Abstract. We describe all the holomorphic Hermitian vector bundles \((E, h)\) over the upper half plane in \(\mathbb{C}\) with the property that \((f^*E, f^*h)\) is holomorphically isometric to \((E, h)\) for any holomorphic automorphism \(f\) of the upper half plane. We give an explicit construction of all such holomorphic Hermitian vector bundles using some linear algebraic data.

1. Introduction

Linear spaces with norms defined by a Hermitian form make an appearance in many areas of mathematics and its applications. However, often the bounded linear operators on such spaces or bundle maps of Hermitian vector bundles are the primary object of study. In [Cowen and Douglas 1978], Cowen and Douglas introduce a class of operators possessing an open set of eigenvalues \(\Omega\) of constant multiplicity, say \(k\). They show that for such an operator \(T\), with the additional assumption that the range \(T - w\) is closed for \(w \in \Omega\), the map \(w \mapsto \ker(T - w)\) is holomorphic. It then follows that \(\pi : E_T \rightarrow \Omega\), where

\[
E_T = \{\ker(T - w) : w \in \Omega\}, \pi(\ker(T - w)) = w, w \in \Omega
\]

defines a holomorphic Hermitian vector bundle on \(\Omega\). Although, we must emphasize that not all holomorphic Hermitian bundles arise this way. One of the main results of [Cowen and Douglas 1978] is that the unitary equivalence class of \(T\) and the equivalence class of the holomorphic Hermitian vector bundle \(E_T\) determine each other. For instance, if the multiplicity \(k = 1\), then the curvature of \(E_T\) is a complete invariant of the holomorphic Hermitian line bundle \(E_T\) and hence that of the operator \(T\). With this bridge between complex geometry and operator theory at our disposal, we raise the question of homogeneity.

Let \(T\) be a bounded linear operator on a Hilbert space \(H\). We will let \(\sigma_T\) denote the spectrum of the operator \(T\). Let \(\text{M"{o}b}\) be the group of bi-holomorphic automorphisms of the unit disc \(\mathbb{D}\). If the function \(\varphi \in \text{M"{o}b}\) is holomorphic on some open set containing \(\sigma_T\), then \(\varphi(T)\) is well-defined via the usual holomorphic functional calculus for \(T\). If \(\varphi(T)\) is unitarily equivalent to \(T\) for all \(\varphi\) in \(\text{M"{o}b}\), then \(T\) is said to be \textit{homogeneous}. It turns out that if \(T\) is homogeneous then \(\sigma_T = \overline{\mathbb{D}}\) and \(\varphi(T)\) is unitarily equivalent to \(T\) for all \(\varphi\) in \(\text{M"{o}b}\). The well-known characterization of systems of imprimitivity due to Mackey amounts to classifying normal operators (or commuting tuples of normal operators) in our context (cf. [Bagchi and Misra 1995]). A systematic study of homogeneous operators when the normality assumption is dropped is of more recent origin (cf. [Bagchi and Misra 2001]). A classification of all homogeneous scalar shift operators has been obtained in [Bagchi and Misra 2003].

2000 Mathematics Subject Classification. Primary 53B35; Secondary 14F05.
Key words and phrases. Upper half plane, homogeneous bundle, Hilbert space.
Both the authors acknowledge financial support from the DST.
Now, assume that $T$ is one of the operators in the Cowen-Douglas class of $\mathbb{D}$ and $E_T$ be the holomorphic Hermitian vector bundle corresponding to $T$. If $T$ is homogeneous then $\varphi(T)$ is evidently in the Cowen-Douglas class as well for all $\varphi$ in Möb. The vector bundle $E_{\varphi(T)}$ for the operator $\varphi(T)$ is the pullback of the bundle $E_T$ under $\varphi$, that is, $\varphi^*E_T = E_{\varphi(T)}$. It is natural to say that a holomorphic Hermitian vector bundle $E$ defined on the unit disc is homogeneous (Möb - equivariant) if its pullback by any $\varphi \in \text{Möb}$ remains (isometrically) isomorphic to $E$, that is, $\varphi^*E_T \cong E_{\varphi(T)}$ for all $\varphi$ in Möb. Thus the classification of homogeneous operators in the Cowen-Douglas class of $\mathbb{D}$ is the same as that of the corresponding homogeneous holomorphic Hermitian bundles defined on $\mathbb{D}$. Wilkins [Wilkins 1993] provides a classification of all, not necessarily the ones arising out of Cowen-Douglas operators, homogeneous holomorphic Hermitian bundles. The main goal of this paper is a classification of homogeneous holomorphic Hermitian bundles on $\mathbb{D}$ as well. However, our methods are somewhat different from that of Wilkins and we give a very explicit construction of all homogeneous holomorphic Hermitian bundles which correspond to homogeneous operators in the Cowen-Douglas class of the unit disc $\mathbb{D}$ is given in [Körányi and Misra 2006, Theorem 4.1].

We point out that the list of $\text{SL}(2,\mathbb{R})$-equivariant vector bundles in [Biswas 2003] is incomplete.

It will be convenient for us to work with the upper half plane in the complex plane rather than the unit disc.

2. Some properties of homogeneous bundles

Let $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half plane of the complex plane. The group $\text{PSL}(2,\mathbb{R})$ acts on $\mathbb{H}$ as

$$A(z) = \frac{az + b}{cz + d},$$

where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2,\mathbb{R})$$

and $z \in \mathbb{H}$. This action identifies the group of all holomorphic automorphisms of $\mathbb{H}$ with $\text{PSL}(2,\mathbb{R})$. For any $A$ as in Eqn. (2.2), let

$$\phi(A) \in \text{Aut}(\mathbb{H})$$

be the holomorphic automorphism defined in Eqn. (2.1)

The universal cover of the group $\text{PSL}(2,\mathbb{R})$ will be denoted by $\widetilde{\text{SL}(2,\mathbb{R})}$. Using the projection

$$p : \widetilde{\text{SL}(2,\mathbb{R})} \rightarrow \text{PSL}(2,\mathbb{R}),$$

the group $\widetilde{\text{SL}(2,\mathbb{R})}$ acts on $\mathbb{H}$.

Let $E$ be a holomorphic vector bundle over $\mathbb{H}$ equipped with a Hermitian structure $h$ such that the holomorphic Hermitian vector bundle $(\phi(A)^*E, \phi(A)^*h)$ is isomorphic to $(E, h)$ for all $A \in \text{Aut}(\mathbb{H})$. By a lift of the action of $\widetilde{\text{SL}(2,\mathbb{R})}$ to $(E, h)$ as vector bundle automorphisms we mean the following:
• The group $\widetilde{\text{SL}}(2, \mathbb{R})$ acts on the total space of $E$ and the natural projection of $E$ to $\mathbb{H}$ is $\widetilde{\text{SL}}(2, \mathbb{R})$-equivariant;
• for any $A \in \text{SL}(2, \mathbb{R})$, the action of $A$ on $E$ is a holomorphic isomorphism of the vector bundle $\phi(p(A^{-1}))^*E$ with $E$, where $\phi$ and $p$ are defined in Eqn. (2.3) and Eqn. (2.4) respectively; and
• the action of $\widetilde{\text{SL}}(2, \mathbb{R})$ on $E$ preserves the Hermitian structure $h$.

The following lemma has appeared as Theorem 2.2 in [Korányi and Misra 2006].

Lemma 2.1. Let $(E, h)$ be a holomorphic Hermitian vector bundle over $\mathbb{H}$ such that the holomorphic Hermitian vector bundle $(\phi(A)^*E, \phi(A)^*h)$ is isomorphic to $(E, h)$ for all $A \in \text{Aut}(\mathbb{H})$. The action of $\widetilde{\text{SL}}(2, \mathbb{R})$ on $\mathbb{H}$ lifts to a holomorphic action on $E$, as vector bundle automorphisms, preserving the Hermitian structure $h$.

