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On the restriction of cuspidal representations to unipotent elements

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1. *Introduction*

Let G be a connected split reductive group defined over a finite field $\mathbb{F}_q$, and $G(\mathbb{F}_q)$ the group of $\mathbb{F}_q$-rational points of G. For each maximal torus T of G defined over $\mathbb{F}_q$ and a complex linear character θ of $T(\mathbb{F}_q)$, let $R_T^G(\theta)$ be the generalized representation of $G(\mathbb{F}_q)$ defined in [**DL**]. It can be seen that the conjugacy classes in the Weyl group W of G are in one-to-one correspondence with the conjugacy classes of maximal tori defined over $\mathbb{F}_q$ in G ([**C1**, 3·3·3]). Let c be the Coxeter conjugacy class of W, and let T_c be the corresponding maximal torus. Then by [**DL**] we know that $\pi_\theta = (-1)^n R_{T_c}^G(\theta)$ (where n is the semisimple rank of G and θ is a character in 'general position') is an irreducible cuspidal representation of $G(\mathbb{F}_q)$. The results of this paper generalize the pattern about the dimensions of cuspidal representations of $GL(n, \mathbb{F}_q)$ as an alternating sum of the dimensions of certain irreducible representations of $GL(n, \mathbb{F}_q)$ appearing in the space of functions on the flag variety of $GL(n, \mathbb{F}_q)$ as shown in the table below.

n	Dimension of cuspidal representation	$\dim(\mathrm{St}_{n,n}) - \dim(\mathrm{St}_{n,n-1})$ $+\dim(\mathrm{St}_{n,n-2}) - \cdots + (-1)^{n-1}\dim(\mathrm{St}_{n,1})$
2	$q-1$	$q-1$
3	$(q^2-1)(q-1)$	$q^3 - (q^2+q) + 1$
4	$(q^3-1)(q^2-1)(q-1)$	$q^6 - (q^5+q^4+q^3) + (q^3+q^2+q) - 1$
5	$(q^4-1)(q^3-1)(q^2-1)(q-1)$	$q^{10} - (q^9 + \cdots q^6) +$ $(q^7+q^6+2q^5+q^4+q^3) - (q^4+\cdots q) + 1$

Here $\mathrm{St}_{n,i}$ is an irreducible representation of $GL(n, F_q)$ appearing in the space of functions on the flag variety of $GL(n, F_q)$; $\mathrm{St}_{n,n}$ is the Steinberg representation, and $\mathrm{St}_{n,1}$ is the trivial representation of $GL(n, F_q)$. We are using the well known formula for the dimension of a cuspidal representation of $GL(n, \mathbb{F}_q)$ as $(q-1)\ldots(q^{n-1}-1)$. We could easily check that this equality remained true for characters of all unipotent elements too for these small values of n by looking into character tables. The aim of the paper is to give a proof of this for $GL(n, \mathbb{F}_q)$ as well as generalizations for other classical groups. For $GL(n, \mathbb{F}_q)$ it seems that this result is well known and can be proved by methods as given in [**L**], but we include a proof in this case too for the sake of completeness.

An irreducible representation ρ of $G(\mathbb{F}_q)$ is called unipotent if it arises as a component of $R_T^G(1)$ for some T. If T is a split torus then $R_T^G(1) = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1)$ where B is a Borel subgroup containing T, defined over $\mathbb{F}_q$. It is well known that $\mathrm{End}_{G(\mathbb{F}_q)}(\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1))$ can be identified with the group algebra $\mathbb{C}[W]$. Therefore the irreducible representations of $G(\mathbb{F}_q)$ occurring in $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1)$ are in one to one correspondence with the irreducible representations of W over $\mathbb{C}$. It is known ([**S**, 14]) that the exterior powers of the reflection representation of W, to be denoted by E throughout this paper, are irreducible and mutually inequivalent. Let π_i be the irreducible component of $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1)$ corresponding to the ith exterior power representation of the reflection representation of W.

By [**L2**] it is known that if G is a classical group, then it can have at most one unipotent cuspidal representation. The groups of type A_n do not have any unipotent cuspidal representation; groups of type B_n, C_n have exactly one if and only if $n = s^2 + s$ for some integer $s \geqslant 1$ and D_n have one if and only if n is an even square. Thus groups of type $B_2 = C_2$, and D_4 have unique unipotent cuspidal representations, and in these cases they occur as a component of $R_{T_c}^G(1)$, where c is the Coxeter conjugacy class of the corresponding root systems. Let us denote these unipotent cuspidal representations by π_{uc}.

Let $G = G_n$ be either Sp_{2n}, SO_{2n+1} $(n \geqslant 2)$, or the split orthogonal group in even number of variables SO_{2n} defined over $\mathbb{F}_q$. For each partition $n = r_1 + r_2 + \cdots + r_k + s$ $(0 \leqslant s < n)$ we have the standard parabolic subgroup P defined over $\mathbb{F}_q$ with Levi subgroup L defined over $\mathbb{F}_q$ and isomorphic to $GL_{r_1} \times GL_{r_2} \times \cdots \times GL_{r_k} \times G_s$. For $G = Sp_{2n}$, or SO_{2n+1} take the partition $n = 1 + \cdots + 1 + 2$, with the corresponding Levi subgroup $(\mathbf{G}_m)^{n-2} \times Sp_4$, or $(\mathbf{G}_m)^{n-2} \times SO_5$. We know that Sp_4 and SO_5 have a unique unipotent cuspidal representation π_{uc}. Extend the representation π_{uc} trivially across $(\mathbf{G}_m(\mathbb{F}_q))^{n-2} = (\mathbb{F}_q^*)^{n-2}$ to construct a representation of $(\mathbb{F}_q^*)^{n-2} \times Sp(4, \mathbb{F}_q)$, or $(\mathbb{F}_q^*)^{n-2} \times SO(5, \mathbb{F}_q)$, as the case may be. We abuse notation to denote this representation of Levi subgroup $L(\mathbb{F}_q)$ again by π_{uc}. Let $\rho = \mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\pi}_{\mathrm{uc}})$, where $\tilde{\pi}_{\mathrm{uc}}$ is the representation of $P(\mathbb{F}_q)$ obtained by composing π_{uc} with the natural homomorphism from $P(\mathbb{F}_q)$ to $L(\mathbb{F}_q)$. By [**L2**, **L5**] we know that $\mathrm{End}_{G(\mathbb{F}_q)}(\rho)$ can be identified with $\mathbb{C}[W(B_{n-2})]$. Therefore the irreducible representations of $G(\mathbb{F}_q)$ occurring in ρ are in one-to-one correspondence with the irreducible representations of $W(B_{n-2})$. Let ρ_i be the irreducible component of ρ corresponding to the i-th exterior power representation of the reflection representation of $W(B_{n-2})$. Similarly, when $G = SO_{2n}$ $(n \geqslant 4)$, take the Levi subgroup $L \cong (\mathbf{G}_m)^{n-4} \times SO_8$. We know that $SO(8, \mathbb{F}_q)$ has a unique unipotent cuspidal representation π_{uc}. Let ρ be constructed as above. It follows by [**L2**, **L5**] that $\mathrm{End}_{G(\mathbb{F}_q)}(\rho)$ can be identified with $\mathbb{C}[W(B_{n-4})]$. Let ρ_i be the irreducible component of ρ corresponding to the ith exterior power representation of the reflection representation of $W(B_{n-4})$. Here is the main theorem of this paper.

THEOREM 1·1. *Let G be a split classical group, and let Θ_π denote the character of a representation π. Let u be a unipotent element of $G(\mathbb{F}_q)$. With the notation as above, we have the following* (*a*) *For* $G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q) \quad (n \geqslant 0)$,

$$\Theta_{\pi_\theta}(u) = \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u). \tag{1·1}$$

(*b*) *For* $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$, *or* $SO(2n+1, \mathbb{F}_q) \quad (n \geqslant 2)$,

$$\Theta_{\pi_\theta}(u) = \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u) + \sum_{i=0}^{i=n-2} (-1)^i \Theta_{\rho_{n-2-i}}(u). \tag{1·2}$$

(*c*) *For* $G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q) \quad (n \geqslant 4)$,

$$\Theta_{\pi_\theta}(u) = \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u) + \sum_{i=0}^{i=n-4} (-1)^i \Theta_{\rho_{n-4-i}}(u). \tag{1·3}$$

To prove the theorem above we shall need some results about the characters of the classical Weyl groups, and their values at the Coxeter conjugacy class. We prove that the character values of irreducible representations of the classical Weyl groups at its Coxeter conjugacy class is $1, -1$, or 0 (Theorem 2·1). We also need the decomposition of $R[\chi]$ (see Theorem 4·1) in terms of irreducible unipotent representations of $G(\mathbb{F}_q)$ (Theorem 4·4).

Remark 1·1. In general, the restriction to unipotent elements of cuspidal representation coming from other maximal tori, can not be expressed as an alternating sum, as in Theorem 1·1. The reason for getting such a small number of representations in Theorem 1·1 is that there are just h (= Coxeter number) irreducible characters of Weyl group that are non-zero at the Coxeter element, and these are either 1 or -1. This is not true in general for other elements of the Weyl group.

2. *Characters of classical Weyl groups*

In this section we shall establish some basic results about the characters of classical Weyl groups.

Let W be the Weyl group corresponding to an irreducible root system Σ in a $\mathbb{Q}$-vector space E (spanned by Σ) with a fixed set of positive roots. Then E is in a natural way an irreducible $\mathbb{Q}[W]$-module, said to be the *reflection representation* of W. Let $\Delta = \{e_1, e_2, \ldots, e_n\}$ be the system of simple roots of Σ. Then Δ is a basis for E. Let $s_i \in GL(E)$ be the reflection with respect to $e_i \in E$. Then W has a presentation as a finite Coxeter group $W = \langle s_1, s_2, \ldots, s_n \mid {s_i}^2 = 1, i = 1, \ldots, n, (s_i s_j)^{n_{ij}} = (s_j s_i)^{n_{ij}} = 1, n_{ij} < \infty \rangle$. The element of the form $c = s_1 \cdots s_n$ is called a Coxeter element of W. The conjugacy class of c does not depend either on the ordering $s_1, \ldots, s_n$, or on the choice of the generating reflections, and therefore defines a well-defined conjugacy class in W, called the Coxeter class. Weyl groups of root systems of type A_n ($n \geqslant 1$), B_n ($n \geqslant 1$) and D_n ($n \geqslant 2$) are called the classical Weyl groups. (The Weyl group of C_n is same as that of B_n.) We will briefly describe these Weyl groups.

