On the restriction of cuspidal representations to unipotent elements

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1. Introduction

Let $G$ be a connected split reductive group defined over a finite field $\mathbb{F}_q$, and $G(\mathbb{F}_q)$ the group of $\mathbb{F}_q$-rational points of $G$. For each maximal torus $T$ of $G$ defined over $\mathbb{F}_q$ and a complex linear character $\theta$ of $T(\mathbb{F}_q)$, let $R^G_T(\theta)$ be the generalized representation of $G(\mathbb{F}_q)$ defined in [DL]. It can be seen that the conjugacy classes in the Weyl group $W$ of $G$ are in one-to-one correspondence with the conjugacy classes of maximal tori defined over $\mathbb{F}_q$ in $G$ ([C1, 3.3-3]). Let $c$ be the Coxeter conjugacy class of $W$, and let $T_c$ be the corresponding maximal torus. Then by [DL] we know that $\pi_\theta = (-1)^n R^G_{T_c}(\theta)$ (where $n$ is the semisimple rank of $G$ and $\theta$ is a character in ‘general position’) is an irreducible cuspidal representation of $G(\mathbb{F}_q)$. The results of this paper generalize the pattern about the dimensions of cuspidal representations of $GL(n, \mathbb{F}_q)$ as an alternating sum of the dimensions of certain irreducible representations of $GL(n, \mathbb{F}_q)$ appearing in the space of functions on the flag variety of $GL(n, \mathbb{F}_q)$ as shown in the table below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Dimension of cuspidal representation</th>
<th>$\dim(\text{St}<em>{n,n}) - \dim(\text{St}</em>{n,n-1}) + \dim(\text{St}<em>{n,n-2}) - \cdots + (-1)^{n-1} \dim(\text{St}</em>{n,1})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$q-1$</td>
<td>$q-1$</td>
</tr>
<tr>
<td>3</td>
<td>$(q^2 - 1)(q - 1)$</td>
<td>$q^3 - (q^2 + q) + 1$</td>
</tr>
<tr>
<td>4</td>
<td>$(q^3 - 1)(q^2 - 1)(q - 1)$</td>
<td>$q^6 - (q^5 + q^4 + q^3) + (q^3 + q^2 + q) - 1$</td>
</tr>
<tr>
<td>5</td>
<td>$(q^4 - 1)(q^3 - 1)(q^2 - 1)(q - 1)$</td>
<td>$q^9 - (q^8 + q^7 + q^5) + (q^7 + q^6 + q^4 + q^3) - (q^4 + \cdots + q) + 1$</td>
</tr>
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</table>

Here $\text{St}_{n,1}$ is an irreducible representation of $GL(n, \mathbb{F}_q)$ appearing in the space of functions on the flag variety of $GL(n, \mathbb{F}_q)$; $\text{St}_{n,n}$ is the Steinberg representation, and $\text{St}_{n,1}$ is the trivial representation of $GL(n, \mathbb{F}_q)$. We are using the well known formula for the dimension of a cuspidal representation of $GL(n, \mathbb{F}_q)$ as $(q - 1) \cdots (q^{n-1} - 1)$. We could easily check that this equality remained true for characters of all unipotent elements too for these small values of $n$ by looking into character tables. The aim of the paper is to give a proof of this for $GL(n, \mathbb{F}_q)$ as well as generalizations for other classical groups. For $GL(n, \mathbb{F}_q)$ it seems that this result is well known and can be
proved by methods as given in [L], but we include a proof in this case too for the sake of completeness.

An irreducible representation $\rho$ of $G(F_q)$ is called unipotent if it arises as a component of $R^G_T(1)$ for some $T$. If $T$ is a split torus then $R^G_T(1) = \text{Ind}_{B(T)}^{G(T)}(1)$ where $B$ is a Borel subgroup containing $T$, defined over $F_q$. It is well known that $\text{End}_{G(F_q)}(\text{Ind}_{B(T)}^{G(T)}(1))$ can be identified with the group algebra $\mathbb{C}[W]$. Therefore the irreducible representations of $G(F_q)$ occurring in $\text{Ind}_{B(T)}^{G(T)}(1)$ are in one to one correspondence with the irreducible representations of $W$ over $C$. It is known ([S, 14]) that the exterior powers of the reflection representation of $W$ to be denoted by $E$ throughout this paper, are irreducible and mutually inequivalent. Let $\pi_i$ be the irreducible component of $\text{Ind}_{B(T)}^{G(T)}(1)$ corresponding to the $i$th exterior power representation of the reflection representation of $W$.

By [L2] it is known that if $G$ is a classical group, then it can have at most one unipotent cuspidal representation. The groups of type $A_n$ do not have any unipotent cuspidal representation; groups of type $B_n, C_n$ have exactly one if and only if $n = s^2 + s$ for some integer $s \geq 1$ and $D_n$ have one if and only if $n$ is an even square. Thus groups of type $B_2, C_2, D_4$ have unique unipotent cuspidal representations, and in these cases they occur as a component of $R^G_T(1)$, where $t$ is the Coxeter conjugacy class of the corresponding root systems. Let us denote these unipotent cuspidal representations by $\pi_{uw}$.

Let $G = G_n$ be either $Sp_{2n}, SO_{2n+1} (n \geq 2)$, or the split orthogonal group in even number of variables $SO_{2n}$ defined over $F_q$. For each partition $n = r_1 + r_2 + \cdots + r_k + s (0 \leq s < n)$ we have the standard parabolic subgroup $L$ defined over $F_q$ with Levi subgroup $L$ defined over $F_q$ and isomorphic to $GL_{r_1} \times GL_{r_2} \times \cdots \times GL_{r_k} \times G_s$. For $G = Sp_{2n}$, or $SO_{2n+1}$ take the partition $n = 1 + \cdots + 1 + 2$, with the corresponding Levi subgroup $(G_m)^{n-2} \times Sp_2$, or $(G_m)^{n-2} \times SO_3$. We know that $Sp_2$ and $SO_3$ have a unique unipotent cuspidal representation $\pi_{uw}$. Extend the representation $\pi_{uw}$ trivially across $(G_m(F_q))^{n-2} = (F_q^*)^{n-2}$ to construct a representation of $(F_q)^{n-2} \times Sp_2, (F_q)^{n-2} \times SO_3$, as the case may be. We abuse notation to denote this representation of Levi subgroup $L(F_q)$ again by $\pi_{uw}$. Let $\rho = \text{Ind}_{B(T)}^{G(T)}(\tilde{\pi}_{uw})$, where $\tilde{\pi}_{uw}$ is the representation of $P(F_q)$ obtained by composing $\pi_{uw}$ with the natural homomorphism from $P(F_q)$ to $L(F_q)$. By [L2, L5] we know that $\text{End}_{G(F_q)}(\rho)$ can be identified with $\mathbb{C}[W(B_{n-2})]$. Therefore the irreducible representations of $G(F_q)$ occurring in $\rho$ are in one-to-one correspondence with the irreducible representations of $W(B_{n-2})$. Let $\rho_i$ be the irreducible component of $\rho$ corresponding to the $i$-th exterior power representation of the reflection representation of $W(B_{n-2})$. Similarly, when $G = SO_{2n} (n \geq 4)$, take the Levi subgroup $L \cong (G_m)^{n-4} \times SO_8$. We know that $SO(8, F_q)$ has a unique unipotent cuspidal representation $\pi_{uw}$. Let $\rho$ be constructed as above. It follows by [L2, L5] that $\text{End}_{G(F_q)}(\rho)$ can be identified with $\mathbb{C}[W(B_{n-2})]$. Let $\rho_i$ be the irreducible component of $\rho$ corresponding to the $i$th exterior power representation of the reflection representation of $W(B_{n-4})$. Here is the main theorem of this paper.

**Theorem 1.1.** Let $G$ be a split classical group, and let $\Theta_{\pi}$ denote the character of a representation $\pi$. Let $u$ be a unipotent element of $G(F_q)$. With the notation as above, we have the following...
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(a) For $G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q)$ $(n \geq 0)$,

$$\Theta_{\pi_s}(u) = \sum_{i=0}^{n-1} (-1)^i \Theta_{\pi_{s_i}}(u). \quad (1.1)$$

(b) For $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$, or $SO(2n+1, \mathbb{F}_q)$ $(n \geq 2)$,

$$\Theta_{\pi_s}(u) = \sum_{i=0}^{n-2} (-1)^i \Theta_{\pi_{s_i}}(u) + \sum_{i=0}^{n-4} (-1)^i \Theta_{\pi_{s_{i-1}}} (u). \quad (1.2)$$

(c) For $G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q)$ $(n \geq 4)$,

$$\Theta_{\pi_s}(u) = \sum_{i=0}^{n-4} (-1)^i \Theta_{\pi_{s_{i-1}}} (u). \quad (1.3)$$

To prove the theorem above we shall need some results about the characters of the classical Weyl groups, and their values at the Coxeter conjugacy class. We prove that the character values of irreducible representations of the classical Weyl groups at its Coxeter conjugacy class is $1, -1, 0$ (Theorem 2.1). We also need the decomposition of $R[\chi]$ (see Theorem 4.1) in terms of irreducible unipotent representations of $G(\mathbb{F}_q)$ (Theorem 4.4).

Remark 1.1. In general, the restriction to unipotent elements of cuspidal representation coming from other maximal tori, can not be expressed as an alternating sum, as in Theorem 1.1. The reason for getting such a small number of representations in Theorem 1.1 is that there are just $h$ (Weyl number) irreducible characters of Weyl group that are non-zero at the Coxeter element, and these are either 1 or $-1$. This is not true in general for other elements of the Weyl group.

2. Characters of classical Weyl groups

In this section we shall establish some basic results about the characters of classical Weyl groups.

Let $W$ be the Weyl group corresponding to an irreducible root system $\Sigma$ in a $\mathbb{Q}$-vector space $E$ (spanned by $\Sigma$) with a fixed set of positive roots. Then $E$ is in a natural way an irreducible $\mathbb{Q}[W]$-module, said to be the reflection representation of $W$. Let $\Delta = \{e_1, e_2, \ldots, e_n\}$ be the system of simple roots of $\Sigma$. Then $\Delta$ is a basis for $E$. Let $s_i \in GL(E)$ be the reflection with respect to $e_i \in E$. Then $W$ has a presentation as a finite Coxeter group $W = \langle s_1, s_2, \ldots, s_n \mid s_i^2 = 1, i = 1, \ldots, n, s_is_j(s_is_j)^{n_{ij}} = (s_js_i)^{n_{ij}} = 1, n_{ij} < \infty \rangle$. The element of the form $c = s_1 \cdots s_n$ is called a Coxeter element of $W$. The conjugacy class of $c$ does not depend either on the ordering $s_1, \ldots, s_n$, or on the choice of the generating reflections, and therefore defines a well-defined conjugacy class in $W$, called the Coxeter class. Weyl groups of root systems of type $A_n$ $(n \geq 1)$, $B_n$ $(n \geq 1)$ and $D_n$ $(n \geq 2)$ are called the classical Weyl groups. (The Weyl group of $C_n$ is same as that of $B_n$.) We will briefly describe these Weyl groups.

