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Hypergeometric series and continued fractions

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Abstract. Ramanujan's results on continued fractions are simple consequences of three-term relations between hypergeometric series. Their q-analogues lead to many of the continued fractions given in the 'Lost' notebook in particular the famous one considered by Andrews and others.

Keywords. Hypergeometric series; continued fractions; q-series; Lost Notebook; three-term relations; modular functions.

1. Introduction

The hypergeometric series

$$f(\alpha, \beta; \gamma, x) = \sum_{m=0}^{\infty} \frac{(\alpha)_m(\beta)_m}{(1)_m \cdot (\gamma)_m} x^m, \tag{1}$$

where, for any α and rational integer m,

$$(\alpha)_m = \frac{\Gamma(\alpha+m)}{\Gamma(\alpha)},$$

was first discussed extensively by Gauss [6] who showed that any three contiguous series (see definition below) satisfy a linear relation. By suitable choice of these contiguous series, Gauss obtained continued fraction representations for ratios of hypergeometric series. Euler earlier had considered the integral

$$F(\alpha, \beta; \gamma, x) = \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tx)^{-\alpha}\,dt \tag{2}$$

and by partial integration obtained a linear relation between three such integrals and then a continued fraction for the ratio of such integrals. This continued fraction is different from that of Gauss. Since it is easy to see that

$$F(\alpha, \beta; \gamma, x) = \frac{\Gamma(\beta)\Gamma(\gamma-\beta)}{\Gamma(\gamma)} f(\alpha, \beta; \gamma, x), \tag{3}$$

Euler's continued fraction is actually one related to contiguous hypergeometric series.

In the middle of the last century Heine, taking his cue from Jacobi and Eisenstein, considered in detail the so-called basic hypergeometric series

$$\varphi(\alpha, \beta; \gamma, q, x) = \sum_{m=0}^{\infty} \frac{(\alpha, q)_m \cdot (\beta, q)_m}{(q, q)_m \cdot (\gamma, q)_m} \cdot x^m, \tag{4}$$

where q is a parameter with $|q|<1$ and for any α and rational integer m,

$$(\alpha, q)_m = (\alpha, q)_\infty/(\alpha q^m, q)_\infty \tag{5}$$

and

$$(\alpha, q)_\infty = (1-\alpha)(1-\alpha q)\cdots = \prod_{m=0}^{\infty}(1-\alpha q^m). \tag{6}$$

Heine [8] found all the three-term relations for φ, analogous to those found by Gauss for f and had obtained related continued fractions.

Ramanujan's researches in mathematics during the years 1903–1913 were recorded in his famous Notebooks. From these it became known that Ramanujan had not only rediscovered many of the classical results of Gauss, Euler, Heine and others on hypergeometric series and continued fractions but also found many new ones on these topics using systematically the three-term relations. There are many such related to Heine's q-series in the 'Lost' Notebook [13] discovered in 1976 by G E Andrews among the Ramanujan papers in the Trinity College, Cambridge.

Our object in this note is to show that many of the results of Ramanujan on continued fractions can be obtained as limiting cases of results of Gauss, Euler and Heine. In particular the famous continued fraction of Rogers–Ramanujan–Schur is a limiting case of Heine's q-analogue of Gauss' continued fraction. Some of the q-continued fractions are related to modular functions. The exploitation of this relationship between modular functions and continued fractions is one of the most beautiful aspects of Ramanujan's work on continued fractions. We do not go into details about this in the present paper, however.

2. Three-term relations

Ramanujan had seen Gauss' continued fraction stated in Carr's book (Synopsis of pure mathematics, London and Cambridge 1886). He studied, perhaps, elements of hypergeometric series from Chrystal's Algebra and Greenhills' Elliptic functions. In any case he must have derived the three-term relations in Gauss' memoir and others which he needed for his work.

From (1) we see that

$$f(\alpha,\beta;\gamma,x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}\cdot x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \cdots.$$

We say that $f(\alpha,\beta;\gamma,x)$ and $f(\alpha',\beta';\gamma',x)$ are contiguous if

$$\alpha\equiv\alpha', \quad \beta\equiv\beta', \quad \gamma\equiv\gamma' \pmod 1.$$

Using three-term relations between contiguous hypergeometric series Gauss obtained a continued fraction for the ratio,

$$\frac{f(\alpha,\beta+1;\gamma+1,x)}{f(\alpha,\beta;\gamma,x)}.$$

Ramanujan gives it in the form

$$\frac{\alpha\beta x}{\gamma}\frac{f(\gamma-\alpha,\beta+1;\gamma+1,-x)}{f(\gamma-\alpha,\beta;\gamma,-x)}=\frac{\alpha\beta x}{\gamma+}\,\frac{(\alpha-\gamma)(\beta-\gamma)x}{\gamma+1+}\,\frac{(\alpha+1)(\beta+1)x}{\gamma+2+}\cdots. \tag{7}$$

The right side of (7) is symmetrical in α and β. This result is Entry 18 in Vol. 1, p. 215.

Entry 19 on the same page consists of three statements, all limiting cases. They are

$$\begin{aligned}
&\frac{\beta x}{\gamma}f(\beta+1,1;\gamma+1,-x)\\
&=\frac{\beta x}{\gamma+}\,\frac{(\beta+1)\gamma x}{\gamma+1+}\,\frac{(\gamma-\beta)x}{\gamma+2+}\cdots\\
&=\frac{\beta x}{\gamma+}\,\frac{(\beta+1)x}{1+}\,\frac{1\cdot(1+x)}{\gamma+}\,\frac{(\beta+2)x}{1+}\cdots\\
&=\frac{\beta x}{\gamma+x(\beta+1)-}\,\frac{1\cdot(\beta+1)x(x+1)}{\gamma+1+x(\beta+3)-}\,\frac{2\cdot(\beta+2)x(x+1)}{\gamma+2+x(\beta+5)-}\cdots.
\end{aligned} \tag{8}$$

Clearly the first of these relations is a special case of (7) obtained by putting $\beta=0,\gamma-\alpha$ for α and $\beta+1$ for α.

In order to obtain the second result, consider the two three-term relations

$$f(\alpha,\beta;\gamma,x)=\begin{cases} f(\alpha,\beta+1;\gamma,x)-\dfrac{\alpha x}{\gamma}f(\alpha+1,\beta+1;\gamma+1,x)\\[2mm] \dfrac{\gamma-\beta}{\gamma}f(\alpha+1,\beta;\gamma+1,x)+\dfrac{\beta(1-x)}{\gamma}f(\alpha+1,\beta+1;\gamma+1,x)\end{cases} \tag{9}$$

of which the first one is due to Gauss. In order to obtain the second one apply the formula

$$f(\alpha,\beta;\gamma,x)=(1-x)^{\gamma-\alpha-\beta}f(\gamma-\alpha,\gamma-\beta;\gamma,x) \tag{10}$$

to both sides of Gauss' relation

$$f(\alpha,\beta;\gamma,x)-f(\alpha,\beta;\gamma+1,x)=\frac{\alpha\beta x}{\gamma(\gamma+1)}f(\alpha+1,\beta+1;\gamma+2,x).$$

One then has

$$\begin{aligned}f(\alpha,\beta;\gamma,x)&=(1-x)f(\alpha+1,\beta+1;\gamma+1,x)\\&\quad+\frac{(\gamma-\alpha)(\gamma-\beta)x}{\gamma(\gamma+1)}f(\alpha+1,\beta+1;\gamma+2,x).\end{aligned} \tag{11}$$

Eliminate $f(\alpha+1,\beta+1;\gamma+2,x)$ between (11) and Gauss' relation

$$f(\alpha,\beta;\gamma,x)=f(\alpha+1,\beta;\gamma+1,x)-\frac{\beta(\gamma-\alpha)x}{\gamma(\gamma+1)}f(\alpha+1,\beta+1;\gamma+2,x).$$

We then obtain the second result in (9).