(1) Let the root system be of type A_n ($n \geqslant 1$). Then $W(A_n) = S_{n+1}$ (the symmetric group on $n+1$ elements). In this case, one can take the transpositions $(i, i+1)$ ($1 \leqslant i \leqslant n$), to be a set of simple reflections. Therefore, the Coxeter element $c = s_1 \cdots s_n$ is the $(n+1)$-cycle $(1, 2, \ldots, n+1)$. We denote the $(n+1)$-cycle by σ_{n+1}.

(2) Let $W_n, (n \geqslant 1)$ be the group of all permutations of the set $\{1, 2, \ldots, n, n', \ldots, 2', 1'\}$ which commute with the involutions $i \to i'$, $i' \to i\,(1 \leqslant i \leqslant n)$. For each j, $1 \leqslant j \leqslant n-1$ let $s_j \in W_n$ be the permutation which interchanges j with $j+1$ and also j' with $(j+1)'$ and leaves the other elements unchanged. Let $s_n \in W_n$ be the permutation which interchanges n with n' and leaves other entries unchanged. Then $W_n = \langle s_1, s_2, \ldots, s_n \rangle$ is the Weyl group of type B_n. A permutation in W_n defines a permutation of the n element set consisting of the unordered pairs $\{(1,1'), (2,2'), \ldots, (n,n')\}$. Thus we have a natural homomorphism of W_n onto S_n. It can be seen that $W_n = \{\pm 1\}^n \rtimes S_n$. The Coxeter element is $c = (1, 1, \ldots, 1, -1) \cdot \sigma_n$, where σ_n is the n-cycle in S_n.

(3) Let $\varepsilon\colon W_n \to \{\pm 1\}$ be the homomorphism defined by the condition that

$$\left.\begin{aligned} \varepsilon(s_i) &= 1 \quad (1 \leqslant i \leqslant n-1), \\ \varepsilon(s_n) &= -1. \end{aligned}\right\} \tag{2·1}$$

Let $\tilde{W}_n = \ker(\varepsilon)$. When $n \geqslant 2$, let $s'_n = s_n s_{n-1} s_n$. Then $\tilde{W}_n = \langle s_1, s_2, \ldots, s_{n-1}, s'_n \rangle$ is the Weyl group of type D_n. The Coxeter element is $c = (1, 1, \ldots, 1, -1, -1) \cdot \sigma_{n-1}$, where σ_{n-1} is the $(n-1)$-cycle in S_n.

LEMMA 2·1. *A Coxeter element commutes only with its powers.*

Proof. See proposition 30 of [**C**].

Let c be a Coxeter element of W and let h be its order, called the Coxeter number of W or of the underlying semisimple group G. Then by the above result $h = |Z_W(c)|$, where $|Z_W(c)|$ denotes the cardinality of the centralizer of c in W.

THEOREM 2·1. *Let W be a classical Weyl group. Then the value of all the irreducible characters of W at its Coxeter conjugacy class is* $1, -1$ *or* 0.

The proof of the theorem above essentially reduces to the case of $W(A_{n-1})$. So we will first prove this for $W(A_{n-1})$.

Let $\{v_1, \ldots, v_n\}$ be the standard basis for $\mathbb{Q}^n$. Let $E = E_n = \{(u_1, \ldots, u_n) \in \mathbb{Q}^n \mid \sum u_i = 0\}$ be the $n-1$ dimensional subspace of $\mathbb{Q}^n$. For the Weyl group $W(A_{n-1}) = S_n$, the generators s_i are the transpositions $(i, i+1)$, $1 \leqslant i \leqslant n-1$, acting as reflections in $GL(E)$ with respect to the root-vectors $\alpha_i = v_i - v_{i+1}$. Thus E_n affords the reflection representation of $W(A_{n-1}) = S_n$. The order h of the Coxeter element $c = s_1 \cdots s_{n-1} = \sigma_n$ is n. The eigenvalues of $\sigma_n = (1, 2, \ldots, n)$ on E are

$$\zeta^i, \quad i = 1, 2, \ldots, n-1, \tag{2·2}$$

where ζ is a primitive nth root of unity. It is easy to see that $\wedge^i E_n$, $i = 0, 1, \ldots, n-1$ are mutually inequivalent irreducible representations of S_n. Let $\chi_i = \Theta_{\wedge^i E_n}$ denote the character of $\wedge^i E_n$, and $\widehat{W}$ denote the set of irreducible characters of W.

PROPOSITION 2·1. *Let $W = W(A_{n-1}) = S_n$. Let $\chi \in \widehat{W}$. Then*

$$\chi(\sigma_n) = \begin{cases} (-1)^i & \text{if } \chi = \chi_i, \quad i \in \{0, 1, \ldots, n-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2·3}$$

Proof. From (2·2) we know that the eigenvalues x_i of σ_n on E satisfy the following equation:

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = 0. \tag{2·4}$$

Therefore,

$$\chi_i(\sigma_n) = \sum_{1\leqslant r_1<r_2<\cdots<r_i\leqslant n-1} x_{r_1}\cdot x_{r_2}\cdots x_{r_i}. \tag{2·5}$$

The expression on the right-hand side is the sum of the product of i distinct roots of (2·4), which is $(-1)^i$ times the coefficient of x^{n-1-i}. Therefore

$$\chi_i(\sigma_n) = (-1)^i, \qquad i = 0, 1, \ldots, n-1. \tag{2·6}$$

We now need an elementary result about the characters of a finite group, according to which for any finite group H and any element $g \in H$,

$$\sum_{\chi\in\widehat{H}} \chi(g)\overline{\chi(g)} = |Z_H(g)| \tag{2·7}$$

where $\widehat{H}$ denotes the set of isomorphism classes of irreducible characters of H, $\overline{\chi(g)}$ denotes the complex conjugate of $\chi(g)$, and $|Z_H(g)|$ denotes the cardinality of the centralizer of g. Applying (2·7) in the situation of W we obtain

$$\sum_{\chi\in\widehat{W}} |\chi(\sigma_n)|^2 = n. \tag{2·8}$$

On the other hand, we have

$$\left.\begin{aligned}\sum_{\chi\in\widehat{W}} |\chi(\sigma_n)|^2 &= \sum_{\chi\notin\{\chi_i\,|\,i=0,1,\ldots,n-1\}} |\chi(\sigma_n)|^2 + \sum_{i=0}^{i=n-1} |\chi_i(\sigma_n)|^2\\ &= \sum_{\chi\notin\{\chi_i\,|\,i=0,1,\ldots,n-1\}} |\chi(\sigma_n)|^2 + \sum_{i=0}^{i=n-1} 1 \qquad \text{(by (2·6))}\\ &= \sum_{\chi\notin\{\chi_i\,|\,i=0,1,\ldots,n-1\}} |\chi(\sigma_n)|^2 + n.\end{aligned}\right\} \tag{2·9}$$

Applying (2·8) we obtain

$$\sum_{\chi\notin\{\chi_i\,|\,i=0,1,\ldots,n-1\}} |\chi(\sigma_n)|^2 = 0. \tag{2·10}$$

Therefore $\chi(\sigma_n) = 0$ if $\chi \notin \{\chi_i \mid i = 0, 1, \ldots, n-1\}$. Hence the proof.

Irreducible characters of S_n are in one-to-one correspondence with the *partitions* of n. A partition of n is any finite sequence

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots]$$

of non-negative integers in increasing order

$$0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_r \leqslant \cdots$$

such that the sum of the parts of λ (denoted by $|\lambda|$ and equal to $\lambda_1+\lambda_2+\cdots+\lambda_r+\cdots$) is n. We shall find it convenient not to distinguish between two such sequences which differ only by a string of zeroes at the beginning. Note that the irreducible character χ_i of S_n corresponds to the partition $[1^i, n-i]$.

LEMMA 2·2. *Let $\mu_i = [1^{i-1}, 2, n-1-i]$, $i = 1, \ldots, n-3$ be a partition of n, and let χ_{μ_i} be the corresponding irreducible character of S_n. Let σ_{n-1} be an $(n-1)$-cycle in S_n. Then $\chi_{\mu_i}(\sigma_{n-1}) = \Theta_{\wedge^i E_{n-1}}(\sigma_{n-1}) = (-1)^i$, where $\wedge^i E_{n-1}$ is the* ith *exterior power representation of the $(n-2)$-dimensional reflection representation of S_{n-1}. Moreover, for any irreducible character χ of S_n, $\chi(\sigma_{n-1}) = 0$ unless $\chi = \chi_{\mu_i}$, or $\chi = 1$, or $\chi = $* sgn.

Proof. It follows from the *Murnaghan–Nakayama* rule (see [**M**, I·7, ex. 5]) that

$$\chi_{\mu_i}(\sigma_{n-1}) = \sum \chi_\nu(\sigma_{n-1}), \tag{2·11}$$

where the sum is over the three partitions $\nu_1 = [1^{i-1}, 2, n-2-i]$, $\nu_2 = [1^i, n-1-i]$, $\nu_3 = [1^{i-2}, 2, n-1-i]$. Now applying Proposition 2·1 in (2·11), we obtain

$$\chi_{\mu_i}(\sigma_{n-1}) = \Theta_{\wedge^i E_{n-1}}(\sigma_{n-1}) = (-1)^i, \quad i = 1, \ldots, n-3. \tag{2·12}$$

As $|Z_{S_n}(\sigma_{n-1})| = n-1$, by applying (2·7) we get that $\chi(\sigma_{n-1}) = 0$ unless $\chi = \chi_{\mu_i}$, or $\chi = 1$, or $\chi = \mathrm{sgn}$. Hence the proof.

Proof of Theorem 2·1

Proof. The proof follows by using (2·7), as soon as we have enumerated $h = |Z_W(c)|$ irreducible characters of W which take value 1 or -1 at the Coxeter class c. The case of $W(A_n)$ is done in Proposition 2·1. We now do this for the Weyl groups of type B_n and D_n.