(1) Let the root system be of type $A_n$ $(n \geq 1)$. Then $W(A_n) = S_{n+1}$ (the symmetric group on $n + 1$ elements). In this case, one can take the transpositions $(i, i + 1)$ $(1 \leq i \leq n)$ to be a set of simple reflections. Therefore, the Coxeter element $c = s_1 \cdots s_n$ is the $(n + 1)$-cycle $(1, 2, \ldots, n + 1)$. We denote the $(n + 1)$-cycle by $\sigma_{n+1}$.
Let $W_n, (n \geq 1)$ be the group of all permutations of the set \{1, 2, \ldots, n, n', \ldots, 2', 1'\} which commute with the involutions $i \rightarrow i', i' \rightarrow i \ (1 \leq i \leq n)$. For each $j, 1 \leq j \leq n - 1$ let $s_j \in W_n$ be the permutation which interchanges $j$ with $j + 1$ and also $j'$ with $(j + 1)'$ and leaves the other elements unchanged. Let $s_n \in W_n$ be the permutation which interchanges $n$ with $n'$ and leaves other entries unchanged. Then $W_n = \langle s_1, s_2, \ldots, s_n \rangle$ is the Weyl group of type $B_n$. A permutation in $W_n$ defines a permutation of the $n$ element set consisting of the unordered pairs \{(1, 1'), (2, 2'), \ldots, (n, n')\}. Thus we have a natural homomorphism of $W_n$ onto $S_n$. It can be seen that $W_n = \{ \pm 1 \}^n \rtimes S_n$. The Coxeter element is $c = (1, 1, \ldots, 1, -1) \cdot \sigma_n$, where $\sigma_n$ is the $n$-cycle in $S_n$.

(3) Let $\varepsilon : W_n \rightarrow \{ \pm 1 \}$ be the homomorphism defined by the condition that
\[
\varepsilon(s_i) = 1 \quad (1 \leq i \leq n - 1), \quad \varepsilon(s_n) = -1.
\]
Let $\widetilde{W}_n = \ker(\varepsilon)$. When $n \geq 2$, let $s'_n = s ns_{n-1} s_n$. Then $\widetilde{W}_n = \langle s_1, s_2, \ldots, s_{n-1}, s'_n \rangle$ is the Weyl group of type $B_n$. The Coxeter element is $c = (1, 1, \ldots, 1, -1, -1) \cdot \sigma_{n-1}$, where $\sigma_{n-1}$ is the $(n - 1)$-cycle in $S_n$.

**Lemma 2.1.** A Coxeter element commutes only with its powers.

**Proof.** See proposition 30 of [C].

Let $c$ be a Coxeter element of $W$ and let $h$ be its order, called the Coxeter number of $W$ or of the underlying semisimple group $G$. Then by the above result $h = |Z_W(c)|$, where $|Z_W(c)|$ denotes the cardinality of the centralizer of $c$ in $W$.

**Theorem 2.1.** Let $W$ be a classical Weyl group. Then the value of all the irreducible characters of $W$ at its Coxeter conjugacy class is 1, −1 or 0.

The proof of the theorem above essentially reduces to the case of $W(A_{n-1})$. So we will first prove this for $W(A_{n-1})$.

Let $\{v_1, \ldots, v_n\}$ be the standard basis for $\mathbb{Q}^n$. Let $E = E_n = \{(u_1, \ldots, u_n) \in \mathbb{Q}^n | \sum u_i = 0\}$ be the $n - 1$ dimensional subspace of $\mathbb{Q}^n$. For the Weyl group $W(A_{n-1}) = S_n$, the generators $s_i$ are the transpositions $(i, i + 1), 1 \leq i \leq n - 1$, acting as reflections in $GL(E)$ with respect to the root-vectors $\alpha_i = v_i - v_{i+1}$. Thus $E_n$ affords the reflection representation of $W(A_{n-1}) = S_n$. The order $h$ of the Coxeter element $c = s_1 \cdots s_{n-1} = \sigma_n$ is $n$. The eigenvalues of $\sigma_n = (1, 2, \ldots, n)$ on $E$ are
\[
\zeta^i, \quad i = 1, 2, \ldots, n - 1,
\]
where $\zeta$ is a primitive $n$th root of unity. It is easy to see that $\wedge^i E_n, i = 0, 1, \ldots, n - 1$ are mutually inequivalent irreducible representations of $S_n$. Let $\chi_i = \Theta_{\wedge^i E_n}$ denote the character of $\wedge^i E_n$, and $\widehat{W}$ denote the set of irreducible characters of $W$.

**Proposition 2.1.** Let $W = W(A_{n-1}) = S_n$. Let $\chi \in \widehat{W}$. Then
\[
\chi(\sigma_n) = \begin{cases} (-1)^i & \text{if } \chi = \chi_i, \quad i \in \{0, 1, \ldots, n - 1\}, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** From (2.2) we know that the eigenvalues $x_i$ of $\sigma_n$ on $E$ satisfy the following
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equation:
\[x^{n-1} + x^{n-2} + \cdots + x + 1 = 0.\]

Therefore,
\[\chi_i(\sigma_n) = \sum_{1 \leq r_1 < r_2 < \cdots < r_i \leq n-1} x_{r_1} \cdot x_{r_2} \cdots x_{r_i}.\]

The expression on the right-hand side is the sum of the product of \(i\) distinct roots of (2-4), which is \((-1)^i\) times the coefficient of \(x^{n-1-i}\). Therefore
\[\chi_i(\sigma_n) = (-1)^i, \quad i = 0, 1, \ldots, n-1.\]

We now need an elementary result about the characters of a finite group, according to which for any finite group \(H\) and any element \(g \in H\),
\[\sum_{\chi \in \hat{H}} |\chi(g)|^2 = |Z_H(g)|\]

where \(\hat{H}\) denotes the set of isomorphism classes of irreducible characters of \(H\), \(\overline{\chi(g)}\) denotes the complex conjugate of \(\chi(g)\), and \(|Z_H(g)|\) denotes the cardinality of the centralizer of \(g\). Applying (2-7) in the situation of \(W\) we obtain
\[\sum_{\chi \in \hat{W}} |\chi(\sigma_n)|^2 = n.\]

On the other hand, we have
\[\sum_{\chi \in \hat{W}} |\chi(\sigma_n)|^2 = \sum_{\chi \notin \{\chi_i \mid i = 0, 1, \ldots, n-1\}} |\chi(\sigma_n)|^2 + \sum_{i=0}^{i=n-1} |\chi_i(\sigma_n)|^2 = \sum_{\chi \notin \{\chi_i \mid i = 0, 1, \ldots, n-1\}} |\chi(\sigma_n)|^2 + \sum_{i=0}^{i=n-1} 1 \quad \text{(by (2-6))}\]
\[= \sum_{\chi \notin \{\chi_i \mid i = 0, 1, \ldots, n-1\}} |\chi(\sigma_n)|^2 + n.\]

Applying (2-8) we obtain
\[\sum_{\chi \notin \{\chi_i \mid i = 0, 1, \ldots, n-1\}} |\chi(\sigma_n)|^2 = 0.\]

Therefore \(\chi(\sigma_n) = 0\) if \(\chi \notin \{\chi_i \mid i = 0, 1, \ldots, n-1\}\). Hence the proof.

Irreducible characters of \(S_n\) are in one-to-one correspondence with the partitions of \(n\). A partition of \(n\) is any finite sequence
\[\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots]\]
of non-negative integers in increasing order
\[0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_r \leq \cdots\]
such that the sum of the parts of \(\lambda\) (denoted by \(|\lambda|\) and equal to \(\lambda_1 + \lambda_2 + \cdots + \lambda_r + \cdots\)) is \(n\). We shall find it convenient not to distinguish between two such sequences which differ only by a string of zeroes at the beginning. Note that the irreducible character \(\chi_i\) of \(S_n\) corresponds to the partition \([1^i, n-i]\).
Lemma 2.2. Let \( \mu_i = [1^{-i}, 2, n - 1 - i] \), \( i = 1, \ldots, n - 3 \) be a partition of \( n \), and let \( \chi_{\mu_i} \) be the corresponding irreducible character of \( S_n \). Let \( \sigma_{n-1} \) be an \( (n-1) \)-cycle in \( S_n \). Then \( \chi_{\mu_i}(\sigma_{n-1}) = \Theta_{\lambda^i,E_{n-1}}(\sigma_{n-1}) = (-1)^{i} \), where \( \lambda^iE_{n-1} \) is the \( i \)-th exterior power representation of the \( (n-2) \)-dimensional reflection representation of \( S_{n-1} \). Moreover, for any irreducible character \( \chi \) of \( S_n \), \( \chi(\sigma_{n-1}) = 0 \) unless \( \chi = \chi_{\mu_i} \), or \( \chi = 1 \), or \( \chi = \text{sgn} \).

Proof. It follows from the Murnaghan–Nakayama rule (see \([M, 1, 7, \text{ex. 5}]\)) that

\[
\chi_{\mu_i}(\sigma_{n-1}) = \sum \chi_{\nu}(\sigma_{n-1}),
\]

where the sum is over the three partitions \( \nu = [1^{i-1}, 2, n - 2 - i], \nu_2 = [1^i, n - 1 - i], \nu_3 = [1^{i-2}, 2, n - 1 - i] \). Now applying Proposition 2.1 in (2.11), we obtain

\[
\chi_{\mu_i}(\sigma_{n-1}) = \Theta_{\lambda^i,E_{n-1}}(\sigma_{n-1}) = (-1)^{i}, \quad i = 1, \ldots, n - 3.
\]

As \( |Z_{S_n}(\sigma_{n-1})| = n - 1 \), by applying (2.7) we get that \( \chi(\sigma_{n-1}) = 0 \) unless \( \chi = \chi_{\mu_i} \), or \( \chi = 1 \), or \( \chi = \text{sgn} \). Hence the proof.

Proof of Theorem 2.1

Proof. The proof follows by using (2.7), as soon as we have enumerated \( h = |ZW(c)| \) irreducible characters of \( W \) which take value 1 or \(-1\) at the Coxeter class \( c \). The case of \( W(A_n) \) is done in Proposition 2.1. We now do this for the Weyl groups of type \( B_n \) and \( D_n \).

