MULTIPLE PROCESSES IN ELECTRON-PHOTON CASCADES

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ABSTRACT

Recent experiments on high energy electron-photon cascades suggest the possibility of the following multiple processes:—

(i) Electron scattering with pair creation,

(ii) Electron scattering with the emission of two photons, and

(iii) Photon scattering with pair creation.

The cross-section for the first process predicted by Bhabha has been recalculated recently by the standard Feynman technique. Using the same technique, we here present the calculations for the cross-sections of processes (ii) and (iii).

INTRODUCTION

THE analysis of high energy electron-photon showers in cosmic rays suggests the necessity of considering third order processes of the following type:

(i) Electron scattering with pair creation, *i.e.*, 'trident' production,

(ii) Electron scattering with emission of two photons. The usual 'bremsstrahlung' of Bethe and Heitler (1934) deals with the emission of one photon,

(iii) Photon scattering with pair creation. In the usual treatment, pair creation is due to the annihilation of the photon. Here we postulate the emergence of a photon after collision.

Pair production by electrons was predicted by Bhabha as early as 1935 while the possibility of process (ii) is mentioned in explicit terms by Heitler (1954). Recently Murota *et al.* (1956) have recalculated the cross-section for the production of 'tridents' (pair + scattered electron) by the Feynman-Dyson technique. The experimental data of Koshiba and Kaplon (1955) seem to confirm the production of electron pairs by electrons. Recently Seeman and Glasser (1956) have observed an event in which a helium nucleus having an energy of the order of 10^5 Gev. gives rise to a shower containing very many pairs of electrons. The results of Fay (1956)* at Gottingen seem to show that the number of pairs is much greater than can be expected from the standard cascade theory. The processes (i) to (iii) we have mentioned above may explain some of the new data relating to high energy electron photon cascades.

The object of the present paper is to obtain the differential cross-sections for processes (ii) and (iii) using the Feynman-Dyson techniques. The standard notation of Feynman will be used and explanation will be given only when necessary.

ELECTRON SCATTERING WITH CREATION OF TWO PHOTONS

(*a*) *Matrix elements and their reduction.*—Throughout this paper, we use the natural units $\hbar = 1$ and $c = 1$ and the following notation:

$a.b \equiv a_t b_t - a_x b_x - a_y b_y - a_z b_z$: The four dimensional scalar product of the vectors*** a and b whose components are (a_t, a_x, a_y, a_z) and (b_t, b_x, b_y, b_z) respectively.

$p \equiv (\mathrm{E}, \vec{p})$: The energy-momentum four vector of electron. Subscripts will be used to distinguish between different electrons.

$q \equiv (\omega, \vec{q})$: The energy-momentum four vector of a photon. Subscripts will be used to distinguish between different photons.

$e \equiv (0, \vec{e})$: The spatial vector representing the direction of polarization of a photon.

m: The rest mass of the electron.

In addition to those we shall use the 'dagger' notation of Feynman (1953). The daggered operator **a**** of any four vector a is defined as

$$\mathbf{a} \equiv a_t\gamma_t - a_x\gamma_x - a_y\gamma_y - a_z\gamma_z$$
$$\equiv a_0\gamma_0 - a_1\gamma_1 - a_2\gamma_2 - a_3\gamma_3 \qquad (1)$$

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* Private communication.

** The daggered operator of a four vector a is represented as $\not a$ by Feynman. For convenience of printing we use the corresponding bold face letter.

*** For convenience ordinary italics are used for four-vectors. No confusion will arise with ordinary numbers since in the paper four-vectors occur only in scalar products.

where γ's are the well-known Dirac matrices (*see* for example, Bethe and Schweber, 1955). The γ's are characterised by the following relations:

$$\begin{aligned}\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= 0 \qquad \mu \neq \nu \\ &= +2 \qquad \mu = \nu = 0 \\ &= -2 \qquad \mu = \nu = 1, 2, 3.\end{aligned} \tag{2}$$

While the matrices γ_x, γ_y and γ_z are anti-Hermitian, γ_t is Hermitian.

In this process, a free electron of momentum p_1 is incident and we obtain after its collision with the nucleus, a free electron of momentum p_2 and two photons of momenta q_1, q_2. The initial and final state electron wave functions are given by

$$\begin{aligned}\psi_1 &= u_1\, e^{-p_1 . r} \\ \psi_2 &= u_2\, e^{-p_2 . r}\end{aligned} \tag{3}$$

where u_1, u_2 are the four component spinor parts of the free particle wave function corresponding to the momenta p_1 and p_2 respectively.

$$\begin{aligned}\mathbf{p}_1 u_1 &= m u_1,\ \mathbf{p}_2 u_2 = m u_2 \\ p_1 \cdot p_1 &= p_2 \cdot p_2 = m^2\end{aligned} \tag{4}$$

r is a four vector with components (t, x, y, z). The potentials corresponding to the emitted photons are given by

$$\begin{aligned}A_{1\mu} &= e_{1\mu}\, e^{-iq_1 . r} \\ A_{2\mu} &= e_{2\mu}\, e^{-iq_2 . r}\end{aligned} \tag{5}$$

where

$$e_1 \cdot q_1 = 0 = e_2 \cdot q_2,\ q_1 \cdot q_1 = 0 = q_2 \cdot q_2 \tag{6}$$

The Coulomb potential due to the nucleus of charge Z is given by

$$\vec{A} = 0,\ A_t = \phi = \frac{Ze}{|\vec{r}|}\,. \tag{7}$$

As we shall be working in momentum representation, it is convenient to define $\mathbf{V}(q)$ as

$$\begin{aligned}\mathbf{V}(q) &= \frac{1}{(2\pi)^4} \int \mathbf{A}(r)\, e^{iq . r} d^4 r \\ &= \frac{4\pi Ze}{\vec{Q} \cdot \vec{Q}}\, \delta(q_4)\gamma_t = v(Q)\, \delta(q_4)\, \gamma_t\end{aligned} \tag{8}$$

where $\vec{Q}$ is the spatial part of the momentum and δ the Dirac delta function.

