proposition 3.9, we can check that the vertical tangent space at $\nabla \in \mathscr{C}_s^\pi$ is

$$B_d^1(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla) = \{\nabla\alpha / \alpha \in A^0(\mathrm{End}(E,h))^\pi\}.$$

Thus the tangent space to N_s^π at $\psi(\nabla)$ is

$$\frac{\text{Tan. space to } \mathscr{C}_s^\pi \text{ at } \nabla}{\text{Vertical tan. sp. at } \nabla} = \frac{Z^1(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla)}{B_d^1(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla)}$$
$$= H_d^1(Y, \pi, \mathrm{End}\,\mathscr{E}_\nabla).$$ ∎

We shall identify the real tangent space of $M_s^\pi(Y)$ with its holomorphic tangent space. Theorem 3.19 implies that if $\nabla \in \mathscr{C}_s^\pi$, then the $(0,1)$-part D_∇ of ∇ belongs to $\mathscr{A}_s^\pi$ and the map $\mathscr{C}_s^\pi \to \mathscr{A}_s^\pi$, $\nabla \mapsto D_\nabla$ induces a bijection $I: N_s^\pi \to M_s^\pi(Y)$ between the quotients. Clearly I is C^∞. The differential of I at $\psi(\nabla)$, $\nabla \in \mathscr{C}_s^\pi$, is given by

$$dI: H_d^1(Y, \pi, \mathrm{End}\,E_\nabla) \to H^1(Y, \pi, \mathrm{End}\,E_{D_\nabla}) = \ker(D^{*,\pi}),$$
$$a \mapsto \alpha''$$

where α is the harmonic representative of the class a and α'' is the $(0,1)$-part of α. Clearly dI is an isomorphism. Thus $I: N_s^\pi \to M_s^\pi(Y)$ is a local diffeomorphism, hence a diffeomorphism.

We now define a Hermitian metric on $M_s^\pi(Y)$. If $\varphi: \mathscr{A}_s^\pi \to M_s^\pi(Y)$ denotes the canonical projection and $D \in \mathscr{A}_s^\pi$, define a Hermitian inner product

$$\begin{array}{ccc} \langle .,. \rangle_{\varphi(D)}: T_{\varphi(D)}(M_s^\pi(Y)) & \times & T_{\varphi(D)}(M_s^\pi(Y)) \to \mathbb{C} \\ \| & & \| \\ \ker(D^{*,\pi}) & \times & \ker(D^{*,\pi}) \to \mathbb{C} \end{array}$$

by

$$\langle \alpha, \beta \rangle = \frac{1}{i} \int_Y \mathrm{tr}(\alpha \wedge \beta^*).$$

As $\varphi(D)$ varies $\langle .,. \rangle_{\varphi(D)}$ gives a smooth Hermitian metric on $M_s^\pi(Y)$. We shall denote its fundamental $(1,1)$-form by Θ^π.

Define a real 2-form $\tilde{\Omega}$ on $\mathscr{C}_s^\pi$ by

$$\tilde{\Omega}_\nabla(\alpha, \beta) = \int_Y tr(\alpha \wedge \beta),$$

where $\nabla \in \mathscr{C}^\pi$ and $\alpha, \beta \in A^1(\mathrm{End}(E, h))^\pi$. It is easily seen that $\tilde{\Omega}$ is a $\mathscr{P}\mathscr{U}^\pi$ invariant symplectic form on $\mathscr{C}^\pi$. Restricted to the submanifold $\mathscr{C}_s^\pi$, the form $\tilde{\Omega}$ vanishes along the vertical direction of ψ. This can be seen as follows: any vertical vector α at $\nabla \in \mathscr{C}_s^\pi$ is of the form $\alpha = \nabla f$ where $f \in A^0 \mathrm{End}(E, h)^\pi$; so if $\beta \in Z_d^1(Y, \pi, \mathrm{End}\, \mathscr{E}_\nabla)$, we have

$$\begin{aligned} \tilde{\Omega}(\alpha, \beta) &= \int_Y \mathrm{tr}(\nabla f \wedge \beta) \\ &= \int_Y \mathrm{tr} \nabla (f \nabla \beta) - \int_Y \mathrm{tr}(f \nabla \beta) \\ &= \int_Y d\, \mathrm{tr}(f\beta) - 0 \quad (\text{since } \nabla\beta = 0) \\ &= 0 \qquad (\text{by Stokes theorem}) \end{aligned}$$

Thus $\tilde{\Omega}$ descends to a symplectic form Ω on N_s^π.

PROPOSITION 3.24

The Hermitian metric on $M_s^\pi(Y)$ is Kähler.

Proof. It is easily seen that $I^*(\Theta) = \Omega$, hence Θ is closed, as Ω is closed and I is a diffeomorphism.

Consider the determinant bundle $\mathscr{L}$ and $\mathscr{A}$ (cf. equation 3.1). For each $D \in \mathscr{A}$, let

$$0 \leqslant \lambda_{D,1} \leqslant \lambda_{D,2} \leqslant \ldots.$$

be the eigenvalues of the Laplacian

$$\square_D := D^* \circ D : A^0(E) \to A^0(E),$$

and define the zeta function

$$\zeta_D(s) = \sum_{\lambda_{D,j} > 0} (\lambda_{D,j})^{-s}.$$

This series converges for $\mathrm{Re}\, s > 1$ and the function $\zeta_D(s)$ extends to a meromorphic function on $\mathbb{C}$ which is holomorphic at 0. Define a Hermitian inner product

$$h_Q : \mathscr{L}_D \times \mathscr{L}_D \to \mathbb{C},$$

by $h_Q = \exp(-\zeta_D'(0)).\langle .,. \rangle_{L^2}$

where $\langle .,. \rangle_{L^2}$ is the L^2-inner product in $\mathscr{L}_D$ induced by the metric in E. It is proved in Quillen [18] that h_Q is a smooth Hermitian metric in $\mathscr{L}$.

There is also a natural Hermitian inner product on the affine space $\mathscr{A} \simeq A^{0,1}(\mathrm{End}\, E)$

which is given by

$$\langle \alpha, \beta \rangle = \frac{1}{i} \cdot \int_Y \mathrm{tr}(\alpha \wedge \beta^*)$$

for $\alpha, \beta \in A^{0,1}(\mathrm{End}\, E)$. This gives rise to a Kähler metric in $\mathscr{A}$ whose Kähler form is denoted here by $\tilde{\Theta}$. We state the following theorem of Quillen [18].

Theorem 3.25. *The Chern form of the canonical connection in the holomorphic Hermitian line bundle* $(\mathscr{L}, h_Q)$ *on* $\mathscr{A}$ *equals the Kähler form* $\tilde{\Theta}$.

Consider the two principal bundles

$$\begin{array}{ccc} \mathscr{C}^\pi_s & \xrightarrow{\tilde{I}} & \mathscr{A}^\pi_s \\ \psi\downarrow & & \downarrow\varphi \\ N^\pi_s & \xrightarrow[I]{} & M^\pi_s(Y). \end{array} \tag{3.3}$$

On the line bundle $\tilde{\mathscr{L}}$ (cf. equation 3.2) we have a metric given by the tensor product of the Quillen metric h_Q in $\mathscr{L}$ and the given metric in E_y. Since $\lambda(E_y)$ is a flat unitary bundle on $\mathscr{A}$, the Chern form of the metrized line bundle $\tilde{\mathscr{L}}$ is r times the Chern form of $\mathscr{L}$ and, hence, by Theorem 3.25, equals $r.\tilde{\Theta}$. Let $\tilde{\mathscr{M}}$ be the pull-back of $\tilde{\mathscr{L}}$ to $\mathscr{C}^\pi_s$ by the map $\tilde{I}$. Then $\tilde{\mathscr{M}}$ is a $C^\infty\, \mathscr{P}\mathscr{U}^\pi$-line bundle on $\mathscr{C}^\pi_s$ and the pull-back metric and the pull-back connection in the line bundle $\tilde{\mathscr{M}}$ are invariant under the action of $\mathscr{P}\mathscr{U}^\pi$ on $\mathscr{C}^\pi_s$. Thus $\tilde{\mathscr{M}}$ descends to a C^∞ Hermitian line bundle M on N^π_s with a Hermitian connection. This bundle M is naturally isomorphic to $I^*(L^\pi)$ (cf. definition 3.11) as a Hermitian line bundle. It follows from this that L^π naturally gets a metric and a connection. This connection is easily seen to be compatible with the metric and the holomorphic structure in L^π. We denote the metric in L^π by g.