(1) The case of \( W(B_n) \). We know that \( W_n = W(B_n) = \{ \pm 1 \}^n \rtimes S_n \) and its Coxeter element is \( c = (1, \ldots, 1, -1) \cdot \sigma_n \). The Coxeter number is \( h = 2n \). The irreducible characters \( \chi \) of \( W_n \) are in one-to-one correspondence with the ordered pairs \((\chi_1, \chi_2)\) of irreducible characters of \( S_k \), \( S_l (k + l = n) \). The correspondence is defined as follows. The subgroup of \( W_n \) consisting of all permutations in \( W_n \) which map \( \{1, 2, \ldots, k, k', \ldots, 2', 1'\} \) into itself and hence also map \( \{k + 1, \ldots, n, n', \ldots, (k + 1)\} \) into itself can be regarded in a natural way as a product \( W_k \times W_l \). The characters \( \chi_1, \chi_2 \) of \( S_k, S_l \) can be regarded as characters \( \bar{\chi}_1, \bar{\chi}_2 \) of \( W_k, W_l \) via the projections \( W_k \to S_k, W_l \to S_l \). Consider the character \( \bar{\chi}_1 \otimes (\varepsilon | W_l \otimes \bar{\chi}_2) \) of \( W_k \times W_l \) where \( \varepsilon \) is as defined in 2.1. We induce it to \( W_n \); the resulting character is irreducible. It is the character corresponding to the ordered pair \((\chi_1, \chi_2)\). Now the irreducible character \( \chi_1 \) of \( S_k \) corresponds to a partition \( \lambda \) of \( k \), and the irreducible character \( \chi_2 \) of \( S_l \) corresponds to a partition \( \mu \) of \( l \). Therefore the irreducible characters \( \chi \) of \( W_n \) are in one-to-one correspondence with the ordered partition \((\lambda, \mu)\) of \( n \).

Let \( (\lambda, 0) \) be an ordered partition of \( n \). Then the corresponding irreducible characters of \( W(B_n) \) are \( \bar{\chi}_{\lambda} \), where \( \bar{\chi}_{\lambda} \) is the irreducible character of \( S_n \) corresponding to partition \( \lambda \) of \( n \). Take \( \lambda = \lambda_i = [1^i, n - i], (i = 0, \ldots, n - 1) \). Applying Proposition 2.1, we get

\[
\bar{\chi}_{\lambda_i}(\varepsilon) = \chi_{\lambda_i}(\sigma_n) = (-1)^i, \quad i = 0, \ldots, n - 1.
\]

Now consider the ordered partitions \((0, \lambda)\) with \( \lambda = \lambda_i = [1^i, n - i], (i = 0, \ldots, n - 1) \). Then \( \varepsilon \otimes \bar{\chi}_{\lambda_i} \) are the irreducible characters of \( W_n \). Therefore

\[
\varepsilon \otimes \bar{\chi}_{\lambda_i}(\varepsilon) = (-1)^i \cdot \chi_{\lambda_i}(\sigma_n) = (-1)^{i+1}, \quad i = 0, \ldots, n - 1.
\]

Thus we have enumerated \( 2n \) characters of \( W_n \)

\[
\{ \bar{\chi}_{\lambda_i}, \varepsilon \otimes \bar{\chi}_{\lambda_i} \mid i = 0, \ldots, n - 1 \}
\]
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with character values 1 or −1 at the Coxeter element. As the Coxeter number is $2n$, using (2.7) we get that these are all the irreducible characters with non-zero value at the Coxeter element $c$.

(2) The case of $W(D_n)$. We know that $\tilde{W}_n = W(D_n)$. The Coxeter element is $c = (1, \ldots, 1, -1, -1) \cdot \sigma_{n-1}$. The Coxeter number is $h = 2(n-1)$. It is easy to see that the irreducible characters of $W_n = W(B_n)$ corresponding to the ordered partition $(\lambda, \mu)$ of $n$ remain irreducible when restricted to $\tilde{W}_n$ except when $\lambda = \mu$, and the restriction of characters corresponding to $(\lambda, \mu)$ and $(\mu, \lambda)$ are the same. Thus the irreducible characters of $\tilde{W}_n$ are in one-to-one correspondence with the unordered partition $(\lambda, \mu)$ of $n$ except when $\lambda = \mu$. When $\lambda \neq \mu$, we call the corresponding irreducible character of $\tilde{W}_n$ non-degenerate. The irreducible character of $W_n$ corresponding to the partition $(\lambda, \lambda)$ decomposes into two distinct irreducible components when restricted to $\tilde{W}_n$. We call these characters degenerate.

Let $\mu_i = [1^{i-1}, 2, n-1-i], (i = 1, \ldots, n-3)$ be a partition of $n$. Let $\chi_{\mu_i}$ be the corresponding irreducible character of $S_n$. Then the irreducible character of $W_n$ corresponding to the ordered partition $(\mu_i, 0)$ is $\tilde{\chi}_{\mu_i}$, whose restriction to $\tilde{W}_n$ is irreducible. By Lemma 2.2 we get that

$$\tilde{\chi}_{\mu_i}(c) = \chi_{\mu_i}(\sigma_{n-1}) = (-1)^i, \quad i = 1, \ldots, n-3.$$  

(2.16)

Now let us take the unordered partitions $(\mu_0, 0)$ and $(\mu_{n-2}, 0)$, where $\mu_0 = [n]$ and $\mu_{n-2} = [1^n]$. Then the corresponding irreducible characters of $\tilde{W}_n$ are $\tilde{\chi}_{\mu_0}$ and $\tilde{\chi}_{\mu_{n-2}}$ respectively. We have,

$$_{} \tilde{\chi}_{\mu_0}(c) = \chi_{\mu_0}(\sigma_{n-1}) = 1,$$


$$_{} \tilde{\chi}_{\mu_{n-2}}(c) = \chi_{\mu_{n-2}}(\sigma_{n-1}) = (-1)^{n-2}. \right \}$$

(2.17)

Therefore we obtain $n-1$ irreducible characters of $\tilde{W}_n$

$$\{ \tilde{\chi}_{\mu_i} \mid i = 0, \ldots, n-2 \}$$

(2.18)

with their character values at the Coxeter element 1 or −1.

Let $\Theta_{\lambda'} \cdot E_{n-1}$ be the character corresponding to the $i$th exterior power representation of the $(n-2)$-dimensional reflection representation of $S_{n-1}$. Then $\Theta_{\lambda'} \cdot E_{n-1}$ corresponds to the partition $\lambda' = [1^i, n-1-i]$ of $n-1$. Let $\chi_{(\lambda', 1)}$ be the irreducible character of $W_n$ corresponding to the ordered partition $(\lambda', 1)$. That is,

$$\chi_{(\lambda', 1)} = \text{Ind}_{W_n \cdot W_{n-1}}^{W_n} (\tilde{\chi}_{\lambda'} \otimes \varepsilon | W_i).$$

(2.19)

Let $n > 2$. Then $\lambda' \neq 1$, and therefore the restriction of $\chi_{(\lambda', 1)}$ to $\tilde{W}_n$ is irreducible. Now using Proposition 2.1 we get that

$$\chi_{(\lambda', 1)}(c) = (-1)^i \cdot \varepsilon | W_i (-1) = (-1)^{i+1}, \quad i = 0, \ldots, n-2.$$  

(2.20)

We have enumerated $n-1$ more irreducible characters

$$\{ \chi_{(\lambda', 1)} \mid i = 0, \ldots, n-2 \}$$

(2.21)

of $\tilde{W}_n$ with values at the Coxeter element 1 or −1. Thus we have obtained $2(n-1)$ irreducible characters of $\tilde{W}_n$ with values 1 or −1 at the Coxeter element $c$. As the Coxeter number is $2(n-1)$, by applying (2.7) we conclude that these are all the irreducible characters with non-zero value at the Coxeter element $c$. Hence the proof.
Remark 2.1. We have been informed by Professor T. A. Springer that the Theorem 2.1 has been known for all Weyl groups by looking at their character tables.

3. Symbols and unipotent representations

In this section we introduce the formalism of symbols due to [L2] which gives a simple combinatorial parameterization of all unipotent representations of classical groups. A symbol is an unordered pair $\Lambda = (S,T)$ of finite subsets (including the empty set $\emptyset$) of \{0, 1, 2, \ldots \}. The rank of $\Lambda$ is defined by

$$\text{rk}(\Lambda) = \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left( \left( \frac{a + b - 1}{2} \right)^2 \right),$$

(3.1)

where $a = |S|, b = |T|$, and for any real number $z$ we denote by $[z]$ the largest integer $m$ such that $m \leq z$. The defect of $\Lambda$ is defined by $\text{def}(\Lambda) = |a - b|$. There is an equivalence relation on such pairs generated by the shift

$$\binom{S}{T} \sim \binom{\{0\} \sqcup (S + 1)}{\{0\} \sqcup (T + 1)}.$$

We shall identify a symbol with its equivalence class. The function $\text{rk}(\lambda)$ and $\text{def}(\lambda)$ are invariant under the shift operation, hence are well-defined on the set of symbol classes. A symbol $\Lambda = (S,T)$ is said to be reduced if $0 \notin S \cap T$; it is called degenerate if $S = T$, and non-degenerate if $S \neq T$. The entries appearing in exactly one row of $\Lambda$ are called singles. Now we shall define special symbols in the sense of [L4] and [L5].

Let $\Lambda = \left( \begin{array}{c} z_0, z_2, \ldots, z_{2m} \\ z_1, z_3, \ldots, z_{2m-1} \end{array} \right)$ be a symbol of rank $n$ and defect 1. Let

$$z_0 \leq z_1 \leq z_2 \leq \cdots \leq z_m \leq \cdots \leq z_{2m-1} \leq z_{2m}$$

(3.3)

are satisfied. It is easy to see that in this case the number of singles is odd.

Let $\Lambda = \left( \begin{array}{c} z_2, z_4, \ldots, z_{2m} \\ z_1, z_3, \ldots, z_{2m-1} \end{array} \right)$ be a symbol of rank $n$ and defect 0. It is so arranged that $0 \leq z_1 < z_3 < \cdots < z_{2m-1}$, and $0 \leq z_2 < z_4 < \cdots < z_{2m}$. A non-degenerate symbol $\Lambda$ is special if and only if

$$\begin{cases} z_1 \leq z_4 \leq z_3 \leq \cdots \leq z_{2m}, & \text{or}, \\ z_2 \leq z_1 \leq z_4 \leq \cdots \leq z_{2m-1}. \end{cases}$$

(3.5)

It is easy to see that in this case the number of singles is even.