(1) *The case of* $W(B_n)$. We know that $W_n = W(B_n) = \{\pm 1\}^n \rtimes S_n$ and its Coxeter element is $c = (1, \ldots, 1, -1) \cdot \sigma_n$. The Coxeter number is $h = 2n$. The irreducible characters χ of W_n are in one-to-one correspondence with the ordered pairs (χ_1, χ_2) of irreducible characters of S_k, S_l $(k + l = n)$. The correspondence is defined as follows. The subgroup of W_n consisting of all permutations in W_n which map $\{1, 2, \ldots, k, k', \ldots, 2', 1'\}$ into itself and hence also map $\{k+1, \ldots, n, n', \ldots, (k+1)'\}$ into itself can be regarded in a natural way as a product $W_k \times W_l$. The characters χ_1, χ_2 of S_k, S_l can be regarded as characters $\bar{\chi}_1, \bar{\chi}_2$ of W_k, W_l via the projections $W_k \to S_k$, $W_l \to S_l$. Consider the character $\bar{\chi}_1 \otimes (\varepsilon\,|_{W_l} \otimes \bar{\chi}_2)$ of $W_k \times W_l$ where ε is as defined in 2·1. We induce it to W_n; the resulting character is irreducible. It is the character corresponding to the ordered pair (χ_1, χ_2). Now the irreducible character χ_1 of S_k corresponds to a partition λ of k, and the irreducible character χ_2 of S_l corresponds to a partition μ of l. Therefore the irreducible characters χ of W_n are in one-to-one correspondence with the ordered partition (λ, μ) of n.

Let $(\lambda, 0)$ be an ordered partition of n. Then the corresponding irreducible characters of $W(B_n)$ are $\bar{\chi}_\lambda$, where χ_λ is the irreducible character of S_n corresponding to partition λ of n. Take $\lambda = \lambda_i = [1^i, n-i]$, $(i = 0, \ldots, n-1)$. Applying Proposition 2·1, we get

$$\bar{\chi}_{\lambda_i}(c) = \chi_{\lambda_i}(\sigma_n) = (-1)^i, \quad i = 0, \ldots, n-1. \tag{2·13}$$

Now consider the ordered partitions $(0, \lambda)$ with $\lambda = \lambda_i = [1^i, n-i]$, $(i = 0, \ldots, n-1)$. Then $\varepsilon \otimes \bar{\chi}_{\lambda_i}$ are the irreducible characters of W_n. Therefore

$$\varepsilon \otimes \bar{\chi}_{\lambda_i}(c) = (-1) \cdot \chi_{\lambda_i}(\sigma_n) = (-1)^{i+1}, \quad i = 0, \ldots, n-1. \tag{2·14}$$

Thus we have enumerated $2n$ characters of W_n

$$\{\bar{\chi}_{\lambda_i}, \varepsilon \otimes \bar{\chi}_{\lambda_i} \mid i = 0, \ldots, n-1\} \tag{2·15}$$

with character values 1 or -1 at the Coxeter element. As the Coxeter number is $2n$, using (2·7) we get that these are all the irreducible characters with non-zero value at the Coxeter element c.

(2) *The case of* $W(D_n)$. We know that $\tilde{W}_n = W(D_n)$. The Coxeter element is $c = (1, \dots, 1, -1, -1) \cdot \sigma_{n-1}$. The Coxeter number is $h = 2(n-1)$. It is easy to see that the irreducible characters of $W_n = W(B_n)$ corresponding to the ordered partition (λ, μ) of n remain irreducible when restricted to $\tilde{W}_n$ except when $\lambda = \mu$, and the restriction of characters corresponding to (λ, μ) and (μ, λ) are the same. Thus the irreducible characters of $\tilde{W}_n$ are in one-to-one correspondence with the unordered partition (λ, μ) of n except when $\lambda = \mu$. When $\lambda \neq \mu$, we call the corresponding irreducible character of $\tilde{W}_n$ *non-degenerate*. The irreducible character of W_n corresponding to the partition (λ, λ) decomposes into two distinct irreducible components when restricted to $\tilde{W}_n$. We call these characters *degenerate*.

Let $\mu_i = [1^{i-1}, 2, n-1-i]$, $(i = 1, \dots, n-3)$ be a partition of n. Let χ_{μ_i} be the corresponding character of S_n. Then the irreducible character of W_n corresponding to the ordered partition $(\mu_i, 0)$ is $\bar{\chi}_{\mu_i}$, whose restriction to $\tilde{W}_n$ is irreducible. By Lemma 2·2 we get that

$$\bar{\chi}_{\mu_i}(c) = \chi_{\mu_i}(\sigma_{n-1}) = (-1)^i, \quad i = 1, \dots, n-3. \tag{2·16}$$

Now let us take the unordered partitions $(\mu_0, 0)$ and $(\mu_{n-2}, 0)$, where $\mu_0 = [n]$ and $\mu_{n-2} = [1^n]$. Then the corresponding irreducible characters of $\tilde{W}_n$ are $\bar{\chi}_{\mu_0}$, and $\bar{\chi}_{\mu_{n-2}}$ respectively. We have,

$$\left.\begin{aligned} \bar{\chi}_{\mu_0}(c) &= \chi_{\mu_0}(\sigma_{n-1}) = 1, \\ \bar{\chi}_{\mu_{n-2}}(c) &= \chi_{\mu_{n-2}}(\sigma_{n-1}) = (-1)^{n-2}. \end{aligned}\right\} \tag{2·17}$$

Therefore we obtain $n-1$ irreducible characters of $\tilde{W}_n$

$$\{\bar{\chi}_{\mu_i} \mid i = 0, \dots, n-2\} \tag{2·18}$$

with their character values at the Coxeter element 1 or -1.

Let $\Theta_{\wedge^i E_{n-1}}$ be the character corresponding to the ith exterior power representation of the $(n-2)$-dimensional reflection representation of S_{n-1}. Then $\Theta_{\wedge^i E_{n-1}}$ corresponds to the partition $\lambda'_i = [1^i, n-1-i]$ of $n-1$. Let $\chi_{(\lambda'_i,1)}$ be the irreducible character of W_n corresponding to the ordered partition $(\lambda'_i, 1)$. That is,

$$\chi_{(\lambda'_i,1)} = \mathrm{Ind}_{W_{n-1}\times W_1}^{W_n}(\bar{\chi}_{\lambda'_i} \otimes \varepsilon\mid_{W_1}). \tag{2·19}$$

Let $n > 2$. Then $\lambda'_i \neq 1$, and therefore the restriction of $\chi_{(\lambda'_i,1)}$ to $\tilde{W}_n$ is irreducible. Now using Proposition 2·1 we get that

$$\chi_{(\lambda'_i,1)}(c) = (-1)^i \cdot \varepsilon\mid_{W_1}(-1) = (-1)^{i+1}, \quad i = 0, \dots, n-2. \tag{2·20}$$

We have enumerated $n-1$ more irreducible characters

$$\{\chi_{(\lambda'_i,1)} \mid i = 0, \dots, n-2\} \tag{2·21}$$

of $\tilde{W}_n$ with values at the Coxeter element 1 or -1. Thus we have obtained $2(n-1)$ irreducible characters of $\tilde{W}_n$ with values 1 or -1 at the Coxeter element c. As the Coxeter number is $2(n-1)$, by applying (2·7) we conclude that these are all the irreducible characters with non-zero value at the Coxeter element c. Hence the proof.

Remark 2·1. We have been informed by Professor T. A. Springer that the Theorem 2·1 has been known for all Weyl groups by looking at their character tables.

3. *Symbols and unipotent representations*

In this section we introduce the formalism of *symbols* due to [**L2**] which gives a simple combinatorial parameterization of all unipotent representations of classical groups. A *symbol* is an unordered pair $\Lambda = \binom{S}{T}$ of finite subsets (including the empty set $\emptyset$) of $\{0, 1, 2, \dots\}$. The rank of Λ is defined by

$$\mathrm{rk}(\Lambda) = \sum_{\lambda \in S} \lambda + \sum_{\mu \in T} \mu - \left[\left(\frac{a+b-1}{2} \right)^2 \right], \tag{3·1}$$

where $a = |S|, b = |T|$, and for any real number z we denote by $[z]$ the largest integer m such that $m \leqslant z$. The defect of Λ is defined by $\mathrm{def}(\Lambda) = |a - b|$. There is an equivalence relation on such pairs generated by the shift

$$\begin{pmatrix} S \\ T \end{pmatrix} \sim \begin{pmatrix} \{0\} \sqcup (S+1) \\ \{0\} \sqcup (T+1) \end{pmatrix}.$$

We shall identify a symbol with its equivalence class. The function $\mathrm{rk}(\lambda)$ and $\mathrm{def}(\lambda)$ are invariant under the shift operation, hence are well-defined on the set of symbol classes. A symbol $\Lambda = \binom{S}{T}$ is said to be *reduced* if $0 \notin S \cap T$; it is called degenerate if $S = T$, and non-degenerate if $S \neq T$.The entries appearing in exactly one row of Λ are called *singles*. Now we shall define *special* symbols in the sense of [**L4**] and [**L5**]. We first consider the case of symbols of rank n and defect 1. Let

$$\Lambda = \begin{pmatrix} z_0, z_2, \dots, z_{2m} \\ z_1, z_3, \dots, z_{2m-1} \end{pmatrix} \tag{3·2}$$

be a symbol of rank n and defect one. We arrange z's in such a way that $0 \leqslant z_0 < z_2 < \cdots < z_{2m}$, $0 \leqslant z_1 < z_3 < \cdots < z_{2m-1}$. The symbol Λ is said to be *special*, if the inequalities

$$z_0 \leqslant z_1 \leqslant z_2 \leqslant z_3 \leqslant \cdots \leqslant z_{2m-1} \leqslant z_{2m} \tag{3·3}$$

are satisfied. It is easy to see that in this case the number of singles is odd.

Let us consider the case of symbols of rank n and defect 0. Let

$$\Lambda = \begin{pmatrix} z_2, z_4, \dots, z_{2m} \\ z_1, z_3, \dots, z_{2m-1} \end{pmatrix} \tag{3·4}$$

be a symbol of rank n and defect 0. It is so arranged that $0 \leqslant z_1 < z_3 < \cdots < z_{2m-1}$, and $0 \leqslant z_2 < z_4 < \cdots < z_{2m}$. A non-degenerate symbol Λ is *special* if and only if

$$\begin{cases} z_1 \leqslant z_4 \leqslant z_3 \leqslant \cdots \leqslant z_{2m}, & \text{or if,} \\ z_2 \leqslant z_1 \leqslant z_4 \leqslant \cdots \leqslant z_{2m-1}. \end{cases} \tag{3·5}$$

It is easy to see that in this case the number of singles is even.

We know that the irreducible characters of $W(B_n)$ are in one-to-one correspondence with the ordered partitions (α, β) of n. Let $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_{m'}]$, $\beta = [\beta_1, \beta_2, \dots, \beta_{m''}]$ with $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{m'}$ and $0 \leqslant \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_{m''}$. Since m' and m'' can be increased at our will (by adding zeroes) we may assume that

$m' = m'' + 1$. We now set $\lambda_i = \alpha_i + i - 1$, $(1 \leqslant i \leqslant m')$, $\mu_i = \beta_i + i - 1$, $(1 \leqslant i \leqslant m'')$. Let $S = \{\lambda_1, \ldots, \lambda_{m'}\}$, $T = \{\mu_1, \ldots, \mu_{m''}\}$. Therefore for each irreducible character χ of $W(B_n)$ corresponding to an ordered partition (α, β) we can associate a symbol $\Lambda = \binom{S}{T}$ given as above. We know that the non-degenerate irreducible characters of $W(D_n)$ are in one-to-one correspondence with the pair of unordered partitions (α, β). In this case we set $m' = m''$, and define Λ as above.