Proof. Let $G_E$ denote the group of all pairs of the form $(\tau, \sigma)$, where $\tau \in \text{Aut}(\mathbb{H})$ is a biholomorphism of $\mathbb{H}$, and
\[
\sigma : E \to \tau^*E
\]
is a holomorphic isomorphism preserving the Hermitian structure $h$. It is easy to see that $G_E$ is a finite dimensional Lie group. Let $H \subset G_E$ be the closed subgroup defined by all pairs of the form $(\text{Id}_\mathbb{H}, \sigma)$; in other words, $H$ is the group of all automorphisms of the holomorphic Hermitian vector bundle $(E, h)$. Therefore, we have an exact sequence of groups
\[
0 \to H \to G_E \to \text{PSL}(2, \mathbb{R}) \to 0. 
\]
We note that the above homomorphism $G_E \to \text{PSL}(2, \mathbb{R})$ is surjective follows from the assumption that the holomorphic Hermitian vector bundle $(\phi(A)^*E, \phi(A)^*h)$ is isomorphic to $(E, h)$ for all $A \in \text{Aut}(\mathbb{H})$.

Let
\[
0 \to \mathfrak{h} \to \mathfrak{g}_E \to \text{sl}(2, \mathbb{R}) \to 0
\]
be the exact sequence of Lie algebras obtained from Eqn. (2.5); here $\mathfrak{h}$ and $\mathfrak{g}_E$ are the Lie algebras of $H$ and $G_E$ respectively. Since the Lie algebra $\text{sl}(2, \mathbb{R})$ is semisimple, the exact sequence of Lie algebras in Eqn. (2.6) is right split [Bourbaki 1960, p. 91, Corollaire 3]. In other words, there is a Lie subalgebra $\mathfrak{l} \subset \mathfrak{g}_E$ that projects isomorphically to $\text{sl}(2, \mathbb{R})$.

Since the group $\widetilde{\text{SL}}(2, \mathbb{R})$ is simply connected, the above isomorphism $\text{sl}(2, \mathbb{R}) \to \mathfrak{l}$ gives a homomorphism of groups $\widetilde{\text{SL}}(2, \mathbb{R}) \to G_E$. Now $\widetilde{\text{SL}}(2, \mathbb{R})$ acts on $E$ using this homomorphism to $G_E$ (the group $G_E$ has a natural action on $E$). Therefore, the action of $\widetilde{\text{SL}}(2, \mathbb{R})$ on $\mathbb{H}$ lifts to a holomorphic action on $E$ as vector bundle automorphisms preserving the Hermitian structure $h$. This completes the proof of the lemma. □

Lemma 2.2. Let $(E, h)$ be as in Lemma 2.1. Assume that the group $H$ in Eqn. (2.5) is a product of copies of the circle group $S^1$. Then there is a unique lift of the action of $\widetilde{\text{SL}}(2, \mathbb{R})$ (on $\mathbb{H}$) to a holomorphic action on $E$ as vector bundle automorphisms preserving the Hermitian structure $h$.

Proof. Consider the adjoint action of the group $G_E$ on the normal subgroup $H$ in Eqn. (2.5). Since both $H$ and $\text{PSL}(2, \mathbb{R})$ are connected, from Eqn. (2.5) it follows that the group $G_E$ is connected; also
the automorphism group of a product of copies of \( S^1 \) is discrete. Therefore, the adjoint action of \( G_E \) on \( H \) is the trivial action. Hence \( H \) is contained in the center of \( G_E \).

Since \( H \) is contained in the center of \( G_E \), any two right splitting of the exact sequence of Lie algebras in Eqn. (2.6) differ by a Lie algebra homomorphism from \( \mathfrak{sl}(2,\mathbb{R}) \) to \( \mathfrak{h} \). As \( \mathfrak{sl}(2,\mathbb{R}) \) is simple and \( \mathfrak{h} \) is abelian, there is no nonzero homomorphism from the Lie algebra \( \mathfrak{sl}(2,\mathbb{R}) \) to \( \mathfrak{h} \). This means that there is a unique right splitting of the exact sequence of Lie algebras in Eqn. (2.6) (we saw in Lemma 2.1 that there is at least one right splitting). From this it follows immediately that there is a unique lift of the action of \( \widetilde{\text{SL}(2,\mathbb{R})} \) on \( H \) to a holomorphic action on \( E \) as vector bundle automorphisms preserving the Hermitian structure \( h \). This completes the proof of the lemma. \( \square \)

Let \( E \) be a holomorphic vector bundle over \( \mathbb{H} \) equipped with a Hermitian structure \( h \). Then \( E \) has a unique Hermitian connection \( \nabla = \nabla^{1,0} + \nabla^{0,1} \) such that

\[
\nabla^{0,1}s = 0
\]

for all locally defined holomorphic sections of \( E \) [Kobayashi 1987, p. 11, Proposition (4.9)]. This connection \( \nabla \) is called the Chern connection on \( E \).

Let \( F \subset E \) be a \( \mathcal{C}^\infty \) subbundle. It is easy to see that the following two statements are equivalent:

1. For any smooth section \( s \) of \( F \) defined over some open subset \( U \subset \mathbb{H} \), the inclusion

\[
\nabla s \in \Omega^1(U, F) = C^\infty(U, F \otimes (T^*_U \mathbb{C}))
\]

holds, where \( \nabla \) is the Chern connection.

2. Both \( F \) and \( F^\perp \) are holomorphic subbundles of \( E \).

Assume that the holomorphic Hermitian vector bundle \( (\phi(A)^* E, \phi(A)^* h) \) is isomorphic to \( (E, h) \) for all \( A \in \text{Aut}(\mathbb{H}) \). From Lemma 2.1 we know that the action of \( \text{SL}(2,\mathbb{R}) \) on \( \mathbb{H} \) lifts to a holomorphic action on \( E \) as vector bundle automorphisms, preserving the Hermitian structure \( h \). Fix such a lift of the action of \( \text{SL}(2,\mathbb{R}) \). Then we have the following proposition:

**Proposition 2.3.** Let \( F \subset E \) be a \( \mathcal{C}^\infty \) subbundle preserved by the Chern connection (i.e., Eqn. (2.7) holds for all smooth sections \( s \) of \( F \) defined over open subsets of \( \mathbb{H} \)). Then the action of \( \text{SL}(2,\mathbb{R}) \) on \( E \) preserves the subbundle \( F \).

**Proof.** Let \( \text{Aut}^0(E) \) denote the connected component, containing the identity element, of the group of all holomorphic Hermitian automorphisms of the holomorphic Hermitian vector bundle \( (E, h) \). In other words, \( \text{Aut}^0(E) \) is the connected component containing the identity element of the group \( H \) in Eqn. (2.5).

Using the action of \( \widetilde{\text{SL}(2,\mathbb{R})} \) on \( E \), the group \( \widetilde{\text{SL}(2,\mathbb{R})} \) acts as automorphisms of \( \text{Aut}^0(E) \) as follows: The action of any \( g \in \text{SL}(2,\mathbb{R}) \) sends any \( T \in \text{Aut}^0(E) \) to the automorphism \( g^{-1} \circ T \circ g \) of \( (E, h) \). Let

\[
\rho_E : \text{SL}(2,\mathbb{R}) \rightarrow \text{Aut}(\text{Aut}^0(E))
\]

be the resulting homomorphism, where \( \text{Aut}(\text{Aut}^0(E)) \) is the group of automorphism of the group \( \text{Aut}^0(E) \).

Since \( \text{Aut}^0(E) \) is a compact Lie group, any element in the connected component of \( \text{Aut}(\text{Aut}^0(E)) \) containing the identity automorphism must be an inner conjugation. As \( \text{SL}(2,\mathbb{R}) \) is connected, and the
connected component of $\text{Aut}(\text{Aut}^0(E))$ containing the identity automorphism is $\text{Aut}^0(E)/Z$, where $Z \subset \text{Aut}^0(E)$ is the center, we conclude that the homomorphism $\rho_E$ in Eqn. (2.8) gives a homomorphism
\begin{equation}
\rho_E : \text{SL}(2, \mathbb{R}) \rightarrow \text{Aut}^0(E)/Z.
\end{equation}
Since $\text{Aut}^0(E)/Z$ is compact, the image of any unipotent element in $\text{SL}(2, \mathbb{R})$ under the homomorphism $\bar{\rho}_E$ in Eqn. (2.9) must be the identity element. The group $\text{SL}(2, \mathbb{R})$ being simple, this implies that $\bar{\rho}_E$ is the trivial homomorphism. Consequently, the homomorphism $\rho_E$ in Eqn. (2.8) is the trivial homomorphism.

