Nonequilibrium Dynamics in the Complex Ginzburg-Landau Equation

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We present results from a comprehensive analytical and numerical study of nonequilibrium dynamics in the 2-dimensional complex Ginzburg-Landau (CGL) equation. In particular, we use spiral defects to characterize the domain growth law and the evolution morphology. An asymptotic analysis of the single-spiral correlation function shows a sequence of singularities – analogous to those seen for time-dependent Ginzburg-Landau (TDGL) models with $O(n)$ symmetry, where n is even.

Much recent interest has focused on pattern formation in the complex Ginzburg-Landau (CGL) equation:

$$\frac{\partial\psi(\vec{r},t)}{\partial t} = \psi(\vec{r},t) + (1+i\alpha)\nabla^2\psi(\vec{r},t) - (1+i\beta)|\psi(\vec{r},t)|^2\psi(\vec{r},t), \tag{1}$$

where $\psi(\vec{r},t)$ is a complex order-parameter field which depends on space ($\vec{r}$) and time (t). In Eq. (1), α and β are real parameters. The CGL equation arises in a range of diverse contexts, as reviewed by Cross and Hohenberg [1]. This universality arises from the fact that the CGL equation provides a generic description of oscillations in a spatially-extended system near a Hopf bifurcation [2].

The CGL equation exhibits rich dynamical behavior with variation of the parameters α and β, and the "phase diagram" has been investigated (mostly numerically) in various studies [3]. In a large range of parameter space, the emergence and interaction of spiral defect structures characterizes the morphology. In this letter, we study the nonequilibrium dynamics of the CGL equation resulting from a small-amplitude random initial condition. In general, this nonequilibrium evolution is referred to as "phase ordering dynamics" or "domain growth", and constitutes a well-studied example of far-from-equilibrium statistical physics [4,5]. Our analytical understanding of phase ordering systems has depended critically upon modeling the dynamics of defects in these systems (e.g., interfaces, vortices, monopoles, etc.) [5]- [8]. In this letter, we use spiral defect structures to characterize the evolution morphology in the CGL equation. Many novel features emerge in our study, which should be of great relevance for both experiments and subsequent numerical simulations.

For simplicity, we will focus on the CGL equation with $\alpha = 0$ and dimensionality $d = 2$. However, the results presented here are also relevant for the cases with $\alpha \neq 0$ and $d > 2$, as the underlying paradigm does not change, i.e., spirals continue to determine the morphology in large regions of parameter space. Following the work of Hagan [9], Aranson *et al.* [10], and Chate and Manneville [3], let us briefly discuss the phase diagram of the $d = 2$ CGL equation with $\alpha = 0$. The limiting case $\beta = 0$ corresponds to the dynamical XY model, which is well understood. The appropriate (point) defects are vortices, and domain growth is driven by the attraction and annihilation of vortex-antivortex pairs. The relevant growth law for the characteristic length scale is $L(t) \sim (t/\ln t)^{1/2}$ [11,5]; and the analytic form of the time-dependent correlation function (which characterizes the evolving morphology) has been obtained by Bray and Puri, and (independently) Toyoki [7]. Without loss of generality, we focus on the case with $\beta \geq 0$. For $0 \leq \beta \leq \beta_1$ ($\beta_1 \simeq 1.397$ [9]), spirals (which are asymptotically plane-waves) are linearly stable to fluctuations. For $\beta_1 < \beta \leq \beta_2$ ($\beta_2 \simeq 1.82$ [10,3]), spirals are linearly unstable to fluctuations, but the growing fluctuations are advected away, i.e., the spiral structure is globally stable. Finally, for $\beta_2 < \beta$, the spirals are globally unstable and cannot exist for extended times [10]. Our results correspond to the parameter regime with $\beta \leq \beta_2$.

Figure 1 shows the typical evolution from a small-amplitude random initial condition for the case with $\beta = 0.75$. Our numerical simulations were performed by implementing an isotropic Euler-discretization of Eq. (1) on N^2-lattices ($N = 256$ for Figure 1), with periodic boundary conditions in both directions. The discretization mesh sizes were $\Delta t = 0.01$ and $\Delta x = 1.0$. In Figure 1, we plot constant-phase regions and the relevant color-coding is provided in the figure caption. The evolving morphology is characterized by spirals and antispirals, and there is a typical length scale L, e.g., inter-spiral spacing or the square root of inverse defect density, which is the definition we will use subsequently.