Let $\Phi_{n,d}$ $(d > 0)$ be the set of symbol classes of rank n and defect d, and $\Phi_{n,0}$ be the set of symbol classes of rank n and defect 0 with each degenerate symbol repeated twice. The following lemma is due to [**L2**, 2·7].

Lemma 3·1. *The above map defines a one-to-one correspondence between the irreducible characters of* $W(B_n)$ (*resp.* $W(D_n)$) *and the set* $\Phi_{n,1}$ (*resp.* $\Phi_{n,0}$).

For any integer $m \geqslant -1$, we denote the set $\{0, 1 \ldots, m\}$ by $[0, m]$. Thus $[0, -1]$ is the empty set. The following proposition is due to [**L2**, 3·2].

Proposition 3·1. *Let d be an integer $\geqslant 1$. The correspondence*

$$\Lambda = \begin{pmatrix} S \\ T \end{pmatrix} \longrightarrow \bar{\Lambda} = \begin{pmatrix} [0, d-2] \cup (S + d - 1) \\ T \end{pmatrix} \tag{3·6}$$

(*where* $|S| = b + 1, |T| = b$) *defines a bijection*

$$\begin{gathered} j : \Phi_{n,1} \longleftrightarrow \Phi_{n',d} \\ \textit{where } n' = n + \left[\left(\frac{d}{2} \right)^2 \right]. \end{gathered} \tag{3·7}$$

Let

$$\left. \begin{aligned} \Phi_n &= \bigsqcup_{d \equiv 1 \pmod 2} \Phi_{n,d}, \\ \Phi_n^+ &= \left(\bigsqcup_{\substack{d \equiv 0 \pmod 4 \\ d > 0}} \Phi_{n,d} \right) \bigsqcup \Phi_{n,0}. \end{aligned} \right\} \tag{3·8}$$

The following theorem due to [**L2**, 8·2] gives a bijection between symbols and all the unipotent representations of $G(\mathbb{F}_q)$ when G is of type B_n, C_n or D_n.

Theorem 3·1. (*a*) *Let G be of type B_n, or C_n $(n \geqslant 1)$. There is a natural one-to-one correspondence $\Lambda \leftrightarrow \rho[\Lambda]$ between the set Φ_n and the set of isomorphism classes of unipotent representations of $G(\mathbb{F}_q)$, extending the correspondence between $\Phi_{n,1}$ and the unipotent representations appearing in the principal series* $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1)$ *given by Lemma* 3·1.

(*b*) *Let G be of type D_n $(n \geqslant 2)$. There is a natural one-to-one correspondence $\Lambda \leftrightarrow \rho[\Lambda]$ between the set Φ_n^+ and the set of isomorphism classes of unipotent representations of $G(\mathbb{F}_q)$, extending the correspondence between $\Phi_{n,0}$ and the unipotent representations appearing in the principal series* $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(1)$ *given by Lemma* 3·1.

The correspondence $\Lambda \leftrightarrow \rho[\Lambda]$ is such that $\rho[\Lambda]$ is cuspidal if and only if

$$n = \mathrm{rk}(\Lambda) = \left[\left(\frac{\mathrm{def}(\Lambda)}{2} \right)^2 \right]. \tag{3·9}$$

4. *The Decomposition of $R[\chi]$*

Let χ be an irreducible character of the Weyl group W of G. We associate a class function $R[\chi]$ of $G(\mathbb{F}_q)$ as follows

$$R[\chi] = \frac{1}{|W|}\sum_{w\in W}\chi(w)R^G_{T_w}(1). \tag{4·1}$$

This is a $\mathbb{Q}$-linear combination of unipotent representations of $G(\mathbb{F}_q)$. The decomposition of $R[\chi]$ is an important and difficult question in general which has been studied extensively by Lusztig ([**L6**]). In this section we decompose $R[\chi]$ for those χ which are non-zero at the Coxeter class. This does not seem to be explicitly available in the literature but follows easily from ([**L4**, **L5**]).

It follows from the orthogonality of $R^G_{T_w}(1)$ [**DL**, 6·8] that

$$\{R[\chi] \,|\, \chi\in\widehat{W}\} \tag{4·2}$$

forms an orthonormal set of class functions of $G(\mathbb{F}_q)$. We have the following inversion relation

$$R^G_{T_w}(1) = \sum_{\chi\in\widehat{W}}\chi(w)R[\chi]. \tag{4·3}$$

Let E be the reflection representation of W, then $\wedge^n E = \mathrm{sgn}$, where n is the semisimple rank of G. By [**DL**, 7·14] we know that,

$$R[\mathrm{Id}] = \mathrm{Id}, \quad R[\wedge^n E] = R[\mathrm{sgn}] = \mathrm{St}, \tag{4·4}$$

where St denotes the 'Steinberg' representation of $G(\mathbb{F}_q)$.

We know that the principal series unipotent representations of $G(\mathbb{F}_q)$ are in bijective correspondence with the irreducible representations of its Weyl group W. The theorem below says that in the case of the groups of type A_n the $R[\chi]$'s are exactly these representations.

THEOREM 4·1. *Let $G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q)$, and let $\pi[\chi]$ be the irreducible principal series unipotent representation of $G(\mathbb{F}_q)$ corresponding to an irreducible character χ of $W = S_{n+1}$. Then $R[\chi] = \pi[\chi]$.*

Proof. By [**A**, 2·3·1] it follows that

$$\langle \pi[\chi], R^G_{T_w}(1)\rangle = \chi(w).$$

Let $\chi, \chi' \in \widehat{W}$. Then using (4·1) we get

$$\langle R[\chi], \pi[\chi']\rangle = \begin{cases} 1 & \text{if } \chi = \chi', \\ 0 & \text{otherwise.} \end{cases} \tag{4·5}$$

Since $R[\chi]$ is a $\mathbb{Q}$-linear combination of unipotent representations of $G(\mathbb{F}_q)$, we have

$$R[\chi] = \sum_{\rho} a_\rho \rho, \tag{4·6}$$

where a_ρs are rational numbers and the sum is over the set of all unipotent representations of $G(\mathbb{F}_q)$. By (4·5), (4·6) and (4·2), we obtain

$$\left.\begin{aligned} a_{\pi[\chi]} &= 1, \\ \sum_{\rho \neq \pi[\chi]} a_\rho^2 &= \langle R[\chi] - \pi[\chi], R[\chi] - \pi[\chi] \rangle \\ &= 0. \end{aligned}\right\} \tag{4·7}$$

Therefore, we have $a_\rho = 0$ for all $\rho \neq \pi[\chi]$. Thus we have $R[\chi] = \pi[\chi]$ for all $\chi \in \widehat{W}$. We remark here that this proves a result in [**LS**] that $R[\chi]$ are all the unipotent representations of groups of type A_n. Hence the proof.

When $G(\mathbb{F}_q)$ is a group of type other than A_n then the $R[\chi]$ are more complicated. They need not be irreducible representations, nor even integral linear combination of irreducible representations. We shall briefly describe the method of decomposition of $R[\chi]$ as outlined in [**L4**] and [**L5**]. We shall first discuss the case of B_n, C_n. In the sequel we shall denote the irreducible characters of the Weyl groups by their corresponding symbol classes.

Let Z be a special symbol of rank n, and let Z_1 be the set of singles of Z. Define d by $2d + 1 = |Z_1|$. We can write $Z_1 = Z_1^* \cup (Z_1)_*$, where Z_1^* is the set of entries of Z_1 appearing in the first row of Z, and $(Z_1)_*$ is the set of entries of Z_1 appearing in the second row of Z. We have $|Z_1^*| = d + 1, |(Z_1)_*| = d$. Let Z_2 be the set of elements which appear in both rows of Z. Thus,

$$Z = \begin{pmatrix} Z_2 \cup Z_1^* \\ Z_2 \cup (Z_1)_* \end{pmatrix}. \tag{4·8}$$

Let

$$\mathscr{S}_Z = \left\{ \Lambda_M = \begin{pmatrix} Z_2 \cup (Z_1 - M) \\ Z_2 \cup M \end{pmatrix} \middle| \, M \subseteq Z_1, |M| \equiv d \pmod 2 \right\}. \tag{4·9}$$

Clearly $|\mathscr{S}_Z| = 2^{2d}$. Associating M to the set $M^\# = (M \cup (Z_1)_*) - (M \cap (Z_1)_*)$, defines a bijection between $\mathscr{S}_Z$ and the set V_{Z_1} of subsets of Z_1 of even cardinality. The set V_{Z_1} has a natural structure of $\mathbb{F}_2$-vector space of dimension $2d$, defined by $M_1^\# + M_2^\# = (M_1^\# \cup M_2^\#) - (M_1^\# \cap M_2^\#)$, with Z as the 0 element. The vector space V_{Z_1} has a natural non-degenerate symplectic form $\langle , \rangle : V_{Z_1} \times V_{Z_1} \to \mathbb{F}_2$, given by

$$\langle M_1^\#, M_2^\# \rangle = |M_1^\# \cap M_2^\#| \mod 2. \tag{4·10}$$

We can regard this also as a symplectic form on $\mathscr{S}_Z$, via the bijection between $\mathscr{S}_Z$ and V_{Z_1}. Let

$$\mathscr{F}_Z = \left\{ \rho[\Lambda_M] \middle| \, \Lambda_M \in \mathscr{S}_Z \right\}. \tag{4·11}$$

Then $\{\mathscr{F}_Z \mid Z \text{ special of rank } n\}$ defines a partition of the set of irreducible unipotent representations of $G(\mathbb{F}_q)$ into disjoint families. The following theorem is due to [**L4**, 5·8].