DEFINITION 3.26

We call g the Quillen metric in L^π.

Theorem 3.27. *The Chern form of the Quillen metric in the holomorphic line bundle* L^π *equals* r *times the Kähler form* Θ^π *on* $M^\pi_s(Y)$.

Proof. Consider the commutative diagram (3.3). We have

$$\begin{aligned} (I\circ\psi)^* c_1(L^\pi, g) &= c_1((I\circ\psi)^* L^\pi, (I\circ\psi)^* g) \\ &= c_1(\tilde{\mathscr{M}}, \tilde{I}^*\nabla), \quad \text{where } \nabla = \text{canonical connection in } \tilde{\mathscr{L}} \\ &= \tilde{I}^* c_1(\tilde{\mathscr{L}}) \\ &= \tilde{I}^*(r\cdot\tilde{\Theta}) \quad \text{(cf. the remarks following the commutative diagram 3.3)} \\ &= r\cdot\tilde{\Omega} \\ &= (I\circ\psi)^*(r\cdot\Theta). \end{aligned}$$

Since $I\circ\psi$ is a submersion, this implies that

$$c_1(L,g) = r\cdot\Theta.$$ ■

4. The relation between the π-determinant and the parabolic determinant

In this section we show that the isomorphism from the moduli space of π-bundles to the moduli space of parabolic bundles carries the π-determinant L^π (cf. Definition 3.11) to a multiple of an "intrinsic" line bundle on the parabolic moduli. By "intrinsic" we mean that the line bundle depends only on the parabolic data and not on the choice of the π-structure. We refer to this intrinsic line bundle as the parabolic determinant bundle. Here we consider only the bundle structures and leave the comparison of the metric structures to the next section. The main results we use here are the Grothendieck–Riemann–Roch theorem and a theorem of Deligne *et al* ([5] and [3]) on the determinant of the tensor product of two families of line bundles.

Consider a compact Riemann surface Y and let π be a finite group acting effectively and holomorphically on Y. Let $X = {}_{\pi\backslash}Y$ be the quotient Riemann surface and $p: Y\to X$ the canonical projection. The map p is a ramified covering. Let I be the set of ramification points of p in Y and let $J = p(I)$ be the set of critical values of p in X. Both I and J are finite sets. For each $x\in J$, choose a point $y_x \in p^{-1}(x)$ and let n_x be the order of π_x, the isotropy subgroup of π at y_x. Fix once and for all, for each $x\in J$, a set $C_x \subseteq \pi$ of left coset representatives of π_x.

DEFINITION 4.1

A holomorphic family of compact Riemann surfaces is a proper surjective holomorphic submersion $f:\mathscr{X}\to T$ of complex manifolds such that for each $t\in T$, the submanifold $X_t = f^{-1}(t)$ is a Reimann surface.

Let $f:\mathscr{X}\to T$ be a family of Riemann surfaces and let $\mathscr{F}$ be a coherent $\mathcal{O}_{\mathscr{X}}$-module. Then, by the Proper Mapping Theorem, all the higher direct images $R^i f_* \mathscr{F}$ are coherent $\mathcal{O}_T$-modules so,

$$f_!(\mathscr{F}) := \sum (-1)^i R^i f_* \mathscr{F}$$

is a well defined element of $K(T)$. Therefore

$$d(\mathscr{F}) = \lambda(f_! \mathscr{F})$$

is a holomorphic line bundle over T, called the determinant bundle of $\mathscr{F}$. The fibre of $d(\mathscr{F})$ at $t\in T$ is canonically isomorphic to

$$\lambda(H^0(X_t, \mathscr{F}_t))^* \otimes \lambda(H^1(X_t, \mathscr{F}_t)),$$

where $\mathscr{F}_t = \mathscr{F}|_{X_t}$.

Remark 4.2. Suppose $f:\mathscr{X}\to T$ is a family of Riemann surfaces. Then, the determinant construction defined above enjoys several nice properties. For instance, if

$$0\to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is an exact sequence of coherent $\mathcal{O}_{\mathcal{X}}$-modules, then

$$d(\mathcal{F}) = d(\mathcal{F}') \otimes d(\mathcal{F}'') = d(\mathcal{F} \oplus \mathcal{F}'').$$

Hence, when we consider the determinants $d(\mathcal{F})$ we will assume that all short exact sequences of $\mathcal{O}_{\mathcal{X}}$-modules split.

DEFINITION 4.3

A family of vector bundles on the Riemann surface Y parametrized by a complex manifold T is, by definition, a holomorphic vector bundle over $Y \times T$. A family of π-bundles on Y parametrized by T is a holomorphic π-bundle on $Y \times T$, where the action of π on T is trivial.

DEFINITION 4.4

A family of parabolic bundles on X parametrized by a manifold T is a holomorphic vector bundle F on $X \times T$ together with the following data: for each $x \in J$, we are given a filtration of the vector bundle $F^x = F|_{x \times T}$ by subbundles

$$F^x = F^1 F^x \supseteq F^2 F^x \supseteq \ldots \supseteq F^{k_x} F^x \supseteq F^{k_x+1} F^x = 0.$$

We denote $\mathrm{Gr}^j F^x = F^j F^x / F^{j+1} F^x$.

Suppose E is a family of π-bundles on Y parametrized by a manifold T. Let r be the rank of E. For each $t \in T$, the local type of the π-bundle $E_t = E|_{Y \times t}$ on Y at each point y_x is independent of t and is given by integers

$$0 \leqslant l_{x,1} < \ldots < l_{x,k_x} \leqslant n_x - 1 \text{ and } (r_{x,1}, \ldots, r_{x,k_x})$$

such that

$$\sum_{j=1}^{k_x} r_{x,j} = r.$$

Define $\alpha_{x,j} = l_{x,j}/n_x$ and $n_{x,j} = n \cdot \alpha_{x,j}$ where $n = |\pi|$. Let $F \to X \times T$ be the family of parabolic bundles induced by E (cf. Proposition 2.11), namely, $F = E^\pi$ is the holomorphic vector bundle on $X \times T$ corresponding to the locally free $\mathcal{O}_{X \times T}$-module $(p \times 1_T)_*^\pi \mathcal{O}(E)$ together with an induced parabolic structure. We now compute the determinant of E in terms of F and the parabolic structure of F:

PROPOSITION 4.5

Consider a family E of π-bundles on Y parametrized by a complex manifold T and let $F = E^\pi$ be the induced parabolic family on $X \times T$, as above. Then as line bundles on T, we have

$$d(F) = d((p \times 1_T)^* F) \otimes \left(\bigotimes_{\substack{x \in J \\ j=1,\ldots,k_x}} \lambda(\mathrm{Gr}^j F^x)^{-n_{x,j}} \right)$$

Proof. We first note that for each $x \in J$ and $j = 1, \ldots, k_x$, $\mathrm{Gr}^j F^x \cong E_j^x$ as vector bundles on T where $E^x = E|_{y_x \times T}$ and $F^x = F|_{x \times T}$ and $E^x = \sum_{j=1}^{k_x} E_j^x$ is the eigenspace decomposition of E^x, i.e, on each E^x_j, π_x acts as a character of weight $l_{x,j}$. We prove this by producing a natural homomorphism

$$h_i: \mathrm{Gr}^j F^x \to E^x_j.$$

The map h_j is defined as follows: Given $0 \neq v \in \mathrm{Gr}^j F^x$, let $\bar{v} \in F^x$ represent v. Now take any section s of F over some open set containing x which satisfies the condition that $s(x) = \bar{v}$. Let $\bar{s}$ be the invariant section E over an open set containing y_x, which represents s. If $\bar{s}$ is the image of $t \otimes \mathbf{1}_i$ where t is a section of E such that $t(y_x) \neq 0$ and $\mathbf{1}_i$ is the canonical section of $\mathcal{O}(-iy_x)$, under the morphism of sheaves

$$0 \to E \otimes \mathcal{O}(-iy_x) \to E,$$

define

$$h_j = t(y_x).$$

From the correspondence between π-bundles and parabolic bundles as described by Mehta and Seshadri [13] it follows that h_j is an isomorphism.