We know that the irreducible characters of $W(B_n)$ are in one-to-one correspondence with the ordered partitions $(\alpha, \beta)$ of $n$. Let $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_{m'}]$, $\beta = [\beta_1, \beta_2, \ldots, \beta_{m''}]$ with $0 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{m'}$ and $0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_{m''}$. Since $m'$ and $m''$ can be increased at our will (by adding zeroes) we may assume that
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Let \( m' = m'' + 1 \). We now set \( \lambda_i = \alpha_i + i - 1, (1 \leq i \leq m'), \mu_i = \beta_i + i - 1, (1 \leq i \leq m'') \).

Let \( S = \{\lambda_1, \ldots, \lambda_{m'}\}, T = \{\mu_1, \ldots, \mu_{m''}\} \). Therefore for each irreducible character \( \chi \) of \( W(B_n) \) corresponding to an ordered partition \((\alpha, \beta)\) we can associate a symbol \( \Lambda = \left(\frac{S}{T}\right) \) given as above. We know that the non-degenerate irreducible characters of \( W(D_n) \) are in one-to-one correspondence with the pair of unordered partitions \((\alpha, \beta)\).

In this case we set \( m' = m'' \), and define \( \Lambda \) as above.

Let \( \Phi_{n,d} (d > 0) \) be the set of symbol classes of rank \( n \) and defect \( d \), and \( \Phi_{n,0} \) be the set of symbol classes of rank \( n \) and defect \( 0 \) with each degenerate symbol repeated twice. The following lemma is due to \([L2, 2.7]\).

**Lemma 3.1.** The above map defines a one-to-one correspondence between the irreducible characters of \( W(B_n) \) (resp. \( W(D_n) \)) and the set \( \Phi_{n,1} \) (resp. \( \Phi_{n,0} \)).

For any integer \( m \geq -1 \), we denote the set \( \{0, 1, \ldots, m\} \) by \([0, m]\). Thus \([0, -1]\) is the empty set. The following proposition is due to \([L2, 3.2]\).

**Proposition 3.1.** Let \( d \) be an integer \( \geq 1 \). The correspondence

\[
\Lambda = \left(\frac{S}{T}\right) \longrightarrow \bar{\Lambda} = \left(\frac{[0, d - 2] \cup (S + d - 1)}{T}\right)
\]

(3.6)

(where \(|S| = b + 1, |T| = b\)) defines a bijection

\[
j: \Phi_{n,1} \longleftrightarrow \Phi_{n',d}
\]

where \( n' = n + \left\lceil \frac{d}{2} \right\rceil \).

(3.7)

Let

\[
\Phi_n = \bigcup_{d \equiv 1 \pmod{2}} \Phi_{n,d},
\]

\[
\Phi_n^+ = \left\{ \bigcup_{d \equiv 0 \pmod{4}} \Phi_{n,d} \bigg| \bigcup_{d > 0} \Phi_{n,0} \right\}
\]

(3.8)

The following theorem due to \([L2, 8.2]\) gives a bijection between symbols and all the unipotent representations of \( G(F_q) \) when \( G \) is of type \( B_n, C_n \) or \( D_n \).

**Theorem 3.1.** (a) Let \( G \) be of type \( B_n, \) or \( C_n \) \((n \geq 1)\). There is a natural one-to-one correspondence \( \Lambda \leftrightarrow \rho[\Lambda] \) between the set \( \Phi_n \) and the set of isomorphism classes of unipotent representations of \( G(F_q) \), extending the correspondence between \( \Phi_{n,1} \) and the unipotent representations appearing in the principal series \( \text{Ind}_{B(F_q)}^{G(F_q)}(1) \) given by Lemma 3.1.

(b) Let \( G \) be of type \( D_n \) \((n \geq 2)\). There is a natural one-to-one correspondence \( \Lambda \leftrightarrow \rho[\Lambda] \) between the set \( \Phi_n^+ \) and the set of isomorphism classes of unipotent representations of \( G(F_q) \), extending the correspondence between \( \Phi_{n,0} \) and the unipotent representations appearing in the principal series \( \text{Ind}_{B(F_q)}^{G(F_q)}(1) \) given by Lemma 3.1.

The correspondence \( \Lambda \leftrightarrow \rho[\Lambda] \) is such that \( \rho[\Lambda] \) is cuspidal if and only if

\[
n = \text{rk}(\Lambda) = \left\lfloor \left(\frac{\text{def}(\Lambda)}{2}\right)^2 \right\rfloor.
\]

(3.9)
4. The Decomposition of $R[\chi]$

Let $\chi$ be an irreducible character of the Weyl group $W$ of $G$. We associate a class function $R[\chi]$ of $G(\mathbb{F}_q)$ as follows

$$R[\chi] = \frac{1}{|W|} \sum_{w \in W} \chi(w) R^G_{T^w}(1). \quad (4\cdot1)$$

This is a $\mathbb{Q}$-linear combination of unipotent representations of $G(\mathbb{F}_q)$. The decomposition of $R[\chi]$ is an important and difficult question in general which has been studied extensively by Lusztig ([L6]). In this section we decompose $R[\chi]$ for those $\chi$ which are non-zero at the Coxeter class. This does not seem to be explicitly available in the literature but follows easily from ([L4, L5]).

It follows from the orthogonality of $R^G_{T^w}(1)$ [DL, 6-8] that

$$\{ R[\chi] | \chi \in \hat{W} \}$$

forms an orthonormal set of class functions of $G(\mathbb{F}_q)$. We have the following inversion relation

$$R^G_{T^w}(1) = \sum_{\chi \in \hat{W}} \chi(w) R[\chi]. \quad (4\cdot3)$$

Let $E$ be the reflection representation of $W$, then $\bigwedge^n E = \text{sgn}$, where $n$ is the semisimple rank of $G$. By [DL, 7-14] we know that,

$$R[\text{Id}] = \text{Id}, \quad R[\bigwedge^n E] = R[\text{sgn}] = \text{St}, \quad (4\cdot4)$$

where St denotes the ‘Steinberg’ representation of $G(\mathbb{F}_q)$.

We know that the principal series unipotent representations of $G(\mathbb{F}_q)$ are in bijective correspondence with the irreducible representations of its Weyl group $W$. The theorem below says that in the case of the groups of type $A_n$ the $R[\chi]$’s are exactly these representations.

**Theorem 4.1.** Let $G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q)$, and let $\pi[\chi]$ be the irreducible principal series unipotent representation of $G(\mathbb{F}_q)$ corresponding to an irreducible character $\chi$ of $W = S_{n+1}$. Then $R[\chi] = \pi[\chi]$.

**Proof.** By [A, 2-3-1] it follows that

$$\langle \pi[\chi], R^G_{T^w}(1) \rangle = \chi(w).$$

Let $\chi, \chi' \in \hat{W}$. Then using (4.1) we get

$$\langle R[\chi], \pi[\chi'] \rangle = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise}. \end{cases} \quad (4\cdot5)$$

Since $R[\chi]$ is a $\mathbb{Q}$-linear combination of unipotent representations of $G(\mathbb{F}_q)$, we have

$$R[\chi] = \sum_{\rho} a_{\rho} \rho, \quad (4\cdot6)$$

where $a_{\rho}$s are rational numbers and the sum is over the set of all unipotent rep-
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sentations of \( G(\mathbb{F}_q) \). By (4.5), (4.6) and (4.2), we obtain

\[
\begin{align*}
\sum_{\rho \neq \pi[\chi]} a^\rho_{\pi[\chi]} &= 1, \\
\sum_{\rho \neq \pi[\chi]} a^\rho_{\pi[\chi]} &= \langle R[\chi] - \pi[\chi], R[\chi] - \pi[\chi] \rangle \quad \text{(4.7)}
\end{align*}
\]

Therefore, we have \( a_\rho = 0 \) for all \( \rho \neq \pi[\chi] \). Thus we have \( R[\chi] = \pi[\chi] \) for all \( \chi \in \hat{W} \). We remark here that this proves a result in \([LS]\) that \( R[\chi] \) are all the unipotent representations of groups of type \( A_n \). Hence the proof.

When \( G(\mathbb{F}_q) \) is a group of type other than \( A_n \) then the \( R[\chi] \) are more complicated. They need not be irreducible representations, nor even integral linear combination of irreducible representations. We shall briefly describe the method of decomposition of \( R[\chi] \) as outlined in \([L4]\) and \([L5]\). We shall first discuss the case of \( B_n, C_n \). In the sequel we shall denote the irreducible characters of the Weyl groups by their corresponding symbol classes.

Let \( Z \) be a special symbol of rank \( n \), and let \( Z_1 \) be the set of singles of \( Z \). Define \( d \) by \( 2d + 1 = |Z_1| \). We can write \( Z_1 = Z_1^* \cup (Z_1)_s \), where \( Z_1^* \) is the set of entries of \( Z_1 \) appearing in the first row of \( Z \), and \((Z_1)_s\) is the set of entries of \( Z_1 \) appearing in the second row of \( Z \). We have \( |Z_1^*| = d + 1, |(Z_1)_s| = d \). Let \( Z_2 \) be the set of elements which appear in both rows of \( Z \). Thus,

\[
Z = \left( \begin{array}{c}
Z_1^* \\
Z_2 \\
\end{array} \right).
\]

Let

\[
\mathcal{F} = \left\{ \Lambda_M = \left( \begin{array}{c}
Z_1^* \\
Z_2 \\
\end{array} \right) \bigm| M \subseteq Z_1, |M| \equiv d \pmod{2} \right\}.
\]

Clearly \( |\mathcal{F}| = 2^{2d} \). Associating \( M \) to the set \( M^\# = (M \cup (Z_1)_s) - (M \cap (Z_1)_s) \), defines a bijection between \( \mathcal{F} \) and the set \( V_{Z_1} \) of subsets of \( Z_1 \) of even cardinality. The set \( V_{Z_1} \) has a natural structure of \( \mathbb{F}_2 \)-vector space of dimension \( 2d \), defined by

\[
\langle M^\#, M''^\# \rangle = |M^\# \cap M''^\#| \pmod{2}.
\]

We can regard this also as a symplectic form on \( \mathcal{F} \), via the bijection between \( \mathcal{F} \) and \( V_{Z_1} \). Let

\[
\mathcal{F} = \left\{ \rho[\Lambda_M] \right\}.
\]

Then \( \{ \mathcal{F} \mid |Z \text{ special of rank } n \} \) defines a partition of the set of irreducible unipotent representations of \( G(\mathbb{F}_q) \) into disjoint families. The following theorem is due to \([L4, 5-8]\).