THEOREM 4·2. *Let $G = Sp_{2n}$, or SO_{2n+1} (defined over $\mathbb{F}_q$). Let Z be a special symbol of rank n, and let d be such that $2d+1$ is the number of singles of Z. Then for any $\Lambda \in \mathscr{S}_Z$*

of defect one, we have

$$R[\Lambda] = 2^{-d} \sum_{\Lambda' \in \mathscr{S}_Z} (-1)^{\langle \Lambda, \Lambda' \rangle} \rho[\Lambda']. \tag{4·12}$$

We now consider the case of symbols for $W(D_n)$. Let Z be a special symbol of rank n, and let Z_1 be the set of singles of Z. Define d by $2d = |Z_1|$. We define Z_1^*, $(Z_1)_*$, Z_2 and $\mathscr{S}_Z$ as before. Also we have $|(Z_1)_*| = |Z_1^*| = d$.

$$\mathscr{S}_Z = \left\{ \Lambda_M = \begin{pmatrix} Z_2 \cup (Z_1 - M) \\ Z_2 \cup M \end{pmatrix} \middle| \, M \subseteq Z_1, |M| \equiv d \pmod 2 \right\}. \tag{4·13}$$

Clearly $|\mathscr{S}_Z| = 2^{2(d-1)}$. Associating M to the set $M^\# = (M \cup (Z_1)_*) - (M \cap (Z_1)_*)$, defines a bijection between $\mathscr{S}_Z$ and the set V_{Z_1} of $\mathbb{F}_2$-vector spaces of dimension $2(d-1)$. As before, we can endow $\mathscr{S}_Z$ with a non-degenerate symplectic form. Let

$$\mathscr{F}_Z = \left\{ \rho[\Lambda_M] \middle| \, \Lambda_M \in \mathscr{S}_Z \right\}. \tag{4·14}$$

Then $\{\mathscr{F}_Z \mid Z \text{ special of rank } n\}$ defines a partition of the set of irreducible unipotent representations of $G(\mathbb{F}_q)$ into disjoint families. The following theorem is also due to [**L5**, 3·15].

THEOREM 4·3. *Let $G = SO_{2n}$ (defined over $\mathbb{F}_q$). Let Z be a special symbol of rank n, and let d be such that $2d$ is the number of singles of Z. Then for any $\Lambda \in \mathscr{S}_Z$ of defect zero, we have*

$$R[\Lambda] = 2^{-(d-1)} \sum_{\Lambda' \in \mathscr{S}_Z} (-1)^{\langle \Lambda, \Lambda' \rangle} \rho[\Lambda']. \tag{4·15}$$

Theorems 4·2 and 4·3 were initially proved by Lusztig for large q, but later on the restriction on the order of q was removed by him in [**L6**].

Let Λ be the symbol corresponding to the irreducible character χ of W. As we are interested in decomposing $R[\chi]$ for those $\chi \in \widehat{W}$ which are non-zero at the Coxeter class, we first determine the corresponding symbol classes.

From the proof of Theorem 2·1 we know that the symbols corresponding to the irreducible characters of $W(B_n)$ which are non-zero at the Coxeter element are

$$\left.\begin{aligned} \Lambda_0 &= \begin{pmatrix} n \\ \emptyset \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} 1,2,\ldots,i,n \\ 0,1,\ldots,i-1 \end{pmatrix} \quad (1 \leqslant i \leqslant n-1), \\ \Lambda'_0 &= \begin{pmatrix} 0,1 \\ n \end{pmatrix}, \quad \Lambda'_i = \begin{pmatrix} 0,1,\ldots,i+1 \\ 1,2,\ldots,i,n \end{pmatrix} \quad (1 \leqslant i \leqslant n-1), \end{aligned}\right\} \tag{4·16}$$

where Λ_0 corresponds to the ordered partition $(n, 0)$, Λ_i corresponds to the ordered partition $(\lambda_i, 0)$ where $\lambda_i = [1^i, n-i]$, $(1 \leqslant i \leqslant n-1)$, while Λ'_0 corresponds to the ordered partition $(0, n)$ and Λ'_i corresponds to the ordered partition $(0, \lambda_i)$. Therefore, we have

$$\left.\begin{aligned} \Lambda_0(c) &= 1, \qquad \Lambda_i(c) = (-1)^i \qquad (1 \leqslant i \leqslant n-1), \\ \Lambda'_0(c) &= -1, \qquad \Lambda'_i(c) = (-1)^{i+1} \qquad (1 \leqslant i \leqslant n-1). \end{aligned}\right\} \tag{4·17}$$

Similarly, the symbol classes corresponding to the irreducible characters of $W(D_n)$

which are non-zero at the Coxeter element are

$$\left.\begin{aligned} &\Lambda_0 = \binom{n}{0}, \quad \Lambda_{n-2} = \binom{1, 2, \ldots, n}{0, 1, \ldots, n-1}, \\ &\Lambda_i = \binom{1, 2, \ldots, i-1, i+1, n-1}{0, 1, \ldots, i}, \qquad (1 \leqslant i \leqslant n-3), \end{aligned}\right\} \tag{4·18}$$

and

$$\Lambda_0' = \binom{n-1}{1}, \quad \Lambda_i' = \binom{1, 2, \ldots, i, n-1}{0, 1, \ldots, i-1, i+1} \qquad (1 \leqslant i \leqslant n-2), \tag{4·19}$$

where Λ_0 corresponds to the unordered partition $(n, 0)$, Λ_{n-2} corresponds to the unordered partition $(1^n, 0)$ and Λ_i corresponds to the unordered partition $(\mu_i, 0)$ where $\mu_i = [1^{i-1}, 2, n-1-i]$, $(1 \leqslant i \leqslant n-3)$, while Λ_0' corresponds to the unordered partition $(n-1, 1)$ and Λ_i' corresponds to the unordered partition $(\lambda_i', 1)$ where $\lambda_i' = [1^i, n-1-i]$, $(1 \leqslant i \leqslant n-2)$. Therefore, we have

$$\left.\begin{aligned} &\Lambda_0(c) = 1, \quad \Lambda_{n-2}(c) = (-1)^{n-2}, \quad \Lambda_i(c) = (-1)^i \qquad (1 \leqslant i \leqslant n-3), \\ &\Lambda_0'(c) = -1, \quad \Lambda_i'(c) = (-1)^{i+1} \qquad (1 \leqslant i \leqslant n-2). \end{aligned}\right\} \tag{4·20}$$

We also need the symbol classes corresponding to the exterior power representation of the reflection representation of $W(B_n)$ and $W(D_n)$. We know that for $W(B_n)$, $\wedge^i E$ is given by the ordered partition $(n-i, 1^i)$. The corresponding symbol classes are

$$\Gamma_{n,0} = \Lambda_0 = \binom{n}{0}, \quad \Gamma_{n,i} = \binom{0, 1, \ldots, i-1, n}{1, 2, \ldots, i} \qquad (1 \leqslant i \leqslant n). \tag{4·21}$$

Similarly, for $W(D_n)$, $\wedge^i E$ is given by the unordered partition $(n-i, 1^i)$. The corresponding symbol classes are

$$\left.\begin{aligned} &\Gamma_{n,0}' = \Lambda_0 = \binom{n}{0}, \quad \Gamma_{n,1}' = \Lambda_0' = \binom{n-1}{1}, \\ &\Gamma_{n,i}' = \binom{0, 1, \ldots, i-2, n-1}{1, 2, \ldots, i}, \qquad (2 \leqslant i \leqslant n). \end{aligned}\right\} \tag{4·22}$$

It is easy to see that $\Gamma_{n,i}$ $(0 \leqslant i \leqslant n)$ and $\Gamma_{n,i}'$ $(0 \leqslant i \leqslant n)$ are special symbols. Apart from these symbols, we also need some symbols of rank n not corresponding to the representations of $W(B_n), W(D_n)$. Define X_i for B_n and C_n, as

$$X_i = \binom{0, 1, \ldots, i+1, n}{1, 2, \ldots, i}, \qquad (0 \leqslant i \leqslant n-2). \tag{4·23}$$

Define the symbols Y_i for D_n as

$$Y_i = \binom{0, 1, \ldots, i+2, n-1}{1, 2, \ldots, i}, \qquad (0 \leqslant i \leqslant n-4). \tag{4·24}$$

Let $\rho[X_i]$ be the unipotent representation of $Sp(2n, \mathbb{F}_q)$ or $SO(2n+1, \mathbb{F}_q)$ corresponding to the symbol X_i, and let $\rho[Y_i]$ be the unipotent representation of $SO(2n, \mathbb{F}_q)$ corresponding to the symbol Y_i. The following theorem gives the decomposition of $R[\Lambda]$ for those Λ whose character at the Coxeter class is non-zero.

Recall (see the proof of Theorem 2·1) that in the case of $W(B_n)$ the symbol Λ_i corresponds to the irreducible representation obtained by composing the $\wedge^i E_n$ with the homomorphism $W(B_n) \to S_n$, Λ'_i corresponds to irreducible representation obtained by tensoring $\wedge^i E_n$ with ε, where ε is as defined in (2·1), and E_n denotes the $(n-1)$-dimensional reflection representation of S_n. In the case of $W(D_n)$ the symbol Λ_i $(1 \leqslant i \leqslant n-3)$ corresponds to the irreducible character $\bar{\chi}_{\lambda_i}$ as defined in (2·16), and Λ_0, Λ_{n-2} correspond to the irreducible characters $\bar{\chi}_{\lambda_0}, \bar{\chi}_{\lambda_{n-2}}$ respectively, as defined in (2·17), and Λ'_i corresponds to the irreducible character $\chi_{(\lambda'_i,1)}$ as defined in (2·19). Also recall that the symbol $\Gamma_{n,i}$ corresponds to the ith exterior power representation of the reflection representation of $W(B_{n-2})$, and the symbol $\Gamma'_{n,i}$ corresponds to the i-th exterior power representation of the reflection representation of $W(D_n)$.