Therefore to prove the proposition, we need to verify that

$$d(E) = d((p \times 1)^* F)\left(\bigotimes_{\substack{x \in J \\ j = 1, \ldots, k_x}} \lambda(E^x_j)^{-n_{x,j}} \right). \tag{4.1}$$

We have a canonical $\mathcal{O}_{Y \times T}$-module monomorphism

$$0 \to (p \times 1)\mathcal{O}(F) \to \mathcal{O}(E).$$

The cokernel $\mathcal{T}$ of this monomorphism is a torsion $\mathcal{O}_{Y \times T}$-module supported on $I \times T$ where I denotes the ramification locus in Y. We thus have an exact sequence of $\mathcal{O}_{y \times T}$-modules

$$0 \to (p \times 1)^* \mathcal{O}(F) \to \mathcal{O}(E) \to \mathcal{T} \to 0. \tag{4.2}$$

By Remark 4.2, we get

$$\begin{aligned} d(E) &= d((p \times 1)^* F) \cdot d(\mathcal{T}) \\ &= d((p \times 1)^* F) \cdot \left(\bigotimes_{y \in I} \lambda(\mathcal{T}^y)^{-1} \right) \end{aligned} \tag{4.3}$$

where $\mathcal{T}^y = \mathcal{T}|_{y \times T}$. But $\mathcal{T}$ is a π-sheaf on $Y \times T$, hence for each $y \in Y$ and $\gamma \in \pi$, $\mathcal{T}^y$ is canonically isomorphic to $\mathcal{T}^{\gamma \cdot y}$ on T. If $x \in J$ and $y \in p^{-1}(x)$, there is a unique $\gamma \in C_x$ such that $y = \gamma \cdot y_x$ (recall that we have fixed a section $C_x \subseteq \pi$ of $\pi \to \pi/\pi_x$); hence we have a canonical isomorphism $\mathcal{T}^y \cong \mathcal{T}^{y_x}$. Thus for each $x \in J$,

$$\bigotimes_{y \in p^{-1}(x)} \lambda(\mathcal{T}^y) = \lambda(\mathcal{T}^{y_x})^{n/n_x}$$

so that

$$\bigotimes_{y \in J} \lambda(\mathcal{T}^y) = \bigotimes_{x \in J} \lambda(\mathcal{T}^{y_x})^{n/n_x}.$$

Using this (4.3) becomes

$$d(E) = d((p \times 1)^* F) \cdot \left(\bigotimes_{x \in J} \lambda(\mathscr{T}^{y_x})^{-n/n_x} \right). \tag{4.4}$$

From (4.1) and (4.4) we see that it suffices to show that for each $x \in J$,

$$\lambda(\mathscr{T}^{y_x})^{n/n_x} = \bigotimes_{j=1}^{k_x} \lambda(E_j^x)^{n_{x,j}}. \tag{4.5}$$

Since $n_{x,j} = n \cdot l_{x,j}/n_x$, this amounts to verifying that

$$\lambda(\mathscr{T}^{y_x}) = \bigotimes_{j=1}^{k_x} \lambda(E_j^x)^{l_{x,j}}. \tag{4.6}$$

Proving (4.6) for a fixed $x \in J$ is a local problem at y_x. By Remark (2.10), using a π-compatible coordinate chart at y_x we are reduced to the following situation: Let π be the group of nth roots of unity acting on the unit disc D by multiplication and let $p\colon \to D = {}_{\pi\backslash}D$ be the quotient map $p(z) = z^n$. Let V be a vector bundle on the variety T and define an action of π on V by $\zeta(v) = \zeta^l \cdot v$ for each $\zeta \in \pi$ and $v \in V$, where $0 \leqslant l \leqslant n-1$ is a fixed integer. Consider the π-bundle $W = p_T^* V$ on $D \times T$ where $p_T\colon \to T$ is the projection map and let $\mathscr{T}$ denote the cokernel of $0 \to (p \times 1)^*(p \times 1)_*^\pi \mathcal{O}(W) \to \mathcal{O}(W)$. Now, prove that

$$\lambda(\mathscr{T}^{\mathrm{o}}) = \lambda(V)^{-l}, \tag{4.7}$$

where o is the centre of D. Let us check that (4.7) is true. We can easily see that $(p \times 1)^*(p \times 1)_*^\pi \mathcal{O}(W) = \mathfrak{m}^l \cdot \mathcal{O}(W)$ where $\mathfrak{m}$ is the ideal sheaf of $\mathrm{o} \times T$ in $D \times T$. Therefore the sheaf $\mathscr{T}$ is isomorphic to the cokernel of

$$0 \to \mathfrak{m}^l \cdot \mathcal{O}(W) \to \mathcal{O}(W).$$

Now we have an exact sequence

$$0 \to \mathfrak{m} \to \mathcal{O}_{D \times T} \to \mathscr{S} \to 0,$$

where $\mathscr{S} = \mathcal{O}_{D \times T}/\mathfrak{m}$. Tensoring this short exact sequence with $\mathfrak{m}^{j-1}\mathcal{O}(W), j = 1, \ldots, l$, we get a system of exact sequences

$$0 \to \mathfrak{m}^j \mathcal{O}(W) \to \mathfrak{m}^{j-1}\mathcal{O}(W) \to \mathfrak{m}^{j-1}\mathcal{O}(W) \otimes \mathscr{S} \to 0, \quad j = 1, \ldots, l.$$

By general functionality, it follows that $\lambda(\mathscr{T}^{\mathrm{o}}) = \bigotimes_{j=1}^{l} \lambda(\mathscr{T}_j^{\mathrm{o}})$ where $\mathscr{T}_j = \mathfrak{m}^{j-1}\mathcal{O}(W) \otimes \mathscr{S}$ and $\mathscr{T}_j^{\mathrm{o}} = \mathscr{T}|_{\mathrm{o} \times T}$. But $\mathscr{T}_j^{\mathrm{o}} = (L^{\mathrm{o}})^{j-1} \otimes V$ where L is the line bundle on $D \times T$ associated to the sheaf $\mathfrak{m}$ and $L^{\mathrm{o}} = L|_{\mathrm{o} \times T}$. Since L^{o} is trivial on T, we get $\mathscr{T}_j^{\mathrm{o}} = V$ for $j = 1, \ldots, l$. Thus $\lambda(\mathscr{T}^{\mathrm{o}}) = \lambda(V)^l$ which is precisely (4.7). This proves the proposition.

We now construct parabolic determinants. For a more detailed description of these determinant bundles we refer the reader to Narasimhan and Ramadas [15]. Let X be a compact Riemann surface and $J \subseteq X$ a finite set. Fix integers k_x and

$(r_{x,1},\ldots,r_{x,k_x})$ such that $\Sigma_{j=1}^{k_x} r_{x,j} = r$. Let $0 \leqslant \alpha_{x,1} \leqslant \ldots \leqslant \alpha_{x,k_x} \leqslant 1$ be rational numbers for each $x \in J$. Let $M_J^p(X)$ denote the moduli space of stable parabolic bundles of rank r and degree d with weights $\alpha_{x,j}$, and multiplicities $r_{x,j}$.