From the two relations in (9) one obtains

$$\frac{f(\alpha,\beta+1;\gamma,x)}{f(\alpha,\beta;\gamma,x)}=\frac{1}{1-}\frac{\alpha x}{\gamma-\beta-1+}\frac{(\beta+1)(1-x)}{1-}\frac{(\alpha+1)x}{\gamma-\beta-1+}\frac{(\beta+2)(1-x)}{1-}\cdots. \tag{12}$$

Putting $\beta=0$, changing α into $\beta+1$, γ into $\gamma+1$ and x into $-x$, we obtain the second formula in (8).

The last relation is obtained easily from the Gauss relation [6],

$$f(\alpha,\beta;\gamma,x)=\frac{\gamma-(\alpha+\beta+1)x}{\gamma}f(\alpha+1,\beta+1;\gamma+1,x)$$
$$+\frac{(\alpha+1)(\beta+1)(x-x^2)}{\gamma(\gamma+1)}f(\alpha+2,\beta+2;\gamma+2,x). \tag{13}$$

This result does not seem to have been noticed. It leads to the continued fraction

$$\frac{f(\alpha+1,\beta+1;\gamma+1,x)}{f(\alpha,\beta;\gamma,x)}=\frac{\gamma}{\gamma-(\alpha+\beta+1)x+}\frac{(\alpha+1)(\beta+1)(x-x^2)}{\gamma+1-(\alpha+\beta+3)x+}$$
$$\frac{(\alpha+2)(\beta+2)(x-x^2)}{\gamma+2-(\alpha+\beta+5)x+\cdots}. \tag{14}$$

If as before we put $\beta=0$, β for α and $-x$ for x, we obtain the third result in (8).

Ramanujan found another continued fraction from a three-term relation between

$$f(\alpha,\beta;\gamma,x),\quad f(\alpha+1,\beta;\gamma+1,x),\quad f(\alpha+2,\beta;\gamma+2,x).$$

It is clear from the form of the three series, that such a relation can be obtained from (13) by using the relation

$$(1-x)^{-\alpha}f\left(\alpha,\beta;\gamma,\frac{x}{x-1}\right)=f(\alpha,\gamma-\beta;\gamma,x) \tag{15}$$

on both sides of the relation (14). We obtain the relation

$$f(\alpha,\beta;\gamma,x)=\frac{\gamma+(\alpha-\beta+1)x}{\gamma}f(\alpha+1,\beta;\gamma+1,x)$$
$$-\frac{(\alpha+1)(\gamma-\beta+1)x}{\gamma(\gamma+1)}f(\alpha+2,\beta;\gamma+2,x). \tag{16}$$

This leads at once to the continued fraction

$$\frac{f(\alpha+1,\beta;\gamma+1,x)}{f(\alpha,\beta;\gamma,x)}=\frac{\gamma}{\gamma+(1+\alpha-\beta)x-}\,\frac{(\alpha+1)(1+\gamma-\beta)x}{\gamma+1+(2+\alpha-\beta)x-}\,\frac{(\alpha+2)(2+\gamma-\beta)x}{\gamma+2+(3+\alpha-\beta)x-\cdots}. \tag{17}$$

This formula is *not* given in chapter XIV of Volume 1 which is devoted to continued fractions. It is given on page 198 of the same volume.

It is to be noted that the two formulae (10) and (15) are also given in Ramanujan's Notebook I (p. 193, Entry 11 and p. 191, Entry 4).

We shall state one another continued fraction whose q-analogue Ramanujan perhaps used. It depends on the three-term relations

$$f(\alpha,\beta;\gamma,x)=f(\alpha,\beta+1;\gamma,x)-\frac{\alpha x}{\gamma}f(\alpha+1,\beta+1;\gamma+1,x) \tag{18}$$

and the relation (16). We then have

$$\frac{f(\alpha,\beta;\gamma,x)}{f(\alpha,\beta+1;\gamma,x)}=1-\frac{\alpha x}{\gamma}\left(\frac{f(\alpha+1,\beta+1;\gamma+1,x)}{f(\alpha,\beta+1;\gamma,x)}\right).$$

We now substitute the continued fraction (17) for

$$\frac{\gamma f(\alpha,\beta+1;\gamma,x)}{f(\alpha+1,\beta+1;\gamma+1,x)}.$$

We then obtain

$$\frac{f(\alpha,\beta+1;\gamma,x)}{f(\alpha,\beta;\gamma,x)}=\frac{1}{1-}\,\frac{\alpha x}{\gamma+(\alpha-\beta)x-}\,\frac{(\alpha+1)(\gamma-\beta)x}{\gamma+1+(\alpha-\beta+1)x-}\,\frac{(\alpha+2)(\gamma-\beta+1)x}{\gamma+2+(\alpha-\beta+2)x-\cdots}. \tag{19}$$

The interest in this is that after the first step everything else is like in the Euler continued fraction (17). In fact this can be easily generalized.

Hence by choosing suitable three-term relations one can get an infinity of continued fraction representations for the same ratio of two contiguous hypergeometric series.

3. Evaluation for special x

A second type of results on continued fractions concerns evaluation of these for special values of the variable x, say for

$$x=-1.$$

It is then clear that one would use the Kummer theorem

$$f(\alpha,\beta;1+\alpha-\beta,-1)=\frac{\Gamma(1+\alpha-\beta)\Gamma\left(1+\frac{\alpha}{2}\right)}{\Gamma(1+\alpha)\Gamma\left(1+\frac{\alpha}{2}-\beta\right)}. \tag{20}$$

This theorem of Kummer's is also in the Notebooks of Ramanujan (Vol. 1, p. 179, chapter XIV, Entry 24).

