SOME APPLICATIONS OF RAMANUJAN'S TRIGONOMETRICAL SUM $C_m (n)$

BY K. G. RAMANATHAN*

University of Madras

Received May 20, 1944
(Communicated by Dr. R. Vaidyanathaswamy, F.A.Sc.)

The object of this paper is to exhibit the application of Ramanujan's trigonometrical sum to two arithmetical theories, the theory of Relative partitions (mod $m$) of Von Sterneck and the theory of the class division of the integers mod $m$ of Dr. R. Vaidyanathaswamy.

If $n = e_1 + e_2 + \ldots + e_r (\text{mod } m)$, $n$ is said to be relatively partitioned (mod $m$). Von Sterneck obtained explicit expressions for various numerical functions in this theory. He showed that these functions assume neat forms when expressed in terms of a certain arithmetic function $f(n, m)$ of two arguments. This function of Von Sterneck was recently proved by me to be identical with Ramanujan's trigonometrical sum $C_m (n)$. Using this fact I prove all of Von Sterneck's results by a method which besides being easy and direct shows clearly the fundamental nature of the trigonometrical sum in this theory.

Dr. R. Vaidyanathaswamy studied a class division of the integers mod $m$, in which these integers are divided into a certain number of classes $C_1, C_2, \ldots$ according to their g.c.d. with $m$. He proved the remarkable theorem that these classes combine by addition, i.e., that they form elements of a linear associative algebra with the scheme

$$C_i C_j = \sum_k \gamma_{ik}^j C_k$$

where $C_i C_j$ means the set of numbers, obtained by adding each number of $C_i$ to each number of $C_j$. It is shown here that $\gamma_{ik}^j$ could be expressed in terms of Ramanujan's sum. In fact

$$\gamma_{ik}^j = \frac{1}{m^2} \sum_{b|m} C_m (\delta) \sum_{t_i | b} C_m (\delta) C_m (t_k)$$

---

* I am indebted to Dr. R. Vaidyanathaswamy for his help in the preparation of this paper.


These references will hereafter be quoted by the numbers given above.
I prove more generally that

**Theorem.** \( C_{a_1}^{a_1} C_{a_2}^{a_2} \ldots = \sum_k A_k C_k \) where

\[
A_k = \frac{1}{m} \sum_{\delta | m} \frac{C^a_m (\delta)}{i_1} \frac{C^a_m (\delta)}{i_2} \ldots \frac{C^a_m (\delta)}{\delta}.
\]

I find also an expression for a certain numerical function connected with the theory of relative partitions (mod \( m \)), the set of integers used being those less than and prime to \( m \). I prove also the interesting results.

1. If \( m \) is even then every odd number is the sum mod \( m \) of three and every even number is the sum mod \( m \) of two numbers less than and prime to \( m \).

2. If \( m \) is odd then every number is the sum mod \( m \) of two numbers less than and prime to \( m \).

Ramanujan's sum is

\[
C_m (n) = \sum_k e^{2\pi i n k / m} = C_m (-n)
\]

where \( k \) runs through all the integers less than and prime to \( m \). Hardy proved that

\[
C_m (n) \cdot C_m' (n) = C_{m m'} (n) \quad (m, m') = 1
\]

and

\[
C_m (n) = \sum_{\delta | m} \mu (\frac{m}{\delta}) \delta
\]

the summation being over the common divisors of \( m \) and \( n \) and \( \mu (n) \) is the Moebius function we shall now prove the

**Lemma A.** \( C_m (n) = \mu (\frac{m}{d}) \frac{\phi (m)}{\phi (\frac{m}{d})} \quad d = (m, n) \)

where \( \phi (n) \) is Euler's function.

**Proof.** \( C_m (n) = \sum_{\delta | d} \mu (\frac{m}{\delta}) \delta = \sum_{\delta | d} \mu (\frac{m \cdot \delta}{d}) \frac{d}{\delta} \).

We might sum, naturally, for those divisors \( d \) of \( d \) which are prime to \( \frac{m}{d} \), for otherwise \( \mu (\frac{m}{d} \cdot \delta) \) vanishes. Thus

\[
C_m (n) = d \sum_{d_1} \mu (\frac{m}{d}) \mu (d_1) d_1^{-1}
\]

\[
= d \mu (\frac{m}{d}) \sum_{d_1} \mu (d_1) d_1^{-1}
\]

which is the right-hand side of the lemma.