For any $\theta \in [0, \pi/2] \subset \mathbb{R}$, consider the automorphism $T_\theta$ of $E$ defined by
\[ T_\theta = \exp(\pi\sqrt{-1}\theta)\text{Id}_F + \exp(-\pi\sqrt{-1}\theta)\text{Id}_{F^\perp}, \]
where $F^\perp \subset E$ is the orthogonal complement of $F$. We noted earlier that the given condition that Eqn. (2.7) holds for all smooth sections $s$ of $F$ defined over open subsets of $\mathbb{H}$ is equivalent to the condition that both $F$ and $F^\perp$ are holomorphic subbundles of $E$. Hence we conclude that $T_\theta \in \text{Aut}^0(E)$.

Since the homomorphism $\rho_E$ in Eqn. (2.8) is the trivial homomorphism, the above automorphism $T_\theta$ of $E$ commutes with the action of $\text{SL}(2, \mathbb{R})$ on $E$. This immediately implies that the action of $\text{SL}(2, \mathbb{R})$ on $E$ preserves the eigenbundles of $T_\theta$. Therefore, the action of $\text{SL}(2, \mathbb{R})$ on $E$ preserves the subbundle $F$. This completes the proof of the proposition. \hfill \Box

In the next section we will investigate the holomorphic Hermitian line bundles over $\mathbb{H}$ equipped with an action of $\text{SL}(2, \mathbb{R})$.

3. \text{SL}(2, \mathbb{R})-HOMOGENEOUS LINE BUNDLES

We start with a simple lemma.

\textbf{Lemma 3.1.} Let $(L_1, h_1)$ and $(L_2, h_2)$ be two holomorphic Hermitian line bundles over $\mathbb{H}$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the curvatures of the Chern connections on $L_1$ and $L_2$ respectively (so $\mathcal{K}_1$ and $\mathcal{K}_2$ are smooth $(1, 1)$-form on $\mathbb{H}$). The two holomorphic Hermitian line bundles $(L_1, h_1)$ and $(L_2, h_2)$ are isomorphic if and only if $\mathcal{K}_1 = \mathcal{K}_2$.

\textbf{Proof.} If $L_1$ is holomorphically isometric to $L_2$, then obviously $\mathcal{K}_1 = \mathcal{K}_2$. To prove the converse, assume that $\mathcal{K}_1 = \mathcal{K}_2$. The Chern connection on $L_1$ and $L_2$ will be denoted by $\nabla_1$ and $\nabla_2$ respectively.

Consider the holomorphic line bundle $L_1^* \otimes L_2$ over $\mathbb{H}$ equipped with the Hermitian structure induced by the pair $h_1$ and $h_2$. The Chern connection on $L_1^* \otimes L_2$ is $\nabla = \nabla_1^* \otimes \text{Id}_{L_2} + \text{Id}_{L_1} \otimes \nabla_2$, where $\nabla_1^*$ is the connection on $L_1^*$ induced by $\nabla_1$. The curvature of $\nabla$ is $\mathcal{K}_2 - \mathcal{K}_1 = 0$.

Since $\mathbb{H}$ is simply connected, and the curvature of $\nabla$ vanishes, the flat line bundle $L_1^* \otimes L_2$ is isomorphic to the trivial line bundle equipped with the constant metric. Fixing a flat section of $L_1^* \otimes L_2$ of (pointwise) norm one, we get a holomorphic isomorphism of $L_1$ with $L_2$ that takes the Hermitian structure $h_1$ to $h_2$. This completes the proof of the lemma. \hfill \Box

We will now construct a family of holomorphic Hermitian line bundle parametrized by the real numbers.
Take any $\lambda \in \mathbb{R}$. Let $\xi$ denote the trivial holomorphic line bundle $\mathbb{H} \times \mathbb{C}$ over $\mathbb{H}$. Consider the Hermitian structure $h_\xi^\lambda$ on $\xi$ defined as follows: For any $z \in \mathbb{H}$ and any $c \in \xi_z = \mathbb{C}$,

$$h_\xi^\lambda(c, c) := \langle c, c \rangle_\lambda := \frac{|c|^2}{\text{Im}(z)^\lambda}.$$  

Let $\xi_\lambda$ denote the holomorphic line bundle $\xi$ over $\mathbb{H}$ equipped with the Hermitian structure $h_\xi^\lambda$ defined above.

It is easy to check that the curvature $\mathcal{K}_\lambda$ of the Chern connection on $\xi_\lambda$ is

$$\mathcal{K}_\lambda = -\lambda \partial \overline{\partial} (\text{Im}(z)) \in \Omega^{1,1}_\mathbb{H},$$

which is a smooth $(1, 1)$-form on $\mathbb{H}$. It is straight-forward to check that the form $\mathcal{K}_\lambda$ is left invariant by the action of $\text{PSL}(2, \mathbb{R})$ on $\mathbb{H}$. Therefore, using Lemma 2.2 and Lemma 3.1 we have the following corollary:

**Corollary 3.2.** There is a unique lift of the action of $\widetilde{\text{SL}(2, \mathbb{R})}$ (on $\mathbb{H}$) to a holomorphic action on $\xi_\lambda$ as vector bundle automorphisms preserving its Hermitian structure $h_\lambda$.

**Proof.** Using Lemma 3.1 it follows that for any $A \in \text{PSL}(2, \mathbb{R})$, the holomorphic Hermitian line bundle $\phi(A)^* \xi_\lambda$ is holomorphically isometric to $\xi_\lambda$, where $\phi$ is defined in Eqn. (2.3). Since the group $H$ (see Lemma 2.2) in this case is $S^1$, the proof is completed using Lemma 2.2. \hfill $\square$

**Lemma 3.3.** Let $(L, h)$ be a holomorphic Hermitian line bundle over $\mathbb{H}$ equipped with a lift of the action of $\widetilde{\text{SL}(2, \mathbb{R})}$ (on $\mathbb{H}$) to a holomorphic action on $L$ preserving the Hermitian structure $h$. Then there is a unique real number $\lambda$ such that $\xi_\lambda$ is holomorphically isometric to $(L, h)$.

**Proof.** Since the curvature of $\xi_\lambda$ is $-\lambda \partial \overline{\partial} (\text{Im}(z))$, and $\partial \overline{\partial} (\text{Im}(z)) \neq 0$, there can be at most one $\lambda$ with the above property. To prove that there is actually one $\lambda$, consider the curvature $\mathcal{K}_h$ of the Chern connection on $(L, h)$. As $(L, h)$ is equipped with a lift of the action of $\text{SL}(2, \mathbb{R})$, we conclude that

$$(3.2) \quad \phi(A)^* \mathcal{K}_h = \mathcal{K}_h$$

for all $A \in \text{PSL}(2, \mathbb{R})$, where $\phi$ is defined in Eqn. (2.3).

From the identity in Eqn. (3.2) it follows that $\mathcal{K}_h = \mu \partial \overline{\partial} (\text{Im}(z))$ for some $\mu \in \mathbb{R}$ (we also use the fact that the form $\mathcal{K}_h$ is real). Now the proof of the lemma is completed using Lemma 3.1. \hfill $\square$

**Proposition 3.4.** If the line bundle $\xi_\lambda$ admits a smooth nonzero section invariant under the action of $\text{SL}(2, \mathbb{R})$, then $\lambda = 0$.

**Proof.** Let $s$ be a smooth section of $\xi_\lambda$ satisfying the following two conditions:

- the action of $\widetilde{\text{SL}(2, \mathbb{R})}$ on $\xi_\lambda$ leaves $s$ invariant, and
- the section $s$ is not identically equal to zero.