We take the continued fraction (19) with $x=-1$. We have

$$\frac{f(\alpha+1,\beta;\gamma+1,-1)}{f(\alpha,\beta;\gamma,-1)}=\frac{\gamma}{\gamma+\beta-\alpha-1+}\,\frac{(\alpha+1)(1+\gamma-\beta)}{\gamma+\beta-\alpha-1+}$$
$$\frac{(\alpha+2)(2+\gamma-\beta)}{\gamma+\beta-\alpha-1+\cdots}. \tag{21}$$

In order to be able to evaluate the left side of (21) using (20) it is necessary to impose some conditions on α,β,γ. Let us take

$$\beta=\alpha+\gamma. \tag{22}$$

Then

$$f(\beta,\alpha+1;\beta-\alpha,-1)=\frac{1}{2}\cdot\frac{\Gamma(\beta-\alpha)\cdot\Gamma\left(\frac{\beta}{2}\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta}{2}-\alpha\right)}.$$

$$f(\beta+1,\alpha+1;\beta-\alpha+1,-1)=\frac{\beta-\alpha}{2\beta}\,\frac{\Gamma(\beta-\alpha)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta+1}{2}-\alpha\right)}.$$

These two lead to

$$f(\beta,\alpha;\beta-\alpha,-1)=\frac{1}{2}\cdot\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}\left(\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}-\alpha\right)}+\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}-\alpha\right)}\right).$$

This along with

$$f(\beta,\alpha;\beta-\alpha+1,-1)=\frac{\beta-\alpha}{2\beta}\,\frac{\Gamma(\beta-\alpha)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma(\beta)\Gamma\left(\frac{\beta+1}{2}-\alpha\right)}$$

gives

$$f(\beta,\alpha+1;\beta-\alpha+1,-1)=\frac{1}{2\alpha}\,\frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}\left(\frac{\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\frac{\beta+1}{2}-\alpha\right)}-\frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}-\alpha\right)}\right).$$

We have hence the following result:

$$\frac{1}{\alpha}\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{\beta}{2}-\alpha\right)-\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\beta+1}{2}-\alpha\right)}{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{\beta}{2}-\alpha\right)+\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\beta+1}{2}-\alpha\right)}$$

$$=\frac{\beta-\alpha}{2(\beta-\alpha)-1+}\,\frac{1^2-\alpha^2}{2(\beta-\alpha)-1+}\,\frac{2^2-\alpha^2}{2(\beta-\alpha)-1+}\cdots. \tag{23}$$

In a similar way, if we put

$$1+\beta=\alpha+\gamma \tag{24}$$

the computations are a little bit simpler. Since

$$\gamma+\beta-\alpha-1=2(\beta-\alpha)$$

$$(\alpha+k)(k+\gamma-\beta)=\tfrac{1}{4}((2k+1)^2-(2\alpha-1)^2)$$

we have

$$\frac{f(\beta,\alpha+1;\beta-\alpha+2,-1)}{f(\beta,\alpha;\beta-\alpha+1,-1)}=\frac{(1+\beta-\alpha)(\beta-2\alpha)}{\alpha(1-\alpha)}\,\frac{\Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{\beta}{2}-\alpha\right)}{\Gamma\left(\frac{\beta+1}{2}-\alpha\right)\Gamma\left(\frac{\beta}{2}\right)}$$

$$+\frac{(1+\beta-\alpha)(\alpha-\beta)}{\alpha(1-\alpha)}.$$

If therefore we set

$$2\alpha-1=N,\quad 2(\beta-\alpha)=x>0,$$

we have Ramanujan's result (Vol. 1, p. 217, Entry 21), due originally to Euler and Stieltjes

$$\frac{4}{x+}\,\frac{1^2-N^2}{2x+}\,\frac{3^2-N^2}{2x+}\,\frac{5^2-N^2}{2x+}\cdots=\frac{\Gamma\left(\frac{x+N+1}{4}\right)\Gamma\left(\frac{x-N+1}{4}\right)}{\Gamma\left(\frac{x+N+3}{4}\right)\Gamma\left(\frac{x-N+3}{4}\right)}. \tag{25}$$

The results (23) and (25) are in a certain sense, complementary

4. Theorems of Perron and Ramanujan

We have seen several continued fraction representions for the same ratio of hypergeometric series. There is also a continued fraction representation which combines both the features of the Gauss method and the Euler method. The Gauss method depends on two three-term relations

$$f(\alpha,\beta;\gamma,x)=\begin{cases} f(\alpha+1,\beta;\gamma+1,x)-\dfrac{\beta(\gamma-\alpha)x}{\gamma(\gamma+1)}f(\alpha+1,\beta+1;\gamma+2,x) \\ f(\alpha,\beta+1;\gamma+1,x)-\dfrac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)}f(\alpha+1,\beta+1;\gamma+2,x) \end{cases} \tag{26}$$

and the Euler method depends on one relation (16). If we use the first equation in (26), then

$$\frac{\gamma f(\alpha,\beta;\gamma,x)}{f(\alpha+1,\beta;\gamma+1,x)}=\gamma-\frac{\beta(\gamma-\alpha)}{\gamma+1}\left(\frac{f(\alpha+1,\beta;\gamma+1,x)}{f(\alpha+1,\beta+1;\gamma+2,x)}\right)^{-1}.$$

Using now the second equation in (26)

$$(\gamma+1)\frac{f(\alpha+1,\beta;\gamma+1,x)}{f(\alpha+1,\beta+1;\gamma+2,\alpha)}=\gamma+1-\frac{(\alpha+1)(\gamma+1-\beta)}{\gamma+2}\times\left(\frac{f(\alpha+1,\beta+1;\gamma+2,x)}{f(\alpha+2,\beta+1,\gamma+3,x)}\right)^{-1}.$$

We now use (16) for

$$(\gamma+2)\frac{f(\alpha+1,\beta+1;\gamma+2,x)}{f(\alpha+2,\beta+1;\gamma+3,x)}$$

with α,β,γ replaced by $\alpha+1$, $\beta+1$ and $\gamma+2$. We then get

$$\gamma+(1+\alpha-\beta)x-\frac{(\alpha+1)(1+\gamma-\beta)x}{\gamma+1+(2+\alpha-\beta)x-}\,\frac{(\alpha+2)(2+\gamma-\beta)x}{\gamma+2+(3+\alpha-\beta)x-}\cdots$$

$$=\gamma-\frac{\beta(\gamma-\alpha)x}{\gamma+1-}\,\frac{(\alpha+1)(\gamma+1-\beta)x}{\gamma+2+(1+\alpha-\beta)x-}\,\frac{(\alpha+2)(3+\gamma-\beta)x}{\gamma+3+(2+\alpha-\beta)x-}\cdots. \tag{27}$$

This is a new equality between continued fractions. One could get a newer result by proceeding as above but carrying the algebra two steps further and then using Euler's continued fraction. This is exactly what Perron first and independently Ramanujan did. If we take $x=-1$, (27) takes the neater form

$$\gamma+\frac{\beta(\gamma-\alpha)}{\gamma+1+}\,\frac{(\alpha+1)(\gamma+1-\beta)}{\gamma+\beta-\alpha+1+}\,\frac{(\alpha+2)(\gamma+2-\beta)}{\gamma+\beta-\alpha+1+}\cdots$$

$$=\gamma+\beta-\alpha-1+\frac{(\alpha+1)(\gamma+1-\beta)}{\gamma+\beta-\alpha-1+}\,\frac{(\alpha+2)(\gamma+2-\beta)}{\gamma+\beta-\alpha-1+}\cdots. \tag{28}$$

This result was first proved by Perron ([9] First edition 1913, p. 223) as a consequence of a theorem of Muir. To identify (28) with Perron's statement, replace α,β,γ by $\alpha/c,\beta/c,\gamma/c$, $c>0$ respectively and put

$$\alpha=a+b,\quad \beta=h+b+c,\quad \gamma=h+a+c. \tag{29}$$

Then (28) takes the form

$$2h+c+\frac{(a+c)^2-b^2}{2h+c+}\,\frac{(a+2c)^2-b^2}{2h+c+}\cdots$$

$$=h+a+c+\frac{(h+c)^2-b^2}{h+a+2c+}\,\frac{(a+c)^2-b^2}{2h+3c+}\,\frac{(a+2c)^2-b^2}{2h+3c+}\cdots. \tag{30}$$