Figure 2(a) plots $\ln[L(t)]$ vs. $\ln t$ for 5 representative values of β. The length-scale data was obtained from 5 independent runs on N^2-lattices with $N = 1024$. After an initial transient period, the length scale $L(t)$ should saturate to an equilibrium value (L_s) because of an effective spiral-antispiral repulsive potential [1]. This should be contrasted with the $\beta = 0$ case, where vortices continue to anneal (at zero temperature) as $t \to \infty$. As a matter of fact, the data for $\beta = 0.25, 0.5$ in Figure 2(a) does not exhibit this morphological freezing on the time-scales of our simulation, though signs of the crossover are evident for $\beta = 0.5$.

To understand this crossover behavior, we recall the analytical solution for an m-armed spiral due to Hagan [9]:

$$\psi(\vec{r},t) = \rho(r)\exp\left[-i\omega t + im\theta - i\phi(r)\right], \tag{2}$$

where $\vec{r} \equiv (r,\theta)$; and $\omega = \beta(1-q^2)$, where q is a constant which is determined by β [9]. The limiting forms of the functions $\rho(r)$ and $\phi(r)$ are

$$\begin{aligned}
&\rho(r) \to \sqrt{1-q^2}, \quad \phi'(r) \to q, \qquad \text{as} \quad r \to \infty, \\
&\rho(r) \to ar^m, \quad \phi'(r) \to r, \qquad\qquad \text{as} \quad r \to 0,
\end{aligned} \tag{3}$$

where a is a constant which is determined by finiteness conditions. We will focus on the case with $m = \pm 1$, as only 1-armed spirals are stable in the evolution [9]. Furthermore, we are only interested in distances $r \gg \xi$, where ξ is the defect core. Thus, we consider the spiral form in Eq. (2) with $\rho(r) = \sqrt{1-q^2}$ and $\phi(r) = qr$ (appropriate for $r \to \infty$).

We expect that spirals behave similarly to vortices for $L < L_c$, where $qL_c \sim O(1)$. Thus, the early evolution should be analogous to that for the XY model, both in terms of the domain growth law and correlation function. In Figure 2(a), the solid line has a slope of 1/2 and the initial growth (at least for $\beta \leq 0.75$) appears to be consistent with the behavior for the XY model, i.e., $L(t) \sim (t/\ln t)^{1/2}$ for $d = 2$. We also expect the saturation length L_s to scale with L_c. Figure 2(b) plots L_s vs. q^{-1} for a range of β-values, and demonstrates that our numerical data is consistent with $L_s \sim q^{-1}$. We can also obtain the scaling law for the crossover time and the corresponding numerical results (not shown here) are in agreement with it.

Next, we consider the correlation function for the evolution morphology shown in Figure 1. It is obviously relevant to first consider the correlation function for a single spiral of length L, as the snapshots in Figure 1 can be thought of as consisting of disjoint spirals of size L. (Of course, this ignores modulations of the order parameter at spiral-spiral boundaries but we will discuss those later.) We have approximated the 1-armed single-spiral solution as $\psi(\vec{r},t) \simeq \sqrt{1-q^2}\exp\left[-i\omega t + i(\theta - qr)\right]$. The correlation function is obtained by considering the correlation between points $\vec{r}_1$ and $\vec{r}_2$ $(= \vec{r}_1 + \vec{r}_{12})$ and integrating over $\vec{r}_1$ as follows:

$$C(r_{12}) = \frac{1}{V}\int d\vec{r}_1 \mathrm{Re}\left\{\psi(\vec{r}_1,t)\psi(\vec{r}_2,t)^*\right\} h(L - r_2)$$

$$= \frac{(1-q^2)}{V} \mathrm{Re} \int d\vec{r}_1 \exp\left[i(\theta_1 - \theta_2 - qr_1 + q|\vec{r}_1 + \vec{r}_{12}|)\right] h(L - |\vec{r}_1 + \vec{r}_{12}|), \tag{4}$$

where V is the spiral volume; and we have introduced the step function $h(x) = 1$ (0) if $x \geq 0$ ($x < 0$). The step function ensures that we do not include points which lie outside the defect of size L.