We shall choose a co-ordinate system in which the nucleus does not move and take the direction of propagation of the photon with momentum q_1 as the z-axis and the plane containing the directions of propagation of the photons as the y-z plane. Let the angle between the directions of propagation of the photons be θ. Thus

$$\mathbf{q}_1 = \omega_1 (\gamma_t - \gamma_z) \tag{9}$$

$$\mathbf{q}_2 = \omega_2 (\gamma_t - \gamma_z \cos\theta - \gamma_y \sin\theta) \tag{10}$$

$$\mathbf{p}_1 = E_1 \gamma_t - p_{1x}\gamma_x - p_{1y}\gamma_y - p_{1z}\gamma_z \tag{11}$$

$$\mathbf{p}_2 = E_2 \gamma_t - p_{2x}\gamma_x - p_{2y}\gamma_y - p_{2z}\gamma_z. \tag{12}$$

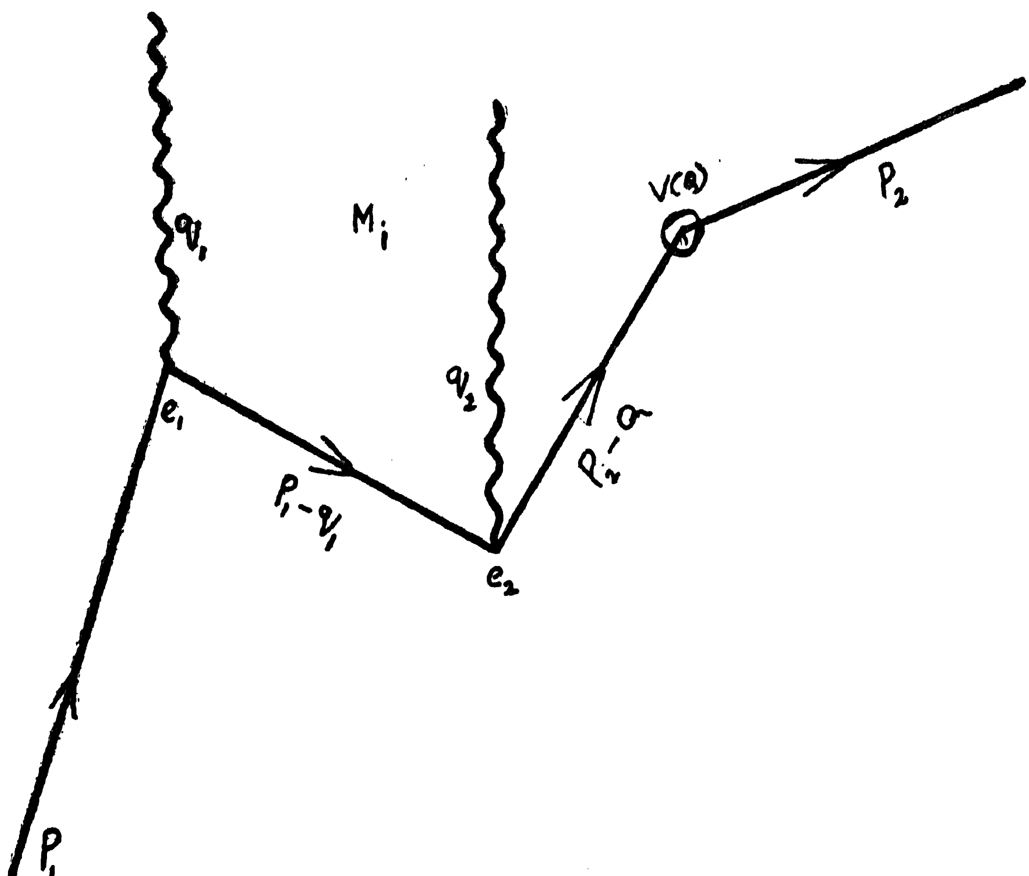
The photon beam corresponding to (9) can be resolved into two types of polarization which will be designated as type A and type B.

(A) $\mathbf{e}_1 = \gamma_x$ (B) $\mathbf{e}_1 = \gamma_y$.

Similarly the beam corresponding to (10) can be resolved into two types of polarization:

(A′) $\mathbf{e}_2 = \gamma_x$ (B′) $\mathbf{e}_2 = \gamma_z \sin\theta - \gamma_y \cos\theta$.

The lowest order Feynman diagrams for the present process are given below.