This result can be generalized leading to a theorem of Ramanujan (Vol. 2, p. 342). We mention, in passing, that for convergence

$$c\geqslant 0,\quad 2h+c>0,\quad a+c\geqslant 0,\quad (a+c)^2>b^2>-\infty. \tag{31}$$

Let us denote the left side of (30) by $\varphi(h,a)$. Then $\varphi(h,a)$ consists of positive terms. Furthermore

$$\varphi(h,a)=h+a+c+\frac{(h+c)^2-b^2}{h+a+2c+}\,\frac{(a+c)^2-b^2}{\varphi(h+c,a+c)}. \tag{32}$$

Since $\varphi(h,a)$ converges for (31), we have, by induction

$$\varphi(h,a)=h+a+c+\frac{(h+c)^2-b^2}{h+a+2c+}\,\frac{(a+c)^2-b^2}{h+a+3c+}\cdots\frac{(h+nc)^2-b^2}{h+a+2nc+}$$

$$\frac{(a+nc)^2-b^2}{h+a+(2n+1)c+\cdots}. \tag{33}$$

As mentioned above Ramanujan has stated this in Vol. 2, p. 342. In order to identify it with Ramanujan's statement, put

$$c=2,\quad h+1=x,\quad a+1=y,\quad b^2=-N. \tag{34}$$

Then

$$x+\frac{(y+1)^2+N}{2x+}\,\frac{(y+3)^2+N}{2x+}\,\frac{(y+5)^2+N}{2x+}\cdots$$

$$=y+\frac{(x+1)^2+N}{x+y+2+}\,\frac{(y+1)^2+N}{x+y+4+}\,\frac{(x+3)^2+N}{x+y+6+}\,\frac{(y+3)^2+N}{x+y+8+}\cdots. \tag{35}$$

We make one last application of a three-term relation. Clearly we have

$$\frac{\gamma f(\alpha,\beta;\gamma,-1)}{f(\alpha+1,\beta;\gamma+1,-1)}=\frac{\gamma f(\alpha,\beta-1;\gamma,-1)}{f(\alpha+1,\beta;\gamma+1,-1)}-\alpha$$

so that on substitution of (29) for α, β, γ,

$$G(h,a)=h+\frac{c}{2}+\frac{(a+c)^2-b^2}{2h+c+}\,\frac{(a+2c)^2-b^2}{2h+c+}\cdots$$

$$=(a+h+c)\frac{f\left(\dfrac{a+b}{c},\dfrac{h+b}{c};\dfrac{h+a+c}{c},-1\right)}{f\left(\dfrac{a+b+c}{c},\dfrac{h+b+c}{c};\dfrac{h+a+2c}{c},-1\right)}$$

$$-\left(a+h+b+\frac{c}{2}\right).$$

Since the hypergeometric series is symmetric in the first two variables and hence in a and h, we get

$$G(h,a) = G(a,h)$$

and therefore

$$G(h,a) = h + \frac{c}{2} + \frac{(a+c)^2 - b^2}{2h+c+}\,\frac{(a+2c)^2 - b^2}{2h+c+}\cdots$$

$$= a + \frac{c}{2} + \frac{(h+c)^2 - b^2}{2a+c+}\,\frac{(h+2c)^2 - b^2}{2a+c+}\cdots = G(a,h). \tag{36}$$

This result which is equivalent to Ramanujan's result given below was first discovered and proved by Perron in 1913 (First edition, p. 228). He used Muir's method. We have obtained it as part of results relating to three-term relations for hypergeometric series. If we use the values of c, x, y, N given earlier we get

$$x + \frac{(y+1)^2 + N}{2x+}\,\frac{(y+3)^2 + N}{2x+}\,\frac{(y+5)^2 + N}{2x+}\cdots$$

$$= y + \frac{(x+1)^2 + N}{2y+}\,\frac{(x+3)^2 + N}{2y+}\,\frac{(x+5)^2 + N}{2y+}\cdots. \tag{37}$$

This formula was communicated by Ramanujan to Hardy in his second letter of February 27, 1913. It is given on page 190 of Volume 1 and on page 148, Entry 27 and page 342 of Volume 2.

5. Continued fractions of Ramanujan

Chapters XII, XIII and XIV of Notebook 1 are devoted to hypergeometric series and continued fractions. One looks in vain in these and other chapters of Volume 1, for results on q-series. However on the left hand side pages 124, 134, 146, 158, 160 (in the pagination of Notebook I [12]) there are a number of results on basic hypergeometric series and continued fractions associated to them.

Heine's work [8] is mentioned in Carr's book only in the section on references at the end of that book. Heine's generalization of Gauss' continued fraction is given as an exercise in Chrystal's Algebra. Ramanujan had perhaps seen Chrystal's book which was widely used in colleges in India. In view of the results mentioned above in the pages of Volume 1, it seems that Ramanujan had thought of q-generalizations of results on hypergeometric series and continued fractions. Chapter XVI of Notebook II, perhaps one of the best chapters in the Notebook shows some aspects of these generalizations.

Set q be a parameter $|q| < 1$ and

$$(a,q)_\infty = (1-a)(1-aq)\cdots = \prod_{n=0}^{\infty}(1-aq^n)$$

and

$$(a,q)_n = (a,q)_\infty/(aq^n,q)_\infty,$$

so that for n a positive integer

$$(a,q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\quad (a,q)_0=1.$$

Heines' basic hypergeometric series is defined by

$$\varphi(\alpha,\beta;\gamma,q,x)=\sum_{m=0}^{\infty}\frac{(\alpha,q)_m(\beta,q)_m}{(q,q)_m(\gamma,q)_m}\cdot x^m$$

and is convergent for $|x|<1$. If we define two series $\varphi(\alpha,\beta;\gamma,q,x)$ and $\varphi(\alpha',\beta';\gamma',q,x)$ to be contiguous if

$$\alpha/\alpha',\quad \beta/\beta',\quad \gamma/\gamma' \tag{38}$$

are integral powers of q, then Heine showed that any three contiguous functions satisfy a linear relation. He obtained the analogues of all the three-term relations of Gauss for the ordinary hypergeometric series. In view of Heine's identity

$$\varphi(\alpha,\beta;\gamma,q,x)-\varphi(\alpha,\beta;\gamma,q,qx)=\frac{(1-\alpha)(1-\beta)x}{1-\gamma}\varphi(\alpha q,\beta q;\gamma q,x) \tag{39}$$

we may include change of x by powers of q in our definition of contiguous functions.