THEOREM 4·4. *Let $G = Sp_{2n}, SO_{2n+1}$, or SO_{2n} defined over $\mathbb{F}_q$. We denote by $\pi[\Lambda]$ the principal series representation of $G(\mathbb{F}_q)$ corresponding to the Weyl group representation denoted by the symbol Λ. With the notations as above, we have the following.*

(1) *Let $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$, or $SO(2n+1, \mathbb{F}_q)$,* $(n \geqslant 2)$.

$$\left.\begin{array}{ll}(a)\ R[\Lambda_0] = \pi[\Lambda_0] = \mathrm{Id}. & \\ (b)\ R[\Lambda_i] = \frac{1}{2}\left(\pi[\Lambda_i] + \pi[\wedge^i E] - \pi[\Lambda'_{i-1}] - \rho[X_{i-1}]\right), & (1 \leqslant i \leqslant n-1). \\ (c)\ R[\Lambda'_i] = \frac{1}{2}\left(-\pi[\Lambda_{i+1}] + \pi[\wedge^{i+1} E] + \pi[\Lambda'_i] - \rho[X_i]\right), & (0 \leqslant i \leqslant n-2). \\ (d)\ R[\Lambda'_{n-1}] = \pi[\wedge^n E] = \mathrm{St}. & \end{array}\right\} \quad (4·25)$$

(2) *Let $G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q)$,* $(n \geqslant 4)$.

$$\left.\begin{array}{ll}(a)\ R[\Lambda_0] = \pi[\Lambda_0] = \mathrm{Id}. & \\ (b)\ R[\Lambda_{n-2}] = \pi[\wedge^n E] = \mathrm{St}. & \\ (c)\ R[\Lambda_i] = \frac{1}{2}\left(\pi[\Lambda_i] - \pi[\wedge^{i+1} E] + \pi[\Lambda'_i] - \rho[Y_{i-1}]\right), & (1 \leqslant i \leqslant n-3). \\ (d)\ R[\Lambda'_i] = \frac{1}{2}\left(\pi[\Lambda_i] + \pi[\wedge^{i+1} E] + \pi[\Lambda'_i] + \rho[Y_{i-1}]\right), & (1 \leqslant i \leqslant n-3). \\ (e)\ R[\Lambda'_0] = \pi[E]. & \\ (f)\ R[\Lambda'_{n-2}] = \pi[\wedge^{n-1} E]. & \end{array}\right\} \quad (4·26)$$

Proof. (1) $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$, or $SO(2n+1, \mathbb{F}_q)$.

(a) Since $\Lambda_0 = \Gamma_0$, $R[\Lambda_0] = R[\Gamma_0] = R[\mathrm{Id}] = \mathrm{Id}$.

(b) We have the following data for $\Gamma_{n,i}$ which is a special symbol of rank n

$$Z_1 = \{0, i, n\}, \quad Z_2 = \{1, 2, \ldots, i-1\}, \quad Z_1^* = \{0, n\}, \quad (Z_1)_* = \{i\}, \quad d = 1. \quad (4·27)$$

It can be seen that,

$$\mathscr{S}_{\Gamma_{n,i}} = \left\{\Lambda_i,\ \Gamma_{n,i},\ \Lambda'_{i-1},\ X_{i-1}\right\}. \quad (4·28)$$

Applying Theorem 4·2, we obtain

$$\left.\begin{aligned} R[\Lambda_i] &= \frac{1}{2} \sum_{\Lambda' \in \mathscr{S}_{\Gamma_{n,i}}} (-1)^{\langle \Lambda_i, \Lambda' \rangle} \rho[\Lambda'] \\ &= \tfrac{1}{2}\left(\pi[\Lambda_i] + \pi[\wedge^i E] - \pi[\Lambda'_{i-1}] - \rho[X_{i-1}]\right). \end{aligned}\right\} \quad (4·29)$$

(c) Similarly, applying Theorem 4·2, we obtain

$$\left.\begin{aligned} R[\Lambda'_i] &= \frac{1}{2}\sum_{\Lambda'\in\mathscr{S}_{\Gamma_{n,i+1}}} (-1)^{\langle\Lambda'_i,\Lambda'\rangle}\rho[\Lambda'] \\ &= \tfrac{1}{2}\left(\pi[\Lambda'_i]+\pi[\wedge^{i+1}E]-\pi[\Lambda_{i+1}]-\rho[X_i]\right). \end{aligned}\right\} \tag{4·30}$$

(d) Since $\Lambda'_{n-1}=\Gamma_{n,n}$, $R[\Lambda'_{n-1}]=R[\Gamma_{n,n}]=R[\wedge^n E]=\pi[\wedge^n E]=\mathrm{St}$.

(2) $G(\mathbb{F}_q)=SO(2n,\mathbb{F}_q)$.

(a) Since $\Lambda_0=\Gamma'_{n,0}$, $R[\Lambda_0]=R[\Gamma'_{n,0}]=R[\mathrm{Id}]=\mathrm{Id}$.
(b) Since $\Lambda_{n-2}=\Gamma'_{n,n}$, $R[\Lambda_{n-2}]=R[\Gamma'_{n,n}]=R[\wedge^n E]=\mathrm{St}$.
(c) We have the following data for $\Gamma'_{n,i+1}$ which is a special symbol of rank n

$$\left.\begin{aligned} &Z_1=\{0,i,i+1,n-1\}, \quad Z_2=\{1,2,\ldots,i-1\}, \\ &Z_1^*=\{0,n-1\}, \quad (Z_1)_*=\{i,i+1\}, \quad d=2. \end{aligned}\right\} \tag{4·31}$$

It can be seen that,

$$\mathscr{S}_{\Gamma'_{n,i+1}}=\left\{\Lambda_i,\,\Gamma'_{n,i+1},\,\Lambda'_i,\,Y_{i-1}\right\}. \tag{4·32}$$

Applying Theorem 4·3, we obtain

$$\left.\begin{aligned} R[\Lambda_i] &= \frac{1}{2}\sum_{\Lambda'\in\mathscr{S}_{\Gamma'_{n,i+1}}} (-1)^{\langle\Lambda_i,\Lambda'\rangle}\rho[\Lambda'] \\ &= \tfrac{1}{2}\left(\pi[\Lambda_i]-\pi[\wedge^{i+1}E]+\pi[\Lambda'_i]-\rho[Y_{i-1}]\right). \end{aligned}\right\} \tag{4·33}$$

(d) Similarly, applying Theorem 4·3, we obtain

$$\left.\begin{aligned} R[\Lambda'_i] &= \frac{1}{2}\sum_{\Lambda'\in\mathscr{S}_{\Gamma'_{n,i+1}}} (-1)^{\langle\Lambda'_i,\Lambda'\rangle}\rho[\Lambda'] \\ &= \tfrac{1}{2}\left(\pi[\Lambda_i]+\pi[\wedge^{i+1}E]+\pi[\Lambda'_i]+\rho[Y_{i-1}]\right). \end{aligned}\right\} \tag{4·34}$$

(e) We have the following data for $\Lambda'_0=\Gamma'_{n,1}$ which is a special symbol of rank n

$$Z_1=\{1,n-1\}, \quad Z_2=\emptyset, \quad Z_1^*=\{n-1\}, \quad (Z_1)_*=\{1\}, \quad d=1. \tag{4·35}$$

It can be seen that,

$$\mathscr{S}_{\Gamma'_{n,1}}=\left\{\Gamma'_{n,1}\right\}. \tag{4·36}$$

Applying Theorem 4·3, we obtain

$$\left.\begin{aligned} R[\Lambda'_0] &= \sum_{\Lambda'\in\mathscr{S}_{\Gamma'_{n,1}}} (-1)^{\langle\Lambda'_0,\Lambda'\rangle}\rho[\Lambda'] \\ &= \pi[E]. \end{aligned}\right\} \tag{4·37}$$

(f) We have the following data for $\Lambda'_{n-2}=\Gamma'_{n,n-1}$ which is a special symbol of rank n

$$Z_1=\{0,n-2\}, \quad Z_2=\{1,2,\ldots,n-3\}, \quad Z_1^*=\{0\}, \quad (Z_1)_*=\{n-2\}, \quad d=1. \tag{4·38}$$

It can be seen that,

$$\mathscr{S}_{\Gamma'_{n,n-1}} = \left\{\Gamma'_{n,n-1}\right\}. \tag{4·39}$$

Applying Theorem 4·3, we obtain

$$\left.\begin{aligned} R[\Lambda'_{n-2}] &= \sum_{\Lambda' \in \mathscr{S}_{\Gamma'_{n,n-1}}} (-1)^{\langle \Lambda'_{n-2}, \Lambda' \rangle} \rho[\Lambda'] \\ &= \pi[\wedge^{n-1} E]. \end{aligned}\right\} \tag{4·40}$$

Hence the proof.

5. *The decomposition of induced representation*

By Theorem 3·1 we know that $\rho[X_i]$ is not cuspidal unless $i = 0, n = 2$. It occurs as an irreducible component of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\rho}_L)$ for some proper parabolic subgroup P defined over $\mathbb{F}_q$ and some irreducible cuspidal representation ρ_L of $L(\mathbb{F}_q)$ where L is the Levi component of P. By $\tilde{\rho}_L$ we mean the lift of ρ_L to $P(\mathbb{F}_q)$ by extending it trivially to the unipotent radical of $P(\mathbb{F}_q)$. As $\rho[X_i]$ is unipotent, we know that the representation ρ_L must also be unipotent (see [**C1**, 12·2]). Since X_i is of defect 3, by [**L2**, 8] we know that $L \cong (\mathbf{G}_m)^{n-2} \times Sp_4$, or $L \cong (\mathbf{G}_m)^{n-2} \times SO_5$, as the case may be. Let $x = (x_1, x'_1, x_2, x'_2, \ldots, x_{n-2}, x'_{n-2}, g) \in (\mathbb{F}_q^*)^{2n-4} \times Sp(4, \mathbb{F}_q)$, or $(x_1, x'_1, x_2, x'_2, \ldots, x_{n-2}, x'_{n-2}, g) \in (\mathbb{F}_q^*)^{2n-4} \times SO(5, \mathbb{F}_q)$, with the condition that $x_1 x'_1 = \cdots = x_{n-2} x'_{n-2} = 1$. Since ρ_L is unipotent cuspidal, and Sp_4 and SO_5 have a unique unipotent cuspidal representation π_{uc} we obtain,

$$\mathrm{Tr}(x, \rho_L) = \mathrm{Tr}(g, \pi_{\mathrm{uc}}). \tag{5·1}$$

We abuse notation to denote ρ_L by π_{uc}. By [**L2**, 5·15] there is a bijective correspondence between irreducible representations of $G(\mathbb{F}_q)$ appearing in $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\pi}_{\mathrm{uc}})$ and irreducible representations of the Weyl group $W(B_{n-2})$ which is given by Proposition 3·1. Since $X_i \in \Phi_{n,3}$ corresponds to $\Gamma_{n-2,i} \in \Phi_{n-2,1}$, and by (4·21) we know that $\Gamma_{n-2,i}$ corresponds to ith exterior power representation of the reflection representation of $W(B_{n-2})$, the irreducible component $\rho[X_i]$ of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\pi}_{\mathrm{uc}})$ corresponds to ith exterior power representation of the reflection representation of $W(B_{n-2})$.