---

6 Hardy and Wright, *Introduction to the Theory of Numbers*, p. 231.
* There is another proof in my paper, Ref. 4. See also S. Holder, *Prace Matematyczne* Fizyczn, 1936, 13-23.
Corollary.—$C_m(n)$ depends on $n$ only through its g.c.d. with $m$ so that 

$$C_m(n) = C_m(d).$$

Lemma B.—If $f(m, r)$ and $\phi(m, r)$ be two arithmetic functions possessing the modulus $m$ and if

$$\sum_{k=0}^{m-1} f(m, k) \rho^{kr} = \phi(m, r) \quad \rho = e^{\frac{2\pi i}{m}}$$

then

$$\sum_{k=0}^{m-1} \phi(m, k) \rho^{-kr} = mf(m, r)$$

Proof.—

$$\sum_{\lambda=0}^{m-1} \phi(m, \lambda) \rho^{-\lambda r} = \sum_{\lambda=0}^{m-1} \sum_{k=0}^{m-1} f(m, k) \rho^{(k-r)\lambda}$$

$$= \sum_{k=0}^{m-1} f(m, k) \sum_{\lambda=0}^{m-1} \rho^{(k-r)\lambda}$$

The inner sum is zero except when $k = r$ when its value is $m$.

Lemma C.—If $\phi(m, r)$ depends on $r$ only through its g.c.d. with $m$ then so does $f(m, r)$ and then each of them can be expressed in terms of the other and Ramanujan's sum.

Proof.—

For if $\phi(m, r) = \phi(m, \delta)$ \quad $\delta = (m, r)$

then

$$\sum_{r=0}^{m-1} \phi(m, r) \rho^{-rk} = \sum_{\delta|m} \phi(m, d) \sum_{t} \rho^{-tk}$$

t running through all the integers mod $m$ having with $m$ a g.c.d. equal to $d$ and

$$\sum_{t} \rho^{-tk} = C_m(k)$$

so that

$$mf(m, k) = \sum_{\delta|m} \phi(m, \delta) C_m(k)$$

$$\phi(m, k) = \sum_{\delta|m} f(m, \delta) C_m(k).$$

3. We shall now prove Von Sterneck's results by using the above lemmas.

**Theorem I.**

$$C_m(n) = \sum_{k=0}^{m-1} (-1)^k (n)_k^{(r)} = \sum_{\nu} (-1)^{\nu}$$

where $(n)_k^{(r)}$ is the number of ways of expressing $n$ as the sum mod $m$ of $k$ different elements of the set $1, 2, 3, \ldots (m - 1)$ and $\nu$ is the number of parts in a relative partition of $n$ mod $m$ into distinct parts not including zero.

Proof.—If $\rho = e^{\frac{2\pi i}{m}}$ then

$$(1 - \rho^r)(1 - \rho^{2r}) \ldots (1 - \rho^{(m-1)r}) = \sum_{k=0}^{m-1} f(m, k) \rho^{rk}$$

where

$$f(m, k) = \sum_{\nu} (-1)^{\nu} (k)_\nu^{(r)}$$
Some Applications of Ramanujan's Trigonometrical Sum \( C_m(n) \) 65

But \((1 - \rho^r)(1 - \rho^{2r})\ldots(1 - \rho^{(m-1)r}) = 0\) \((m, r) > 1\)
\[ = m \quad (m, r) = 1 \]

so that using lemma (c) we have

\[
m f(m, k) = \sum_{r} m \rho^{-rk} \quad (r, m) = 1 \quad 1 \leq r < m
\]

\[ = m C_m(k).\]

This is a direct and simple proof of the identity of Ramanujan’s sum and Von Sterneck’s function.

**Theorem 2.**

\[
A(m, n) = \sum_{t=0}^{m-1} \frac{(n)^{(t)}}{2m} = \frac{1}{2m} \sum \delta C_\delta(n)
\]

**Summation being for all odd divisors of** \( m \).

**Proof.**—It is easy to see that

\[
\sum_{k=0}^{m-1} A(m, k) \rho^{kr} = \prod_{\lambda=1}^{m-1} (1 + \rho^{\lambda r}) = \frac{1}{2} \phi(m, r)
\]

where \( \phi(m, r) = \prod_{\lambda=1}^{m} (1 + \rho^{\lambda r}) \).