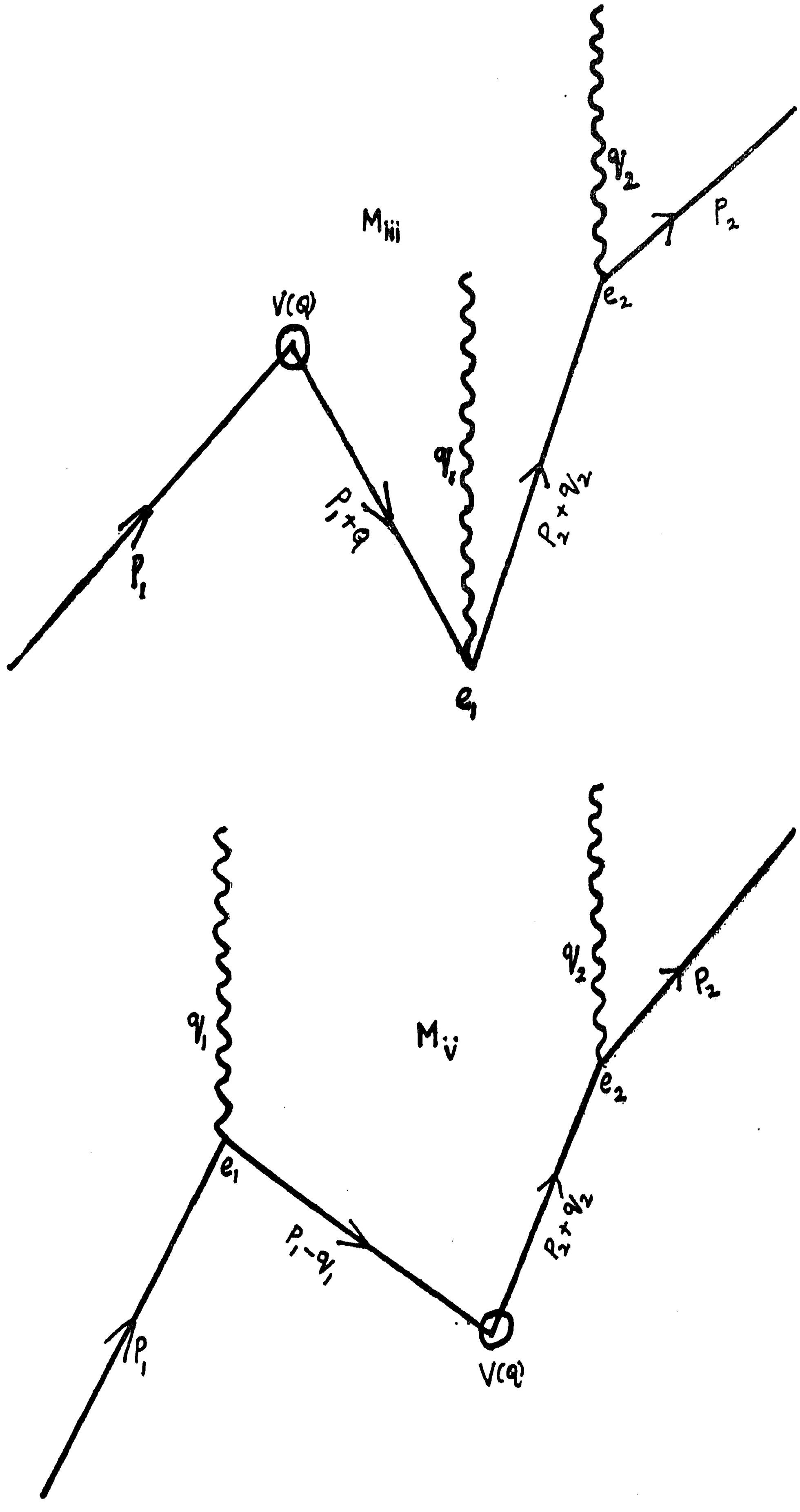


FIG. 1. Electron scattering with emission of two photons.

There are six different Feynman diagrams for the process yielding six matrix elements, M_i, M_{ii}, M_{iii}, M_{iv}, M_v, M_{vi}. In the figure, we have shown only three and the other three can be obtained by interchanging the photons. For convenience we pair the matrix elements corresponding to the interchange of photons. The matrix elements are given by

$$M_i + M_{ii} = -4\pi i e^3 v(Q)\,\tilde{u}_2 \left\{\gamma t \frac{1}{\mathbf{p}_2 - \mathbf{Q} - m} \mathbf{e}_2 \frac{1}{\mathbf{p}_1 - \mathbf{q}_1 - m} \mathbf{e}_1 + \gamma t \frac{1}{\mathbf{p}_2 - \mathbf{Q} - m} \mathbf{e}_1 \frac{1}{\mathbf{p}_1 - \mathbf{q}_2 - m} \mathbf{e}_2 \right\} u_1 \tag{13}$$

$$M_{iii} + M_{iv} = -4\pi i e^3 v(Q)\,\tilde{u}_2 \left\{\mathbf{e}_2 \frac{1}{\mathbf{p}_2 + \mathbf{q}_2 - m} \mathbf{e}_1 \frac{1}{\mathbf{p}_1 + \mathbf{Q} - m} \gamma t + \mathbf{e}_1 \frac{1}{\mathbf{p}_2 + \mathbf{q}_1 - m} \mathbf{e}_2 \frac{1}{\mathbf{p}_1 + \mathbf{Q} - m} \gamma t \right\} u_1 \tag{14}$$

$$M_v + M_{vi} = -4\pi i e^3 v(Q)\,\tilde{u}_2 \left\{\mathbf{e}_2 \frac{1}{\mathbf{p}_2 + \mathbf{q}_2 - m} \gamma t \frac{1}{\mathbf{p}_1 - \mathbf{q}_1 - m} \mathbf{e}_1 + \mathbf{e}_1 \frac{1}{\mathbf{p}_2 + \mathbf{q}_1 - m} \gamma t \frac{1}{\mathbf{p}_1 - \mathbf{q}_2 - m} \mathbf{e}_2 \right\} u_1. \tag{15}$$