It is convenient to introduce variables $\theta_1 - \theta_{12} = \theta$; $x = r_1/L$; $r = r_{12}/L$, to obtain

$$C(r_{12}) = \frac{(1-q^2)}{\pi} \mathrm{Re} \int_0^1 dx x \int_0^{2\pi} d\theta \frac{x + re^{i\theta}}{(x^2 + r^2 + 2xr\cos\theta)^{1/2}} \times$$
$$\exp\left[-iqL\left\{x - (x^2 + r^2 + 2xr\cos\theta)^{1/2}\right\}\right] h[1 - (x^2 + r^2 + 2xr\cos\theta)^{1/2}], \tag{5}$$

where we have used $V = \pi L^2$ in $d = 2$. Thus, the scaling form of the single-spiral correlation function is $C(r_{12})/C(0) \equiv g(r_{12}/L, q^2L^2)$. In general, there is no scaling with the spiral size because of the additional factor qL. We recover scaling only in the limit $q = 0$ ($\beta = 0$), which corresponds to the case of a vortex. Essentially, spirals of different sizes are not morphologically equivalent because there is more rotation in the phase as one goes out further from the core.

Figure 3 plots $C(r_{12})/C(0)$ vs. r_{12}/L for the case with $\beta = 0.75$ ($q \simeq 0.203$). These results are obtained by a direct numerical integration of Eq. (5). We consider 4 different values of L. The functional form in Figure 3 exhibits near-monotonic behavior for small values of L (i.e., in the vortex or XY limit); and pronounced oscillatory behavior for larger values of L, as is expected from the integral expression. Notice that $r_{12}/L \leq 2$ – larger values of r_{12} correspond to the point $\vec{r}_2$ lying outside the defect.

The asymptotic behavior of the correlation function in the limit $r = r_{12}/L \to 0$ (though $r_{12}/\xi \gg 1$) is of considerable importance as it determines the tail of the momentum-space structure factor [5]. In particular, we are interested in the singular part of the correlation function as $r \to 0$. In this limit, we can discard the step function in Eq. (5) as it only provides corrections at the edge of the defect. The asymptotic analysis of the integral in Eq. (5) involves considerable algebra, which we will report in detail elsewhere. Here, we confine ourselves to quoting the final result for the singular part of $C(r_{12})$:

$$C_{\text{sing}}(r_{12}) = \frac{1}{2}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{p+m}\frac{(qL)^{2(p+m)}}{(2p)!(2m)!}\frac{\Gamma\left(\frac{1}{2}+m\right)^2}{\Gamma\left(\frac{1}{2}-p\right)^2(m+p+1)!^2}\times$$
$$(2m+1)(2p+1)r^{2(m+p+1)}\ln r. \tag{6}$$

Eq. (6) is one of the central results of this paper and we would like to briefly discuss its implications. The leading-order singularity is the same as that for the XY model ($\beta = q = 0$), $C_{\text{sing}}(r_{12}) = \frac{1}{2}r^2\ln r$ [12], as expected. However, there is also a sequence of sub-dominant singularities proportional to $(qL)^2r^4\ln r$, $(qL)^4r^6\ln r$, etc., and these become increasingly important as the length scale L increases. These sub-dominant terms in $C_{\text{sing}}(r_{12})$ are reminiscent of the leading-order singularities in models with $O(n)$ symmetry, where n is even [5,12]. Of course, in the context of $O(n)$ models, these singularities only arise for $n \leq d$ as there are no topological defects unless this condition is satisfied. In the present context, all these terms are already present for $d = 2$. The implication for the structure-factor tail is a sequence of power-law decays with $S(k) \sim (qL)^{2(m-1)}L^d/(kL)^{d+2m}$, where $m = 1, 2$, etc. Thus, though the true asymptotic behavior in $d = 2$ is still the generalized Porod tail, $S(k) \sim L^2(kL)^{-4}$, it may be difficult to disentangle this from other power-law decays.

Finally, Figure 4 compares our numerical data for the correlation function with the functional form of the single-spiral correlation function. Recall that the correlation function does not scale with the characteristic length because of the spiral nature of the defects. In Figures 4(a)-(c), we have plotted numerical data for $C(r_{12}, t)/C(0, t)$ vs. r_{12} at $t = 500$, and $\beta = 0.75, 1.0, 1.25$. For the comparison with Eq. (5), the length scale L is taken to be an adjustable parameter. In each case, the best-fit value of L matches the length scale obtained from the inverse defect density (see Figure 2(a)) within 10 percent. As is seen from Figure 4, the single-spiral correlation function is in good agreement with the numerical data for the multi-spiral morphology upto (approximately) the first minimum. As a matter of fact, the agreement is excellent (perhaps fortuitously) for $\beta = 1.25$, shown in Figure 4(c).