Suppose $F \to X \times T$ is a family of parabolic bundles of type $\{\alpha_{x,j}, r_{x,j}\}_{x\in J}$, i.e., $rk(\mathrm{Gr}^j F^x) = r_{x,j}$ and the weights are $\alpha_{x,j}$. Then a possible candidate $pd(F)$ for the parabolic determinant is the usual determinant minus a factor involving the flags and the weights; thus, we define $\overline{pd}(F)$ to be the element of $\mathrm{Pic}(T)_{\mathbb{Q}} = \mathrm{Pic}(T) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ given by

$$\overline{pd}(F) = d(F) - \sum_{\substack{x\in J\\ j=1,\ldots,k_x}} \alpha_{x,j} \cdot \lambda(\mathrm{Gr}^j F^x).$$

Now if L is a line bundle on T and $G = F \otimes p_T^*(L)$ where $p_T: X \times T$ is the projection, then F and G are, by definition, equivalent families. Therefore we would like to have $\overline{pd}(F) = \overline{pd}(G)$ in $\mathrm{Pic}(T)_{\mathbb{Q}}$. But $d(G) = d(f) - \chi(E)\cdot L$ and $\lambda(\mathrm{Gr}^j G^x) = \lambda(\mathrm{Gr}^j F^x) + r_{x,j}\cdot L$ in $\mathrm{Pic}(T)_{\mathbb{Q}}$where $\chi(E) = \chi(E_t)$ is independent of t. Hence $\overline{pd}(G) = \overline{pd}(F) - \chi_p(F)\cdot L$, where

$$\chi_p(F) = \chi(F) + \sum_{\substack{x\in J\\ j=1,\ldots,k_x}} r_{x,j}\cdot \alpha_{x.j}.$$

We thus see that the definition of $\overline{pd}(F)$ has to be modified to get an invariant determinant. Fix once and for all a point $x_o \in X\backslash J$.

DEFINITION 4.7

Let $F \to X \times T$ be a family of parabolic bundles of type $\{r_{x,j}, \alpha_{x,j}\}$ on X parametrized by a variety T. Then the parabolic determinant $\overline{pd}(F)$, of F is the element of $\mathrm{Pic}(T)_{\mathbb{Q}} = \mathrm{Pic}(T) \otimes \mathbb{Q}$ defined by the formula

$$\overline{pd}(F) = \mathrm{d}(F) - \sum_{\substack{x\in J\\ j=1,\ldots,k_x}} \alpha_{x,j}\lambda(\mathrm{Gr}^j F^x) + \frac{\chi_p(F)}{r}\lambda(F^{x_o})$$

where

$$\chi_p(F) = \chi(F_t) + \sum_{x,j} \alpha_{x,j} r_{x,j}.$$

The above discussion shows that if F and G are equivalent families of parabolic bundles on $X \times T$, then $\overline{pd}(F) = \overline{pd}(G)$ in $\mathrm{Pic}(T)_{\mathbb{Q}}$.

We would like to think of the parabolic determinant as a concrete line bundle representing an element of $\mathrm{Pic}(T)$, rather than as an abstract entity in $\mathrm{Pic}(T)_{\mathbb{Q}}$. To achieve this we have to multiply the $\overline{pd}(F)$ of Definition 4.7 by an integer to remove the denominators from the rationals $\alpha_{x,j}$ and $\chi_p(F)/r$. Write $\alpha_{x,j} = p_{x,j}/q_{x,j}$ where $p_{x,j}$ and $q_{x,j}$ are relatively prime positive integers if $\alpha_{x,j} > 0$ and $q_{x,j} = 1$ if $\alpha_{x,j} = 0$. Denote by m the l.c.m of $\{q_{x,j}/x\in J, 1 \leqslant j \leqslant k_x\}$ and let $m_{x,j} = m\cdot\alpha_{x,j}$. Then $0 \leqslant m_{x,1} < m_{x,2} < \ldots < m_{x,k_x} \leqslant m-1$ are integers for each $x \in J$.

DEFINITION 4.8

Let $F \to X \times T$ be a family of parabolic bundles of type $\{\alpha_{x,j}, r_{x,j}\}$. Then the parabolic determinant bundle, $pd(F)$, of F is the line bundle on T defined by

$$pd(F) = \mathrm{d}(E)^{rm} \cdot \left\{ \bigotimes_{\substack{x \in J \\ j=1,\dots,k_x}} \lambda(\mathrm{Gr}^j F^x)^{-m_{x,j} \cdot r} \right\} \cdot \lambda(F^{x_0})^{m \cdot \chi_p(F)}.$$

Note that in $\mathrm{Pic}(T)_{\mathbb{Q}}$, $pd(F) = rm \cdot \overline{pd}(F)$.

Remark 4.9. The above arguments again imply that $pd(F) = pd(G)$ if F and G are equivalent families of parabolic bundles on $X \times T$.

We recall a proof from Narasimhan and Ramadas [15, § 2.c] that there is a parabolic determinant bundle on the moduli space $M_s^p(X)$. By tensoring with a fixed line bundle we may assume that the degree d is large enough to ensure that for every $E \in M_s^p(X)$, $H^0(E)$ generates E and $H^1(E) = 0$. Let Q denote the Quot scheme of coherent $\mathcal{O}_X$-modules which are quotients of $\mathcal{O}_X^k$, $k = d + r(1-g)$, with Hilbert polynomial $P(m) = d + r(m - g + 1)$. Then there is a Poincaré family of sheaves on $X \times Q$. Denote by R the open subset of Q consisting of locally free sheaves E such that $H^0(\mathcal{O}_X^k) \to H^0(E)$ is an isomorphism. Let $\mathscr{F}$ denote the restriction of the Poincaré family to $X \times R$. For each $x \in J$, let Flag $\mathscr{F}^x$ denote the flag bundle of type $(r_{x,1}, \dots, r_{x,1})$ associated to the vector bundle $\mathscr{F}^x$ on R; Flag $\mathscr{F}^x$ is a fibre bundle over R. Let $\tilde{R}$ denote the fibre product $\tilde{R} = \prod_{\substack{R \\ x \in J}} \mathrm{Flag}\, \mathscr{F}^x$. Then $\tilde{R}$ parametrizes a family $\tilde{\mathscr{F}}$ of parabolic bundles on X. Let $\tilde{R}_s$ denote the subset of $\tilde{R}$ corresponding to stable parabolic bundles. There is a natural action of $\mathrm{PGL}(k)$ on $\tilde{R}_s$ and the canonical map $\tilde{R}_s \to M_s^p(X)$ is a holomorphic principal $\mathrm{PGL}(k)$-bundle. Let $pd(\tilde{\mathscr{F}})$ denote the determinant bundle (cf. Definition 4.8.) on R_s defined by the family $\tilde{\mathscr{F}} \to X \times \tilde{R}_s$. Clearly the action of $GL(k)$ on $\tilde{R}_s$ lifts to an action on $pd(\tilde{\mathscr{F}})$. Let us see how the scalars $\mathbb{C}^* \subseteq GL(k)$ act on $pd(\mathscr{F})$. From Definition 4.8 and the properties of R, we see that

$$pd(\mathscr{F}) = \lambda(\mathcal{O}_{\tilde{R}_s})^{-rm} \cdot \left\{ \bigotimes_{\substack{x \in J \\ j=1,\dots,k_x}} \lambda(\mathrm{Gr}^j \mathscr{F}^x)^{-rm_{x,j}} \right\} \cdot \lambda(\mathscr{F}^{x_0})^{m \cdot \chi_p(\mathscr{F})}.$$

Thus a scalar $\lambda \in \mathbb{C}^*$ acts on $pd(\tilde{\mathscr{F}})$ as multiplication by λ^c, where

$$c = -rmk - r \sum_{\substack{x \in J \\ j=1,\dots,k_x}} m_{x,j} \cdot r_{x,j} + rm\chi_p(\tilde{\mathscr{F}});$$

on simplifying the right side, we see that $c = 0$. Therefore $pd(\tilde{\mathscr{F}})$ is a $PGL(k)$-line bundle on $\tilde{R}_s$ and hence descends to a holomorphic line bundle L^p on $M_s^p(X)$.