If we take q^2 instead of q for the step, then

$$(\alpha,q^2)_m(\beta,q^2)_m=(1-\alpha)(1-\alpha q^2)\cdots(1-\alpha q^{2m-2})$$
$$\times(1-\beta)(1-\beta q^2)\cdots(1-\beta q^{2m-2})$$

and hence

$$(\alpha,q^2)_m(\alpha q,q^2)_m=(\alpha,q)_{2m}.$$

Therefore

$$\varphi(\alpha,\alpha q;q,q^2,x^2)=\sum_{m=0}^{\infty}\frac{(\alpha,q)_{2m}}{(q,q)_{2m}}x^{2m}.$$

The q-binomial theorem states

$$1+\frac{1-\alpha}{1-q}x+\frac{(1-\alpha)(1-\alpha q)}{(1-q)(1-q^2)}x^2+\cdots=\frac{(\alpha x,q)_\infty}{(x,q)_\infty}$$

and hence

$$\varphi(\alpha q,\alpha;q,q^2,x^2)=\frac{1}{2}\left(\frac{(\alpha x,q)_\infty}{(x,q)_\infty}+\frac{(-\alpha x,q)_\infty}{(-x,q)_\infty}\right)$$

and

$$\frac{(1-\alpha)x}{1-q}\varphi(\alpha q,\alpha q^2;q^3,q^2,x^2)=\frac{1}{2}\left(\frac{(\alpha x,q)_\infty}{(x,q)_\infty}-\frac{(-\alpha x,q)_\infty}{(-x,q)_\infty}\right).$$

Heine's generalization of Gauss' continued fraction states

$$\frac{\varphi(\alpha,\beta q;\gamma q,q,x)}{\varphi(\alpha,\beta;\gamma,q,x)}=\frac{1}{1-}\frac{\beta_1}{1-}\frac{\beta_2}{1-}\cdots,$$

where

$$\beta_{2r} = \frac{(1-\beta q^r)(1-\gamma q^r/\alpha)}{(1-\gamma q^{2r-1})(1-\gamma q^{2r})}\alpha x q^{r-1}, \quad r \geqslant 1$$

$$\beta_{2r+1} = \frac{(1-\alpha q^r)(1-\gamma q^r/\beta)}{(1-\gamma q^{2r})(1-\gamma q^{2r+1})}\beta x q^r, \quad r \geqslant 0.$$

We then have Ramanujan's result (Chapter XVI, Vol. 2, Entry 11)

$$\frac{a-b}{1-q+}\,\frac{(a-bq)(aq-b)}{1-q^3+}\,\frac{(a-bq^3)(aq^3-b)}{1-q^5+}\cdots$$

$$= \frac{(b,q)_\infty(-a,q)_\infty - (-b,q)_\infty(a,q)_\infty}{(b,q)_\infty(-a,q)_\infty + (-b,q)_\infty(a,q)_\infty}. \tag{40}$$

Ramanujan obtained a q-generalization of Euler's continued fraction from the three-term relation

$$\varphi(\alpha,\beta;\gamma,q,x) = \left(1+\frac{\beta-\alpha q}{q(1-\gamma)}x\right)\varphi(\alpha q,\beta;\gamma q,q,x) - \frac{(1-\alpha q)(\beta-\gamma q)x}{q(1-\gamma)(1-\gamma q)}$$

$$\times\, \varphi(\alpha q^2,\beta;\gamma q^2,q,x), \tag{41}$$

namely

$$\frac{\varphi(\alpha q,\beta;\gamma q,q,x)}{\varphi(\alpha,\beta;\gamma,q,x)} = \frac{q(1-\gamma)}{q(1-\gamma)+(\beta-\alpha q)x+}\,\frac{(1-\alpha q)(\gamma q-\beta)xq}{q(1-\gamma q)+(\beta-\alpha q^2)x+}$$

$$\frac{(1-\alpha q^2)(\gamma q^2-\beta)xq}{q(1-\gamma q^2)+(\beta-\alpha q^3)x+\cdots}. \tag{42}$$

If $x=-q/\alpha$, then q-analogue of Kummer's theorem can be applied to evaluate $\varphi(\alpha,\beta;\gamma,q,-q/\alpha)$ provided α,β,γ satisfy some conditions. The q-Kummer theorem was proved by Daum. A simple proof based on the q-binomial theorem was given by G E Andrews [2].

If we put

$$\beta q = \gamma\alpha, \quad x = -q/\alpha \tag{43}$$

then

$$\varphi(\alpha,\beta;\gamma,q,-q/\alpha) = \frac{(\beta q,q^2)_\infty(-q,q)_\infty\left(\frac{\beta q^2}{\alpha^2},q^2\right)_\infty}{\left(\frac{\beta q^2}{\alpha},q\right)_\infty(-q/\alpha,q)_\infty}. \tag{44}$$

We apply this to the left side of equation (42). The denominator is easily evaluated by (44). In order to evaluate the numerator we use the three-term relations

$$\varphi(\alpha,\beta;\gamma,q,x) = \varphi(\alpha q,\beta;\gamma q,q,x) - \frac{\alpha x(1-\beta)(1-\gamma/\alpha)}{(1-\gamma)(1-\gamma q)}\varphi(\alpha q,\beta q;\gamma q^2,q,x)$$

$$= \varphi(\alpha,\beta q;\gamma q,q,x) - \frac{\beta x(1-\alpha)(1-\gamma/\beta)}{(1-\gamma)(1-\gamma q)}\varphi(\alpha q,\beta q;\gamma q^2,q,x).$$

We eliminate $\varphi(\alpha q, \beta q; \gamma q^2, q, x)$ from these two relations and use the fact that $\varphi(\alpha, \beta; \gamma, q, x)$ and $\varphi(\alpha, \beta q; \gamma q, q, x)$ can be evaluated for $x = -q/\alpha$ using (43). (Note that the main difficulty is due to the fact that $x = -q/\alpha$ depends on α and is not independent of α, β, γ as in the case of ordinary hypergeometric series.) We thus obtain

$$\frac{\varphi(\alpha q, \beta; \gamma q, q, -q/\alpha)}{\varphi(\alpha, \beta; \gamma, q, -q/\alpha)}$$

$$= \frac{(\alpha - \beta q)(\beta - \alpha)}{\beta(1-\alpha)(\alpha - q)} + \frac{\alpha(\alpha - \beta q)}{\beta(1-\alpha)(\alpha - q)} \cdot \frac{(\beta, q^2)_\infty \left(\frac{\beta q}{\alpha^2}, q^2\right)_\infty}{\left(\frac{\beta q^2}{\alpha^2}, q^2\right)_\infty (\beta q, q^2)_\infty}. \tag{45}$$

Ramanujan obtains a symmetric relation by taking

$$\alpha = aq/b, \quad \beta = a^2 q.$$

Thus

$$\frac{1}{1-ab} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q)^4} + \frac{(a-bq^5)(b-aq^5)}{(1-ab)(1+q^6)+\cdots}$$

$$= \frac{(a^2q^3, q^4)_\infty (b^2q^3, q^4)_\infty}{(a^2q, q^4)_\infty (b^2q, q^4)_\infty}. \tag{46}$$

Compare this with (25) using q-gamma functions of Jackson (see Askey [4]).

One might also consider the case

$$\beta = \gamma\alpha \tag{47}$$

one would obtain an analogue of (23). One would have to use (39) in order to evaluate the various hypergeometric series. We shall not proceed with this further.