Note that these two conditions imply that $s$ is nowhere vanishing. Let $\overline{\partial}_{\xi_\lambda}$ be the Dolbeault operator defining the holomorphic structure of $\xi_\lambda$. So $\overline{\partial}_{\xi_\lambda}s$ is a smooth section of $\xi_\lambda \otimes (T_\mathbb{H}^{0,1})^*$ which is left invariant by the action of $\text{SL}(2, \mathbb{R})$.

Since $s$ is nowhere vanishing, we have

$$\overline{\partial}_{\xi_\lambda}s = s \otimes \omega,$$
where ω is a (0, 1)-form on \( \mathbb{H} \) which is left invariant by the action of \( \widetilde{SL(2, \mathbb{R})} \) on \( \mathbb{H} \). But there is no nonzero (0, 1)-form on \( \mathbb{H} \) which is left invariant by the action of \( SL(2, \mathbb{R}) \). Indeed, this follows immediately from the fact that the isotropy subgroup of \( SL(2, \mathbb{R}) \) for any point \( z \in \mathbb{H} \) acts nontrivially on the line \( (T^0_{z})^* \) (the fiber of \( (T^1_{z})^* \) over \( z \)).

Therefore, the (0, 1)-form \( \omega \) vanishes identically. Hence \( \partial\xi_\lambda s = 0 \). In other words, the section \( s \) is holomorphic. Since \( s \) is left invariant by the action of \( SL(2, \mathbb{R}) \), the inner product \( \langle s(z), s(z) \rangle_\lambda \) is independent of \( z \in \mathbb{H} \) (see Eqn. (3.1)). As \( s \) is a holomorphic section of \( \xi_\lambda \) whose norm is a constant function on \( \mathbb{H} \), the Chern connection on the holomorphic Hermitian line bundle \( \xi_\lambda \) is flat. Therefore, \( \lambda = 0 \). This completes the proof of the proposition. \( \square \)

Proposition 3.4 has the following corollary:

**Corollary 3.5.** Let \( f : \xi_\lambda \longrightarrow \xi_{\lambda'} \) be a \( C^\infty \) homomorphism of line bundles that intertwines the actions of \( SL(2, \mathbb{R}) \). If \( f \) is not identically zero, then \( \lambda = \lambda' \). In that case, \( f \) is a constant scalar multiplication.

**Proof.** Consider the Hermitian structure on the holomorphic line bundle \( \xi_\lambda \bigotimes \xi_{\lambda'} \) induced by the Hermitian structures on \( \xi_\lambda \) and \( \xi_{\lambda'} \). The curvature of the Chern connection on \( \xi_\lambda \bigotimes \xi_{\lambda'} \) is \( -(\lambda' - \lambda)\partial\partial(\text{Im}(z)) \). Using Lemma 3.3 it follows that the Holomorphic Hermitian line bundle \( \xi_\lambda \bigotimes \xi_{\lambda'} \) holomorphically isometric to \( \xi_{\lambda' - \lambda} \). Now the proof is completed by applying Proposition 3.4 to \( \xi_{\lambda' - \lambda} \). \( \square \)

4. DATA FOR HIGHER RANK VECTOR BUNDLES

For any point \( z \in \mathbb{H} \), let \( \Gamma_z \subset \widetilde{SL(2, \mathbb{R})} \) be the subgroup that fixes \( z \). It is easy to see that \( \Gamma_z \) is isomorphic to the additive group \( \mathbb{R} \). Fix a point \( z_0 \in \mathbb{H} \), and fix an isomorphism

\[
\alpha : \Gamma_{z_0} \longrightarrow \mathbb{R}.
\]

Let \((E, h)\) be a holomorphic Hermitian vector bundle over \( \mathbb{H} \) equipped with an action of \( \widetilde{SL(2, \mathbb{R})} \) compatible with both holomorphic and Hermitian structures. The fiber of the vector bundle \( E \) over a point \( z \in \mathbb{H} \) will be denoted by \( E_z \). The isotropy subgroup \( \Gamma_z \subset \widetilde{SL(2, \mathbb{R})} \) acts on \( E_z \) preserving the Hermitian structure on \( E_z \). Since \( \Gamma_{z_0} \) preserves the Hermitian structure of \( E_{z_0} \), using the isomorphism \( \alpha \) in Eqn. (4.1) we have a positive integer \( \ell \), real number \( r_1, \ldots, r_\ell \) and positive integers \( m_1, \ldots, m_\ell \) satisfying the following conditions:

- \( \sum_{i=1}^\ell m_i = \text{rank}(E) \), and
- for each \( i \in [1, \ell] \), there is a subspace \( V_{z_0, r_i} \subset E_{z_0} \) of dimension \( m_i \) such that \( \alpha^{-1}(t) \) acts on \( V_{z_0, r_i} \) as multiplication by \( \exp(\sqrt{-1}r_it) \) for all \( t \in \mathbb{R} \).

Using the action of \( \widetilde{SL(2, \mathbb{R})} \) on \( E \), the above decomposition

\[
E_{z_0} = \bigoplus_{i=1}^\ell V_{z_0, r_i}
\]

induces a \( C^\infty \) decomposition of the vector bundle \( E \)

\[
E = \bigoplus_{i=1}^\ell V_{r_i}.
\]
In other words, $V_r$ is the orbit of $V_{z_0,r}$ for the action of $\widetilde{\text{SL}}(2,\mathbb{R})$ on $E$. It is easy to see that for any $z \in \mathbb{H}$, the decomposition

$$E_z = \bigoplus_{i=1}^\ell (V_r)_z$$

obtained from Eqn. (4.2) is the isotypical decomposition of the $\Gamma_z$-module $E_z$.

We will interpret the decomposition in Eqn. (4.2) using the line bundles $\xi_\lambda$ constructed in Section 3.

In Corollary 3.2 we noted that there is a unique lift of the action of $\widetilde{\text{SL}}(2,\mathbb{R})$ on $H$ to a holomorphic action on $\xi_\lambda$ preserving its Hermitian structure. For any $\lambda \in \mathbb{R}$, let

$$W_{E,\lambda} := C^\infty(\mathbb{H}, \xi_\lambda^* \otimes E)^{\widetilde{\text{SL}}(2,\mathbb{R})}$$

be the space of all smooth homomorphisms from $\xi_\lambda$ to $E$ that intertwine the actions of $\widetilde{\text{SL}}(2,\mathbb{R})$ on $\xi_\lambda$ and $E$ (these homomorphisms need not be holomorphic). Since $\text{SL}(2,\mathbb{R})$ acts transitively on $\mathbb{H}$, for any point $z \in \mathbb{H}$, the evaluation at $z$

$$W_{E,\lambda} \rightarrow (\xi_\lambda^* \otimes E)_z$$

is an injective homomorphism. In particular, $W_{E,\lambda}$ is a finite dimensional vector space.

We will construct an inner product on the vector space $W_{E,\lambda}$. Consider the inner product on $W_{E,\lambda}$ given by the Hermitian structure on the fiber $(\xi_\lambda^* \otimes E)_z$ using the injective homomorphism in Eqn. (4.4). It is now an easy exercise to check that this inner product on $W_{E,\lambda}$ does not depend on the choice of the point $z$. Therefore, $W_{E,\lambda}$ is canonically a finite dimensional Hilbert space.

The Hermitian structure of $\xi_\lambda$ and the inner product on $W_{E,\lambda}$ together induce a Hermitian structure on the holomorphic vector bundle

$$E_\lambda := \xi_\lambda \otimes_\mathbb{C} W_{E,\lambda}.$$

The actions of $\widetilde{\text{SL}}(2,\mathbb{R})$ on $\xi_\lambda$ and the trivial action of $\text{SL}(2,\mathbb{R})$ on $W_{E,\lambda}$ together induce an action of $\text{SL}(2,\mathbb{R})$ on the vector bundle $E_\lambda$ constructed in Eqn. (4.5). This action of $\text{SL}(2,\mathbb{R})$ on $E_\lambda$ clearly preserves the holomorphic structure as well as the Hermitian structure.