The value of \( \phi(m, r) \) depends on \( r \) only through its g.c.d. with \( m \) so that using lemma C we have

\[
m A(m, n) = \frac{1}{2} \sum_{\delta|m} \phi(m, \delta) C_m(n).
\]

But \( \phi(m, \delta) = \left[1 + e\left(\frac{1}{\delta}\right)\right] \left[1 + e\left(\frac{2}{\delta}\right)\right] \ldots \left[1 + e\left(\frac{\delta}{\delta}\right)\right]^{m/\delta}
\]

where \( e(x) = e^{2\pi i x} \).

Since \( \frac{\sin m\theta}{\sin \theta} = 2^{m-1} \sin(\theta + \beta) \ldots \sin(\theta + m-1\beta) \) where \( \beta = \frac{\pi}{m} \)

we see that, by putting \( \theta = \frac{\pi}{2} \)

\[
\left[1 + e\left(\frac{1}{\delta}\right)\right] \ldots \left[1 + e\left(\frac{\delta-1}{\delta}\right)\right] = \sin \frac{\delta\pi}{2} \times (-1)^{\frac{\delta-1}{2}}.
\]

Substituting this value we have the required result.

**Theorem 3.**—If \( \binom{n}{k} \) denotes the number of ways of expressing \( n \) as the sum \((mod m)\) of \( k \) integers of the set \( 0, 1, 2 \ldots m-1 \) repetitions being allowed then

\[
[n]_k = \frac{1}{m} \sum_{\delta|m} \left(\frac{m+k}{\delta} - 1\right) C_\delta(n)
\]

where \( \binom{m}{n} \) is the usual coefficient which vanishes if \( n \) or \( m \) is non integral.
Proof.—It is easily seen that if \( n \leq m \) and
\[
A_a(r) = \frac{\sum_{k=0}^{m-1} [k]_{a} \rho^{kr}}{\rho = e^{m}},
\]
then \( A_a(r) \) is the coefficient of \( x^a \) in the expansion of \( [(1 - x\rho^r)(1 - x\rho^{2r}) \ldots (1 - x\rho^{mr})]^{-1} \) as a power series in \( x \). But \( (1 - x\rho^r)(1 - x\rho^{2r}) \ldots (1 - x\rho^{mr}) = (1 - x^{m\delta})^\delta \) where \( \delta = (m, r) \) so that \( A_a(r) \) is the coefficient of \( x^a \) in the binomial expansion of \( (1 - x^{m\delta})^\delta \)
\[
\therefore A_a(r) = 0 \text{ if } m/\delta \text{ does not divide } a
\]
\[
= (-1)^{\lambda} \binom{-\delta}{\lambda} \text{ when } \lambda = \frac{a\delta}{m}.
\]
Thus \( A_k(r) = \sum_{n=0}^{m-1} [n]_{r} \rho^{nr} \) and by lemma C we have the result.

Theorem 4.—If \( (n)_k \) denotes the number of ways of expressing \( n \) as the sum \( \text{(mod } m) \) of \( k \) distinct integers of the set 0, 1, \ldots, \( m - 1 \) then
\[
(n)_k = \frac{(-1)^{\delta}}{m} \sum_{\delta|m} (-1)^{\lambda} \binom{m/\delta}{k/\delta} C_\delta(n)
\]
Proof.—As in the previous theorem we see easily that \( B_a(r) = \sum_{k=0}^{m-1} (k)_{a} \rho^{kr} \) is the coefficient of \( x^a \) in the expansion of \( (1 + x\rho^r)(1 + x\rho^{2r}) \ldots (1 + x\rho^{mr}) \).
\[
\therefore B_a(r) = 0 \text{ if } m/\delta \text{ does not divide } a.
\]
\[
= (-1)^{\lambda} \binom{-\delta}{\lambda} \times (-1)^{m\lambda/\delta} \text{ when } \lambda = \frac{a\delta}{m}.
\]
\[
= (-1)^{\delta/\delta + \lambda} \binom{-\delta}{\lambda} = (-1)^{\delta} \binom{-\delta}{\lambda}
\]
\[
\therefore B_k(r) = \sum_{n=0}^{m-1} (n)_k \rho^{nr}.
\]
By lemma C we have the theorem.