In the context of phase ordering dynamics, the Gaussian auxiliary field (GAF) ansatz [5]- [8] has proven particularly useful for the characterization of multi-defect morphologies.

We have critically examined the utility of the GAF ansatz in the present context [13] and find that it is only reasonable at early times – where, in any case, the ordering process is analogous to that for the XY model. We are presently studying methods of improving the GAF ansatz for the CGL equation and will discuss this elsewhere.

More generally, the utility of the GAF ansatz arises from the summation over phases from many defects, which results in a near-Gaussian distribution for the auxiliary field. However, in the present context, the shocks between spirals effectively isolate one spiral region from the influence of other regions. As a matter of fact, the waves from other spirals decay exponentially through the shock and the phase of a point is always dominated by the nearest spiral. Therefore, we expect that the correlation function will be dominated by the single-spiral result – in accordance with our numerical results.

To summarize: we have undertaken a detailed analytical and numerical study of nonequilibrium dynamics in the CGL equation. For early times ($L < L_c \sim q^{-1}$), the domain growth process is analogous to that for the XY model, which is well understood. At later times, distinct effects due to spirals are seen and the evolving system freezes (in a statistical sense) into a multi-spiral morphology. We have undertaken an asymptotic analysis of the correlation function $C(r_{12})$ for a single spiral. It exhibits a sequence of singularities as $r_{12}/L \to 0$. Furthermore, this correlation function is in good agreement with the numerical data for multi-spiral morphologies, over an extended range of distances.

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FIGURE CAPTIONS

Figure 1: Evolution of the CGL equation from a small-amplitude random initial condition. The evolution pictures were obtained from an Euler-discretized version of Eq. (1) with $\alpha = 0$, $\beta = 0.75$, implemented on an N^2-lattice ($N = 256$). The discretization mesh sizes were $\Delta t = 0.01$, $\Delta x = 1.0$; and periodic boundary conditions were imposed in both directions. The snapshots show regions of constant phase $\theta_\psi = \tan^{-1}(\mathrm{Im}\psi/\mathrm{Re}\psi)$, measured in radians, with the following color coding: $\theta_\psi \in [1.85, 2.15]$ (black); $\theta_\psi \in [3.85, 4.15]$ (red); $\theta_\psi \in [5.85, 6.15]$ (green). The snapshots are labeled by the appropriate evolution times.

Figure 2: (a) Plot of $\ln[L(t)]$ vs. $\ln t$ for $\alpha = 0$ and $\beta = 0.25, 0.5, 0.75, 1.0, 1.25$ – denoted by the specified symbols. The characteristic length scale, $L(t)$, is obtained from the square root of the inverse defect density – measured directly from snapshots as shown in Figure 1. The numerical data shown here was obtained as an average over 5 independent runs for N^2-lattices (with $N = 1024$). The solid line has a slope of 1/2.
(b) Plot of saturation length L_s vs. q^{-1} for a range of β-values. The corresponding values of q (as a function of β) are obtained from Hagan's solution, cf. Figure 5 of Ref. [9]. The solid line denotes the best linear fit to the numerical data.

Figure 3: Correlation function for the 1-armed spiral solution when $\beta = 0.75$ ($q \simeq 0.203$). We plot $C(r_{12})/C(0)$ vs. r_{12}/L for different spiral sizes, $L = 15, 25, 50, 100$ – denoted by the specified line-types. These results are obtained from a direct numerical integration of Eq. (5).

Figure 4: Numerical data for the correlation function $C(r_{12}, t)/C(0, t)$ vs. r_{12} at $t = 500$ for the cases $\alpha = 0$ and (a) $\beta = 0.75$; (b) $\beta = 1.0$; (c) $\beta = 1.25$. The numerical data was obtained as an average over 5 independent runs for N^2-lattices (with $N = 1024$). The

solid line refers to the numerical integration of Eq. (5) with L as an adjustable parameter. Subsequently, the r_{12}-axis is scaled so that the point $C(r_{12}, t)/C(0, t) = 1/2$ is matched for the numerical data and the analytical expression.