We shall demonstrate the reduction of R.H.S. of (13). Rationalisation of the denominators yields

$$M_i + M_{ii} = -4\pi i e^3 v(Q)\,\tilde{u}_2 \left\{\frac{\gamma t (\mathbf{p}_2 - \mathbf{Q} + m)\,\mathbf{e}_2 (\mathbf{p}_1 - \mathbf{q}_1 + m)\,\mathbf{e}_1}{[(\mathbf{p}_2 - \mathbf{Q})^2 - m^2]\,[(\mathbf{p}_1 - \mathbf{q}_1)^2 - m^2]} + \frac{\gamma t (\mathbf{p}_2 - \mathbf{Q} + m)\,\mathbf{e}_1 (\mathbf{p}_1 - \mathbf{q}_2 + m)\,\mathbf{e}_2}{[(\mathbf{p}_2 - \mathbf{Q})^2 - m^2]\,[(\mathbf{p}_1 - \mathbf{q}_2)^2 - m^2]}\right\} u_1. \tag{16}$$

To simplify the numerator, we write

$$\begin{aligned} &\tilde{u}_2 \gamma t (\mathbf{p}_2 - \mathbf{Q} + m)\,\mathbf{e}_2 (\mathbf{p}_1 - \mathbf{q}_1 + m)\,\mathbf{e}_1 u_1 \\ &= \tilde{u}_2 \gamma t (\mathbf{p}_2 - \mathbf{Q} + m)\,\mathbf{e}_2 (-\mathbf{e}_1 \mathbf{p}_1 + 2p_1 \cdot e_1 - \mathbf{q}_1 \mathbf{e}_1 + m\mathbf{e}_1)\,u_1 \\ &= \tilde{u}_2 \gamma t (\mathbf{p}_2 - \mathbf{Q} + m)\,\mathbf{e}_2 (2p_1 \cdot e_1 - \mathbf{q}_1 \mathbf{e}_1)\,u_1. \end{aligned} \tag{17}$$

We next use the relations

$$\mathbf{p}_2 - \mathbf{Q} = \mathbf{p}_1 - \mathbf{q}_1 - \mathbf{q}_2. \tag{18}$$

$$\mathbf{p}_1 \mathbf{a} = -\mathbf{a}\mathbf{p}_1 + 2p_1 \cdot a \tag{19}$$

and eliminate $\mathbf{p}_1$ from (16) with the help of (4). Reducing the other part in a similar manner, we obtain after some simplification

$$\mathrm{M_i} + \mathrm{M_{ii}} = -4\pi e^3 v\,(\mathrm{Q}) \frac{u_2\gamma_t}{\mathrm{Q.Q} - 2p_2\,\mathrm{Q}} \left\{ 2\frac{p_1\cdot e_1}{p_1\cdot q_1}(q_1\cdot e_2 - p_1\cdot e_2) \right.$$

$$+ \frac{2p_1\cdot e_1}{p_1\cdot q_2}(q_2\cdot e_1 - p_1\cdot e_1) + 2e_1\cdot e_2 + \mathbf{q}_1\,\mathbf{e}_1\left(\frac{p_1.e_2}{p_1.q_2}\right.$$

$$\left. + \frac{p_1\cdot e_2}{p_1\cdot q_1} - \frac{e_2\cdot q_1}{p_1\cdot q_1}\right) + \mathbf{q}_2\,\mathbf{e}_2\left(\frac{p_1\cdot e_1}{p_1\cdot q_1} + \frac{p_1\cdot e_1}{p_1\cdot q_2} - \frac{e_1\cdot q_2}{p_1\cdot q_2}\right)$$

$$\left. + \frac{\mathbf{q}_1\mathbf{e}_1\mathbf{e}_2\mathbf{q}_2}{2p_1\cdot q_2} + \frac{\mathbf{q}_2\mathbf{e}_2\mathbf{e}_1\mathbf{q}_1}{2p_1\cdot q_1}\right\} u_1. \tag{20}$$