4. We now proceed to the class division of the integers \( \text{mod } m \). Let \( t_1 (= 1), t_2, \ldots, t_\lambda (= m) \) \( [\lambda = d(m)] \) the number of divisors of \( m \) be the distinct divisors of \( m \). Dr. R. Vaidyanathaswamy divides the integers 1, 2 \ldots, \( m \) into \( \lambda \) classes \( C_1, C_2, \ldots, C_\lambda \) in such a way that \( C_r \) contains those integers \( \text{mod } m \) which have with \( m \) a g.c.d. equal to \( t_r \). Thus the number of elements in any set \( C_r \) is \( \phi \left( \frac{m}{t_r} \right) \). These classes combine among themselves by means of addition. Let \( C_r \) consist of the integers \( \beta_{1r}, \beta_{2r}, \ldots, \beta_{gr} \) where \( g_r = \phi \left( \frac{m}{t_r} \right) \). We shall prove the following:
**Theorem 5.**

\[ C_i C_j = \sum_k \gamma_{i\delta}^k C_k \text{ where} \]

\[ \gamma_{i\delta}^k = \frac{1}{m} \sum_{\delta \mid m} C_m (\delta) C_m (\delta) C_m (t_k). \]

**Proof.**—If \( \rho = e^m \) and

\[ (\rho r^\beta_{1i} + \rho r^\beta_{2i} + \ldots \rho r^\beta_{gi}) (\rho r^\beta_{1j} + \rho r^\beta_{2j} + \ldots + \rho r^\beta_{gj}) \]

\[ = \sum_{n=0}^{m-1} f(m, n) \rho^{nr} \]

then \( f(m, n) \) is the number of ways of expressing \( n \) as the sum (mod \( m \)) of two numbers one from each of the sets \( C_i \) and \( C_j \).

It is easy to see that \( \sum \rho^{ar} \) where \( a \) runs through all the elements of the set \( C_k \) has the value \( C_m (r) \). Thus

\[ \sum_{n=0}^{m-1} f(m, n) \rho^{nr} = C_m (r) C_m (r). \]

By lemma C we have the result.

**Theorem 6.**

\[ C_{a_1} C_{a_2} \ldots = \sum_k A_k C_k \text{ where} \]

\[ A_k = \frac{1}{m} \sum_{\delta \mid m} C_m (\delta) \ldots C_m (t_k). \]

**Proof.**—If \( f(m, n) \) represents the number of ways of expressing \( n \) as the sum (mod \( m \)) of \( a_1 \) numbers of the set \( C_1 \), \( a_2 \) numbers of the set \( C_2 \), \ldots then

\[ \phi(m, r) = \sum_{n=0}^{m-1} f(m, n) \rho^{nr} = \prod_{t=1}^{r} \left( \prod_{s=1}^{a_t} \rho^{r^s at} \right). \]

By an argument similar to the one used in the previous theorem we have the result.

**Theorem 7.**

\[ C_i = \sum A_k C_k \text{ where} \]

\[ A_k = \frac{\varphi^* (m)}{m} \prod_{p \mid m \atop p \neq \pm t_k} \left( \frac{p - 1^r - (-1)^r}{p - 1^r} \right) \prod_{p \mid t_k} \left( \frac{(p - 1)^{r-1} - (-1)^{r-1}}{(p - 1)^{r-1}} \right) \]

\( p \) being a prime number, \( \varphi^* (m) = [\phi(m)]^r \) and \( \phi(m) \) is Euler’s totient function.

**Proof.**—As before if \( f(m, n) \) is the number of representations of \( n \) as the sum (mod \( m \)) of \( r \) integers of the set \( C_1 \) then we have easily

\[ \sum_{n=0}^{m-1} f(m, n) \rho^{nr} = C_m (\lambda). \]
and by the usual inversion (lemma C) we have
\[ f(m, n) = \frac{1}{m} \sum_{\delta | m} C_m^{\prime}(\delta) C_m(n). \]

By lemma A we have
\[ f(m, n) = \frac{\phi^r(m)}{m} \sum_{\delta | m} \frac{\mu^r(\delta)}{\phi^r(\delta)} C_\delta(n) \]
\[ = \frac{\phi^r(m)}{m} \prod_{p | m} \left[ 1 + \frac{(-1)^r}{\phi^r(p)} C_p(n) \right] \]

But \( C_p(n) = -1 \) if \( p \nmid n \) and \( = p - 1 \) if \( p | n \). Using this we have the required result.