**Theorem 4.2.** Let \( G = Sp_{2n} \), or \( SO_{2n+1} \) (defined over \( \mathbb{F}_q \)). Let \( Z \) be a special symbol of rank \( n \), and let \( d \) be such that \( 2d + 1 \) is the number of singles of \( Z \). Then for any \( \Lambda \in \mathcal{F} \).
of defect one, we have

\[ R[\Lambda] = 2^{-d} \sum_{\Lambda' \in \mathcal{F}_Z} (-1)^{\langle \Lambda, \Lambda' \rangle} \rho[\Lambda']. \] (4.12)

We now consider the case of symbols for \( W(D_n) \). Let \( Z \) be a special symbol of rank \( n \), and let \( Z_1 \) be the set of singles of \( Z \). Define \( d \) by \( 2d = |Z_1| \). We define \( Z_1^* \), \((Z_1)_*\), \( Z_2 \) and \( \mathcal{F}_Z \) as before. Also we have \(|(Z_1)_*| = |Z_1^*| = d\).

\[ \mathcal{F}_Z = \left\{ \Lambda_M = \left( \frac{Z_2 \cup (Z_1 - M)}{Z_2 \cup M} \right) \left| M \subseteq Z_1, |M| \equiv d \mod 2 \right. \right\}. \] (4.13)

Clearly \(|\mathcal{F}_Z| = 2^{2(d-1)}\). Associating \( M \) to the set \( M^* = (M \cup (Z_1)_*) - (M \cap (Z_1)_*) \), defines a bijection between \( \mathcal{F}_Z \) and the set \( V_{Z_1} \) of \( \mathbb{F}_q \)-vector spaces of dimension \( 2(d - 1) \). As before, we can endow \( \mathcal{F}_Z \) with a non-degenerate symplectic form. Let

\[ \mathcal{F}_Z = \left\{ \rho[\Lambda_M] \left| \Lambda_M \in \mathcal{F}_Z \right. \right\}. \] (4.14)

Then \( \{\mathcal{F}_Z \mid Z \text{ special of rank } n\} \) defines a partition of the set of irreducible unipotent representations of \( G(\mathbb{F}_q) \) into disjoint families. The following theorem is also due to [L5, 3:15].

**Theorem 4.3.** Let \( G = SO_{2n} \) (defined over \( \mathbb{F}_q \)). Let \( Z \) be a special symbol of rank \( n \), and let \( d \) be such that \( 2d \) is the number of singles of \( Z \). Then for any \( \Lambda \in \mathcal{F}_Z \) of defect zero, we have

\[ R[\Lambda] = 2^{-2(d-1)} \sum_{\Lambda' \in \mathcal{F}_Z} (-1)^{\langle \Lambda, \Lambda' \rangle} \rho[\Lambda']. \] (4.15)

Theorems 4.2 and 4.3 were initially proved by Lusztig for large \( q \), but later on the restriction on the order of \( q \) was removed by him in [L6].

Let \( \Lambda \) be the symbol corresponding to the irreducible character \( \chi \) of \( W \). As we are interested in decomposing \( R[\chi] \) for those \( \chi \in \hat{W} \) which are non-zero at the Coxeter class, we first determine the corresponding symbol classes.

From the proof of Theorem 2.1 we know that the symbols corresponding to the irreducible characters of \( W(B_n) \) which are non-zero at the Coxeter element are

\[
\begin{align*}
\Lambda_0 &= \begin{pmatrix} n \\ 0 \end{pmatrix}, & \Lambda_i &= \begin{pmatrix} 1, 2, \ldots, i, n \\ 0, 1, \ldots, i-1 \end{pmatrix} & (1 \leq i \leq n-1), \\
\Lambda_0' &= \begin{pmatrix} 0, 1 \\ n \end{pmatrix}, & \Lambda_i' &= \begin{pmatrix} 0, 1, \ldots, i+1 \\ 1, 2, \ldots, i, n \end{pmatrix} & (1 \leq i \leq n-1),
\end{align*}
\] (4.16)

where \( \Lambda_0 \) corresponds to the ordered partition \((n, 0)\), \( \Lambda_i \) corresponds to the ordered partition \((\lambda_i, 0)\) where \( \lambda_i = [1^i, n-i] \), \( (1 \leq i \leq n-1) \), while \( \Lambda_0' \) corresponds to the ordered partition \((0, n)\) and \( \Lambda_i' \) corresponds to the ordered partition \((0, \lambda_i)\). Therefore, we have

\[
\begin{align*}
\Lambda_0(c) &= 1, & \Lambda_i(c) &= (-1)^i & (1 \leq i \leq n-1), \\
\Lambda_0'(c) &= -1, & \Lambda_i'(c) &= (-1)^{i+1} & (1 \leq i \leq n-1).
\end{align*}
\] (4.17)

Similarly, the symbol classes corresponding to the irreducible characters of \( W(D_n) \)
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which are non-zero at the Coxeter element are

\[
\begin{align*}
\Lambda_0 &= \binom{n}{0}, & \Lambda_{n-2} &= \binom{1, 2, \ldots, n}{0, 1, \ldots, n-1}, \\
\Lambda_i &= \binom{1, 2, \ldots, i-1, i+1, n-1}{0, 1, \ldots, i}, & (1 \leq i \leq n-3),
\end{align*}
\]

(4.18)

and

\[
\begin{align*}
\Lambda'_0 &= \binom{n-1}{1}, & \Lambda'_i &= \binom{1, 2, \ldots, i, n-1}{0, 1, \ldots, i-1, i+1}, & (1 \leq i \leq n-2),
\end{align*}
\]

(4.19)

where \(\Lambda_0\) corresponds to the unordered partition \((n, 0)\), \(\Lambda_{n-2}\) corresponds to the unordered partition \((1^n, 0)\) and \(\Lambda_i\) corresponds to the unordered partition \((\mu_i, 0)\) where \(\mu_i = [1^{i-1}, 2, n-1-i]\), \((1 \leq i \leq n-3)\), while \(\Lambda'_0\) corresponds to the unordered partition \((n-1, 1)\) and \(\Lambda'_i\) corresponds to the unordered partition \((\lambda'_i, 1)\) where \(\lambda'_i = [1^i, n-1-i]\), \((1 \leq i \leq n-2)\). Therefore, we have

\[
\begin{align*}
\Lambda_0(c) &= 1, & \Lambda_{n-2}(c) &= (-1)^{n-2}, & \Lambda_i(c) &= (-1)^i \quad (1 \leq i \leq n-3), \\
\Lambda'_0(c) &= -1, & \Lambda'_i(c) &= (-1)^{i+1} \quad (1 \leq i \leq n-2).
\end{align*}
\]

(4.20)

We also need the symbol classes corresponding to the exterior power representation of the reflection representation of \(W(B_n)\) and \(W(D_n)\). We know that for \(W(B_n)\), \(\wedge^i E\) is given by the ordered partition \((n-i, 1^i)\). The corresponding symbol classes are

\[
\Gamma_{n,0} = \Lambda_0 = \binom{n}{0}, & \quad \Gamma_{n,i} = \binom{0, 1, \ldots, i-1, n}{1, 2, \ldots, i} \quad (1 \leq i \leq n).
\]

(4.21)

Similarly, for \(W(D_n)\), \(\wedge^i E\) is given by the unordered partition \((n-i, 1^i)\). The corresponding symbol classes are

\[
\begin{align*}
\Gamma'_{n,0} &= \Lambda_0 = \binom{n}{0}, & \Gamma'_{n,1} &= \Lambda'_0 = \binom{n-1}{1}, \\
\Gamma'_{n,i} &= \binom{0, 1, \ldots, i-1, n-1}{1, 2, \ldots, i} \quad (2 \leq i \leq n).
\end{align*}
\]

(4.22)

It is easy to see that \(\Gamma_{n,i} (0 \leq i \leq n)\) and \(\Gamma'_{n,i} (0 \leq i \leq n)\) are special symbols. Apart from these symbols, we also need some symbols of rank \(n\) not corresponding to the representations of \(W(B_n), W(D_n)\). Define \(X_i\) for \(B_n\) and \(C_n\), as

\[
X_i = \binom{0, 1, \ldots, i+1, n}{1, 2, \ldots, i}, \quad (0 \leq i \leq n-2).
\]

(4.23)

Define the symbols \(Y_i\) for \(D_n\) as

\[
Y_i = \binom{0, 1, \ldots, i+2, n-1}{1, 2, \ldots, i}, \quad (0 \leq i \leq n-4).
\]

(4.24)

Let \(\rho[X_i]\) be the unipotent representation of \(Sp(2n, \mathbb{F}_q)\) or \(SO(2n + 1, \mathbb{F}_q)\) corresponding to the symbol \(X_i\), and let \(\rho[Y_i]\) be the unipotent representation of \(SO(2n, \mathbb{F}_q)\) corresponding to the symbol \(Y_i\). The following theorem gives the decomposition of \(R[\Lambda]\) for those \(\Lambda\) whose character at the Coxeter class is non-zero.

Recall (see the proof of Theorem 2.1) that in the case of \(W(B_n)\) the symbol \(\Lambda_i\) corresponds to the irreducible representation obtained by composing the \(\wedge^i E_n\) with
the homomorphism $W(B_n) \to S_n$, $\Lambda'_i$ corresponds to irreducible representation obtained by tensoring $\wedge^i E_n$ with $\varepsilon$, where $\varepsilon$ is as defined in (2.1), and $E_n$ denotes the $(n-1)$-dimensional reflection representation of $S_n$. In the case of $W(D_n)$ the symbol $\Lambda_i (1 \leq i \leq n-3)$ corresponds to the irreducible character $\chi_\lambda$, as defined in (2.16), and $\Lambda_0, \Lambda_{n-2}$ correspond to the irreducible characters $\chi_{\lambda_0, \lambda_{n-2}}$ respectively, as defined in (2.17), and $\Lambda'_i$ corresponds to the irreducible character $\chi_{(\lambda', 1)}$ as defined in (2.19).

Also recall that the symbol $\Gamma_{n,i}$ corresponds to the $i$th exterior power representation of the reflection representation of $W(B_{n-2})$, and the symbol $\Gamma_{n,i}$ corresponds to the $i$th exterior power representation of the reflection representation of $W(D_n)$.

**Theorem 4.4.** Let $G = Sp_{2n}, SO_{2n+1},$ or $SO_{2n}$ defined over $\mathbb{F}_q$. We denote by $\pi[\Lambda]$ the principal series representation of $G(\mathbb{F}_q)$ corresponding to the Weyl group representation denoted by the symbol $\Lambda$. With the notations as above, we have the following.

