SOME APPLICATIONS OF RAMANUJAN'S TRIGONOMETRICAL SUM $C_m(n)$

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THE object of this paper is to exhibit the application of Ramanujan's trigonometrical sum[1] to two arithmetical theories, the theory of Relative partitions (mod m) of Von Sterneck[2] and the theory of the class division of the integers mod m of Dr. R. Vaidyanathaswamy.[3]

If $n \equiv e_1 + e_2 \ldots$ (mod m), n is said to be relatively partitioned (mod m). Von Sterneck obtained explicit expressions for various numerical functions in this theory. He showed that these functions assume neat forms when expressed in terms of a certain arithmetic function $f(n, m)$ of two arguments. This function of Von Sterneck was recently proved by me[4] to be identical with Ramanujan's trigonometrical sum $C_m(n)$. Using this fact I prove all of Von Sterneck's results by a method which besides being easy and direct shows clearly the fundamental nature of the trigonometrical sum in this theory.

Dr. R. Vaidyanathaswamy studied a class division of the integers mod m, in which these integers are divided into a certain number of classes $C_1, C_2 \ldots$ according to their g.c.d. with m. He proved the remarkable theorem that these classes combine by addition, *i.e.*, that they form elements of a linear associative algebra with the scheme

$$C_i C_J = \sum_k \gamma_{iJ}^k C_k$$

where $C_i C_J$ means the set of numbers, obtained by adding each number of C_i to each number of C_J. It is shown here that γ_{iJ}^k could be expressed in terms of Ramanujan's sum. In fact

$$\gamma_{iJ}^k = \frac{1}{m} \sum_{\delta/m} C_{\frac{m}{t_i}}(\delta)\, C_{\frac{m}{t_J}}(\delta)\, C_{\frac{m}{\delta}}(t_k)$$

* I am indebted to Dr. R. Vaidyanathaswamy for his help in the preparation of this paper.

1 Collected papers of S. Ramanujan (Cambridge), 1927, p. 179.

2 Bachmann, *Niedere Zahlentheorie*, Bd. 2, 222–41.

3 *Proc. Ind. Acad. Sci.*, 1937, **5**, 63–75.

4 *Journal of the Madras University*, 1943, **15**, 1–9.

These references will hereafter be quoted by the numbers given above.

I prove more generally that

THEOREM.—$C_1^{\alpha_1} C_2^{\alpha_2} \ldots\ldots = \sum_k A_k C_k$ where

$$A_k = \frac{1}{m} \sum_{\delta/m} C^{\alpha_1}_{\frac{m}{t_1}}(\delta)\, C^{\alpha_2}_{\frac{m}{t_2}}(\delta) \ldots\ldots C_{\frac{m}{\delta}}(t_k).$$

I find also an expression for a certain numerical function connected with the theory of relative partitions (mod m), the set of integers used being those less than and prime to m. I prove also the interesting results.

(1) *If* m *is even then every odd number is the sum mod* m *of three and every even number is the sum mod* m *of two numbers less than and prime to* m.

(2) *If* m *is odd then every number is the sum mod* m *of two numbers less than and prime to* m.

2. Ramanujan's sum is

$$C_m(n) = \sum_k e^{2\pi i k n/m} = C_m(-n)$$

where k runs through all the integers less than and prime to m. Hardy[5] proved that

$$C_m(n) \cdot C_{m'}(n) = C_{mm'}(n) \qquad (m, m') = 1$$

and

$$C_m(n) = \sum \mu\left(\frac{m}{\delta}\right) \delta$$

the summation being over the common divisors of m and n and $\mu(n)$ is the Moebius function[6] we shall now prove the

Lemma A*.—$C_m(n) = \mu\left(\frac{m}{d}\right) \dfrac{\phi(m)}{\phi\left(\frac{m}{d}\right)} \qquad d = (m, n)$

where $\phi(n)$ is Euler's function.

Proof.—$C_m(n) = \sum_{\delta/d} \mu\left(\frac{m}{\delta}\right) \delta = \sum_{\delta/d} \mu\left(\frac{m}{d} \cdot \delta\right) \frac{d}{\delta}.$

We might sum, naturally, for those divisors d_1 of d which are prime to $\frac{m}{d}$, for otherwise $\mu\left(\frac{m}{d} \cdot \delta\right)$ vanishes. Thus

$$C_m(n) = d \sum_{d_1} \mu\left(\frac{m}{d}\right) \mu(d_1)\, d_1^{-1}$$

$$= d\, \mu\left(\frac{m}{d}\right) \sum_{d_1} \mu(d_1)\, d_1^{-1}$$

which is the right-hand side of the lemma.