**Corollaries:**

1. If \( m \) is even then every odd number is the sum \((\text{mod } m)\) of 3 and every even number is the sum \((\text{mod } m)\) of 2 numbers less than and prime to \( m \).

2. If \( m \) is odd then every number is the sum \((\text{mod } m)\) of two numbers less than and prime to \( m \).

These follow easily from the above theorem because we have merely to find the least \( r \) for which no \( A_k \) is zero when \( m \) is even all \( A_k \)'s, for which the corresponding \( t_k \)'s are even, are zeroes. When \( m \) is odd, \( r = 2 \), no \( A_k \) is zero.

\[ \sum_{k=1}^{m} C_m^{\prime}(k) = \phi^r(m) \prod_{p | m} \left[ 1 - \frac{(-1)^{r-1}}{(p-1)^{r-1}} \right] \]

This follows easily from the result \( \sum_{n=0}^{m} f(m, n) \rho^{n\lambda} = C_m^{\prime}(\lambda) \) by using lemma A and putting \( n = m \).

\[ \sum_{k=1}^{m} C_m^{\prime}(k) = m\phi(m). \]

5. We shall study the problem similar to that considered by Von Sterneck but confining ourselves to the integers less than and prime to \( m \).

**THEOREM 8.**—If \( f(m, n) \) denotes the excess of the number of relative portions of \( n \) \((\text{mod } m)\) into an even number of parts over those into an odd number, the parts being all distinct and chosen from the set of integers less than and prime to \( m \) then

\[ f(m, n) = \frac{1}{m} \sum \text{Exp.} \left( \frac{\left( \frac{m}{\delta} \right) \phi(m)}{\phi\left( \frac{m}{\delta} \right)} \right) C_m(n) \]

where \( \text{Exp.} (x) \) means \( e \) and \( \wedge (n) \) is the arithmetic function defined by
\[ -\frac{d}{ds} \log \zeta(s) = \sum_{n=1}^{\infty} \frac{\wedge(n)}{n^s}. \]

\( \zeta(s) \) being Riemann zeta function.
Some Applications of Ramanujan's Trigonometrical Sum $C_m(n)$

Proof.—Using the notation of Section 3 we have

$$(1 - \rho_1^2)(1 - \rho_2^2)\ldots(1 - \rho_n^2) = \sum_{k=0}^{m-1} f(m, k) \rho^k.$$  

But it is known⁷ that

$$(1 - \rho_1^2)(1 - \rho_2^2)\ldots(1 - \rho_n^2) = e^\wedge(m)$$

and therefore

$$\sum_{k=0}^{m-1} f(m, k) \rho^k = \exp \left[ \wedge \left( \frac{m}{d} \right) \phi(m) \right] \phi\left( \frac{m}{d} \right)$$

using lemma C we have the required result.

6. It is well known⁷ that if $\rho_1, \rho_2, \ldots, \rho_\lambda \lambda = \phi(m)$ be the primitive $m$th roots of unity then

$$(x - \rho_1)(x - \rho_2)\ldots(x - \rho_\lambda) = \prod_{d|m} (x - 1)^{\mu(m)}.$$  

It is known from Newton's theorem that the coefficients in the product could be expressed in terms of the sums of powers of roots. But

$$\rho_1^k + \rho_2^k + \ldots + \rho_\lambda^k = C_m(k)$$

and thus we get

$$\pi (x - 1)^{\mu(m)} = \sum_{r=0}^{\lambda} A_r x^r$$

where

$$A_r = \frac{(-1)^r}{r!} \begin{vmatrix} C_m(1) & 1 & 0 & \ldots & 0 \\ C_m(2) & C_m(1) & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ C_m(r-1) & C_m(r-2) & \ldots & C_m(1) & (r-1) \\ C_m(r) & C_m(r-1) & \ldots & C_m(1) & \end{vmatrix}$$

which shows that if $m$ has a square factor (at least) prime to $r!$ $A_r$ and all the previous ones therefore vanish. It is also of interest to notice that since

$$(1 - \rho_1)(1 - \rho_2)\ldots(1 - \rho_\lambda) = e^\wedge(m)$$

we have

$$A_0 + A_1 + A_2 + \ldots + A_\lambda = e^\wedge(m)$$

which shows that a cyclotomic equation (Kreistellungsgleichung) of degree $\phi(m)$ has the sum of the positive coefficients greater than the negative ones always. Since $e^\wedge(m)$ is never negative.

---