1. Let $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$, or $SO(2n+1, \mathbb{F}_q)$, $(n \geq 2)$.

   (a) $R[\Lambda_0] = \pi[\Lambda_0] = Id.$

   (b) $R[\Lambda_i] = \frac{1}{2} \left( \pi[\Lambda_i] + \pi[\wedge^i E] - \pi[\Lambda'_{i-1}] - \rho[X_{i-1}] \right), \quad (1 \leq i \leq n - 1). (4.25)$

   (c) $R[\Lambda'_i] = \frac{1}{2} \left( -\pi[\Lambda_{i+1}] + \pi[\wedge^{i+1} E] + \pi[\Lambda'_i] - \rho[X_i] \right), \quad (0 \leq i \leq n - 2).$

   (d) $R[\Lambda'_{n-1}] = \pi[\wedge^n E] = St.$

2. Let $G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q)$, $(n \geq 4)$.

   (a) $R[\Lambda_0] = \pi[\Lambda_0] = 1.$

   (b) $R[\Lambda_{n-2}] = \pi[\wedge^n E] = St.$

   (c) $R[\Lambda_i] = \frac{1}{2} \left( \pi[\Lambda_i] - \pi[\wedge^{i+1} E] + \pi[\Lambda'_i] - \rho[Y_{i-1}] \right), \quad (1 \leq i \leq n - 3). (4.26)$

   (d) $R[\Lambda'_i] = \frac{1}{2} \left( \pi[\Lambda_i] + \pi[\wedge^{i+1} E] + \pi[\Lambda'_i] + \rho[Y_{i-1}] \right), \quad (1 \leq i \leq n - 3).$

   (e) $R[\Lambda'_{n-2}] = \pi[\wedge^{n-1} E].$

   (f) $R[\Lambda'_{n-1}] = \pi[\wedge^{n-1} E].$

**Proof.** (1) $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$, or $SO(2n+1, \mathbb{F}_q)$.

(a) Since $\Lambda_0 = \Gamma_0, R[\Lambda_0] = R[\Gamma_0] = R[1d] = 1d.$

(b) We have the following data for $\Gamma_{n,i}$ which is a special symbol of rank $n$

$$Z_1 = \{0, i, n\}, \quad Z_2 = \{1, 2, \ldots, i - 1\}, \quad Z^*_i = \{0, n\}, \quad (Z_1)_* = \{i\}, \quad d = 1. \quad (4.27)$$

It can be seen that,

$$\mathcal{S}_{\Gamma_{n,i}} = \{\Lambda_i, \Gamma_{n,i}, \Lambda'_{i-1}, X_{i-1}\}. \quad (4.28)$$

Applying Theorem 4.2, we obtain

\begin{equation}
R[\Lambda_i] = \frac{1}{2} \sum_{\Lambda' \in \mathcal{S}_{\Gamma_{n,i}}} (-1)^{\langle\Lambda, \Lambda'\rangle} \rho[\Lambda'] \quad (4.29)
\end{equation}

\begin{equation}
= \frac{1}{2} \left( \pi[\Lambda_i] + \pi[\wedge^i E] - \pi[\Lambda'_{i-1}] - \rho[X_{i-1}] \right). \quad (4.29)
\end{equation}
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(c) Similarly, applying Theorem 4.2, we obtain

\[ R[\Lambda'_i] = \frac{1}{2} \sum_{\Lambda' \in \mathcal{G}_{n+1}} (-1)^{(|\Lambda|,|\Lambda'|)} \rho[\Lambda'] \]

\[ = \frac{1}{2} \left( \pi[\Lambda'_i] + \pi[\wedge^{i+1}E] - \pi[\Lambda_{n+1}] - \rho[X_i] \right) \]  

(4.30)

(d) Since \( \Lambda'_{n-1} = \Gamma_{n,n} \), \( R[\Lambda'_{n-1}] = R[\Gamma_{n,n}] = R[\wedge^nE] = \pi[\wedge^nE] = \text{St} \).

(2) \( G(\mathbb{F}_q) = SO(2n,\mathbb{F}_q) \).

(a) Since \( \Lambda_0 = \Gamma_{n,0} \), \( R[\Lambda_0] = R[\Gamma_{n,0}] = R[\text{Id}] = \text{Id} \).

(b) Since \( \Lambda_{n-2} = \Gamma_{n,n-2} \), \( R[\Lambda_{n-2}] = R[\Gamma_{n,n-2}] = R[\wedge^nE] = \text{St} \).

(c) We have the following data for \( \Gamma'_{n,i+1} \) which is a special symbol of rank \( n \)

\[
Z_i = \{0, i, i + 1, n - 1\}, \quad Z_2 = \{1, 2, \ldots, i - 1\}, \\
Z'_i = \{0, n - 1\}, \quad (Z_i)_s = \{i, i + 1\}, \quad d = 2.
\]

(4.31)

It can be seen that,

\[ \mathcal{G}_{\Gamma'_{n,i+1}} = \{\Lambda_i, \Gamma'_{n,i+1}, \Lambda'_i, Y_{i-1}\} \]  

(4.32)

Applying Theorem 4.3, we obtain

\[ R[\Lambda_i] = \frac{1}{2} \sum_{\Lambda' \in \mathcal{G}_{\Gamma'_{n,i+1}}} (-1)^{(|\Lambda|,|\Lambda'|)} \rho[\Lambda'] \]

\[ = \frac{1}{2} \left( \pi[\Lambda_i] - \pi[\wedge^{i+1}E] + \pi[\Lambda'_i] - \rho[Y_{i-1}] \right) \]

(4.33)

(d) Similarly, applying Theorem 4.3, we obtain

\[ R[\Lambda'_i] = \frac{1}{2} \sum_{\Lambda' \in \mathcal{G}_{\Gamma'_{n,i+1}}} (-1)^{(|\Lambda|,|\Lambda'|)} \rho[\Lambda'] \]

\[ = \frac{1}{2} \left( \pi[\Lambda'_i] - \pi[\wedge^{i+1}E] + \pi[\Lambda_{n+1}] + \rho[Y_{i-1}] \right) \]

(4.34)

(e) We have the following data for \( \Lambda'_0 = \Gamma'_{n,1} \) which is a special symbol of rank \( n \)

\[
Z_1 = \{1, n - 1\}, \quad Z_2 = 0, \quad Z'_1 = \{n - 1\}, \quad (Z_1)_s = \{1\}, \quad d = 1.
\]

(4.35)

It can be seen that,

\[ \mathcal{G}_{\Gamma'_{n,1}} = \{\Gamma'_{n,1}\} \]

(4.36)

Applying Theorem 4.3, we obtain

\[ R[\Lambda'_0] = \sum_{\Lambda' \in \mathcal{G}_{\Gamma'_{n,1}}} (-1)^{(|\Lambda|,|\Lambda'|)} \rho[\Lambda'] \]

\[ = \pi[E]. \]

(4.37)

(f) We have the following data for \( \Lambda'_{n-2} = \Gamma'_{n,n-1} \) which is a special symbol of rank \( n \)

\[
Z_1 = \{0, n - 2\}, \quad Z_2 = \{1, 2, \ldots, n - 3\}, \quad Z'_1 = \{0\}, \quad (Z_1)_s = \{n - 2\}, \quad d = 1.
\]

(4.38)
It can be seen that,
\[ \mathcal{R}_{\Gamma_{n,n-1}} = \{ \Gamma'_{n,n-1} \}. \]  
(4.39)

Applying Theorem 4.3, we obtain
\[ R[\Lambda_{n-2}] = \sum_{\Lambda' \in \mathcal{R}_{\Gamma_{n,n-1}}} (-1)^{[\Lambda' - \Lambda]} \rho[\Lambda'] \right \} \]
\[ = \pi[\Lambda^{n-1} E]. \]
(4.40)

Hence the proof.

5. The decomposition of induced representation

By Theorem 3.1 we know that \( \rho[X_i] \) is not cuspidal unless \( i = 0, n = 2 \). It occurs as an irreducible component of \( \text{Ind}_{P(F_q)}^G(\tilde{\rho}_L) \) for some proper parabolic subgroup \( P \) defined over \( F_q \) and some irreducible cuspidal representation \( \rho_L \) of \( L(F_q) \) where \( L \) is the Levi component of \( P \). By \( \tilde{\rho}_L \) we mean the lift of \( \rho_L \) to \( P(F_q) \) by extending it trivially to the unipotent radical of \( P(F_q) \). As \( \rho[X_i] \) is unipotent, we know that the representation \( \rho_L \) must also be unipotent (see [C1, 12-2]). Since \( X_i \) is of defect 3, by [L2, 8] we know that \( L \cong (G_m)^{n-2} \times Sp_4 \) or \( L \cong (G_m)^{n-2} \times SO_5 \), as the case may be. Let \( x = (x_1, x_1', x_2, x_2', \ldots, x_{n-2}, x_1' \), \( x_2' ) \in (F_q)^{2n-4} \times Sp(4, F_q) \), or \( (x_1, x_1', x_2, x_2', \ldots, x_{n-2}, x_1' \), \( x_2' ) \in (F_q)^{2n-4} \times SO(5, F_q) \), with the condition that \( x_1x_1' = \cdots = x_{n-2}x_1' = 1 \). Since \( \rho_L \) is unipotent cuspidal, and \( Sp_4 \) and \( SO_5 \) have a unique unipotent cuspidal representation \( \pi_{uc} \), we obtain
\[ \text{Tr}(x, \rho_L) = \text{Tr}(g, \pi_{uc}). \]
(5.1)

We abuse notation to denote \( \rho_L \) by \( \pi_{uc} \). By [L2, 5-15] there is a bijective correspondence between irreducible representations of \( G(F_q) \) appearing in \( \text{Ind}_{P(F_q)}^G(\tilde{\rho}_L) \) and irreducible representations of the Weyl group \( W(B_{n-2}) \) which is given by Proposition 3.1. Since \( X_1 \in \Phi_{n,3} \) corresponds to \( \Gamma_{n-2,1} \in \Phi_{n-2,1} \) and by (4.21) we know that \( \Gamma_{n-2,1} \) corresponds to \( i \)th exterior power representation of the reflection representation of \( W(B_{n-2}) \), the irreducible component \( \rho[X_1] \) of \( \text{Ind}_{P(F_q)}^G(\tilde{\rho}_L) \) corresponds to \( i \)th exterior power representation of the reflection representation of \( W(B_{n-2}) \).