Proceeding in an exactly same manner, we reduce the R.H.S. of (14) and (15). The total matrix element M is given by

$$\mathrm{M} = -4\pi i e^3 v\,(\mathrm{Q})\,\tilde{u}_2 \{\mathrm{A}_1\gamma_t + \mathrm{A}_2\gamma_t\,\mathbf{e}_1\mathbf{q}_1 + \mathrm{A}_3\gamma_t\,\mathbf{e}_2\mathbf{q}_2 + \mathrm{A}_4\mathbf{e}_1$$

$$+ \mathrm{A}_5\mathbf{e}_2 + \mathrm{A}_6\mathbf{e}_1\mathbf{q}_1\gamma_t\mathbf{e}_2\mathbf{q}_2 + \mathrm{A}_7\mathbf{e}_2\mathbf{q}_2\gamma_t\mathbf{e}_1\mathbf{q}_1 + \mathrm{A}_8\omega_2\mathbf{e}_1\mathbf{q}_1\mathbf{e}_2$$

$$+ \mathrm{A}_9\omega_2\mathbf{e}_2\mathbf{e}_1\mathbf{q}_1 + \mathrm{A}_{10}\omega_1\mathbf{e}_1\mathbf{e}_2\mathbf{q}_2 + \mathrm{A}_{11}\omega_1\mathbf{e}_2\mathbf{q}_2\mathbf{e}_1\}\,u_1 \tag{21}$$

where

$$\mathrm{A}_1 = -\frac{p_2\cdot e_2}{p_1\cdot q_1}\frac{p_1\cdot e_1}{p_2\cdot q_1} - \frac{p_2\cdot e_1}{p_1\cdot q_2}\frac{p_1\cdot e_2}{p_2\cdot q_1} + \left[\frac{p_2\cdot e_1}{p_2\cdot q_1}e_2\cdot q_1 + \frac{p_2\cdot e_2}{p_2\cdot q_2}e_1\cdot q_2\right.$$

$$\left. - e_1\cdot e_2 + \frac{p_2\cdot e_2}{p_2\cdot q_2}p_2\cdot e_1 + \frac{p_2\cdot e_2}{p_2\cdot q_1}p_2\cdot e_1\right]\frac{2}{\mathrm{Q.Q} + 2p_1\cdot\mathrm{Q}}$$

$$+ \left[\frac{p_1\cdot e_1}{p_1\cdot q_1}e_2\cdot(q_1 - p_1) + \frac{p_1\cdot e_2}{p_1\cdot q_2}e_1\cdot(q_2 - p_1)\right.$$

$$\left. + e_1\cdot e_2\right]\frac{2}{\mathrm{Q.Q} - 2p_2\cdot\mathrm{Q}}$$

$$\mathrm{A}_2 = \frac{1}{\mathrm{Q.Q} + 2p_1\cdot\mathrm{Q}}\left(\frac{p_2\cdot e_2}{p_2\cdot q_2} + \frac{p_2\cdot e_2}{p_2\cdot q_1} + \frac{e_2\cdot q_1}{p_2\cdot q_1}\right) + \frac{1}{\mathrm{Q.Q} - 2p_2\cdot\mathrm{Q}}$$

$$\times\left(\frac{e_2\cdot q_1}{p_1\cdot q_1} - \frac{p_1\cdot e_2}{p_1\cdot q_2} - \frac{p_1\cdot e_2}{p_1\cdot q_1}\right) - \left(\frac{p_1\cdot e_2}{2p_1\cdot q_2\,p_2\cdot q_1} + \frac{p_2\cdot e_2}{2p_1\cdot q_1\,p_2\cdot q_2}\right)$$

$$\mathrm{A}_3 = \frac{1}{\mathrm{Q.Q} + 2p_1\cdot\mathrm{Q}}\left(\frac{p_2\cdot e_1}{p_2\cdot q_1} + \frac{p_2\cdot e_1}{p_2\cdot q_2} + \frac{e_1\cdot q_2}{p_2\cdot q_2}\right)$$

$$+ \frac{1}{\mathrm{Q.Q} - 2p_2\cdot\mathrm{Q}}\left(-\frac{p_1\cdot e_1}{p_1\cdot q_1} - \frac{p_1\cdot e_1}{p_1\cdot q_2} + \frac{e_1\cdot q_2}{p_1\cdot q_2}\right)$$

$$-\left(\frac{p_2\cdot e_1}{2p_1\cdot q_2\,p_2\cdot q_1} + \frac{p_1\cdot e_1}{2p_1\cdot q_1\,p_2\cdot q_2}\right)$$

$$A_4 = \frac{2\omega_1}{Q.Q + 2p_1.Q}\left(\frac{p_2.e_2}{p_2.q_2} + \frac{p_2.e_2}{p_2.q_1} + \frac{e_2.q_1}{p_2.q_1}\right) - \frac{p_1.e_2\omega_1}{p_1.q_1\, p_2.q_2}$$

$$A_5 = \frac{2\omega_2}{Q.Q + 2p_1.Q}\left(\frac{p_2.e_1}{p_2.q_1} + \frac{p_2.e_1}{p_2.q_2} + \frac{e_1.q_2}{p_2.q_2}\right) - \frac{p_1.e_1}{(p_1.q_1\ p_1.q_2)}\,\omega_2$$

$$A_6 = -\frac{1}{(2p_1.q_2)(Q.Q - 2p_2.Q)} + \frac{1}{(2p_2.q_1)(Q.Q + 2p_1.Q)} - \frac{1}{4(p_1.q_2)(p_2.q_1)}$$

$$A_7 = -\frac{1}{(2p_1.q_1)(Q.Q - 2p_2.Q)} + \frac{1}{(2p_2.q_2)(Q.Q + 2p_1.Q)} - \frac{1}{(4p_1.q_1)(p_2.q_2)}$$