DEFINITION 4.10

The above line bundle L^p on $M_s^p(X)$ is called the parabolic determinant bundle on $M_s^p(X)$.

Consider a family of parabolic bundles on X, $F \to X \times T$, parametrized by a variety

T. Choose a compact Riemann surface Y and a group π of order n acting on Y such that (i) the quotient ${}_{\pi}\backslash Y$ is biholomorphic to X; and (ii) if $p\colon Y\to X$ denotes the quotient projection, then for each $x\in J$ and $y\in p^{-1}(x)$, the order of the isotropy subgroup of π at y equals m. Then n is a multiple of m and we let $a=n/m$. Define

$$n_{x,j}=n\cdot\alpha_{x,j}=a\cdot m_{x,j}\quad\text{for } x\in J \text{ and } j=1,\ldots,k_x.$$

By Proposition 2.11, the family F induces a family of π-bundles on Y, $E\to Y\times T$, parametrized by T. Let $\pi d(E)$ denote the determinant bundle of E as in §3. Fix a point $y\in p(x)$. Since $x\in X\backslash J$, y is not a ramification point of p and

$$\pi d(E)=d(E)\otimes\lambda(E^{y_0})^{\chi(E)}$$

where $\chi(E)=\chi(E_t)$ is independent of t. The two line bundles $pd(F)$ and $\pi d(E)$ on T are related by

PROPOSITION 4.11

Let F and E be as above. Then as elements of $Pic(T)_{\mathbb{Q}}=Pic(T)\otimes\mathbb{Q}$,

$$\{\pi d(E)\cdot pd(F)^{-a}\}^2=\bigotimes_{x\in J}\{\lambda(F^x)\lambda(F^{x_0})^{-1}\}^{ra(m-1)}.$$

Before proving the above proposition we make a simple observation whose proof is due to V. Balaji:

Lemma 4.12. *Let* $f\colon\mathscr{X}\to T$ *be a family of compact Riemann surfaces and suppose that F and G are holomorphic vector bundles on* $\mathscr{X}$ *with ranks r and n, respectively. Then*

$$d(F\otimes G)d(F)^{-n}d(G)^{-r}=d(\lambda(F)\cdot\lambda(G))d(\lambda(F))^{-1}d(\lambda(G))^{-1}$$

in $Pic(T)\otimes\mathbb{Q}$.

Proof. Denote by $A^*(T)=\sum_{i=0}^{\infty}A^i(T)$ the Chow ring of T and for any $\alpha\in A^*(T)$, let $\{\alpha\}_i$ denote the degree i component of α in $A^i(T)$. Also let $f_*\colon A^*(\mathscr{X})\to A^*(T)$ denote integration along the fibre. Then the Grothendieck–Riemann–Roch (G-R-R) theorem implies that for any vector bundle V on $\mathscr{X}$,

$$c_1(d(V))=f_*(\{ch(V)\cdot td(T_f)\}_2),$$

where T_f denotes the tangent bundle of $\mathscr{X}$ along the fibres of f. Note that T_f is a line bundle on $\mathscr{X}$. Let $\alpha_1,\ldots,\alpha_r$ be the Chern roots of F and $\beta_1,\ldots,\beta_n$ the Chern roots of G. Then $(\alpha_i+\beta_j)$ are the Chern roots of $F\otimes G$. Using the splitting principle (cf. Fulton [8], page 54) we formally compute

$$\{ch(F)\cdot td(T_f)\}_2=\frac{r\mu^2}{12}+\left(\sum_{i=1}^{r}\alpha_i\right)\frac{\mu}{2}+\frac{1}{2}\sum_{i=1}^{r}\alpha_i^2$$

where $\mu=c_1(T_f)$. Similarly

$$\{ch(F)\cdot td(T_f)\}_2=\frac{r\mu^2}{12}+\left(\sum_{j=1}^{r}\beta_j\right)\frac{\mu}{2}+\frac{1}{2}\sum_{j=1}^{r}\beta_i^2.$$

Finally,

$$\{ch(F\otimes G)\cdot td(T_f)\}_2 = \frac{nr\mu^2}{12} + \left(\sum_{i=1}^{r}\sum_{j=1}^{n}(\alpha_i+\beta_j)\right)\cdot\frac{\mu}{2} + \frac{1}{2}\left(\sum_{i=1}^{r}\sum_{j=1}^{n}(\alpha_i+\beta_j)\right)^2$$

$$= \frac{nr\mu^2}{12} + n(\Sigma\alpha_i)\frac{\mu}{2} + r(\Sigma\beta_j)\frac{\mu}{2} + r(\Sigma\beta_j)\frac{\mu}{2} + \frac{n}{2}\sum_i \alpha_i^2 + \frac{r}{2}\sum_j \beta_j^2 + (\Sigma\alpha_i)(\Sigma\beta_j)$$

i.e.,

$$\{ch(F\otimes G)\cdot td(T_f)\} = \{n\cdot ch(F)\cdot td(T_f) + rch(G)\cdot td(T_f)\}_2 + c_1(F)\cdot c_1(G) - \frac{nr\mu^2}{12}.$$

Applying G-R-R and noting that $f_*(\mu^2)=0$, we get

$$c_1(d(F\otimes G)) = n\cdot c_1(d(F)) + rc_1(d(G)) + f_*(c_1(F)\cdot c_1(G)),$$

i.e.,

$$c_1(d(F\otimes G)\cdot d(F)^{-n}d(G)^{-r}) = f_*(c_1(F)\cdot c_1(G)).$$

Replacing F by $\lambda(F)$ and G by $\lambda(G)$ in the above relation, we get

$$c_1(d(\lambda(F)\cdot\lambda(G))\cdot d(\lambda(F))^{-1}d(\lambda(G))^{-1}) = f_*(c_1(F)\cdot c_1(G))$$

since $c_1(\lambda(F)) = c_1(F)$. Thus

$$c_1(d(F\otimes G)\cdot d(F)^{-n}\cdot d(G)^{-r}) = c_1(d(\lambda(F)\lambda(G))d(\lambda(F)^{-1}d(\lambda(G)^{-1}).$$

Since $c_1:\mathrm{Pic}(T)\otimes\mathbb{Q}\to A^*(T)\otimes\mathbb{Q}$ is an isomorphism (Fulton [8], page 294) our lemma is proved.

We now recall a formalism due to Deligne [5]; we quote below Proposition 2.5 and Lemma 2.6 from Beilinson and Manin [3] as a brief summary of this formalism.

PROPOSITION 4.13

Let $f:\mathscr{X}\to T$ be a holomorphic family of compact Riemann surfaces. Then for each pair L, M of line bundles on $\mathscr{X}$, there is a line bundle $\langle L,M\rangle$ on T with the following properties:

(i) *There are canonical isomorphisms*

$$\langle L,\mathcal{O}_{\mathscr{X}}\rangle = \mathcal{O}_T,\quad \langle L_1\otimes L_2, M\rangle = \langle L_1,M\rangle\langle L_2,M\rangle,$$

$$\langle L,M\rangle = \langle M,L\rangle,\quad \langle L,M^{-1}\rangle = \langle L,M\rangle^{-1}$$

(ii) *If $M=\mathcal{O}_{\mathscr{X}}(D)$, where D is a relatively positive divisor on $\mathscr{X}$, locally free on T, then*

$$\langle L,M\rangle = d(L\otimes\mathcal{O}_D)^{-1}\cdot d(\mathcal{O}_D).$$

(iii) *There is a canonical isomorphism*

$$\langle L,M\rangle^{-1} = d(L\otimes M)\cdot d(\mathcal{O}_{\mathscr{X}})\cdot d(L)^{-1}d(M)^{-1}.$$

Now we proceed to the Proof of Proposition 4.11. Using Proposition 4.5 we see that

$$d(E) = d((p \times 1_T)^* F) \cdot \bigotimes_{x \in J} \lambda(\mathrm{Gr}^j F^x)^{-n_{x,j}}. \tag{4.3}$$