Associate to $(E,h)$ the holomorphic Hermitian vector bundle

$$E := \bigoplus_{\lambda \in \mathbb{R}} E_\lambda$$

Note that $W_{E,\lambda} = 0$ for all but finitely many $\lambda$s in $\mathbb{R}$. The Hermitian structure on $E$, which we will denote by $\mathcal{H}$, is the above mentioned Hermitian structure on each $E_\lambda$, and the direct sum in Eqn. (4.6) is an orthogonal direct sum.

The actions of $\text{SL}(2,\mathbb{R})$ on $E_\lambda$, $\lambda \in \mathbb{R}$, together induce an action of $\text{SL}(2,\mathbb{R})$ on the holomorphic Hermitian vector bundle $(E,\mathcal{H})$. This action clearly preserves the holomorphic structure as well as the the Hermitian structure.

The evaluation homomorphisms in Eqn. (4.4) for different values of $\lambda$ together define a $C^\infty$ isomorphism

$$\gamma_E : E \rightarrow E$$
which takes the metric $\mathcal{H}$ to $h$. This homomorphism $\gamma_E$ clearly intertwines the actions of $\text{SL}(2, \mathbb{R})$ on $\mathcal{E}$ and $E$. However, the homomorphism $\gamma_E$ need not be holomorphic.

Let

$$\overline{\partial}_E : E \rightarrow E \otimes \Omega^0_{\mathbb{H}}$$

be the Dolbeault operator defining the holomorphic structure of the holomorphic vector bundle $E$. Let

$$\overline{\partial}_E^\phi := (\gamma_E^{-1} \otimes \text{Id}_{\Omega^0_{\mathbb{H}}}) \circ \overline{\partial}_E \circ \gamma_E : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^0_{\mathbb{H}}$$

be the Dolbeault operator obtained by transporting $\overline{\partial}_E$ to $\mathcal{E}$ using the isomorphism $\gamma_E$ in Eqn. (4.7).

Let $\overline{\partial}_E : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^0_{\mathbb{H}}$ be the Dolbeault operator defining the holomorphic structure of the holomorphic vector bundle $\mathcal{E}$. Therefore, (4.9)

$$\Theta := \overline{\partial}_E^\phi - \overline{\partial}_E \in C^\infty(\mathbb{H}, \text{End} \otimes (T^0_{\mathbb{H}})^*) = \Omega^0_{\mathbb{H}}(\text{End}(\mathcal{E}))$$

is a $(0, 1)$-form on $\mathbb{H}$ with values in the vector bundle $\text{End}(\mathcal{E})$, where $\overline{\partial}_E^\phi$ is defined in Eqn. (4.8).

Since the $C^\infty$ isomorphism $\gamma_E$ (constructed in Eqn. (4.7)) intertwines the actions of $\text{SL}(2, \mathbb{R})$, the section $\Theta$ in Eqn. (4.9) is preserved by the action of $\text{SL}(2, \mathbb{R})$ (the action of $\text{SL}(2, \mathbb{R})$ on $\mathcal{E}$ induces an action of $\text{SL}(2, \mathbb{R})$ on $\text{End}(\mathcal{E})$).

**Proposition 4.1.** For any $\lambda \in \mathbb{R}$, the homomorphism $\Theta$ in Eqn. (4.9) sends the direct summand $\mathcal{E}_\lambda \subset \mathcal{E}$ (see Eqn. (4.6)) to the direct summand $\mathcal{E}_{\lambda - 2} \bigotimes (T^0_{\mathbb{H}})^*$.

**Proof.** We recall that the holomorphic tangent bundle $T^1_{\mathbb{H}}$ has the Poincaré metric defined by $\|\frac{\partial}{\partial z}\| = 1/\text{Im}(z)$. Hence the holomorphic Hermitian line bundle $T^1_{\mathbb{H}}$ is holomorphically isometric to $\xi_2$.

The $(1, 1)$-form on $\mathbb{H}$ given by the Poincaré metric, which is a $\text{SL}(2, \mathbb{R})$-invariant Kähler form, gives a smooth $\text{SL}(2, \mathbb{R})$-equivariant homomorphism of $T^1_{\mathbb{H}}$ with $(T^0_{\mathbb{H}})^*$.

Therefore, from Corollary 3.5 it follows that if there is a nonzero $\text{SL}(2, \mathbb{R})$-equivariant homomorphism from the $C^\infty$ line bundle $\xi_\lambda$ to the $C^\infty$ line bundle $\xi_{\lambda'} \bigotimes (T^0_{\mathbb{H}})^*$, then $\lambda' = \lambda - 2$.

Since the action of $\text{SL}(2, \mathbb{R})$ on $\xi_{\lambda}$ is defined using the trivial action of $\text{SL}(2, \mathbb{R})$ on $W_{E, \lambda}$ (see Eqn. (4.5)), applying the above assertion to the $\text{SL}(2, \mathbb{R})$-invariant section $\Theta$, the proof of the proposition is complete. \hfill \Box

In the proof of Proposition 4.1 we saw that the holomorphic tangent bundle $T^1_{\mathbb{H}}$ equipped with the Poincaré metric is holomorphically isometric to $\xi_2$. On the other hand, $\xi_{\lambda_1 + \lambda_2}$ is holomorphically isometric to $\xi_{\lambda_1} \bigotimes \xi_{\lambda_2}$ equipped with the Hermitian structure induced by the Hermitian structures on $\xi_{\lambda_1}$ and $\xi_{\lambda_2}$. Therefore, $\xi_{\lambda} \bigotimes T^1_{\mathbb{H}}$ is holomorphically isometric to $\xi_{\lambda + 2}$ for all $\lambda \in \mathbb{R}$.

Fix once and for all a holomorphic bundle map

$$f_\lambda : \xi_{\lambda} \rightarrow \xi_{\lambda - 2} \otimes T^1_{\mathbb{H}}, \lambda \in \mathbb{R}$$

which is isometric. The isomorphism $f_\lambda$ clearly intertwines the actions of $\text{SL}(2, \mathbb{R})$.

We also noted in the proof of Proposition 4.1 that the Kähler form on $\mathbb{H}$ for the Poincaré metric gives a $\text{SL}(2, \mathbb{R})$-equivariant smooth isomorphism of $T^1_{\mathbb{H}}$ with $(T^0_{\mathbb{H}})^*$. Combining this with the isomorphism
In Eqn. (4.10) we get a $\widetilde{\text{SL}(2, \mathbb{R})}$-equivariant smooth isomorphism

\begin{equation}
(4.11) \quad f'_\lambda : \xi_\lambda \to \xi_{\lambda-2} \otimes (T_H^{0,1})^*.
\end{equation}

Let

\begin{equation}
\Theta_\lambda : \mathcal{E}_\lambda \to \mathcal{E}_{\lambda-2} \otimes (T_H^{0,1})^*
\end{equation}

be the homomorphism obtained by restricting the smooth homomorphism $\Theta$ constructed in Eqn. (4.9) (see Proposition 4.1). This homomorphism $\Theta_\lambda$ and the smooth isomorphism $f'_\lambda$ in Eqn. (4.11) together give a $\widetilde{\text{SL}(2, \mathbb{R})}$-equivariant smooth homomorphism of vector bundles

\begin{equation}
(4.12) \quad \Theta'_\lambda : \mathcal{E}_\lambda := \xi_\lambda \otimes \mathcal{C} \mathcal{W}_{E,\lambda} \to \xi_{\lambda} \otimes \mathcal{C} \mathcal{W}_{E,\lambda-2}
\end{equation}

(see Eqn. (4.5)).

**Lemma 4.2.** The homomorphism $\Theta'_\lambda$ in Eqn. (4.12) is of the form $\text{Id}_{\xi_\lambda} \otimes T$, for some linear transformation $T \in \text{Hom}(W_{E,\lambda}, W_{E,\lambda-2})$.