6. Generalized Eisenstein fraction

Consider now the following identity due to Ramanujan stated as Entry 9 in Chapter XVI of Volume 2 of the Notebooks. It is

$$(-bq, q)_\infty \cdot \sum_{m=0}^{\infty} \frac{\lambda^m q^{m^2}}{(q,q)_m(-bq,q)_m}$$

$$= \sum_{m=0}^{\infty} \frac{(b+\lambda)(b+\lambda q)\cdots(b+\lambda q^{m-1})q^{(m(m+1)/2)}}{(q,q)_m}. \tag{48}$$

This can be proved in several ways. It is obvious that each side is a limiting case of a basic hypergeometric series.

Let $h(b, \lambda)$ denote the right side of (48). Then

$$h(b, \lambda) = \lim_{\substack{a\to 0\\ c\to 0}} \varphi\left(-\frac{\lambda}{b}, -\frac{q}{c}; -aq, q, bc\right) \tag{49}$$

as can be easily verified. We now consider the Gauss–Heine continued fractions for

$$\frac{\varphi(\alpha,\beta q;\gamma q,q,x)}{\varphi(\alpha,\beta;\gamma,q,x)} \quad \text{and} \quad \frac{\varphi(\alpha q,\beta;\gamma q,q,x)}{\varphi(\alpha,\beta;\gamma,q,x)}$$

with

$$\alpha=-\lambda/b,\quad \beta=-q/c,\quad \gamma=-aq,\quad x=bc \tag{50}$$

and take limits as $c\to 0$, $a\to 0$. We will then have continued fractions for

$$\frac{h(bq,\lambda q)}{h(b,\lambda)} \quad \text{and} \quad \frac{h(b,\lambda q)}{h(b,\lambda)}. \tag{51}$$

We take the Gauss–Heine continued fraction

$$\frac{\varphi(\alpha,\beta q;\gamma q,q,x)}{\varphi(\alpha,\beta;\gamma,q,x)}=\frac{1}{1-}\,\frac{\beta_1}{1-}\,\frac{\beta_2}{1-}\cdots$$

where

$$\beta_{2r}=\frac{(1-\beta q^r)(\alpha-\gamma q^r)q^{r-1}x}{(1-\gamma q^{2r-1})(1-\gamma q^{2r})},\qquad r\geqslant 1$$

$$\beta_{2r+1}=\frac{(1-\alpha q^r)(\beta-\gamma q^r)q^r x}{(1-\gamma q^{2r})(1-\gamma q^{2r+1})},\qquad r\geqslant 0$$

substitute from (50), take limits as $c\to 0$, $a\to 0$ and use Satz 3.10, P112 in Perron [9]. We obtain

$$\lim\beta_{2r}=-\lambda q^{2r},\quad \lim\beta_{2r+1}=-q^{r+1}(b+\lambda q^r)$$

and hence

$$\frac{h(bq,\lambda q)}{h(b,\lambda)}=\frac{1}{1+}\,\frac{q(b+\lambda)}{1+}\,\frac{\lambda q^2}{1+}\,\frac{q^2(b+\lambda q)}{1+}\,\frac{\lambda q^4}{1+}\cdots. \tag{52}$$

In a similar way

$$\frac{h(b,\lambda q)}{h(b,\lambda)}=\frac{1}{1+}\,\frac{\lambda q}{1+}\,\frac{q(b+\lambda q)}{1+}\,\frac{\lambda q^3}{1+}\,\frac{q^2(b+\lambda q^2)}{1+}\cdots. \tag{53}$$

We now take the left side of (48) and call it $g(b,\lambda)$. It is clear that

$$g(b,\lambda)=\lim_{\substack{a\to 0\\ c\to 0}}(-bq;q)_\infty\,\varphi\left(-\frac{\lambda q}{c},-\frac{1}{a};-bq,q,ac\right).$$

We now take the q-Euler continued fraction as given in (42) and substitute

$$\alpha=-\lambda q/c,\quad \beta=-1/a,\quad \gamma=-bq,\quad x=ac. \tag{54}$$

If we take limits as $c\to 0$, $a\to 0$, then

$$g(b,\lambda)=(1+bq)\cdot g(bq,\lambda q)+\lambda q\cdot g(bq^2,\lambda q^2) \tag{55}$$

and the coresponding continued fraction

$$\frac{g(bq, \lambda q)}{g(b, \lambda)} = \frac{1}{1+bq+} \frac{\lambda q}{1+bq^2+} \frac{\lambda q^2}{1+bq^3+} \cdots. \tag{56}$$

Since $h(b, \lambda) = g(b, \lambda)$ by (48), we have from (52) and (56)

$$\frac{1}{1+bq+} \frac{\lambda q}{1+bq^2+} \frac{\lambda q^2}{1+bq^3+} \cdots = \frac{1}{1+} \frac{q(b+\lambda)}{1+} \frac{\lambda q^2}{1+} \frac{q^2(b+\lambda q)}{1+} \cdots. \tag{57}$$

From (55) we have

$$g(b, \lambda q) = (1+bq) \cdot g(bq, \lambda q^2) + \lambda q^2 \cdot g(bq^2, \lambda q^3).$$

On the other hand one can prove easily that

$$g(b, \lambda) = g(b, \lambda q) + \lambda q g(bq, \lambda q^2).$$

From these two relations we have

$$\frac{g(b, \lambda q)}{g(b, q)} = \frac{h(b, \lambda q)}{h(b, q)} = \frac{1}{1+} \frac{\lambda q}{1+bq+} \frac{\lambda q^2}{1+bq^2+} \frac{\lambda q^3}{1+bq^3+} \cdots$$

$$= \frac{1}{1+} \frac{\lambda q}{1+} \frac{q(b+\lambda q)}{1+} \frac{\lambda q^3}{1+} \frac{q^2(b+\lambda q^2)}{1+} \cdots$$

$$= \sum_{m=0}^{\infty} \frac{\lambda^m \cdot q^{m^2+m}}{(q,q)_m(-bq,q)_m} \Bigg/ \sum_{m=0}^{\infty} \frac{\lambda^m q^{m^2}}{(q,q)_m(-bq,q)_m}. \tag{58}$$

The equality of the second and fourth results was stated by Ramanujan in his Notebook II, chapter XVI, Entry 15. If $b=0$, it asserts the equality

$$\frac{1}{1+} \frac{\lambda q}{1+} \frac{\lambda q^2}{1+} \cdots = \sum_0^{\infty} \frac{\lambda^m q^{m^2+m}}{(q,q)_m} \Bigg/ \sum_0^{\infty} \frac{\lambda^m q^{m^2}}{(q,q)_m}, \tag{59}$$

which is already due to L J Rogers. It was also independently discovered by Issai Schur in 1917.

If $b = -\lambda$ then by definition of $h(b, \lambda)$, we have $h(-\lambda, \lambda) = 1$ and hence

$$1 - \lambda q + \lambda^2 q^3 - \lambda^3 q^6 + \lambda^4 q^{10} - \cdots = \frac{1}{1+} \frac{\lambda q}{1+} \frac{\lambda(q^2-q)}{1+} \frac{\lambda q^3}{1+} \frac{\lambda(q^4-q^2)}{1+} \cdots. \tag{60}$$

This is a result due originally to Eisenstein. It is also stated by Ramanujan as Entry 13 of chapter XVI, Volume 2.