The case of \( \rho[Y_1] \) is similar. Since \( Y_1 \) is of defect 4, it can be seen that \( \rho[Y_1] \) is an irreducible component of \( \text{Ind}_{P(F_q)}^G(\tilde{\rho}_L) \) corresponding to the \( i \)th exterior power representation of the reflection representation of \( W(B_{n-4}) \), where \( P \) is the parabolic subgroup with Levi component \( L \cong (G_m)^{n-4} \times SO_8 \), and \( \pi_{uc} \) is the unique unipotent cuspidal representation of \( SO(8, F_q) \). Therefore, we have

**Proposition 5.1.** Let \( X_i \) and \( Y_i \) be defined as above. The unipotent representation \( \rho[X_i] \) of \( Sp(2n, F_q) \) or \( SO(2n + 1, F_q) \) is the irreducible component of \( \text{Ind}_{P(F_q)}^G(\tilde{\rho}_L) \) corresponding to the \( i \)th exterior power representation of the reflection representation of \( W(B_{n-2}) \), where \( \pi_{uc} \) is the unique unipotent cuspidal representation of \( Sp(4, F_q) \) or \( SO(5, F_q) \), as the case may be. Similarly, the unipotent representation \( \rho[Y_i] \) of \( SO(2n, F_q) \) is the irreducible component of \( \text{Ind}_{P(F_q)}^G(\tilde{\rho}_L) \) corresponding to the \( i \)th exterior power representation of the reflection representation of \( W(B_{n-4}) \), where \( \pi_{uc} \) is the unique unipotent cuspidal representation of \( SO(8, F_q) \).
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6. Proof of the Main Theorem

Let \( G \) be a split group of semisimple rank \( n \), \( W \) its Weyl group, \( c \) the Coxeter conjugacy class, and \( T_c \) the corresponding torus. By [DL, 8-3] we know that \( \Psi_\theta = (-1)^n R_{T_c}^G(\theta) \) is an irreducible cuspidal representation of \( G(\mathbb{F}_q) \) for a character \( \theta \) of \( T_c(\mathbb{F}_q) \) in general position. By [DL, 4-2] we also know that for an unipotent element \( u \) of \( G(\mathbb{F}_q) \), \( \text{Tr}(u, R_{T_c}^G(\theta)) \) is independent of \( \theta \). Therefore,

\[
\Theta u, R_{T_c}^G(1)) = (-1)^n \text{Tr}(u, R_{T_c}^G(1)). \tag{6-1}
\]

Let \( G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q) \). Let us denote \( \pi[\wedge^i E] \) by \( \pi_i \). Applying Proposition 2-1 in (4-3), we obtain

\[
R_{T_c}^G(1) = \sum_{i=0}^{i=n} (-1)^j R[\chi_i]. \tag{6-2}
\]

Now applying Theorem 4-1, we obtain

\[
R_{T_c}^G(1) = \sum_{i=0}^{i=n} (-1)^j \pi[\wedge^i E] \left\{ \begin{array}{l}
= \sum_{i=0}^{i=n} (-1)^j \pi_i, \end{array} \right. \tag{6-3}
\]

By (6-3) and (6-1), we obtain

\[
\Theta u, R_{T_c}^G(1) = (-1)^n \sum_{i=0}^{i=n} (-1)^j \Theta \pi_i(u) \left\{ \begin{array}{l}
= \sum_{i=0}^{i=n} (-1)^j \Theta \pi_{n-i}(u). \end{array} \right. \tag{6-4}
\]

This completes the proof of Theorem 1-1 for \( GL(n+1, \mathbb{F}_q) \).

We now take up the case of \( G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q) \) or \( SO(2n+1, \mathbb{F}_q) \). Applying (4-17) in (4-3), we obtain

\[
R_{T_c}^G(1) = \sum_{i=0}^{i=n-1} (-1)^j R[\Lambda_i] + \sum_{i=0}^{i=n-1} (-1)^{i+1} R[\Lambda'_i] \left\{ \begin{array}{l}
= (R[\Lambda_0] + (-1)^n R[\Lambda'_{n-1}]) + \sum_{i=1}^{i=n-1} (-1)^j (R[\Lambda_i] + R[\Lambda'_{i-1}]). \end{array} \right. \tag{6-5}
\]

From Theorem 4-4 it follows that

\[
R[\Lambda_0] + (-1)^n R[\Lambda'_{n-1}] = \pi[\wedge^0 E] + (-1)^n \pi[\wedge^n E], \tag{6-6}
\]

\[
R[\Lambda_i] + R[\Lambda'_{i-1}] = \pi[\wedge^i E] - \rho[X_{i-1}].
\]

Let us denote \( \pi[\wedge^i E] \) by \( \pi_i \) and \( \rho[X_i] \) by \( \rho_i \). By Proposition 5-1, \( \rho_i \) is the irreducible component of \( \text{Ind}_{P(\mathbb{F}_q)}(\pi_{m_i}) \) corresponding to the \( i \)th exterior power representation of the reflection representation of \( W(B_{n-2}) \), where \( P \) is the parabolic subgroup defined over \( \mathbb{F}_q \) with the Levi component \((G_m)^{n-2} \times Sp_4 \) or \((G_m)^{n-2} \times SO_5 \), and \( \pi_{m_i} \) is the unique unipotent cuspidal representation of \( Sp(4, \mathbb{F}_q) \) or \( SO(5, \mathbb{F}_q) \), as the case may
be. Using (6.6), we obtain

\[
R^{\mathbb{G}}_{T_1} (1) = \left( \pi \wedge E \right) + (-1)^n \pi [\wedge^n E] + \sum_{i=1}^{i=n-1} (-1)^i \left( \pi [\wedge^i E] - \rho [X_{i-1}] \right)
\]

\[
= \sum_{i=0}^{i=n} (-1)^i \pi [\wedge^i E] - \sum_{i=1}^{i=n-1} (-1)^i \rho [X_{i-1}]
\]

(6.7)

\[
= \sum_{i=0}^{i=n} (-1)^i \pi_i + \sum_{i=0}^{i=n-2} (-1)^i \rho_i.
\]

This completes the proof of Theorem 1.1 for \( Sp(2n, \mathbb{F}_q) \) and \( SO(2n + 1, \mathbb{F}_q) \).

Finally, let \( G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q) \). Again denote \( \pi [\wedge^i E] \) by \( \pi_i \) and \( \rho [Y_i] \) by \( \rho_i \). By Proposition 5.1, \( \rho_i \) is the irreducible component of \( \text{Ind}_{S_3(F_q)}^{G(\mathbb{F}_q)}(\pi_w) \) corresponding to the \( i \)-th exterior power representation of the reflection representation of \( W(B_{n-4}) \), where \( P \) is the parabolic subgroup defined over \( \mathbb{F}_q \) with the Levi component \( (G_n)^n \times \text{SO}_4 \) and \( \pi_w \) is the unique unipotent cuspidal representation of \( SO(8, \mathbb{F}_q) \). By calculations similar to those done in the previous case yields the following result for \( SO(2n, \mathbb{F}_q) \).

\[
R^{\mathbb{G}}_{T_1} (1) = \sum_{i=0}^{i=n} (-1)^i \pi_i + \sum_{i=0}^{i=n-4} (-1)^i \rho_i.
\]

(6.9)

By (6.9) and (6.1), we obtain

\[
\Theta_{\mathbb{G}}(u) = \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u) + \sum_{i=0}^{i=n-4} (-1)^i \Theta_{\rho_{n-i-1}}(u).
\]

(6.10)

This completes the proof of Theorem 1.1.

**Remark 6.1.** The only property of \( R^{\mathbb{G}}_{T_1}(\theta) \) used in the proof of Theorem 1.1 in this section is that

\[
\text{Tr}(g, R^{\mathbb{G}}_{T_1}(\theta)) = \text{Tr}(g, R^{\mathbb{G}}_{T_f}(1)).
\]

This property is known for all elements \( g \in G(\mathbb{F}_q) \) with Jordan decomposition \( g = su \) such that either \( s = 1 \) or \( s \) is not conjugate in \( G(\mathbb{F}_q) \) to any element of \( T(\mathbb{F}_q) \). Therefore the character identity contained in Theorem 1.1 is true for all such elements. In particular, the identity in Theorem 1.1 is true for any element of \( (n, 1) \) parabolic in \( GL_{n+1} \).

**Remark 6.2.** In the case of \( GL(n+1, \mathbb{F}_q) \) all the irreducible cuspidal representations are given by \((-1)^n R^{\mathbb{G}}_{T_1}(\theta)\) where \( c = \sigma_{n+1} \) is the Coxeter element and \( \theta \) is a regular character of \( T_c(\mathbb{F}_q) = \mathbb{F}_q^{n+1} \). Thus the character value of any irreducible cuspidal representation of \( GL(n+1, \mathbb{F}_q) \) at any element of \( (n, 1) \) parabolic is given by (6.4).
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7. The case of exceptional groups

In this section we shall give the decomposition of \( R_{L_7}^G(1) \) in the case of split simple exceptional algebraic groups. We shall follow the notations of [C1, 13-9] for the unipotent cuspidal representations of \( G(\mathbb{F}_q) \).

The class functions \( R_\chi \) can be decomposed using the method of non-abelian Fourier transforms as introduced in [L3]. By [L3, theorem 1-5] and [L3, corollary 1-13] the multiplicities \( \langle \rho, R_\chi \rangle \) can be explicitly described, where \( \rho \) is any unipotent representation of \( G(\mathbb{F}_q) \). See [C1, 13-6] for the Fourier transform matrices.

Let \( G \) be a split exceptional simple algebraic group. Let \( (P, \phi) \) be a pair of parabolic subgroup in \( G \) containing a fixed Borel subgroup \( B \) with Levi decomposition \( P = MN \), and a unipotent cuspidal representation \( \phi \) of \( M(\mathbb{F}_q) \). The irreducible components of \( \text{Ind}_{P(PF_q)}^G(\phi) \) are in one to one correspondence with the irreducible representations of the Weyl group \( W' \) of the quotient root system which is a root system of a simple group of rank \( r(G) - r(P) \), where \( r(G) \) and \( r(P) \) denote the semisimple ranks of \( G \) and \( P \) respectively. Denote by \( \phi_i \) the irreducible components of \( \text{Ind}_{P(PF_q)}^G(\phi) \) corresponding to the \( i \)-th exterior power representation of the reflection representation of \( W' \).

Theorem 7.1. With the notations as above, (i) if \( G \) is a simple algebraic group of type \( E_6, E_7 \), then,

\[
R_{L_7}^G(1) = \sum_{(P, \phi)} (-1)^{r(P)} \sum_{i=0}^{i=r(G) - r(P)} (-1)^i \phi_i,
\]

where \( \phi \) runs over all the unipotent cuspidal representations of \( M(\mathbb{F}_q) \).