**Proof.** Recall that the action of $\widetilde{\text{SL}(2, \mathbb{R})}$ on $\mathcal{E}_\lambda$ was constructed using the trivial action of $\widetilde{\text{SL}(2, \mathbb{R})}$ on the vector space $W_{E,\lambda}$. Therefore, the lemma follows immediately from the fact that $\Theta'_\lambda$ is $\text{SL}(2, \mathbb{R})$-equivariant. \qed

Thus, given a holomorphic Hermitian vector bundle $(E, h)$ over $\mathbb{H}$ equipped with an action of $\text{SL}(2, \mathbb{R})$ compatible with both holomorphic and Hermitian structures, we have constructed the following data:

1. A direct sum of finite dimensional Hilbert spaces indexed by real numbers

\begin{equation}
(4.13) \quad H(E) := \bigoplus_{\lambda \in \mathbb{R}} W_{E,\lambda}
\end{equation}

such that $W_{E,\lambda} = 0$ for all but finitely many real numbers, and

2. a linear map

\begin{equation}
(4.14) \quad \Theta' : H(E) \to H(E)
\end{equation}

such that $\Theta'(W_{E,\lambda}) \subset W_{E,\lambda-2}$ for all $\lambda$.

By making Eqn. (4.13) an orthogonal direct sum we get an inner product on the vector space $H(E)$.

In the next section we will invert the above construction in the sense that starting from a data of the above type, we will construct a holomorphic Hermitian vector bundle over $\mathbb{H}$ equipped with a compatible action of $\text{SL}(2, \mathbb{R})$.

## 5. Higher Rank Equivariant Bundles on $\mathbb{H}$

Suppose we are given the following data:

1. An orthogonal direct sum of finite dimensional complex Hilbert spaces indexed by real numbers

\begin{equation}
H := \bigoplus_{\lambda \in \mathbb{R}} H_\lambda
\end{equation}

such that $H_\lambda = 0$ for all but finitely many real numbers, and
(2) a linear map

\[ \rho : H \rightarrow H \]  

such that \( \rho(H_\lambda) \subset H_{\lambda-2} \) for all \( \lambda \in \mathbb{R} \).

Let \( F \) denote the smooth vector bundle

\[ F := \bigoplus_{\lambda \in \mathbb{R}} \xi_\lambda \otimes \mathbb{C} H_\lambda \]

(recall that \( H_\lambda \) is nonzero for only finitely many \( \lambda \)). The Hermitian structures on the line bundles \( \xi_\lambda \) and the inner product on the vector spaces \( H_\lambda \) together give a Hermitian structure on the vector bundle \( F \) defined in Eqn. (5.2). This Hermitian structure on \( F \) will be denoted by \( h_F \).

Equip each \( H_\lambda \) with the trivial action of \( \widetilde{SL(2, \mathbb{R})} \). Now using the action of \( \widetilde{SL(2, \mathbb{R})} \) on the line bundles \( \xi_\lambda \) we get an action of \( \widetilde{SL(2, \mathbb{R})} \) on \( F \) that preserves the above defined Hermitian structure \( h_F \).

The holomorphic structures on the line bundles \( \xi_\lambda \) together induce a holomorphic structure on \( F \). This holomorphic structure on \( F \) is clearly preserved by the above action of \( \widetilde{SL(2, \mathbb{R})} \).

Using \( \rho \) (in Eqn. (5.1)) we will construct a new holomorphic structure on \( F \).

Let \( \overline{\partial}_F \) denote the Dolbeault operator on \( F \) giving the holomorphic structure of \( F \) obtained using the holomorphic structures on the line bundles \( \xi_\lambda \).

For any \( \lambda \in \mathbb{R} \), let \( \rho(\lambda) : H_\lambda \rightarrow H_{\lambda-2} \) be the restriction of the homomorphism \( \rho \) in Eqn. (5.1). Using the isomorphism \( f'_\lambda \) in Eqn. (4.11), this homomorphism \( \rho(\lambda) \) gives a smooth homomorphism of vector bundles

\[ S_\lambda = f'_\lambda \otimes \rho(\lambda) : \xi_\lambda \otimes \mathbb{C} H_\lambda \rightarrow (\xi_{\lambda-2} \otimes (T_{\mathbb{H}}^{0,1})^*) \otimes \mathbb{C} H_{\lambda-2} = (\xi_{\lambda-2} \otimes \mathbb{C} H_{\lambda-2}) \otimes (T_{\mathbb{H}}^{0,1})^* . \]

It is easy to see that \( S_\lambda \) intertwines the actions of \( SL(2, \mathbb{R}) \).

Let

\[ \overline{\partial}'_F := \overline{\partial}_F + \left( \bigoplus_{\lambda \in \mathbb{R}} S_\lambda \right) \]

be the Dolbeault operator on \( F \), where \( \overline{\partial}_F \) is the earlier defined Dolbeault operator on \( F \), and \( S_\lambda \) is defined in Eqn. (5.3). Since both \( \overline{\partial}_F \) and \( S_\lambda \) are preserved by the action of \( SL(2, \mathbb{R}) \) on \( F \), the holomorphic structure defined by \( \overline{\partial}_F \) (see Eqn. (5.4)) is also preserved by the action of \( SL(2, \mathbb{R}) \).

Therefore, \(((F, \overline{\partial}_F), h_F)\) is a holomorphic Hermitian vector bundle over \( \mathbb{H} \) equipped with an action of \( SL(2, \mathbb{R}) \) which is compatible with both the holomorphic and the Hermitian structures. It is straightforward to check the following: If we substitute \(((F, \overline{\partial}_F), h_F)\) for \(((E, h), \Theta')\) in Section 4, then the data we get from it (see the end of Section 4) coincide with the above data \(((H, \rho), \Theta')\).

Conversely, if we perform the above construction using the data \(((H(E), \Theta'), \Theta')\) in Eqn. (4.13) and Eqn. (4.14), then we get back the holomorphic Hermitian vector bundle \((E, h)\) from which the data \((H(E), \Theta', \Theta')\) were constructed.

Therefore, we have proved the following theorem.

**Theorem 5.1.** Once the isomorphisms \( f'_\lambda \) in Eqn. (4.11) are fixed, there is a natural bijective correspondence between the following two collections:
(1) holomorphic Hermitian vector bundle over \( \mathbb{H} \) equipped with an action of \( \widetilde{\text{SL}(2,\mathbb{R})} \) which is compatible with both the holomorphic and the Hermitian structures, and

(2) pairs of the form \((H, \rho)\), where

\[
H := \bigoplus_{\lambda \in \mathbb{R}} H_{\lambda}
\]

is an orthogonal direct sum of finite dimensional Hilbert spaces with \( H_{\lambda} = 0 \) for all but finitely many real numbers, and

\[
\rho : H \longrightarrow H
\]

is a linear map with \( \rho(H_{\lambda}) \subset H_{\lambda-2} \) for all \( \lambda \in \mathbb{R} \).

6. Classification of isomorphism classes

Let \((E_1, h_1)\) and \((E_2, h_2)\) be two holomorphic Hermitian vector bundles over \( \mathbb{H} \) equipped with actions of \( \widetilde{\text{SL}(2,\mathbb{R})} \) which are compatible with both the holomorphic and the Hermitian structures. They will be called equivalent if there is a holomorphic isometry \( E_1 \longrightarrow E_2 \) that intertwines the actions of \( \text{SL}(2,\mathbb{R}) \).

Our aim in this section is to investigate the conditions on the data of the type \((H, \rho)\) in Theorem 5.1 that translate into the above equivalence condition on equivariant holomorphic Hermitian vector bundles.