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[5] *Proc. Camb. Phil. Soc.*, 1921, 263–71.

[6] Hardy and Wright, *Introduction to the Theory of Numbers*, p. 231.

* There is another proof in my paper, Ref. 4. See also O. Holder, *Prace, Matematyzno Fizycyn*, 1936, 13-23.

Corollary.—$C_m(n)$ depends on n only through its g.c.d. with m so that

$$C_m(n) = C_m(d).$$

Lemma B.—If f (m, r) *and* ϕ (m, r) *be two arithmetic functions possessing the modulus* m *and if*

$$\sum_{k=0}^{m-1} f(m, k)\, \rho^{kr} = \phi(m, r) \qquad \rho = e^{\frac{2\pi i}{m}}$$

then

$$\sum_{k=0}^{m-1} \phi(m, k)\, \rho^{-kr} = m f(m, r)$$

Proof.—

$$\sum_{\lambda=0}^{m-1} \phi(m, \lambda)\, \rho^{-\lambda r} = \sum_{\lambda=0}^{m-1} \sum_{k=0}^{m-1} f(m, k)\, \rho^{(k-r)\lambda}$$

$$= \sum_{k=0}^{m-1} f(m, k) \sum_{\lambda=0}^{m-1} \rho^{(k-r)\lambda}$$

The inner sum is zero except when $k = r$ when its value is m.

Lemma C.—If ϕ (m, r) *depends on* r *only through its g.c.d. with* m *then so does* f (m, r) *and then each of them can be expressed in terms of the other and Ramanujan's sum.*

Proof.— For if $\phi(m, r) = \phi(m, \delta) \qquad \delta = (m, r)$

then

$$\sum_{r=0}^{m-1} \phi(m, r)\, \rho^{-rk} = \sum_{d/m} \phi(m, d) \sum_{t} \rho^{-tk}$$

t running through all the integers mod m having with m a g.c.d. equal to d and

$$\sum_{t} \rho^{-tk} = C_{\frac{m}{d}}(k)$$

so that

$$m f(m, k) = \sum_{\delta/m} \phi(m, \delta)\, C_{\frac{m}{\delta}}(k)$$

$$\phi(m, k) = \sum_{\delta/m} f(m, \delta)\, C_{\frac{m}{\delta}}(k).$$

3. We shall now prove Von Sterneck's results by using the above lemmas.

THEOREM I. $$C_m(n) = \sum_{k=0}^{m-1} (-1)^k\, (n)_k^{(0)} = \sum_{\nu} (-1)^{\nu}$$

where $(n)_k^{(0)}$ *is the number of ways of expressing* n *as the sum mod* m *of* k *different elements of the set* 1, 2, 3....(m − 1) *and* ν *is the number of parts in a relative partition of* n *mod* m *into distinct parts not including zero.*

Proof.—If $\rho = e^{\frac{2\pi i}{m}}$ then

$$(1 - \rho^{r})(1 - \rho^{2r}) \ldots (1 - \rho^{(m-1)r}) = \sum_{k=0}^{m-1} f(m, k)\, \rho^{rk}$$

where

$$f(m, k) = \sum_{t=0}^{m-1} (-1)^t\, (k)_t^{(0)}$$

But $(1-\rho^r)(1-\rho^{2r})\ldots.(1-\rho^{(m-1)r}) = 0 \qquad (m, r) > 1$

$\qquad\qquad\qquad\qquad = m \qquad (m, r) = 1$

so that using lemma (c) we have

$$m\, f(m, k) = \sum_r m\, \rho^{-rk} \qquad (r, m) = 1 \quad 1 \leqslant r < m$$

$$= m\, C_m(k).$$

This is a direct and simple proof of the identity of Ramanujan's sum and Von Sterneck's function.

THEOREM 2.—

$$A(m, n) = \sum_{t=0}^{m-1} (n)_t^{(0)} = \frac{1}{2m} \Sigma\, 2^{m/\delta}\, C_\delta(n)$$

summation being for all odd divisors of m.

Proof.—It is easy to see that

$$\sum_{k=0}^{m-1} A(m, k)\, \rho^{kr} = \prod_{\lambda=1}^{m-1} (1+\rho^{\lambda r}) = \tfrac{1}{2}\phi(m, r)$$

where $\phi(m, r) = \prod_{\lambda=1}^{m} (1+\rho^{\lambda r})$.