$$A_8 = \frac{1}{(p_2.q_1)(Q.Q + 2p_1.Q)}$$

$$A_9 = \frac{1}{(p_1.q_1)(Q.Q - 2p_2.Q)}$$

$$A_{10} = \frac{1}{(p_1.q_2)(Q.Q - 2p_2.Q)}$$

$$A_{11} = \frac{1}{(p_2.q_2)(Q.Q + 2p_1.Q)}\,. \tag{22}$$

Using the relations (9) to (12) we can still further reduce the matrix element corresponding to various combinations of polarizations and express it finally in terms of the γ-matrices and the products of γ-matrices. Let M_1, M_2, M_3 and M_4 be the matrix elements corresponding to the polarizations AA′, AB′, BA′ and BB′ respectively. After considerable calculation and simplification, we obtain

$$M_i = -4\pi e^3 v(Q)\,\tilde{u}_2\{a_{i1}\gamma_t + a_{i2}\gamma_x + a_{i3}\gamma_y + a_{i4}\gamma_z + a_{i5}\gamma_t\gamma_x\gamma_y + a_{i6}\gamma_t\gamma_y\gamma_z + a_{i7}\gamma_t\gamma_z\gamma_x + a_{i8}\gamma_x\gamma_y\gamma_z\}\,u_1 \tag{23}$$

where

$$a_{11} = A_1 - \omega_1\omega_2(A_6 + A_7)(1 + \cos\theta) + \omega_1\omega_2(A_8 - A_9) - A_{10} + A_{11}$$

$$a_{12} = A_2 + A_3 - \omega_2 A_5 - \omega_1 A_4$$

$$a_{13} = \omega_1\omega_2(A_6 + A_7)\sin\theta + \omega_1\omega_2(A_{10} - A_{11})\sin\theta$$

$$a_{14} = \omega_1\omega_2(A_6 + A_7)(1 + \cos\theta) + \omega_1\omega_2(A_9 - A_8) + \omega_1\omega_2(A_{10} - A_{11})\cos\theta$$

$$a_{15} = -\ \omega_2 A_5 \sin\theta$$

$$a_{16} = \omega_1\omega_2 (A_7 - A_6) \sin\theta$$

$$a_{17} = \omega_1 A_4 + \omega_2 A_5 \cos\theta$$

$$a_{18} = 0 \tag{24}$$

$$a_{21} = A_1$$

$$a_{22} = A_2 - \omega_1 A_4 - \omega_1\omega_2 (A_6 + A_7 - A_8 + A_9) \sin\theta$$

$$a_{23} = (\omega_2 A_5 - A_3) \cos\theta$$

$$a_{24} = (A_3 - \omega_2 A_5) \sin\theta$$

$$a_{25} = \omega_1\omega_2 (-\ A_6 + A_7)(1 + \cos\theta) + \omega_1\omega_2 (A_8 + A_9 - A_{10} - A_{11}) \cos\theta$$

$$a_{26} = \omega_2 A_5$$

$$a_{27} = \omega_1 A_4 + \omega_1\omega_2 (-\ A_6 + A_7 + A_8 + A_9 - A_{10} - A_{11}) \sin\theta$$

$$a_{28} = \omega_1\omega_2 A_6 (1 + \sin\theta) - \omega_1\omega_2 A_7 (1 + \cos\theta) - \omega_1\omega_2 (A_8 + A_9) \cos\theta + \omega_1\omega_2 (A_{10} + A_{11}) \tag{25}$$

$$a_{31} = A_1$$

$$a_{32} = A_3 + \omega_2 A_5 - \omega_1\omega_2 A_6 \sin\theta - \omega_1\omega_2 A_7 \sin\theta - \omega_1\omega_2 A_{10} \sin\theta + \omega_1\omega_2 A_{11} \sin\theta$$

$$a_{33} = A_2 - \omega_1 A_4$$

$$a_{34} = 0$$

$$a_{35} = \omega_2 A_5 \sin\theta - \omega_1\omega_2 A_6 (1 + \cos\theta) + \omega_1\omega_2 A_7 (1 + \cos\theta) + \omega_1\omega_2 (A_8 + A_9 - A_{10} - A_{11})$$

$$a_{36} = -\ \omega_1 A_4$$

$$a_{37} = -\ \omega_2 A_5 \cos\theta - \omega_1\omega_2 A_6 \sin\theta + \omega_1\omega_2 A_7 \sin\theta$$

$$a_{38} = \omega_1\omega_2 A_6 (1 + \cos\theta) - \omega_1\omega_2 A_7 (1 + \cos\theta) + \omega_1\omega_2 (-\ A_8 - A_9 + A_{10}) + \omega_1\omega_2 A_{11} \cos\theta \tag{26}$$

$$a_{41} = A_1 + \omega_1\omega_2 A_6 (1 + \cos\theta) + \omega_1\omega_2 A_7 (1 + \cos\theta) + \omega_1\omega_2 (-\ A_8 + A_9 + A_{10} - A_{11}) \cos\theta$$

$$a_{42} = a_{45} = a_{46} = a_{48} = 0$$

$$a_{43} = A_2 - A_3 \cos\theta + \omega_2 A_5 \cos\theta + \omega_1\omega_2 (-\ A_6 - A_7 + A_8 - A_9) \sin\theta$$

$$a_{44} = A_3 \sin\theta - \omega_2 A_5 \sin\theta - \omega_1\omega_2 A_6 (1 + \cos\theta) - \omega_1\omega_2 A_7 (1 + \cos\theta) + \omega_1\omega_2 (A_8 - A_9) \cos\theta + \omega_1\omega_2 (-\ A_{10} + A_{11})$$

$$a_{47} = -\ \omega_1 A_4 + \omega_2 A_5 + \omega_1\omega_2 (A_6 - A_7 - A_8 - A_9 + A_{10} + A_{11}) \sin\theta. \tag{27}$$