Take a pair \((H, \rho)\), where

\[
H := \bigoplus_{\lambda \in \mathbb{R}} H_{\lambda}
\]

is an orthogonal direct sum of finite dimensional Hilbert spaces with \( H_{\lambda} = 0 \) for all but finitely many real numbers, and

\[
\rho : H \longrightarrow H
\]

is a linear map with \( \rho(H_{\lambda}) \subset H_{\lambda-2} \) for all \( \lambda \in \mathbb{R} \). Let

\[
H' := \bigoplus_{\lambda \in \mathbb{R}} H'_{\lambda}
\]

be another finite dimensional Hilbert space of above type, and let

\[
\rho' : H' \longrightarrow H'
\]

be a linear map such that \( \rho'(H'_{\lambda}) \subset H'_{\lambda-2} \) for all \( \lambda \in \mathbb{R} \). Assume that there is a linear isometry

\[
(6.1) \quad \psi : H \longrightarrow H'
\]

such that \( \psi(H_{\lambda}) = H'_{\lambda} \) for all \( \lambda \in \mathbb{R} \), and furthermore,

\[
\rho' \circ \psi = \psi \circ \rho.
\]

Let \(((F, \bar{\partial}_F), h_F)\) (respectively, \(((F', \bar{\partial}_{F'})', h_{F'})\)) be the vector bundle corresponding to \((H, \rho)\) (respectively, \((H', \rho')\)) given by Theorem 5.1. Therefore,

\[
F := \bigoplus_{\lambda \in \mathbb{R}} \xi_{\lambda} \otimes \mathbb{C} H_{\lambda},
\]

and

\[
F' := \bigoplus_{\lambda \in \mathbb{R}} \xi_{\lambda} \otimes \mathbb{C} H'_{\lambda}
\]
Consider the smooth isomorphism between vector bundles

\[ \tilde{\psi} := \bigoplus_{\lambda \in \mathbb{R}} (\text{Id}_{\xi_\lambda} \otimes \psi_\lambda) : F = \bigoplus_{\lambda \in \mathbb{R}} \xi_\lambda \otimes_{\mathbb{C}} H_\lambda \rightarrow \bigoplus_{\lambda \in \mathbb{R}} \xi_\lambda \otimes_{\mathbb{C}} H'_\lambda = F', \]

where \( \psi_\lambda \) is the restriction to \( H_\lambda \) of the isomorphism \( \psi \) in Eqn. (6.1). It is straightforward to check that this isomorphism \( \tilde{\psi} \) takes the holomorphic structure \( \partial F \) to \( \partial F' \), and it takes the Hermitian structure \( h_F \) to \( h_{F'} \). Furthermore, \( \tilde{\psi} \) intertwines the actions of \( \tilde{\text{SL}}(2, \mathbb{R}) \).

Therefore, \((F, \partial F, h_F)\) and \((F', \partial F', h_{F'})\) are equivalent.

Conversely, let \((E, h)\) and \((E', h')\) be two holomorphic Hermitian vector bundles over \( \mathbb{H} \) equipped with actions of \( \tilde{\text{SL}}(2, \mathbb{R}) \) that are compatible with the holomorphic and Hermitian structures. Let

\[ \delta : E \rightarrow E' \]

be a holomorphic isometry intertwining the actions of \( \tilde{\text{SL}}(2, \mathbb{R}) \). In other words, \( \delta \) gives an equivalence.

For any \( \tilde{\text{SL}}(2, \mathbb{R}) \)-equivariant smooth homomorphism of vector bundles

\[ \alpha : \xi_\lambda \rightarrow E, \]

the composition \( \delta \circ \alpha \) is a \( \tilde{\text{SL}}(2, \mathbb{R}) \)-equivariant smooth homomorphism from \( \xi_\lambda \) to \( E' \), where \( \delta \) is the homomorphism in Eqn. (6.2). The map defined by \( \alpha \mapsto \delta \circ \alpha \) gives a linear isometry of the Hilbert space \( W_{E, \lambda} \) (defined in Eqn. (4.3)) with the Hilbert space \( W_{E', \lambda} \), where \( W_{E', \lambda} \) denotes the Hilbert space obtained by substituting \( (E', h') \) for \( (E, h) \) in the construction of \( W_{E, \lambda} \); see Eqn. (4.3). Let

\[ \psi_\lambda : W_{E, \lambda} \rightarrow W_{E', \lambda} \]

be the isometry obtained this way.

Let

\[ \rho : \bigoplus_{\lambda \in \mathbb{R}} W_{E, \lambda} \rightarrow \bigoplus_{\lambda \in \mathbb{R}} W_{E, \lambda} \]

and

\[ \rho' : \bigoplus_{\lambda \in \mathbb{R}} W_{E', \lambda} \rightarrow \bigoplus_{\lambda \in \mathbb{R}} W_{E, \lambda} \]

be the homomorphisms constructed as in Eqn. (4.14) from \( (E, h) \) and \( (E', h') \) respectively. The restriction of \( \rho \) (respectively, \( \rho' \)) to \( W_{E, \lambda} \) (respectively, \( W_{E', \lambda} \)) will be denoted by \( \rho_\lambda \) (respectively, \( \rho'_\lambda \)). Now, it is easy to check that \( \psi_\lambda \) constructed in Eqn. (6.3) has the property that

\[ \rho'_\lambda \circ \psi_\lambda = \psi_{\lambda - 2} \circ \rho_\lambda, \]

Therefore, we have proved the following:

**Theorem 6.1.** Let \((E, h)\) and \((E', h')\) be two holomorphic Hermitian vector bundles over \( \mathbb{H} \) equipped with actions of \( \tilde{\text{SL}}(2, \mathbb{R}) \) which are compatible with both the holomorphic and the Hermitian structures. Let \((H = \bigoplus_{\lambda \in \mathbb{R}} H_\lambda, \rho)\) and \((H' = \bigoplus_{\lambda \in \mathbb{R}} H'_\lambda, \rho')\) be the data for \((E, h)\) and \((E', h')\) respectively given by Theorem 5.1. Then the following two statements are equivalent:

1. There is a holomorphic isometry \( E \rightarrow E' \) intertwining the actions of \( \tilde{\text{SL}}(2, \mathbb{R}) \).
2. There is a linear isometry \( H \rightarrow H' \) that takes \( H_\lambda \) to \( H'_\lambda \) for all \( \lambda \), and it intertwines \( \rho \) and \( \rho' \).
Let \((E, h)\) be a holomorphic Hermitian vector bundle over \(\mathbb{H}\) equipped with an action of \(\widetilde{\text{SL}}(2, \mathbb{R})\) which is compatible with both the holomorphic and the Hermitian structures. We will say that \((E, h)\) is \textit{decomposable} if there is a smooth nonzero subbundle \(F \subset E\) such that
\begin{itemize}
  \item \(\text{rank}(F) < \text{rank}(E)\), and
  \item the Chern connection on \(E\) leaves \(F\) invariant.
\end{itemize}

These two conditions are equivalent to the condition that the orthogonal complement \(F^\perp\) is nonzero with both \(F\) and \(F^\perp\) being left invariant by the Chern connection on \(E\). From the equivalence of statements prior to Proposition 2.3 it follows that the above two conditions are equivalent to the condition that both \(F\) and \(F^\perp\) are holomorphic nonzero subbundles. If a smooth subbundle \(F \subset E\) is left invariant by the Chern connection on \(E\), then Proposition 2.3 says that \(F\) is also left invariant by the action of \(\widetilde{\text{SL}}(2, \mathbb{R})\) on \(E\).

We will say that \((E, h)\) is \textit{indecomposable} if it is not decomposable.

Theorem 5.1 associates to \((E, h)\) data of following type:
\begin{itemize}
  \item A Hilbert space \(H\) and an orthogonal decomposition
    \[ H := \bigoplus_{\lambda \in \mathbb{R}} H_\lambda \]
    with \(H_\lambda = 0\) for all but finitely many real numbers, and
  \item a linear map
    \[ \rho : H \longrightarrow H \]
    with \(\rho(H_\lambda) \subset H_{\lambda-2}\) for all \(\lambda \in \mathbb{R}\).
\end{itemize}

We will call \((H = \bigoplus_{\lambda \in \mathbb{R}} H_\lambda, \rho)\) to be \textit{decomposable} if the following conditions hold:
\begin{enumerate}
  \item for each \(\lambda \in \mathbb{R}\), there is a linear subspace \(H'_\lambda \subset H_\lambda\) such that
    \[ 0 \neq \bigoplus_{\lambda \in \mathbb{R}} H'_\lambda =: H' \neq H \]
    and
  \item the subspaces \(H' \subset H\) and its orthogonal complement \(H'^\perp\) are both preserved by the homomorphism \(\rho\).
\end{enumerate}

We will say that \((H = \bigoplus_{\lambda \in \mathbb{R}} H_\lambda, \rho)\) is \textit{indecomposable} if it is not decomposable.