The value of $\phi(m, r)$ depends on r only through its g.c.d. with m so that using lemma C we have

$$m\, A(m, n) = \tfrac{1}{2} \sum_{\delta/m} \phi(m, \delta)\, C_{\frac{m}{\delta}}(n).$$

$$\text{But } \phi\left(m, \frac{m}{\delta}\right) = \left\{\left[1+e\left(\frac{1}{\delta}\right)\right]\left[1+e\left(\frac{2}{\delta}\right)\right]\ldots.\left[1+e\left(\frac{\delta}{\delta}\right)\right]\right\}^{\frac{m}{\delta}}$$

$$\text{where } e(x) = e^{2\pi i x}.$$

Since $\dfrac{\sin m\theta}{\sin \theta} = 2^{m-1}\sin(\theta+\beta)\ldots.\sin(\theta+\overline{m-1}\beta)$ where $\beta = \dfrac{\pi}{m}$

we see that, by putting $\theta = \dfrac{\pi}{2}$.

$$\left[1+e\left(\frac{1}{\delta}\right)\right]\ldots.\left[1+e\left(\frac{\delta-1}{\delta}\right)\right] = \sin\frac{\delta\pi}{2} \times (-1)^{\frac{\delta-1}{2}}.$$

Substituting this value we have the required result.

THEOREM 3.—*If* $(n)_k$ *denotes the number of ways of expressing* n *as the sum (mod* m) *of* k *integers of the set* 0, 1, 2....m − 1 *repetitions being allowed then*

$$[n]_k = \frac{1}{m} \sum_{\delta/m} \binom{\frac{m+k}{\delta} - 1}{\frac{k}{\delta}} C_\delta(n)$$

where $\binom{m}{n}$ *is the usual coefficient which vanishes if* n *or* m *is non integral.*

Proof.—It is easily seen that if $n \leqslant m$ and

$$A_a(r) = \sum_{k=0}^{m-1} [k]_a \rho^{kr} \qquad \rho = e^{\frac{2\pi i}{m}}$$

then $A_a(r)$ is the coefficient of x^a in the expansion of $[(1 - x\rho^r)(1 - x\rho^{2r}) \ldots (1 - x\rho^{mr})]^{-1}$ as a power series in x. But $(1 - x\rho^r) \ldots (1 - x\rho^{mr}) = (1 - x^{m/\delta})^{\delta}$ where $\delta = (m, r)$ so that $A_a(r)$ is the coefficient of x^a in the binomial expansion of $(1 - x^{m/\delta})^{-\delta}$

$$\therefore \quad A_a(r) = 0 \qquad \text{if } m/\delta \text{ does not divide } a$$

$$= (-1)^{\lambda} \binom{-\delta}{\lambda} \text{ when } \lambda = \frac{a\delta}{m}.$$

Thus $A_k(r) = \sum_{n=0}^{m-1} [n]_k \rho^{nr}$ and by lemma C we have the result.

THEOREM 4.—*If* $(n)_k$ *denotes the number of ways of expressing* n *as the sum* (*mod* m) *of* k *distinct integers of the set* $0, 1, \ldots m - 1$ *then*

$$(n)_k = \frac{(-1)^k}{m} \sum_{\delta \mid m} (-1)^{k/\delta} \binom{m/\delta}{k/\delta} C_\delta(n)$$

Proof.—As in the previous theorem we see easily that $B_a(r) = \sum_{k=0}^{m-1} (k)_a \rho^{kr}$ is the coefficient of x^a in the expansion of $(1 + x\rho^r)(1 + x\rho^{2r}) \ldots (1 + x\rho^{mr})$. But $(1 - x\rho^r)(1 - x\rho^{2r}) \ldots (1 - x\rho^{mr}) = (1 - x^{m/\delta})^{\delta}$.

$$\therefore \quad B_a(r) = 0 \text{ if } m/\delta \text{ does not divide } a.$$

$$= (-1)^{\lambda} \binom{\delta}{\lambda} \times (-1)^{m\lambda/\delta} \text{ when } \lambda = \frac{a\delta}{m}.$$

$$= (-1)^{\frac{m\lambda}{\delta} + \lambda} \binom{\delta}{\lambda}$$

$$\therefore \quad B_k(r) = \sum_{n=0}^{m-1} (n)_k \rho^{nr}.$$

By lemma C we have the theorem.