But $H^i(X, p_* V) = H^i(Y, V)$ canonically, so

$$d((p \times 1_T)^* F) = d((p \times 1)_* (p \times 1)^* F) \tag{4.4}$$

Now, by the projection formula

$$(p \times 1)_* (p \times 1)^* F = F \otimes G,$$

where

$$G = (p \times 1)_* \mathcal{O}_{Y \times T} = p_X^* p_* \mathcal{O}_Y \otimes p_T^* \mathcal{O}_T,$$

and $p_X : X \times T \to X$ and $p_T : X \times T \to T$ are the projections. Thus (4.4) becomes

$$d((p \times 1)^* F) = d(F \otimes G). \tag{4.5}$$

We now prove from Lemma 4.12 that

$$d(F \otimes G) \cdot d(F)^{-n} \cdot d(G)^{-r} = \langle \lambda(F), \lambda(G) \rangle^{-1} \cdot d(\lambda(F))^{-1} d(\lambda(G))^{-1}$$

in $\mathrm{Pic}(T)_{\mathbb{Q}}$. Using Proposition 4.13 (iii), we write this as

$$d(F \otimes G) \cdot d(F)^{-n} \cdot d(G)^{-r} = \langle \lambda(F), \lambda(G) \rangle^{-1} \cdot d(\mathcal{O}_{X \times T})^{-1}.$$

But $d(G)$ and $d(\mathcal{O}_{X \times T})$ are trivial on T. Thus

$$\langle \lambda(F), \lambda(G) \rangle^{-1} = d(F \otimes G) \cdot d(F)^{-n} \tag{4.6}$$

in $\mathrm{Pic}(T)_{\mathbb{Q}}$. Squaring both sides of (4.6) and using Proposition 4.13, we get

$$\langle \lambda(F), \lambda(G)^2 \rangle = \{ d(F \otimes G) \cdot d(F)^{-n} \}^2. \tag{4.7}$$

But $\lambda(G) = \lambda(p_X^* p_* \mathcal{O}_Y) \cdot p_T^* (\mathcal{O}_T)^n$. Therefore,

$$\begin{aligned} \langle \lambda(F), \lambda(G)^2 \rangle &= \langle \lambda(F), \lambda(p_X^* p_* \mathcal{O}_Y)^2 \cdot (p_T^* \mathcal{O}_T)^{2n} \rangle \\ &= \langle \lambda(F), \lambda(p_X^* p_* \mathcal{O}_Y)^2 \rangle \langle \lambda(F), p_T^* \mathcal{O}_T \rangle^{2n} \\ &= \langle \lambda(F), p_X^* \lambda(p_* \mathcal{O}_Y P)^2 \rangle \end{aligned}$$

because $\langle \lambda(F), p_T^* \mathcal{O}_T \rangle$ is trivial by virtue of Proposition 4.13. Now $\lambda(p_* \mathcal{O}_Y)^2 = \mathcal{O}_X(D)$ where $D = -\sum_{x \in J} a(m-1) \cdot x$ (cf. Beauville *et al* [2], §2.5). Therefore, by Proposition 4.13(ii)

$$\langle \lambda(F), \lambda(G)^2 \rangle = \langle \lambda(F), p_X^* \mathcal{O}_X(D) \rangle = d(\lambda(F) \otimes p_X^* \mathcal{O}_D)^{-1} d(p_X^* \mathcal{O}_D)$$

i.e.,

$$\langle \lambda(F), \lambda(G)^2 \rangle = \bigotimes_{x \in J} \lambda(F^x)^{a(m-1)} \tag{4.8}$$

where we drop $d(p_X^*\mathcal{O}_D)$ as it is trivial on T. Combining (4.7) and (4.8), and using (4.5),

$$\{d((p\times 1)^*F)\cdot d(F)^{-n}\}^2=\bigotimes_{x\in J}\lambda(F^x)^{a(m-1)}.$$

Now apply (4.3) we obtain

$$\left\{d(E)\cdot\bigotimes_{x\in J}\lambda(\mathrm{Gr}^jF^x)^{n_{x,j}}\cdot d(F)^{-n}\right\}^2=\bigotimes_{x\in J}\lambda(F^x)^{a(m-1)}.$$

Raising the above equation to the rth power and noting that $n_{x,j}=a\cdot m_{x,j}$ and $n=a\cdot m$, we see that

$$\{\pi d(E)\cdot\lambda(E^{y_0})^{-\chi(E)})\cdot(pd(F)^{-a}\cdot\lambda(F^{x_0})^{\chi_p(F)\cdot n})\}^2=\bigotimes_{x\in J}\lambda(F^x)^{ra(m-1)}. \tag{4.9}$$

Since y_o is not a ramification point of p and $p(y_o)=x_o$, we have $F^{y_0}\cong E^{x_0}$. So we can write (4.9) as

$$\{\pi d(E)\cdot pd(F)^{-a}\}^2\cdot\lambda(F^{x_0})^{2n\chi_p(F)-2\chi(E)}=\bigotimes_{x\in J}\lambda(F^x)^{ra(m-1)}. \tag{4.10}$$

The Riemann–Hurwitz formula gives

$$2n\chi_p(F)-2\chi(E)=|J|ra(m-1).$$

So we may express (4.10) as

$$\{\pi d(E)\cdot pd(F)^{-a}\}^2=\bigotimes_{x\in J}(\lambda(F^x)\lambda(F^{x_0})^{-1})^{ra(m-1)}$$

in $\mathrm{Pic}(T)_{\mathbb{Q}}$. This completes the proof of Proposition 4.11.

Let us see what Proposition 4.11 implies for our moduli spaces. Fix a line bundle δ of degree d on X and let $M_s^p(X,\delta)$ be the smooth subvariety of $M_s^p(X)$ consisting of parabolic bundles F such that $\lambda(F)\cong\delta$. Similarly, let $M_s^\pi(Y,\delta)$ be the smooth submanifold of $M_s^\pi(Y)$ consisting of π-bundles E such that $\lambda(E^\pi)\cong\delta$. Then Proposition 2.11 gives a natural isomorphism $\theta: M_s^p(X,\delta)\to M_s^\pi(Y,\delta)$. In §3, we constructed a π-determinant L^π on $M_s^\pi(Y,\delta)$ using Quillen's theorem. In this section we defined the parabolic determinant bundle L^p on $M_s^p(X,\delta)$. These determinants are related to each other by

PROPOSITION 4.14

The two line bundles $\theta^*(L^\pi)$ and $(L^p)^a$, where a is the integer defined in the paragraph following Definition 4.10, *are isomorphic on $M_s^p(X,\delta)$.*

Proof. Consider the variety $\tilde{R}_s$ mentioned earlier on; $\tilde{R}_s$ sits as a principal $PGL(k)$-bundle over $M_s^p(X)$ and parametrizes a family $\tilde{\mathscr{F}}$ of parabolic bundles on X. Let $\tilde{V}$ be the corresponding family of π-bundles on Y parametrized by $\tilde{R}_s$. For each pair of points $x, y \in X$, let $\tilde{G}_{x,y}$ be the line bundle on $\tilde{R}_s$ given by

$$\tilde{G}_{x,y} = \lambda(\tilde{\mathscr{F}}^x)\lambda(\tilde{\mathscr{F}}^x)^{-1}.$$