The left side of (60) is what Rogers calls a "false theta function".

If we put q^2 for q and λq^{-1} for λ, then (60) gives

$$1 - \lambda q + \lambda^2 q^4 - \lambda^3 q^9 + \lambda^4 q^{16} - \cdots = \frac{1}{1+} \frac{\lambda q}{1+} \frac{\lambda q(1-q^2)}{1+} \frac{\lambda q^5}{1+} \cdots \tag{61}$$

a theorem also due to Eisenstein.

We shall call the continued fraction

$$\frac{1}{1+}\frac{\lambda q}{1+}\frac{q(b+\lambda q)}{1+}\frac{\lambda q^3}{1+}\frac{q^2(b+\lambda q^2)}{1+}\cdots$$

a *generalized Eisenstein continued fraction.*

7. Modular functions

There are several examples of generalized Eisenstein continued fraction in Ramanujan's Notebooks as well as in the 'Lost' Notebook. For example if $b=\lambda=1$, we have ([ ] p. 277)

$$\sum_0^{\infty}\frac{q^{m^2}}{(q^2,q^2)_m}=\prod_{m=1}^{\infty}(1+q^{2m-1})=\frac{\prod_1^{\infty}(1-q^{2(2m-1)})}{\prod_1^{\infty}(1-q^{2m-1})}$$

$$\sum_0^{\infty}\frac{q^{m^2+m}}{(q^2,q^2)_m}=\prod_1^{\infty}(1+q^{2m}).$$

If we put

$$q=\exp(\pi i\tau),\quad \tau=x+iy,\quad y>0$$

then with

$$\eta(\tau)=q^{1/12}\prod_1^{\infty}(1-q^{2m}),$$

we have

$$\frac{\eta(\tau/2)}{\eta(\tau)}=f_1(\tau)=q^{-1/24}\prod_1^{\infty}(1-q^{2m-1}).$$

We then have Ramanujan's result (chapter XIX, Entry 1, Vol. 2)

$$\frac{q^{1/8}}{1+}\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+}\cdots=\frac{q^{1/8}}{1+}\frac{q}{1+q+}\frac{q^2}{1+q^2+}\frac{q^3}{1+q^3+}\cdots=\frac{f_1(\tau)}{(f_1(2\tau))^2}. \tag{62}$$

The function $f_1(\tau)$ is a modular function due to Schläfli. It is remarkable that Ramanujan gives the correct power $q^{1/8}$ with which to multiply the continued fraction to make it into a modular function.

Ramanujan came across the modular function of Schläfli in his work on modular equations, singular moduli etc. If

$$\tau=iK'/K=\sqrt{-n},\quad n>0$$

n is an even integer, then $f_1(\tau)$ can be evaluated. Further $f_1(2\sqrt{-n})$ can be easily evaluated using a nice lemma due to Ramanujan (Notebook I, p. 320). For example if $n=2$

$$\frac{\exp(-\pi/\sqrt{2})/8}{1+}\,\frac{\exp(-\pi\sqrt{2})}{1+}\,\frac{\exp(-\pi\sqrt{2})+\exp(-2\pi\sqrt{2})}{1+}\,\frac{\exp(-3\pi\sqrt{2})}{1+}\cdots$$

$$=\frac{(\sqrt{2}-1)^{1/4}}{\sqrt{2}}.$$

Our second example is also from chapter XIX of Volume 2 of the Notebooks. It states that

$$\frac{q^{1/2}}{1+q+}\,\frac{q^2}{1+q^3+}\,\frac{q^4}{1+q^5+}\cdots=q^{1/2}\prod_1^\infty\frac{(1-q^{8m-1})(1-q^{8m-7})}{(1-q^{8m-3})(1-q^{8m-5})}. \tag{63}$$

The factor $q^{1/2}$ makes the right side a modular function. If in (57) we put q^2 for q, $\lambda=1$ and $b=q^{-1}$ we see that

$$\frac{q^{1/2}}{1+}\,\frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+}\cdots=q^{1/2}\prod_1^\infty\frac{(1-q^{8m-1})(1-q^{8m-7})}{(1-q^{8m-3})(1-q^{8m-5})} \tag{64}$$

a result obtained by Selberg [14]. From the definition of $h(b,\lambda)$, (63) and (64) are equal to

$$q^{1/2}\sum_{m=0}^\infty\frac{(-q,q^2)_m q^{m^2+2m}}{(q^2,q^2)_m}\Bigg/\sum_{m=0}^\infty\frac{(-q,q^2)_m\cdot q^{m^2}}{(q^2,q^2)_m}. \tag{65}$$

The infinite product for (65) was also proved by Andrews [2].

In his Notebooks Ramanujan has explicitly given an expression for (64) and (65) in terms of theta functions which exhibits it as a modular function.

8. The general continued fraction

A more general continued fraction was stated by Ramanujan in the 'Lost' Notebook [13]. He considers the function

$$G(a,\lambda)=G(a,\lambda,b,q)=\sum_{m=0}^\infty\frac{(\alpha+\lambda)(a+\lambda q)\cdots(a+\lambda q^{m-1})q^{(m(m+1)/2)}}{(q,q)_m(-bq,q)_m},$$

and states the very interesting result

$$\frac{G(aq,\lambda q)}{G(a,\lambda)}=\frac{1}{1+}\,\frac{(a+\lambda)q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+}\cdots. \tag{66}$$

The continued fractions mentioned earlier are special cases of (66).

In § 2 we had given three continued fractions of which two are due to Euler and Gauss. We shall consider the q-analogue of the second result in (8). Let us observe that

$$\varphi\left(\frac{-\lambda}{a},\frac{-q}{c};-bq,q,ac\right)=\sum_{m=0}^\infty\frac{\left(\frac{-\lambda}{a},q\right)_m\left(\frac{-q}{c},q\right)_m}{(q,q)_m(-bq,q)_m}a^mc^m$$

so that

$$\lim_{c\to 0} \varphi\left(\frac{-\lambda}{a}, \frac{-q}{c}; -bq, q, ac\right) = G(a, \lambda, b, q).$$

We now have the following q-analogues of the two results in (9).

$$\begin{aligned} &\varphi(\alpha, \beta; \gamma, q, x) \\ &= \varphi(\alpha, \beta q; \gamma, q, x) - \frac{\beta x(1-\alpha)}{1-\gamma} \varphi(\alpha q, \beta q; \gamma q, q, x) \\ &= \frac{\beta - \gamma}{\beta(1-\gamma)} \varphi(\alpha q, \beta; \gamma q, q, x) + \frac{(1-\beta)(\gamma - \alpha\beta x)}{\beta(1-\gamma)} \varphi(\alpha q, \beta q; \gamma q, q, x). \end{aligned} \tag{67}$$