The case of $\rho[Y_i]$ is similar. Since Y_i is of defect 4, it can be seen that $\rho[Y_i]$ is an irreducible component of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\pi}_{\mathrm{uc}})$ corresponding to the ith exterior power representation of the reflection representation of $W(B_{n-4})$, where P is the parabolic subgroup with Levi component $L \cong (\mathbf{G}_m)^{n-4} \times SO_8$, and π_{uc} is the unique unipotent cuspidal representation of $SO(8, \mathbb{F}_q)$. Therefore, we have

PROPOSITION 5·1. *Let X_i and Y_i be defined as above. The unipotent representation $\rho[X_i]$ of $Sp(2n, \mathbb{F}_q)$ or $SO(2n+1, \mathbb{F}_q)$ is the irreducible component of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\pi}_{\mathrm{uc}})$ corresponding to the* ith *exterior power representation of the reflection representation of $W(B_{n-2})$, where π_{uc} is the unique unipotent cuspidal representation of $Sp(4, \mathbb{F}_q)$ or $SO(5, \mathbb{F}_q)$, as the case may be. Similarly, the unipotent representation $\rho[Y_i]$ of $SO(2n, \mathbb{F}_q)$ is the irreducible component of $\mathrm{Ind}_{P(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\pi}_{\mathrm{uc}})$ corresponding to the* ith *exterior power representation of the reflection representation of $W(B_{n-4})$, where π_{uc} is the unique unipotent cuspidal representation of $SO(8, \mathbb{F}_q)$.*

6. *Proof of the Main Theorem*

Let G be a split group of semisimple rank n, W its Weyl group, c the Coxeter conjugacy class, and T_c the corresponding torus. By [**DL**, 8·3] we know that $\Psi_\theta = (-1)^n R^G_{T_c}(\theta)$ is an irreducible cuspidal representation of $G(\mathbb{F}_q)$ for a character θ of $T_c(\mathbb{F}_q)$ in general position. By [**DL**, 4·2] we also know that for an unipotent element u of $G(\mathbb{F}_q)$, $\mathrm{Tr}(u, R^G_T(\theta))$ is independent of θ. Therefore,

$$\Theta_{\Psi_\theta}(u) = (-1)^n \mathrm{Tr}(u, R^G_{T_c}(1)). \tag{6·1}$$

Let $G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q)$. Let us denote $\pi[\wedge^i E]$ by π_i. Applying Proposition 2·1 in (4·3), we obtain

$$R^G_{T_c}(1) = \sum_{i=0}^{i=n} (-1)^i R[\chi_i]. \tag{6·2}$$

Now applying Theorem 4·1, we obtain

$$\left.\begin{aligned} R^G_{T_c}(1) &= \sum_{i=0}^{i=n} (-1)^i \pi[\wedge^i E] \\ &= \sum_{i=0}^{i=n} (-1)^i \pi_i. \end{aligned}\right\} \tag{6·3}$$

By (6·3) and (6·1), we obtain

$$\left.\begin{aligned} \Theta_{\Psi_\theta}(u) &= (-1)^n \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_i}(u) \\ &= \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u). \end{aligned}\right\} \tag{6·4}$$

This completes the proof of Theorem 1·1 for $GL(n+1, \mathbb{F}_q)$.

We now take up the case of $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$ or $SO(2n+1, \mathbb{F}_q)$. Applying (4·17) in (4·3), we obtain

$$\left.\begin{aligned} R^G_{T_c}(1) &= \sum_{i=0}^{i=n-1} (-1)^i R[\Lambda_i] + \sum_{i=0}^{i=n-1} (-1)^{i+1} R[\Lambda'_i] \\ &= \left(R[\Lambda_0] + (-1)^n R[\Lambda'_{n-1}]\right) + \sum_{i=1}^{i=n-1} (-1)^i \left(R[\Lambda_i] + R[\Lambda'_{i-1}]\right). \end{aligned}\right\} \tag{6·5}$$

From Theorem 4·4 it follows that

$$\left.\begin{aligned} R[\Lambda_0] + (-1)^n R[\Lambda'_{n-1}] &= \pi[\wedge^0 E] + (-1)^n \pi[\wedge^n E], \\ R[\Lambda_i] + R[\Lambda'_{i-1}] &= \pi[\wedge^i E] - \rho[X_{i-1}]. \end{aligned}\right\} \tag{6·6}$$

Let us denote $\pi[\wedge^i E]$ by π_i and $\rho[X_i]$ by ρ_i. By Proposition 5·1, ρ_i is the irreducible component of $\mathrm{Ind}^{G(\mathbb{F}_q)}_{P(\mathbb{F}_q)}(\pi_{\mathrm{uc}})$ corresponding to the ith exterior power representation of the reflection representation of $W(B_{n-2})$, where P is the parabolic subgroup defined over $\mathbb{F}_q$ with the Levi component $(G_m)^{n-2} \times Sp_4$ or $(G_m)^{n-2} \times SO_5$, and π_{uc} is the unique unipotent cuspidal representation of $Sp(4, \mathbb{F}_q)$ or $SO(5, \mathbb{F}_q)$, as the case may be. Using (6·6), we obtain

$$\left.\begin{aligned} R^G_{T_c}(1) &= \left(\pi[\wedge^0 E] + (-1)^n \pi[\wedge^n E]\right) + \sum_{i=1}^{i=n-1} (-1)^i \left(\pi[\wedge^i E] - \rho[X_{i-1}]\right) \\ &= \sum_{i=0}^{i=n} (-1)^i \pi[\wedge^i E] - \sum_{i=1}^{i=n-1} (-1)^i \rho[X_{i-1}] \\ &= \sum_{i=0}^{i=n} (-1)^i \pi_i + \sum_{i=0}^{i=n-2} (-1)^i \rho_i. \end{aligned}\right\} \qquad (6·7)$$

By (6·7) and (6·1), we obtain

$$\left.\begin{aligned} \Theta_{\Psi_\theta}(u) =& (-1)^n \left(\sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_i}(u) + \sum_{i=0}^{i=n-2} (-1)^i \Theta_{\rho_i}(u)\right) \\ =& \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u) + \sum_{i=0}^{i=n-2} (-1)^i \Theta_{\rho_{n-2-i}}(u). \end{aligned}\right\} \qquad (6·8)$$

This completes the proof of Theorem 1·1 for $Sp(2n, \mathbb{F}_q)$ and $SO(2n+1, \mathbb{F}_q)$.

Finally, let $G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q)$. Again denote $\pi[\wedge^i E]$ by π_i and $\rho[Y_i]$ by ρ_i. By Proposition 5·1, ρ_i is the irreducible component of $\mathrm{Ind}^{G(\mathbb{F}_q)}_{P(\mathbb{F}_q)}(\pi_{\mathrm{uc}})$ corresponding to the i-th exterior power representation of the reflection representation of $W(B_{n-4})$, where P is the parabolic subgroup defined over $\mathbb{F}_q$ with the Levi component $(G_m)^{n-4} \times SO_8$ and π_{uc} is the unique unipotent cuspidal representation of $SO(8, \mathbb{F}_q)$. By calculations similar to those done in the previous case yields the following result for $SO(2n, \mathbb{F}_q)$.

$$R^G_{T_c}(1) = \sum_{i=0}^{i=n} (-1)^i \pi_i + \sum_{i=0}^{i=n-4} (-1)^i \rho_i. \qquad (6·9)$$

By (6·9) and (6·1), we obtain

$$\Theta_{\Psi_\theta}(u) = \sum_{i=0}^{i=n} (-1)^i \Theta_{\pi_{n-i}}(u) + \sum_{i=0}^{i=n-4} (-1)^i \Theta_{\rho_{n-4-i}}(u). \qquad (6·10)$$

This completes the proof of Theorem 1·1.

Remark 6·1. The only property of $R^G_T(\theta)$ used in the proof of Theorem 1·1 in this section is that

$$\mathrm{Tr}(g, R^G_T(\theta)) = \mathrm{Tr}(g, R^G_T(1)).$$

This property is known for all elements $g \in G(\mathbb{F}_q)$ with Jordan decomposition $g = su$ such that either $s = 1$ or s is not conjugate in $G(\mathbb{F}_q)$ to any element of $T(\mathbb{F}_q)$. Therefore the character identity contained in Theorem 1·1 is true for all such elements. In particular, the identity in Theorem 1·1 is true for any element of $(n, 1)$ parabolic in GL_{n+1}.

Remark 6·2. In the case of $GL(n+1, \mathbb{F}_q)$ all the irreducible cuspidal representations are given by $(-1)^n R^G_{T_c}(\theta)$ where $c = \sigma_{n+1}$ is the Coxeter element and θ is a regular character of $T_c(\mathbb{F}_q) = \mathbb{F}^*_{q^{n+1}}$. Thus the character value of any irreducible cuspidal representation of $GL(n+1, \mathbb{F}_q)$ at any element of $(n, 1)$ parabolic is given by (6·4).

7. *The case of exceptional groups*

In this section we shall give the decomposition of $R^G_{T_c}(1)$ in the case of split simple exceptional algebraic groups. We shall follow the notations of [**C1**, 13·9] for the unipotent cuspidal representations of $G(\mathbb{F}_q)$.

The class functions R_χ can be decomposed using the method of non-abelian Fourier transforms as introduced in [**L3**]. By [**L3**, theorem 1·5] and [**L3**, corollary 1·13] the multiplicities $\langle \rho, R_\chi \rangle$ can be explicitly described, where ρ is any unipotent representation of $G(\mathbb{F}_q)$. See [**C1**, 13·6] for the Fourier transform matrices.

Let G be a split exceptional simple algebraic group. Let (P, ϕ) be a pair of parabolic subgroup in G containing a fixed Borel subgroup B with Levi decomposition $P = MN$, and a unipotent cuspidal representation ϕ of $M(\mathbb{F}_q)$. The irreducible components of $\mathrm{Ind}^{G(\mathbb{F}_q)}_{P(\mathbb{F}_q)}(\phi)$ are in one to one correspondence with the irreducible representations of the Weyl group W' of the quotient root system which is a root system of a simple group of rank $= r(G) - r(P)$, where $r(G)$ and $r(P)$ denote the semisimple ranks of G and P respectively. Denote by ϕ_i the irreducible components of $\mathrm{Ind}^{G(\mathbb{F}_q)}_{P(\mathbb{F}_q)}(\phi)$ corresponding to the i-th exterior power representation of the reflection representation of W'.

THEOREM 7·1. *With the notations as above,* (i) *if G is a simple algebraic group of type E_6, E_7, then,*

$$R^G_{T_c}(1) = \sum_{(P,\phi)} (-1)^{r(P)} \sum_{i=0}^{i=r(G)-r(P)} (-1)^i \phi_i, \tag{7·1}$$

where ϕ runs over all the unipotent cuspidal representations of $M(\mathbb{F}_q)$.