From the construction of the bijective correspondence in Theorem 5.1, we have the following corollary:

\textbf{Corollary 6.2.} Let \((E, h)\) be a holomorphic Hermitian vector bundle over \(\mathbb{H}\) equipped with an action of \(\widetilde{\text{SL}}(2, \mathbb{R})\) which is compatible with both the holomorphic and the Hermitian structures. Let \((H = \bigoplus_{\lambda \in \mathbb{R}} H_\lambda, \rho)\) be the corresponding data given by Theorem 5.1. Then the following two statements are equivalent:
\begin{enumerate}
  \item The holomorphic Hermitian vector bundle \((E, h)\) is indecomposable.
  \item The data \((H = \bigoplus_{\lambda \in \mathbb{R}} H_\lambda, \rho)\) is indecomposable.
\end{enumerate}
7. Some numerical invariants

Let \((E, h)\) be a holomorphic Hermitian vector bundle over \(\mathbb{H}\) equipped with an action of \(\widetilde{\text{SL}(2, \mathbb{R})}\) which is compatible with both the holomorphic and the Hermitian structures. Let \(\nabla\) denote the Chern connection on \(E\). The curvature of \(\nabla\) will be denoted by \(\mathcal{K}(\nabla)\). Therefore, \(\mathcal{K}(\nabla)\) is a smooth \((1, 1)\)-form on \(\mathbb{H}\), with values in \(\text{End}(E)\), which is left invariant by the action of \(\widetilde{\text{SL}(2, \mathbb{R})}\).

Let \(\omega\) denote the Kähler form on \(\mathbb{H}\) given by the Poincaré metric. So, we have

\[
\mathcal{K}(\nabla) = K_E \otimes \omega,
\]

where \(K_E\) is a \(\widetilde{\text{SL}(2, \mathbb{R})}\)-invariant smooth section of \(\text{End}(E)\). For any point \(z \in \mathbb{H}\), consider the eigenvalues (along with multiplicities) of \(K_E(z) \in \text{End}(E_z)\). Since \(K_E\) is \(\widetilde{\text{SL}(2, \mathbb{R})}\)-invariant, these eigenvalues are independent of the choice of the point \(z\). Therefore, they can be considered as numerical invariants for \((E, h)\).

We will describe \(\mathcal{K}(\nabla)\) in terms of the data for \((E, h)\) given by Theorem 5.1.

Let \(\nabla_E\) denote the Chern connection of the holomorphic Hermitian vector bundle \((E, h)\) constructed in Eqn. (4.6) from \((E, h)\). Therefore, \(\nabla^{1,0}_E\) is the Dolbeault operator \(\partial^E\) on \(E\) (see Eqn. (4.9)).

Let

\[
\nabla^{1,0} := (\gamma^{-1}_E \otimes \text{Id}_{\Omega^1_{\mathbb{H}}}) \circ \nabla^{1,0} \circ \gamma_E : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^1_{\mathbb{H}}
\]

be the differential operator, where \(\nabla^{1,0}\) is the \((1, 0)\)-component of the Chern connection \(\nabla\) on \((E, h)\), and \(\gamma_E\) is the smooth isomorphism constructed in Eqn. (4.7). We know that

\[
\partial^E := (\gamma^{-1}_E \otimes \text{Id}_{\Omega^0_{\mathbb{H}}}) \circ \nabla^{1,0} \circ \gamma_E = \partial_E + \Theta
\]

(see Eqn. (4.8) and Eqn. (4.9)). Since the homomorphism \(\gamma_E\) is an isometry, the connection \(\gamma^*E\nabla = \nabla^{1,0} + \partial_E\) on \(E\) is preserves its Hermitian structure \(\mathcal{H}\). Now from Eqn. (7.3), and the fact the connection \(\nabla_E\) is Hermitian, it follows that

\[
\nabla^{1,0}_E = \nabla^{1,0}_E - \Theta^*,
\]

where \(\nabla^{1,0}_E\) is the \((1, 0)\)-component of the Chern connection \(\nabla_E\) on \(E\).

In other words, we have

\[
\gamma^*_E \nabla = \nabla_E - \Theta^* + \Theta.
\]

Therefore, the curvature \(\mathcal{K}(\gamma^*_E \nabla)\) of \(\gamma^*_E \nabla\) has the following expression:

\[
\mathcal{K}(\gamma^*_E \nabla) = \gamma^*_E \mathcal{K}(\nabla) = \mathcal{K}(\nabla_E) - \Theta^* \wedge \Theta - \Theta \wedge \Theta^*,
\]

where \(\mathcal{K}(\nabla_E)\) is the curvature of the connection \(\nabla_E\).

The curvature \(\mathcal{K}(\nabla_E)\) has the following expression:

\[
\mathcal{K}(\nabla_E) = - \bigoplus_{\lambda \in \mathbb{R}} \lambda \cdot \text{Id}_{\mathcal{E}_\lambda} \cdot \omega,
\]

where \(\omega\) is the Kähler form on \(\mathbb{H}\) for the Poincaré metric (here the decomposition of \(\mathcal{E}\) given in Eqn. (4.6) is used). Indeed, this follows from the fact that the decomposition in Eqn. (4.6) is an orthogonal decomposition into holomorphic subbundles.
Let \( (H = \bigoplus_{\lambda \in \mathbb{R}} H_{\lambda}, \rho) \) be the data for \((E, h)\) given by Theorem 5.1. Using Eqn. (7.4) and the above expression for \(K(\nabla)\) it follows that the \(\gamma_{E}^{C}(\nabla)\) corresponds to the endomorphism
\[
T = -\bigoplus_{\lambda \in \mathbb{R}}((\rho_{\lambda})^{*}\rho_{\lambda} - \rho_{\lambda+2}(\rho_{\lambda+2})^{*} + \lambda \cdot \text{Id}_{H_{\lambda}})
\]
of \(H\), where \(\rho_{\lambda}\) is the restriction of \(\rho\) to \(H_{\lambda}\).

It may be noted that the eigenvalues of the operator
\[
\hat{\rho} := \bigoplus_{\lambda \in \mathbb{R}}((\rho_{\lambda})^{*}\rho_{\lambda})
\]
on \(H\) are also numerical invariants of \((E, \rho)\).

We will now give an example to show that these numerical invariants and the numerical invariants given by the eigenvalues of \(K_{E}\) (constructed in Eqn. (7.1)) do not determine \((E, h)\). Let \(a, b\) and \(x\) be any three real numbers. Consider
\[
\rho = \begin{pmatrix}
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\quad \text{and} \quad
\rho' = \begin{pmatrix}
0 & 0 & a & 0 & 0 & 0 \\
0 & 0 & 0 & b & 0 & 0 \\
0 & 0 & 0 & 0 & x & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Here \(H = H_{0} \bigoplus H_{2} \bigoplus H_{4}\) with \(H_{0} = H_{2} = H_{4} = \mathbb{C}^{2}\) equipped with the standard metric. Since \(\rho^{3} = 0\) and \(\rho^{3} \neq 0\), it follows that \(\rho\) cannot be unitarily equivalent to \(\rho'\). Furthermore, \(\rho^{*}\rho\) as well as \(\rho^{*}\rho'\) are diagonal with eigenvalues \(a^{2}, b^{2}, x^{2}, 0\). Similarly, \(\rho'\rho^{*}\) and \(\rho'\rho'\) are diagonal with the same set of eigenvalues \(a^{2}, b^{2}, x^{2}, 0\). The eigenvalues of \(\rho\rho^{*} - \rho'\rho\) are \(a^{2}, b^{2}, x^{2} - a^{2}, -b^{2}, -x^{2}, 0\) while those of \(\rho'\rho^{*} - \rho'\rho'\) are \(a^{2}, b^{2}, -a^{2}, x^{2} - b^{2}, -x^{2}, 0\). These two sets of eigenvalues are equal for \(a = b\).

**References**


School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India

*E-mail address*: indranil@math.tifr.res.in

Indian Statistical Institute, R. V. College Post Office, Bangalore 560059, India

*E-mail address*: gm@isibang.ac.in