4. We now proceed to the class division of the integers mod m. Let $t_1 (= 1), t_2, \ldots t_\lambda (= m)$ $[\lambda = d(m)]$ the number of divisors of m be the distinct divisors of m. Dr. R. Vaidyanathaswamy[3] divides the integers $1, 2 \ldots m$ into λ classes $C_1, C_2, \ldots C_\lambda$ in such a way that C_r contains those integers mod m which have with m a g.c.d. equal to t_r. Thus the number of elements in any set C_r is $\phi\left(\frac{m}{t_r}\right)$. These classes combine among themselves by means of addition. Let C_r consist of the integers $\beta_{1r}, \beta_{2r}, \ldots \beta_{g_r}{}^r$ where $g_r = \phi\left(\frac{m}{t_r}\right)$. We shall prove the following :

THEOREM 5.

$$C_i\, C_J = \sum_k \gamma_{iJ}^k\, C_k \text{ where}$$

$$\gamma_{iJ}^k = \frac{1}{m} \sum_{\delta | m} C_{\frac{m}{t_i}}(\delta)\, C_{\frac{m}{t_J}}(\delta)\, C_{\frac{m}{\delta}}\ (t_k).$$

Proof.—If $\rho = e^{\frac{2\pi i}{m}}$ and

$$\left(\rho^{r\beta_{1i}} + \rho^{r\beta_{2i}} + \ldots .\ \rho^{\gamma\beta g_i i}\right)\left(\rho^{r\beta_{1J}} + \rho^{r\beta_{2J}} + \ldots . + \rho^{r\beta g_J J}\right)$$

$$= \sum_{n=0}^{m-1} f(m, n)\, \rho^{nr}$$

then $f(m, n)$ is the number of ways of expressing n as the sum (mod m) of two numbers one from each of the sets C_i and C_J.

It is easy to see that $\Sigma\, \rho^{\alpha r}$ where α runs through all the elements of the set C_k has the value $C_{\frac{m}{t_k}}(r)$. Thus

$$\sum_{n=0}^{m-1} f(m, n)\ \rho^{nr} = C_{\frac{m}{t_i}}(r)\, C_{\frac{m}{t_J}}(r).$$

By lemma C we have the result.

THEOREM 6.

$$C_1^{\alpha_1}\, C_2^{\alpha_2} \ldots . = \sum_k A_k C_k \quad \text{where}$$

$$A_k = \frac{1}{m} \Sigma\, C^{\alpha_1}_{\frac{m}{t_i}}(\delta) \ldots . C_{\frac{m}{\delta}}(t_k).$$

Proof.—If $f(m, n)$ represents the number of ways of expressing n as the sum (mod m) of α_1 numbers of the set C_1, α_2 numbers of the set C_2. . . .then

$$\phi(m, r) = \sum_{n=0}^{m-1} f(m, n)\ \rho^{nr} = \prod_{t=1}^{\lambda} \left(\sum_{a=1}^{gt} \rho^{r\beta at}\right)^{\alpha_t}.$$

By an argument similar to the one used in the previous theorem we have the result.

THEOREM 7.

$$C_1^r = \Sigma\, A_k\, C_k \text{ where}$$

$$A_k = \frac{\phi^r(m)}{m} \prod_{\substack{p|m \\ p \nmid t_k}} \left[\frac{(p-1)^r - (-1)^r}{(p-1)^r}\right] \prod_{p|t_k} \left[\frac{(p-1)^{r-1} - (-1)^{r-1}}{(p-1)^{r-1}}\right]$$

p *being a prime number,* ϕ^r (m) $= [\phi\,(\text{m})]^r$ *and* ϕ (m) *is Euler's totient function.*

Proof.—As before if $f(m, n)$ is the number of representations of n as the sum (mod m) of r integers of the set C_1 then we have easily

$$\sum_{n=0}^{m-1} f(m, n)\, \rho^{n\lambda} = C_m^r(\lambda)$$

and by the usual inversion (lemma C) we have

$$f(m, n) = \frac{1}{m} \sum_{\delta | m} C^r_m(\delta)\, C_{\frac{m}{\delta}}(n).$$

By lemma A we have

$$f(m, n) = \frac{\phi^r(m)}{m} \sum_{\delta | m} \frac{\mu^r(\delta)}{\phi^r(\delta)} C_\delta(n)$$

$$= \frac{\phi^r(m)}{m} \prod_{p | m} \left[1 + \frac{(-1)^r}{\phi^r(p)} C_p(n)\right]$$

But[5] $C_p(n) = -1$ if $p \nmid n$ and $= p - 1$ if $p | n$. Using this we have the required result.

Corollaries.—(1) *If* m *is even then every odd number is the sum* (*mod* m) *of* 3 *and every even number is the sum* (*mod* m) *of* 2 *numbers less than and prime to* m.