(ii) If \( G = G_2 \), then same as in (i) except that the term corresponding to \( P = G \) has instead of all the 4 unipotent cuspidal representations of \( G_2(\mathbb{F}_q) \), only 3 which can be specified as \( G_2[-1] + G_2[\theta] + G_2[\theta^2] \) following Carter’s notation [C1, 13-9].

(iii) If \( G = F_4 \), then same as in (i) except that the term corresponding to \( P = G \) has instead of all the 7 unipotent cuspidal representations of \( F_4(\mathbb{F}_q) \), only 4 which can be specified as \( F_4[\theta], F_4[\theta^2], F_4[i], F_4[-i] \) following the notations in [C1, 13-9].

(iv) If \( G = E_6 \), then same as in (i) except that the term corresponding to \( P = G \) has instead of all the 13 unipotent cuspidal representations of \( E_6(\mathbb{F}_q) \), only 6 which can be specified as \( E_6[\zeta^i](i = 1, \ldots, 4), E_6[\theta], E_6[\theta^2] \) following the notations in [C1, 13-9].

To illustrate Theorem 7.1 we take the case of \( G = E_7 \). The Levi subgroups of \( E_7 \) which have unipotent cuspidal representations are \( L_0 \cong (G_m)^7, L_1 \cong SO_8 \times (G_m)^3, L_2 \cong E_6 \times G_m \) and \( L_3 = G \).

The quotient root system arising from \( L_0 \) is the root system of type \( E_7 \), and \( \phi = 1 \) is the unique unipotent cuspidal representation of \( (\mathbb{F}_q^*)^7 \). Hence, \( \phi_i = \pi_i \) is the irreducible component of \( \text{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1) \) corresponding to the \( i \)-th exterior power representation of the reflection representation of \( W(E_7) \).

The quotient root system arising from \( L_1 \) is of type \( C_3 \). Let \( \phi = \pi_{uc} \) be the unique unipotent cuspidal representation of \( SO(8, \mathbb{F}_q) \). Let \( \phi_i[D_4] \) be the irreducible component of \( \text{Ind}_{B(\mathbb{F}_q)}^{D_4(\mathbb{F}_q)}(\phi) \) corresponding to the \( i \)-th exterior power representation of the reflection representation of \( W(C_3) \) for \( i = 0, 1, 2, 3 \).

The quotient root system arising from \( L_2 = E_6 \times G_m \) is of type \( A_1 \). Let \( \phi' = E_6[\theta] \) and \( \phi'' = E_6[\theta^2] \) be the two unipotent cuspidal representations \( E_6(\mathbb{F}_q) \). Let \( \phi'_i[E_6] \) and
\(\phi_i^c[E_\alpha]\) be the irreducible components of Ind_{P_{L_i}}^{G(F_q)}(\phi') and Ind_{P_{L_i}}^{G(F_q)}(\phi') respectively corresponding to the \(i\)-th exterior power of the reflection representation of \(W(A_n)\) for \(i = 0, 1\).

When \(L = L_\Delta = G\), we have two unipotent cuspidal representations of \(E_\gamma(\mathbb{F}_q)\) denoted by \(E_\gamma[\zeta]\) and \(E_\gamma[-\zeta]\) as in [C1, 13.9]. Then, by Theorem 7.1 we get,

\[
R_{E_i}^G(1) = \sum_{i=0}^{i=3} (-1)^i \pi_i + \sum_{i=0}^{i=3} (-1)^i \phi_i [D_4] \\
+ \sum_{i=0}^{i=3} (-1)^i \phi_i' [E_6] + \sum_{i=0}^{i=3} (-1)^i \phi_i'' [E_6] \\
- (E_\gamma[\zeta] + E_\gamma[-\zeta]).
\]

(7.2)

8. Cohomological interpretation

The virtual representation \(R_{E_i}^G(1)\) is defined in [DL] as

\[
R_{E_i}^G(1) = \sum (-1)^i H^i(X(w), \mathbb{Q})
\]

where \(X(w)\) is the Deligne-Lusztig variety as defined in [DL, 1.4]. When \(w = c\) is the Coxeter element, Lusztig ([L1]) has given the decomposition of \(H^i(X(c), \mathbb{Q})\) as irreducible \(G(\mathbb{F}_q)\)-modules. In the sequel, we will denote \(H^i(X(c), \mathbb{Q})\) by \(H^i\). When \(G\) is simple, he proves that \([L1, 6.1] \bigoplus H^i\) is a multiplicity free \(G(\mathbb{F}_q)\)-module, and it has \(h\) irreducible components, where \(h\) is the Coxeter number of \(G\).

Let \(\Delta\) be the set of simple roots of \(G\) and \(I\) be a subset of \(\Delta\). Let \(L_I\) be the Levi subgroup of the parabolic subgroup \(P_I\) corresponding to \(I\). We put \(H_I^i = H^i(X(c_I), \mathbb{Q})\), where \(c_I\) is the Coxeter element of \(W_I\), and \(X(c_I)\) is the corresponding Deligne-Lusztig variety of \(L_I\). Let \(n\) be the semisimple rank of \(G\). Let \((H^i)^{(0)}\) denote the cuspidal part of \(H^i\). The following results are due to Lusztig.

**Proposition 8.1** [L1, 2.9]. \(H^i = 0\), unless \(n \leq i \leq 2n\).

**Proposition 8.2** [L1, 4.3]. \((H^i)^{(0)} = 0\) for \(i \neq n\).

Let \(I\) be a proper subset of \(\Delta\). Let \(\rho\) be an irreducible cuspidal \(L_I(\mathbb{F}_q)\)-module. Let \(M\) be a \(G(\mathbb{F}_q)\)-module. We define \(M[\rho]\) to be the largest submodule of \(M\) such that each irreducible representation in \(M[\rho]\) is contained in \(\text{Ind}_{P_I(\mathbb{F}_q)}^{G(F_q)}(\rho)\). The irreducible components of \(\text{Ind}_{P_I(\mathbb{F}_q)}^{G(F_q)}(\rho)\) are in one to one correspondence with the irreducible representations of the Weyl group \(\tilde{W}\) of the quotient root system corresponding to \((L_I, \rho)\) (see [L1, 5.9]).

**Proposition 8.3** [L1, 6.7, 7.8]. Let \(I\) be a subset of \(\Delta\), and let \(\rho\) be an irreducible cuspidal representation of \(L_I(\mathbb{F}_q)\) occurring in \(H_I^i\). Then the \(G(\mathbb{F}_q)\)-module \(H_I^i[\rho]\), \((n \leq i \leq 2n - |I|)\) is irreducible and corresponds to the \((2n - i - |I|)\)th exterior power representation of the reflection representation of \(W\).

**Proposition 8.4.** Let \(G\) be a split classical group. We preserve the notations of Section 1.

(a) For \(G(\mathbb{F}_q) = GL(n + 1, \mathbb{F}_q)\) \((n \geq 0)\),

\[
H^{n+i} = \pi_{n-i} \quad (0 \leq i \leq n).
\]

(8.2)
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(b) For $G(\mathbb{F}_q) = \text{Sp}(2n, \mathbb{F}_q)$ or $\text{SO}(2n + 1, \mathbb{F}_q)$, $(n \geq 2)$,
\[
\begin{align*}
H^\pi_{n+i} &= \pi_{n-i} \oplus \rho_{n-2-i} & (0 \leq i \leq n-2), \\
H^\pi_{n+i} &= \pi_{n-i} & (n-1 \leq i \leq n).
\end{align*}
\]  (8.3)

(c) For $G(\mathbb{F}_q) = \text{SO}(2n, \mathbb{F}_q)$, $(n \geq 4)$.
\[
\begin{align*}
H^\pi_{n+i} &= \pi_{n-i} \oplus \rho_{n-4-i} & (0 \leq i \leq n-4), \\
H^\pi_{n+i} &= \pi_{n-i} & (n-3 \leq i \leq n).
\end{align*}
\]  (8.4)

Proof. (a) Let $I$ be the empty subset of $\Delta$, and let $\rho$ be the trivial representation of $T(\mathbb{F}_q)$, where $T$ is the split maximal torus of $G$. Then by Proposition 8.3 we know that $H^{\pi_1}[1] = \pi_{n-1}$ $(0 \leq i \leq n)$. As there are $n+1$ irreducible components of $\bigoplus H^i$, we have $H^i = \pi_{n-i}$ $(0 \leq i \leq n)$.

(b) We choose $I$ such that the corresponding Levi component $L_I$ of the parabolic subgroup $P_I$ is of the form $(\mathbb{G}_m)^{n-2} \times \text{Sp}_4$, or $(\mathbb{G}_m)^{n-2} \times \text{SO}_5$ as the case may be. Then $|I| = 2$. By [L1, 7-3], we know that $\pi_{u_i}$ is a component of $H^i$. We know that $W = W(B_{n-2})$. By proposition 8.3, we know that $H^{\pi_1}[\pi_{u_i}] = \rho_{n-2-i}$ $(0 \leq i \leq n-2)$. Now, if we take $I$ to be the empty subset of $\Delta$ and $\rho$ the trivial representation of $T(\mathbb{F}_q)$, where $T$ is the split maximal torus of $G$, we get $H^{\pi_1}[1] = \pi_{n-1}$. Since there are $2n$ irreducible components of $\bigoplus H^i$, we have
\[
\begin{align*}
H^\pi_{n+i} &= \pi_{n-i} \oplus \rho_{n-2-i} & (0 \leq i \leq n-2), \\
H^\pi_{n+i} &= \pi_{n-i} & (n-1 \leq i \leq n).
\end{align*}
\]  (8.5)

(c) We choose $I$ such that the corresponding Levi component $L_I$ of the parabolic subgroup $P_I$ is of the form $(\mathbb{G}_m)^{n-4} \times \text{SO}_6$. Then $|I| = 4$. By [L1, 7-3] we know that $\pi_{u_i}$ appears in $H^I$. We know that $W = W(D_{n-4})$. By Proposition 8.3 we know that $H^{\pi_1}[\pi_{u_i}] = \rho_{n-4-i}$ $(0 \leq i \leq n-4)$. Now, if we take $I$ to be the empty subset of $\Delta$ and $\rho$ the trivial representation of $T(\mathbb{F}_q)$, where $T$ is the split maximal torus of $G$, we get $H^{\pi_1}[1] = \pi_{n-1}$. Since there are $2(n-1)$ irreducible components of $\bigoplus H^i$, we have
\[
\begin{align*}
H^\pi_{n+i} &= \pi_{n-i} \oplus \rho_{n-4-i} & (0 \leq i \leq n-4), \\
H^\pi_{n+i} &= \pi_{n-i} & (n-3 \leq i \leq n).
\end{align*}
\]  (8.6)

Hence the proof.

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