(ii) *If $G = G_2$, then same as in* (i) *except that the term corresponding to $P = G$ has instead of all the* 4 *unipotent cuspidal representations of $G_2(\mathbb{F}_q)$, only* 3 *which can be specified as $G_2[-1] + G_2[\theta] + G_2[\theta^2]$ following Carter's notation* [**C1**, 13·9].

(iii) *If $G = F_4$, then same as in* (i) *except that the term corresponding to $P = G$ has instead of all the* 7 *unipotent cuspidal representations of $F_4(\mathbb{F}_q)$, only* 4 *which can be specified as $F_4[\theta], F_4[\theta^2], F_4[i], F_4[-i]$ following the notations in* [**C1**, 13·9].

(iv) *If $G = E_8$, then same as in* (i) *except that the term corresponding to $P = G$ has instead of all the* 13 *unipotent cuspidal representations of $E_8(\mathbb{F}_q)$, only* 6 *which can be specified as $E_8[\zeta^i] (i = 1, \ldots, 4), E_8[\theta], E_8[\theta^2]$ following the notations in* [**C1**, 13·9].

To illustrate Theorem 7·1 we take the case of $G = E_7$. The Levi subgroups of E_7 which have unipotent cuspidal representations are $L_0 \cong (\mathbf{G}_m)^7$, $L_1 \cong SO_8 \times (\mathbf{G}_m)^3$, $L_2 \cong E_6 \times \mathbf{G}_m$ and $L_3 = G$.

The quotient root system arising from L_0 is the root system of type E_7, and $\phi = 1$ is the unique unipotent cuspidal representation of $({\mathbb{F}_q}^*)^7$. Hence, $\phi_i = \pi_i$ is the irreducible component of $\mathrm{Ind}^{G(\mathbb{F}_q)}_{B(\mathbb{F}_q)}(1)$ corresponding to the ith exterior power representation of the reflection representation of $W(E_7)$.

The quotient root system arising from L_1 is of type C_3. Let $\phi = \pi_{\mathrm{uc}}$ be the unique unipotent cuspidal representation of $SO(8, \mathbb{F}_q)$. Let $\phi_i[D_4]$ be the irreducible component of $\mathrm{Ind}^G_{P_1}(\phi)$ corresponding to the ith exterior power representation of the reflection representation of $W(C_3)$ for $i = 0, 1, 2, 3$.

The quotient root system arising from $L_2 = E_6 \times \mathbf{G}_m$ is of type A_1. Let $\phi' = E_6[\theta]$ and $\phi'' = E_6[\theta^2]$ be the two unipotent cuspidal representations $E_6(\mathbb{F}_q)$. Let $\phi'_i[E_6]$ and

$\phi_i''[E_6]$ be the irreducible components of $\mathrm{Ind}_{P_2(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\phi')$ and $\mathrm{Ind}_{P_2(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\phi')$ respectively corresponding to the i-th exterior power of the reflection representation of of $W(A_1)$ for $i = 0, 1$.

When $L = L_3 = G$, we have two unipotent cuspidal representations of $E_7(\mathbb{F}_q)$ denoted by $E_7[\zeta]$ and $E_7[-\zeta]$ as in [**C1**, 13·9]. Then, by Theorem 7·1 we get,

$$\left.\begin{aligned} R_{T_c}^G(1) = &\sum_{i=0}^{i=7}(-1)^i\pi_i + \sum_{i=0}^{i=3}(-1)^i\phi_i[D_4] \\ &+ \sum_{i=0}^{i=1}(-1)^i\phi_i'[E_6] + \sum_{i=0}^{i=1}(-1)^i\phi_i''[E_6] \\ &- (E_7[\zeta] + E_7[-\zeta]). \end{aligned}\right\} \quad (7{\cdot}2)$$

8. *Cohomological interpretation*

The virtual representation $R_{T_w}^G(1)$ is defined in [**DL**] as

$$R_{T_w}^G(1) = \sum(-1)^i H_c^i(X(w), \bar{\mathbb{Q}}_l) \quad (8{\cdot}1)$$

where $X(w)$ is the Deligne-Lusztig variety as defined in [**DL**, 1·4]. When $w = c$ is the Coxeter element, Lusztig ([**L1**]) has given the decomposition of $H_c^i(X(c), \bar{\mathbb{Q}}_l)$ as irreducible $G(\mathbb{F}_q)$-modules. In the sequel, we will denote $H_c^i(X(c), \bar{\mathbb{Q}}_l)$ by H^i. When G is simple, he proves that ([**L1**, 6·1]) $\bigoplus H^i$ is a multiplicity free $G(\mathbb{F}_q)$-module, and it has h irreducible components, where h is the Coxeter number of G.

Let Δ be the set of simple roots of G and I be a subset of Δ. Let L_I be the Levi subgroup of the parabolic subgroup P_I corresponding to I. We put $H_I^i = H_c^i(X(c_I), \bar{\mathbb{Q}}_l)$, where c_I is the Coxeter element of W_I, and $X(c_I)$ is the corresponding Deligne-Lusztig variety of L_I. Let n be the semisimple rank of G. Let $(H^i)^{(0)}$ denote the cuspidal part of H^i. The following results are due to Lusztig.

Proposition 8·1 [**L1**, 2·9]. *$H^i = 0$, unless $n \leqslant i \leqslant 2n$.*

Proposition 8·2 [**L1**, 4·3]. *$(H^i)^{(0)} = 0$ for $i \neq n$.*

Let I be a proper subset of Δ. Let ρ be an irreducible cuspidal $L_I(\mathbb{F}_q)$-module. Let M be a $G(\mathbb{F}_q)$-module. We define $M[\rho]$ to be the largest submodule of M such that each irreducible representation in $M[\rho]$ is contained in $\mathrm{Ind}_{P_I(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\rho})$. The irreducible components of $\mathrm{Ind}_{P_I(\mathbb{F}_q)}^{G(\mathbb{F}_q)}(\tilde{\rho})$ are in one to one correspondence with the irreducible representations of the Weyl group $\bar{W}$ of the quotient root system corresponding to (L_I, ρ) (see [**L1**, 5·9]).

Proposition 8·3 [**L1**, 6·7, 7·8]. *Let I be a subset of Δ, and let ρ be an irreducible cuspidal representation of $L_I(\mathbb{F}_q)$ occurring in $H_I^{|I|}$. Then the $G(\mathbb{F}_q)$-module $H^i[\rho]$, $(n \leqslant i \leqslant 2n - |I|)$ is irreducible and corresponds to the* $(2n - i - |I|)$th *exterior power representation of the reflection representation of $\bar{W}$.*

Proposition 8·4. *Let G be a split classical group. We preserve the notations of Section* 1.

(*a*) *For $G(\mathbb{F}_q) = GL(n+1, \mathbb{F}_q) \quad (n \geqslant 0)$,*

$$H^{n+i} = \pi_{n-i} \qquad (0 \leqslant i \leqslant n). \quad (8{\cdot}2)$$

(*b*) *For* $G(\mathbb{F}_q) = Sp(2n, \mathbb{F}_q)$ *or* $SO(2n+1, \mathbb{F}_q)$ $(n \geqslant 2)$,

$$\left.\begin{aligned} H^{n+i} &= \pi_{n-i} \oplus \rho_{n-2-i} \qquad (0 \leqslant i \leqslant n-2), \\ H^{n+i} &= \pi_{n-i} \qquad (n-1 \leqslant i \leqslant n). \end{aligned}\right\} \tag{8·3}$$

(*c*) *For* $G(\mathbb{F}_q) = SO(2n, \mathbb{F}_q)$ $(n \geqslant 4)$,

$$\left.\begin{aligned} H^{n+i} &= \pi_{n-i} \oplus \rho_{n-4-i} \qquad (0 \leqslant i \leqslant n-4), \\ H^{n+i} &= \pi_{n-i} \qquad (n-3 \leqslant i \leqslant n). \end{aligned}\right\} \tag{8·4}$$

Proof. (*a*) Let I be the empty subset of Δ, and let ρ be the trivial representation of $T(\mathbb{F}_q)$, where T is the split maximal torus of G. Then by Proposition 8·3 we know that $H^{n+i}[1] = \pi_{n-i}$ $(0 \leqslant i \leqslant n)$. As there are $n+1$ irreducible components of $\bigoplus H^i$, we have $H^i = \pi_{n-i}$ $(0 \leqslant i \leqslant n)$.

(*b*) We choose I such that the corresponding Levi component L_I of the parabolic subgroup P_I is of the form $(\mathbf{G}_m)^{n-2} \times Sp_4$, or $(\mathbf{G}_m)^{n-2} \times SO_5$ as the case may be. Then $|I| = 2$. By [**L1**, 7·3], we know that π_{uc} is a component of H_I^2. We know that $\bar{W} = W(B_{n-2})$. By proposition 8·3, we know that $H^{n+i}[\pi_{\mathrm{uc}}] = \rho_{n-2-i}$ $(0 \leqslant i \leqslant n-2)$. Now, if we take I to be the empty subset of Δ and ρ the trivial representation of $T(\mathbb{F}_q)$, where T is the split maximal torus of G, we get $H^{n+i}[1] = \pi_{n-i}$. Since there are $2n$ irreducible components of $\bigoplus H^i$, we have

$$\left.\begin{aligned} H^{n+i} &= \pi_{n-i} \oplus \rho_{n-2-i} \qquad (0 \leqslant i \leqslant n-2), \\ H^{n+i} &= \pi_{n-i} \qquad (n-1 \leqslant i \leqslant n). \end{aligned}\right\} \tag{8·5}$$

(*c*) We choose I such that the corresponding Levi component L_I of the parabolic subgroup P_I is of the form $(\mathbf{G}_m)^{n-4} \times SO_8$. Then $|I| = 4$. By [**L1**, 7·3] we know that π_{uc} appears in H_I^4. We know that $\bar{W} = W(D_{n-4})$. By Proposition 8·3 we know that $H^{n+i}[\pi_{\mathrm{uc}}] = \rho_{n-4-i}$ $(0 \leqslant i \leqslant n-4)$. Now, if we take I to be the empty subset set of Δ and ρ the trivial representation of $T(\mathbb{F}_q)$, where T is the split maximal torus of G, we get $H^{n+i}[1] = \pi_{n-i}$. Since there are $2(n-1)$ irreducible components of $\bigoplus H^i$, we have

$$\left.\begin{aligned} H^{n+i} &= \pi_{n-i} \oplus \rho_{n-4-i} \qquad (0 \leqslant i \leqslant n-4), \\ H^{n+i} &= \pi_{n-i} \qquad (n-3 \leqslant i \leqslant n). \end{aligned}\right\} \tag{8·6}$$

Hence the proof.

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