We easily see that $\tilde{G}_{x,y}$ is a $PGL(k)$-line bundle and descends to a line bundle $G_{x,y}$ on $M_s^p(X)$. By Proposition 4.11,

$$(\pi d(\tilde{V})\cdot pd(\tilde{\mathscr{F}})^{-a})^2 = \bigotimes_{x\in J}(\tilde{G}_{x,x_0})^{ra(m-1)} \tag{4.11}$$

in $\mathrm{Pic}(\tilde{R}_s)\otimes\mathbb{Q}$. But (cf. Drezet and Narasimhan [7], Lemma 4.2) $\mathrm{Pic}_G(\tilde{R}_s)\to\mathrm{Pic}(\tilde{R}_s)$ is injective, where $\mathrm{Pic}_G(\tilde{R}_s)$ is the group of $PGL(k)$-isomorphism classes of $PGL(k)$-line bundles on $\tilde{R}_s$. Therefore, since all the bundles in (4.11) lie in $\mathrm{Pic}_G(\tilde{R}_s)$, we see that (4.11) holds in $\mathrm{Pic}_G(\tilde{R}_s)\otimes\mathbb{Q}$. By descending to $M_s^p(X)$, we get

$$(\theta^* L^\pi (L^p)^{-a})^2 = \bigotimes_{x\in J}(G_{x,x_0})^{ra(m-1)}. \tag{4.12}$$

Consider a fixed $x\in X$. As y varies in X, $G_{x,y}$ varies in a continuous family; since $\mathrm{Pic}_0 M_s^p(X,\delta)$ is trivial, we conclude that $G_{x,y}$ is trivial on $M_s^p(X,\delta)\forall x,y\in X$. Thus (3.12) implies that

$$(\theta^* L^\pi (L^p)^{-a})^2 = 1$$

in $\mathrm{Pic}(M_s^p(X,\delta))\otimes\mathbb{Q}$. Since $\mathrm{Pic}\, M_s^p(X,\delta)$ is torsion-free, we conclude that

$$\theta^* L^\pi = (L^p)^a$$

on $M^p(X,\delta)$. ■

5. The metric on the parabolic determinant

In this section we define a Hermitian metric on the parabolic determinant bundle and give a proof of Theorem 1.2.

Let $M_s^p(X)$ and $M_s^p(X,\delta)$ be as in §4. Again, choose a covering $p: Y\to X$ of degree n. We continue using the notation of §4.

Suppose $E\in M_s^p$. Then $\mathrm{End}\, E$ is local system on $X_0 = X\backslash J$. We denote this local system by $\underline{\mathrm{End\,E}}$. Let $H_c^1(X_0, \underline{\mathrm{End\,E}})$ denote the compactly supported de Rham cohomology of X_0 with values in the local system $\underline{\mathrm{End\,E}}$. The real tangent space to $M_s^p(X)$ at E is canonically isomorphic to the image I_E of the natural map

$$H_c^1(X_0, \underline{\mathrm{End\,E}}) \to H_d^1(X_0, \underline{\mathrm{End\,E}}),$$

where H_d^1 denotes usual de Rham cohomology. We shall prove this fact in Lemma 5.5 below; for the moment let us assume it.

We define a sympletic form Ω_E^p on $T_E(M_s^p(X))$ by

$$\Omega_E^p(a,b) = \int_{X_0} \mathrm{tr}(\alpha\wedge\beta),$$

where $a, b \in I_E$ and α, β are compactly supported forms representing a and b. As E varies over $M_s^p(X)$, Ω_E^p vary to define a C^∞ symplectic form Ω^p on $M_s^p(X)$.

Recall that we have an isomorphism $\theta: M_s^p(X) \to M_s^\pi(Y)$. In §3 we defined a symplectic form, which we now denote as Ω^π, on $M_s^\pi(Y)$. The two forms Ω^π and Ω^p are related as follows.

Lemma 5.1 $\theta^*(\Omega^\pi) = n \cdot \Omega^p$.

Proof. Recall that the tangent space to $M_s^\pi(Y)$ at a point $E \in M_s^\pi(Y)$ is given by $H_d^1(Y, \pi, \underline{\text{End E}})$ where $\underline{\text{End E}}$ is the local system defined by the Einstein-Hermitian connection in E. Let $F = E^\pi$. Then the differential of θ at E is described as follows. Let $a \in T_E(M_s^\pi(Y))$. Choose a π-invariant 1-form α on y with values in $\underline{\text{End E}}$ representing a. Then $\alpha|_{Y \setminus I}$ descends to a form on $X_0 = X \setminus J$ which is cohomologous to a compactly supported form $\alpha_\#$ on X_0 with values in $\underline{\text{End E}}$. Thus $\alpha_\#$ defines an element $a_\#$ in $I_F = T_F(M_s^p(X))$. The differential of θ^{-1} at E is given by

$$d\theta^{-1}: T_E(M_s^\pi(Y)) \to T_F(M_s^p(X))$$
$$a \to a_\#.$$

Now, let $a, b \in T_E(M_s^\pi(Y))$. Then

$$\begin{aligned}(\theta^{-1})^*(\Omega^p)(a,b) &= \Omega^p(d\theta^{-1}(a), d\theta^{-1}(b)) \\ &= \Omega^p(a_\#, b_\#) \\ &= \int_{X_0} \operatorname{tr}(\alpha_\# \wedge \beta_\#) \\ &= \frac{1}{n}\int_Y \operatorname{tr}(\alpha \wedge \beta) \\ &= \frac{1}{n}\Omega^\pi(a,b).\end{aligned}$$

Thus

$$(\theta^{-1})^*\Omega^p = \frac{1}{n}\Omega^\pi,$$

i.e.,

$$\Omega^p = \frac{1}{n}\theta^*\Omega^\pi. \qquad \blacksquare$$

Let us now state the obvious.

Lemma 5.2. *Let V and W be two complex manifolds and suppose $f: V \to W$ is a biholomorphism. Let $\beta: T(W)_{\mathbb{C}} \times T(W)_{\mathbb{C}} \to \mathbb{C}$ be a real positive* (1, 1,)*-form on W and let $\beta_{\mathbb{R}}$ denote $\beta|_{T(W) \times T(W)}$. Suppose $\alpha: T(V) \times T(V) \to \mathbb{R}$ is a real 2-form on V such that $f^*\beta_{\mathbb{R}} = \alpha$. Then the complexification $\alpha_{\mathbb{C}}$ of α is a real, positive* (1, 1)*-form on V.*

Thus the complexification Θ^p of the 2-form Ω^p on $M_s^p(X)$ is real, positive and of type (1, 1), so it defines a Kähler structure on $M_s^p(X)$.

In §3, we defined a natural Hermitian metric h^π in the determinant bundle L^π on

$M_s^p(Y)$. This metric induces a metric $\theta^* h^\pi$ in the pull-back $\theta^* L^\pi$ over $M_s^p(X)$. But Proposition 4.14 implies that $\theta^* L^\pi = (L^p)^a$ on $M_s^p(X, \delta)$. Thus we have obtained a metric in $(L^p)^a$ on $M_s^p(X, \delta)$. This defines a metric h^p in L^p. We now have Theorem 1.1 of the Introduction.

Theorem 5.3. *The Chern form of the metrized line bundle (L^p, h^p) equals mr times the Kähler form Θ^p on $M_s^p(X, \delta)$.*

Proof. Since both $c_1(L^p, h^p)$ and $mr \cdot \Theta^p$ are real (1, 1)-form, it suffices to show that they are equal on the real tangent space of $M_s^p(X, \delta)$. But as real forms

$$\begin{aligned} a \cdot c_1(L^p, h^p)_{\mathbb{R}} &= c_1((L^p)^a, (h^p)^a) = c_1(\theta^* L^\pi, \theta^* h^\pi)_{\mathbb{R}} = \theta^* c_1(L^\pi, h^\pi)_{\mathbb{R}} \\ &= \theta^*(r \cdot \Omega^\pi) = rn \cdot \Omega^p = rn \cdot \Theta^p_{\mathbb{R}}, \end{aligned}$$

using Theorem 3.27. Dividing by a and noting that $m \cdot a = n$, we get

$$c_1(L^p, h^p)_{\mathbb{R}} = rm \cdot \Theta^p_{\mathbb{R}}.$$

This proves the Theorem. ∎

The metric a priori depends on the choice of the covering (Y, π). But we now see that in fact it is independent of π up to a positive constant.

PROPOSITION 5.4

Let (Y_1, π_1) and (Y_2, π_2) be two ramified coverings of X with the properties of Y as above. Let h_1^p and h_2^p be the induced metrics in L^p. Then there is a constant $c > 0$ such that $h_1^p = c \cdot h_2^p$ in L^p over $M_s^p(X, \delta)$..