SPUR CALCULATION

The differential cross-section for any particular polarization is given by

$$d\sigma_i = 2\pi \frac{E_1}{p_1} \cdot (\text{normalisation factor}) \cdot \tfrac{1}{2}.$$

$$\sum_{\text{spins 1}} \sum_{\text{spins 2}} |M_i|^2 \cdot (\text{density of final states}). \qquad (28)$$

where we have summed over the final spin states of the electron and averaged over the initial spin states. In this section, we shall evaluate

$$\sum_{\text{spins 1}} \sum_{\text{spins 2}} |M_i|^2.$$

It is well known that if

$$M = \tilde{u}_2 M u_1 \qquad (29)$$

then

$$\sum_{\text{spins 1}} \sum_{\text{spins 2}} |M^2| = Sp\,|(\mathbf{p}_2 + m)\, M\, (\mathbf{p}_1 + m)\, M'| \qquad (30)$$

where

$$M' = \gamma_t \tilde{M}. \qquad (31)$$

Now M_i is given by

$$M_i = a_{i1}\gamma_t + a_{i2}\gamma_x + a_{i3}\gamma_y + a_{i4}\gamma_z + a_{i5}\gamma_t\gamma_x\gamma_y + a_{i6}\gamma_t\gamma_y\gamma_z + a_{i7}\gamma_t\gamma_z\gamma_x + a_{i8}\gamma_x\gamma_y\gamma_z. \qquad ‡(32)$$

It is easily verified by direct multiplication that

$$\mathbf{p}_2 M_i = l_{i1}\,\mathrm{I} + l_{i2}\gamma_t\gamma_x + l_{i3}\gamma_t\gamma_y + l_{i4}\gamma_t\gamma_z + l_{i5}\gamma_x\gamma_y + l_{i6}\gamma_y\gamma_z + l_{i7}\gamma_z\gamma_x + l_{i8}\gamma_t\gamma_x\gamma_y\gamma_z \qquad (33)$$

$$\mathbf{p}_1 M_i' = m_{i1}\,\mathrm{I} + m_{i2}\gamma_t\gamma_x + m_{i3}\gamma_t\gamma_y + m_{i4}\gamma_t\gamma_z + m_{i5}\gamma_x\gamma_y + m_{i6}\gamma_y\gamma_z + m_{i7}\gamma_z\gamma_x + m_{i8}\gamma_t\gamma_x\gamma_y\gamma_z \qquad (34)$$

where

$$\begin{aligned}
l_{i1} &= E_2 a_{i1} + p_{2x} a_{i2} + p_{2y} a_{i3} + p_{2z} a_{i4}\\
l_{i2} &= E_2 a_{i2} + p_{2x} a_{i1} + p_{2y} a_{i5} - p_{2z} a_{i7}\\
l_{i3} &= E_2 a_{i3} - p_{2x} a_{i5} + p_{2y} a_{i1} + p_{2z} a_{i6}\\
l_{i4} &= E_2 a_{i4} + p_{2x} a_{i7} - p_{2y} a_{i6} + p_{2z} a_{i1}\\
l_{i5} &= E_2 a_{i5} - p_{2x} a_{i3} + p_{2y} a_{i2} + p_{2z} a_{i8}\\
l_{i6} &= E_2 a_{i6} + p_{2x} a_{i8} - p_{2y} a_{i4} + p_{2z} a_{i3}\\
l_{i7} &= E_2 a_{i7} + p_{2x} a_{i4} + p_{2y} a_{i8} - p_{2z} a_{i2}\\
l_{i8} &= E_2 a_{i8} + p_{2x} a_{i6} + p_{2y} a_{i7} + p_{2z} a_{i5}
\end{aligned} \qquad (35)$$

‡ We omit the factor $-4\pi i e^3 \nu(Q)$ since it has no role in the calculation of the spur. We shall restore the factor when we substitute for $|M_i|^2$ in equation (28).

$$\begin{aligned}
m_{i1} &= E_1 a_{i1} + p_{1x} a_{i2} + p_{1y} a_{i3} + p_{1z} a_{i4} \\
m_{i2} &= E_1 a_{i2} + p_{1x} a_{i1} - p_{1y} a_{i5} + p_{1z} a_{i7} \\
m_{i3} &= E_1 a_{i3} + p_{1x} a_{i5} + p_{1y} a_{i1} - p_{1z} a_{i6} \\
m_{i4} &= E_1 a_{i4} - p_{1x} a_{i7} + p_{1y} a_{i6} + p_{1z} a_{i1} \\
m_{i5} &= - E_1 a_{i5} - p_{1x} a_{i3} + p_{1y} a_{i2} - p_{1z} a_{i8} \\
m_{i6} &= - E_1 a_{i6} - p_{1x} a_{i8} + p_{1y} a_{i4} + p_{1z} a_{i3} \\
m_{i7} &= - E_1 a_{i7} + p_{1x} a_{i4} - p_{1y} a_{i8} - p_{1z} a_{i2} \\
m_{i8} &= - E_1 a_{i8} - p_{1x} a_{i6} - p_{1y} a_{i7} - p_{1z} a_{i5}
\end{aligned} \tag{36}$$

and I is the unit matrix.

