CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

BY K. G. RAMANATHAN
(Research Scholar, University of Madras)

Received January 26, 1944
(Communicated by Dr. R. Vaidyanathaswamy, F.A.S.C.)

Ramanujan's multiplicative arithmetic function $\tau(n)$* is defined by

$$\sum_{n=1}^{\infty} \tau(n) x^n = x [(1-x)(1-x^2)\ldots].$$

He stated, some congruence properties of $\tau(n)$ for the moduli 5, 7 and 23. These were later proved by L. J. Mordell. In this paper I prove congruences to moduli which are divisors of 24, viz.,

(a) $\tau(n)$ is odd if and only if $n$ is the square of an odd number.
(b) $\tau(3n-1) \equiv 0 \pmod{3}$.
(c) $\tau(4n-1) \equiv 0 \pmod{4}$.
(d) $\tau(6n-1) \equiv 0 \pmod{6}$.
(e) $\tau(8n-1) \equiv 0 \pmod{8}$.

These imply in turn

(f) $\tau(12n-1) \equiv 0 \pmod{12}$.
(g) $\tau(24n-1) \equiv 0 \pmod{24}$.

I think, these congruences have not been stated before [except (a) and (e) proved by Mr. Hansraj Gupta]. These resemble the congruences to $\sigma(n)$ proved by me namely,

If $\sigma(n)$ is the sum of the divisors of the positive integer $n$ then

$$\sigma(kn-1) \equiv 0 \pmod{k}$$

for $k = 3, 4, 6, 8, 12$ and 24 and $n \geq 1$.

2. We shall now prove the congruences of $\tau(n)$. It is well known* that

$$[(1-x)(1-x^2)\ldots]^3 = 1 - 3x + 5x^3 - 7x^6 + \ldots.$$ 

$$= \sum_{n=1}^{\infty} (-1)^n (2n+1) x^{n\cdot n+1}$$

* It might be pointed out that Hardy's statement that $r(23n) \equiv 0 \pmod{23}$ given on p. 165 is false since $\tau(23) = 18643272 \equiv 0 \pmod{23}$. 

146
Congruence Properties of Ramanujan's Function $\tau(n)$  

so that  

$$
\sum_{1}^{\infty} \tau(n) x^n = x [1 - 3x + 5x^2 - 7x^3 + \ldots]^{8}  
$$

(3)

But if $f(x)$ is any rational integral function of $x$ then  

$$
[f(x)]^{8} = f(x^{8}) \pmod{2}.  
$$

(4)

Therefore  

$$
\sum_{1}^{\infty} \tau(n) x^n = x \sum_{0}^{\infty} (-1)^{n} (2n + 1) x^{(2n+1)^{8}} \pmod{2}.  
$$

$$
\equiv \sum_{0}^{\infty} (-1)^{n} (2n + 1) x^{(2n+1)^{8}} \pmod{2}  
$$

which shows that: $\tau(n)$ is odd if and only if $n$ is the square of an odd number.  

(5)

3. Differentiating logarithmically both sides of (1) we obtain  

$$
\sum_{1}^{\infty} n \tau(n) x^n = P(x) \sum_{1}^{\infty} \tau(n) x^n  
$$

(6)

where  

$$
P(x) = 1 - 24 \left( \frac{x}{1-x} + \frac{2x^3}{1-x^3} + \ldots \right)  
$$

$$
= 1 - 24 \sum_{x}^{\infty} \sigma(n) x^n  
$$

(7)

where $\sigma(n)$ is the sum of the divisors of $n$.

Equating coefficients of like powers of $x$ we get  

$$(1 - m) \tau(m) = 24 \sum_{r=1}^{m-1} \sigma(r) \tau(m-r)  
$$

(8)

which shows that since $\tau(n)$ and $\sigma(n)$ are integers,  

$$
(m - 1) \tau(m) \equiv 0 \pmod{24} m \geq 1  
$$

(9)

Putting in succession $m = 2n, 3n, 3n - 1, 4n, 4n - 1$ we get  

$$
\tau(2n) \equiv 0 \pmod{8}  
$$

(10)

$$
\tau(3n) \equiv \tau(3n - 1) \equiv 0 \pmod{3}  
$$

(11)

$$
\tau(4n) \equiv \tau(4n - 1) \equiv \tau(4n - 2) \equiv 0 \pmod{4}  
$$

(12)

As regards $\tau(6n - 1)$ we see that, since $-1$ is a quadratic non-residue of $6$, $6n - 1$ cannot be a square and by (5) $\tau(6n - 1)$ is even.

Also by (11) $\tau(6n - 1)$ is divisible by $3$. Thus:  

$$
\tau(6n - 1) \equiv 0 \pmod{6}  
$$

(13)

For $1 < r < 8n - 1$ we have  

$$
\tau(8n - r) \sigma(r - 1) \equiv 0 \pmod{2}.  
$$

(14)
Because if \( r \equiv 7 \, (\text{mod} \, 8) \) then \( r - 1 \equiv 6 \, (\text{mod} \, 8) \), \( i.e., \, -1 \, (\text{mod} \, 4) \) so that
\[
\sigma(r - 1) \equiv 0 \, (\text{mod} \, 2).
\]
For all other \( r \)'s we see that (14) holds in virtue of 10, (11) and (12) and the congruences of \( \sigma(n) \) stated by me.\(^9\)

\[
\therefore 2(1 - 4n) \tau(8n - 1) = 24 \sum_{r=2}^{8n-1} \sigma(r - 1) \tau(8n - r)
\]
\[
\equiv 48 \, (\text{mod} \, 8)
\]
\[
\therefore \tau(8n - 1) \equiv 0 \, (\text{mod} \, 8) \tag{15}
\]

(11) and (12) imply \( \tau(12n - 1) \equiv 0 \, (\text{mod} \, 12) \) and
(11) and (15) imply \( \tau(24n - 1) \equiv 0 \, (\text{mod} \, 24) \).

REFERENCES