Proof. Since h_1^p and h_2^p are two Hermitian metrics in the same line bundle L^p, there exists a positive c^∞ function $f: M_s^p(X, \delta) \to \mathbb{R}^+$ such that $h_1 = f \cdot h_2$. By theorem 5.3, h_1 and h_2 have the same curvature. So, locally, $\partial\bar{\partial} \log h_1 = \partial\bar{\partial} \log h_2$, and hence, $\partial\bar{\partial} \log f = 0$. Thus $\log f$ is a plurisubharmonic function on $M_s^p(X, \delta)$. Now the moduli space $M^p(X, \delta)$ of semistable parabolic bundles is a normal projective variety and codim of the complement of $M_s^p(X, \delta) \geqslant 2$, so $\log f$ extends to a plurisubharmonic function on the whole of $M^p(X, \delta)$. Since $M^p(X, \delta)$ is compact, $\log f = a$ is constant, and $h_1 = c h_2$ where $c = e^a$. ∎

In the rest of the section we prove.

Lemma 5.5.. The real tangent space to $M_s^p(X)$ at E is canonically isomorphic to the image I_E of the natural map

$$H_c^1(X_c, \underline{\mathrm{End}\, E}) \to H_d^1(X_0, \underline{\mathrm{End}\, E}).$$

Proof. It is enough to prove the lemma in the case where parabolic degree is zero. If the parabolic degree is not zero then choose sufficiently large number of points $a_1, \ldots, a_n \in X_0$ and include them among the parabolic points and give on the fibre of each a_i the trivial quasi-parabolic structure; i.e. the only flag is the whole fibre (cf.

Mehta and Seshadri [13], Remark 1.17). Assign non-zero weights to the flags so that the new parabolic degree is zero. This gives an isomorphism of $M_s^p(X)$ to some other moduli of parabolic stable bundles with parabolic degree 0. Let V denote the image of E under this isomorphism. Stable bundles of parabolic degree 0 arise as representations of the fundamental group of the complement of the parabolic points in $U(n)$. So, as V is a degree 0 stable bundle End V has a structure of a local system. But around each a_i the monodromy is trivial; scalars are in the kernel of the adjoint representation. So End E and End V are naturally isomorphic on X_0; in particular End E has a structure of a local system on X_0. Now onwards we will assume the parabolic degree to be zero.

So, let X be a compact Riemann surface of genus $g \geqslant 2$ and $J \subseteq X$ a finite set. Fix integers $d \in \mathbb{Z}$ and $r \geqslant 1$. Let us recall our earlier notation. For each $x \in J$, let $0 \leqslant \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,k_x} < 1$ be rational numbers and let $(r_{x,1}, \ldots, r_{x,k_x})$ be positive integers such that $\sum_{j=1}^{k_x} r_{x,j} = r$. Assume that

$$d + \sum_{\substack{x \in J \\ j=1,\ldots,r}} \alpha_{x,j} \cdot r_{x,j} = 0.$$

Let $M_s^p(X)^0$ denote the moduli space of stable parabolic bundles of rank r, degree d and weights $\alpha_{x,j}$ with multiplicities $r_{x,j}$. For each $x \in J$, define a function $c_x : \{1, \ldots, r\} \to [0, 1]$ by

$$c_x(i) = \alpha_{x,j}, \quad \text{if } r_{x,1} + \cdots + r_{x,j-1} + 1 \leqslant r_{x,1} + \cdots + r_{x,j}.$$

Let W_x denote the conjugacy class in $U(r)$ of the diagonal matrix

$$\begin{pmatrix} e^{2\pi i c_x(1)} & & 0 \\ & \ddots & \\ 0 & & e^{2\pi i c_x(r)} \end{pmatrix}.$$

Express the $\alpha_{x,j}$ with a common denominator m, $\alpha_{x,j} = m_{x,j}/m$. Then there exists a discrete subgroup Γ of $PSL_2(\mathbb{R})$ acting on the upper half-plane H with the following properties: (i) Γ has no parabolic fixed points; (ii) $\Gamma \backslash H \cong X$; (iii) Γ is generated by $2g + |J|$ elements $\{a_1, \ldots, a_g, b_1, \ldots, b_g, c_x, x \in J\}$ with the relations

$$\left(\prod_{i=1}^{g} [a_i, b_i]\right)\left(\prod_{x \in J} c_x\right) = 1 \text{ and } c_x^m = 1,\ x \in J.$$

Let R denote the manifold of irreducible unitary representations $\rho : \Gamma \to V(r)$ such that $\rho(c_x) \in W_x$ for all $x \in J$. Then by Mehta and Seshadri [13], $R \cong M_s^p(X)^0$.

The real tangent space to R at a representation ρ is equal to $H^1(\Gamma, \mathrm{ad}\rho)$ where $\mathrm{ad}: U(r) \to \mathrm{Lie}\, U(r)$ is the adjoint representation. Using an argument of Prasad [17], we shall identify this group cohomology with compactly supported cohomology. From Prasad [17, Lemma 3 and §6] we get $H^1(\Gamma, \mathrm{ad}\rho) = H^1(X, A)$ where A is the invariant direct image of the constant sheaf Lie $U(r)$ on H. Let $X_0 = X \backslash J$ and let $i: X_0 \to X$ denote the inclusion map. It is easily seen that $i_*(A|_{X_0}) = A$. Therefore the edge homomorphism associated to the Leray spectral sequence of i implies that the canonical map $g: H^1(X, A) \to H^1(X_0, A)$ is injective. On the other hand Theorem 4.10.1 of Godement [9] applied to the closed set J gives an exact sequence

$$\cdots \to H_c^1(X_0, A) \to H^1(X, A) \to H^1(J, A) \to \cdots$$

where H_c^1 denotes cohomology with compact supports. But J being a finite set, we get $H^1(J, A) = 0$. Thus the map $f: H_c^1(X_0, A) \to H^1(X, A)$ is surjective. Let

$$h: H_c^1(X_0, A) \to H^1(X_0, A)$$

denote the canonical map. Then we have a commutative diagram

$$\begin{array}{ccc} H_c^1(X, A) & \xrightarrow{f} & H^1(X, A) \\ h\downarrow & & \downarrow g \\ H^1(X_0, A) & \underset{\text{identity}}{\rule[0.5ex]{3em}{0.4pt}} & H^1(X_0, A) \end{array}$$

Since f is surjective and g injective, we conclude that $H^1(X, A) \cong \mathrm{Im}(h)$, i.e., $T_\rho(R)$ is the image of $H_c^1(X_0, A)$ in $H^1(X_0, A)$ under the canonical map h. This completes the proof of the Lemma. ■

6. An application

In this section we deduce a finiteness statement from our earlier results.

Let Y be a compact Riemann surface of genus at least 2. Fix two relatively prime integers r and d with $r \geqslant 1$. Let $M(Y)$ denote the moduli space of stable bundles of rank r and degree d on Y. Suppose a finite group π acts holomorphically and effectively on Y. Let $M^\pi(Y)$ denote the moduli space of stable π-bundles of rank r, degree d and fixed local type on Y. Then both $M(Y)$ and $M^\pi(Y)$ are projective algebraic manifolds and there is a canonical forgetful map

$$\alpha: M^\pi(Y) \to M(Y).$$

PROPOSITION 6.1

The morphism α is a finite map.

Proof. As in § 3, we can construct a determinant line bundle L on M and show that L is positive (cf. Theorem 3.27). In fact, to see this we put π to be the trivial group everywhere in § 3. Since $M(Y)$ is compact, the Kodaira embedding theorem implies that L is ample. But the determinant bundle L^π on $M^\pi(Y)$ is exactly equal to $\alpha^*(L)$. Thus α pulls back the ample line bundle L on $M(Y)$ to an ample line bundle on $M^\pi(Y)$. As a result α is a finite morphism. ■

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