The first of these two formulae is already in Heine's paper [8]. As for the second, eliminate $\varphi(\alpha q^2, \beta q; \gamma q^2, q, x)$ from the following two results of Heine:

$$\left.\begin{aligned} &\varphi(\alpha q, \beta; \gamma q, q, x) - \varphi(\gamma q, \beta; \gamma, q, x) \\ &= -\frac{\gamma x(1-\alpha q)(1-\beta)}{(1-\gamma)(1-\gamma q)} \varphi(\alpha q^2, \beta q; \gamma q^2, q, x) \end{aligned}\right\}$$

$$\left.\begin{aligned} &\varphi(\alpha q, \beta q; \gamma q, q, x) - \varphi(\alpha q, \beta; \gamma, q, x) \\ &= \frac{\beta x(1-\alpha q)(1-\gamma/\beta)}{(1-\gamma)(1-\gamma q)} \varphi(\alpha q^2, \beta q; \gamma q^2, q, x) \end{aligned}\right\}$$

to obtain

$$\begin{aligned} &(\beta - \gamma)(\varphi(\alpha q, \beta; \gamma q, q, x) - \varphi(\alpha q, \beta; \gamma, q, x)) \\ &\quad + \gamma(1-\beta)(\varphi(\alpha q, \beta q; \gamma q, q, x) - \varphi(\alpha q, \beta; \gamma, q, x)) = 0. \end{aligned}$$

Now substitute for $\varphi(\alpha q, \beta; \gamma, q, x)$ from Heine's result

$$\varphi(\alpha q, \beta; \gamma, q, x) = \varphi(\alpha, \beta; \gamma, q, x) + \frac{\alpha x(1-\beta)}{1-\gamma} \varphi(\alpha q, \beta q; \gamma q, q, x),$$

we then obtain the second result in (67).

We now use the two three-term relations alternately and obtain the continued fraction

$$\begin{aligned} &\frac{\varphi(\alpha, \beta q; \gamma, q, x)}{\varphi(\alpha, \beta; q, x)} \\ &= \frac{1}{1-} \frac{\beta^2 qx(1-\alpha)(1-\beta q)(\gamma - x\alpha\beta q)}{\beta q - \gamma +} \frac{\beta^2 q^3 x(1-\alpha q)}{1-} \frac{}{\beta q^2 - \gamma q +} \cdots. \end{aligned} \tag{68}$$

Let us now make the substitution

$$\alpha = \frac{-\lambda}{a}, \quad \beta = \frac{-q}{c}, \quad \gamma = -bq, \quad x = ac$$

and take the limit as $c \to 0$. Then by a theorem of Perron we obtain (66).

We now again take two more three-term relations

$$\begin{aligned}\varphi(\alpha,\beta;\gamma,q,x)\\ &= \varphi(\alpha,\beta q;\gamma,q,x) - \frac{\beta x(1-\alpha)}{1-\gamma}\varphi(\alpha q,\beta q;\gamma q,q,x)\\ &= \left(1+\frac{(\beta-\alpha q)x}{q(1-\gamma)}\right)\varphi(\alpha q,\beta;\gamma q,q,x) - \frac{(1-\alpha q)(\beta-\gamma q)x}{q(1-\gamma)(1-\gamma q)}\\ &\quad\times \varphi(\alpha q^2,\beta;\gamma q^2,q,x).\end{aligned}$$

The first relation is the same as that in (67) and the second relation is the same as (41). In the first place we have

$$\frac{\varphi(\alpha,\beta;\gamma,q,x)}{\varphi(\alpha,\beta q;\gamma,q,x)} = 1 - \frac{\beta x(1-\alpha)}{1-\gamma}\left(\frac{\varphi(\alpha,\beta q;\gamma,q,x)}{\varphi(\alpha q,\beta q;\gamma q,q,x)}\right)^{-1}. \tag{69}$$

For the ratio of the basic hypergeometric series on the right of (69) we use the continued fraction (42) after changing β into βq in (42). We have

$$\begin{aligned}&\frac{\varphi(\alpha,\beta q;\gamma,q,x)}{\varphi(\alpha,\beta;\gamma,q,x)}\\ &= \frac{1}{1-}\,\frac{\beta(1-\alpha)xq}{q(1-\gamma)+(\beta-\alpha)xq-}\,\frac{(1-\alpha q)(\beta q-\gamma q)xq}{q(1-\gamma q)+(\beta q-\gamma q^2)x-}\cdots.\end{aligned} \tag{70}$$

Inserting the values of α,β,γ,x as given above and taking the limit when $c\to 0$ we have

$$\frac{G(aq,\lambda q)}{G(a,\lambda)} = \frac{1}{1+}\,\frac{q(a+\lambda)}{1-aq+bq+}\,\frac{q(a+\lambda q)}{1-aq+bq^2+}\,\frac{q(a+\lambda q^2)}{1-aq+bq^3+}\cdots. \tag{71}$$

It is clear that this is a generalization of (58). One can also obtain a generalization of (57).

The interesting problem now is to find values of a, b, λ for which $G(aq,\lambda q)/G(a,\lambda)$ is an infinite product. Ramanujan put q^2 for q and aq^{-1} for a and $\lambda=1$ to obtain

$$\begin{aligned}R(a,b,q) &= \frac{1}{1+}\,\frac{aq+q^2}{1+}\,\frac{bq^2+q^4}{1+}\,\frac{aq^3+q^6}{1+}\cdots\\ &= \frac{\displaystyle\sum_{m=0}^{\infty}\frac{(a+q)(a+q^3)\cdots(a+q^{2m-1})q^{m^2+2m}}{(q^2,q^2)_m(-bq^2,q^2)_m}}{\displaystyle\sum_{m=0}^{\infty}\frac{(a+q)(a+q^3)\cdots(a+q^{2m-1})q^{m^2}}{(q^2,q^2)_m(-bq^2,q^2)_m}}.\end{aligned} \tag{72}$$

Ramanujan now considers the 9 cases.

$$a=0,\pm 1,\quad b=0,\pm 1. \tag{73}$$

Precisely these 9 cases were independently considered by Selberg [14]. Selberg, Andrews and Ramanujan found four cases in which the right side of (72) is a product.

One of these is

$$q^{1/5}R(0,0,\sqrt{q})=\frac{q^{1/5}}{1+}\frac{q}{1+}\frac{q^2}{1+}\cdots=q^{1/5}\prod_{m=0}^{\infty}\frac{(1-q^{5m+1})(1-q^{5m+4})}{(1-q^{5m+2})(1-q^{5m+3})}$$

which Ramanujan discussed extensively in the Notebooks and the 'Lost' Notebook. The other which Ramanujan discussed at length in the unpublished manuscripts (now published [13]) is

$$q^{1/3}R(1,1,q)=\frac{q^{1/3}}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\cdots=\frac{f_1(\tau)}{(f_1(3\tau))^3}. \tag{74}$$

Actually a result equivalent to this was first proved by G N Watson [15] where he used his q-generalization of Whipple's theorem on the ${}_7F_6$. Selberg's method, as mentioned earlier, is elementary. Andrews uses some theorems of Slater.

From the form of the right side in (74) it follows that one can evaluate the continued fraction on the left of (74) for $\tau=i\sqrt{n}$, where n is even idoneal number of Euler. Ramanujan in his Notebook I, p. 318 has given a formula for $f_1(3\sqrt{-n})$ in terms of $f_1(\sqrt{-n})$. This shows that the continued fraction on the left of (74) can be evaluated for an infinity of $3^k\sqrt{-n}$, k any rational integer. Ramanujan himself stated [12] the result for $n=10$.

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