(2) *If* m *is odd then every number is the sum* (*mod* m) *of two numbers less than and prime to* m.

These follow easily from the above theorem because we have merely to find the least r for which no A_k is zero when m is even all A_k's, for which the corresponding t_k's are even, are zeroes. When m is odd, $r = 2$, no A_k is zero.

(3) $$\sum_{k=1}^{m} C^r_m(k) = \phi^r(m) \prod_{p | m} \left[1 - \frac{(-1)^{r-1}}{(p-1)^{r-1}}\right].$$

This follows easily from the result $\sum_{n=0}^{m-1} f(m, n)\, \rho^{n\lambda} = C^r_m(\lambda)$ by using lemma A and putting $n = m$.

(4) $$\sum_{k=1}^{m} C^2_m(k) = m\phi(m).$$

5. We shall study the problem similar to that considered by Von Sterneck but confining ourselves to the integers less than and prime to m.

THEORFM 8.—*If* f (m, n) *denotes the excess of the number of relative partions of* n (*mod* m) *into an even number of parts over those into an odd number, the parts being all distinct and chosen from the set of integers less than and prime to* m *then*

$$f(m, n) = \frac{1}{m} \sum \text{Exp.} \left(\frac{\Lambda\left(\frac{m}{\delta}\right) \phi(m)}{\phi\left(\frac{m}{\delta}\right)} \right) C_{\frac{m}{\delta}}(n)$$

where Exp. (x) *means* e *and* Λ (n) *is the arithmetic function defined by*

$$-\frac{d}{ds} \log \zeta(s) = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}.$$

ζ (s) *being Riemann zeta function.*

Proof.—Using the notation of Section 3 we have

$$\left(1-\rho^{\beta_{11}r}\right)\left(1-\rho^{\beta_{21}r}\right)\ldots\left(1-\rho^{\beta_{\sigma,1}r}\right)=\sum_{k=0}^{m-1} f(m,k)\,\rho^{kr}.$$

But it is known[4] that

$$\left(1-\rho^{\beta_{11}}\right)\left(1-\rho^{\beta_{21}}\right)\ldots\left(1-\rho^{\beta_{\sigma_1}1}\right)=e^{\Lambda(m)} \text{ and therefore}$$

$$\sum_{k=0}^{m-1} f(m,k)\,\rho^{kr}=\operatorname{Exp}\left[\frac{\Lambda\left(\frac{m}{d}\right)\phi(m)}{\phi\left(\frac{m}{d}\right)}\right]$$

using lemma C we have the required result.

6. It is well known[7] that if $\rho_1, \rho_2 \ldots \rho_\lambda$ $\lambda=\phi(m)$ be the primitive mth roots of unity then

$$(x-\rho_1)(x-\rho_2)\ldots(x-\rho_\lambda)=\prod_{d\mid m}\left(x^{d}-1\right)^{\mu\left(\frac{m}{d}\right)}.$$

It is known from Newton's theorem that the coefficients in the product could be expressed in terms of the sums of powers of roots. But

$$\rho_1^k+\rho_2^k+\ldots+\rho_\lambda^k=C_m(k)$$

and thus we get

$$\prod_{d\mid m}\left(x^{d}-1\right)^{\mu\left(\frac{m}{d}\right)}=\sum_{r=0}^{\lambda} A_r x^r \quad \text{where}$$

$$A_r=\frac{(-1)^r}{r\,!}\begin{vmatrix} C_m(1)\ 1 & 0\ldots\ldots\ldots\ldots\ldots 0 \\ C_m(2)\,C_m(1) & 2\ldots\ldots\ldots\ldots\ldots 0 \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\ C_m(r-1),\ C_m(r-2)\ldots\ldots & C_m(1)\ \ (r-1) \\ C_m(r),\ C_m(r-1) & \ldots\ldots\ldots\ldots C_m(1) \end{vmatrix}$$

which shows that if m has a square factor (at least) prime to $r\,!$ A_r and all the previous ones therefore vanish. It is also of interest to notice that since

$$(1-\rho_1)(1-\rho_2)\ldots(1-\rho_\lambda)=e^{\Lambda(m)};$$

we have
$$A_0+A_1+A_2+\ldots+A_\lambda=e^{\Lambda(m)}.$$

which shows that a cyclotomic equation (Kreisteilungsgleichung) of degree $\phi(m)$ has the sum of the positive coefficients greater than the negative ones always. since $e^{\Lambda(m)}$ is never negative.

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[7] B. L. Van der Waerden, *Moderne Algebra*, Bd. 1, p. 108.