Now (30) can be written as

$$\begin{aligned}
\sum_{\text{spins 1}} \sum_{\text{spins 2}} |M_i|^2 &= Sp\,|(\mathbf{p}_2 + m)\, M_i\, (\mathbf{p}_1 + m)\, M_i'| \\
&= Sp\,|\mathbf{p}_2\, M_i\, \mathbf{p}_1\, M_i'| + m^2\, Sp\,|M_i M_i'| \\
&\quad + m\, Sp\,|\mathbf{p}_2\, M_i M_i'| + m\, Sp\,|M_i \mathbf{p}_1 M_i'|.
\end{aligned} \tag{37}$$

Using the well-known properties of γ-matrices, it can be proved that

$$Sp\,|M_i \mathbf{p}_1 M_i'| = 0$$

$$Sp\,|\mathbf{p}_2 M_i M_i'| = 0$$

$$Sp\,|\mathbf{p}_2 M_i \mathbf{p}_1 M_i'| = 4 \sum_{j=1}^{8} l_{ij} m_{ij}\, \epsilon(j)$$

$$Sp\,|M_i M_i'| = 4 \sum_{j=1}^{8} a_{ij}^2\, \epsilon'(j)$$

$$\begin{aligned}
\epsilon(j) &= +1 \text{ for } j = 1, 2, 3, 4. & \epsilon'(j) &= +1 \text{ for } j = 1, 5, 6, 7. \\
&= -1 \text{ for } j = 5, 6, 7, 8. & &= -1 \text{ for } j = 2, 3, 4, 8.
\end{aligned} \tag{38}$$

Thus we obtain

$$\sum_{\text{spins 1}} \sum_{\text{spins 2}} |M_i|^2 = 4\,|4\pi e^3 v\,(Q)|^2 \sum_{j=i}^{8} [l_{ij} m_{ij}\, \epsilon(j) + m^2 a_{ij}^2 \epsilon'(j)]. \tag{39}$$

Density of States

The density of states for a 4-particle system can be easily derived as

$$D = (2\pi)^{-9} \frac{p_3^{\,2} dp_3 d\Omega_3 p_4^{\,2} dp_4 d\Omega_4 E_5 E_6 p_5^{\,3} d\Omega_5}{(E - E_3 - E_4)\, p_5^{\,2} - E_5 \{(p - p_3 - p_4) \cdot p_5\}} \qquad \S(40)$$

§ We use the subscripts 3, 4, 5 and 6 to denote the four particles instead of 1, 2, 3, 4 just to avoid any confusion with the use of subscripts 1 and 2 for the initial and final states of the electron in the problem.

Adapting (40) to our problem, we make the substitutions

$$E_6 = \infty, \quad p_3 \equiv p_2, \quad p_4 \equiv \omega_1, \quad p_5 \equiv \omega_2.$$

We then obtain

$$D = (2\pi)^{-9} p_2^2 dp_2 d\Omega_2 \omega_1^2 d\omega_1 d\Omega_{\omega_1} \omega_2^2 d\Omega_{\omega_2}. \tag{41}$$

Hence $d\sigma_i$ is given by

$$d\sigma_i = \frac{2\pi}{2E_1 . 2\omega_1 . 2\omega_2 . 2E_2} \cdot \frac{E_1}{p_1} (2\pi)^{-9} p_2^2 dp_2 \omega_1^2 d\omega_1 d\Omega_{\omega 1} \omega_2^2 d\Omega_{\omega 2}$$

$$\times 2 [4\pi e^3 v(Q)]^2 \sum_{j=1}^{8} [l_{ij} m_{ij} \epsilon(j) + m^2 a_{ij}^2 \epsilon'(j)]. \tag{42}$$

If we are not interested in any particular polarization, we can sum over all polarizations. This is done by asumming $d\sigma_i$ over i.

We shall consider a particular case when the two photons are weak. In this case, the cross-section reduces to a very simple form. The matrix element (13) can be expressed purely in terms of $\mathbf{q}_1$ and $\mathbf{q}_2$ as

$$M_1 + M_{11} = -4\pi i e^3 v(Q) \tilde{u}_2 \gamma_t \frac{1}{\mathbf{p}_1 - \mathbf{q}_1 - \mathbf{q}_2 - m}$$

$$\times \left(\mathbf{e}_2 \frac{1}{\mathbf{p}_1 - \mathbf{q}_1 - m} \mathbf{e}_1 + \mathbf{e}_1 \frac{1}{\mathbf{p}_1 - \mathbf{q}_2 - m} \mathbf{e}_2 \right) u_1. \tag{43}$$

Rationalising the denominator and neglecting $\mathbf{q}_1$ and $\mathbf{q}_2$ in the numerator and $q_1 \cdot q_2$ in the denominator, we obtain

$$M_1 + M_{11} = -4\pi i e^3 v(Q) \frac{\tilde{u}_2 \gamma_t (\mathbf{p}_1 - m)}{4 p_1 \cdot (q_1 + q_2)}$$

$$\times \left[\mathbf{e}_2 \frac{(\mathbf{p}_1 - m)\, \mathbf{e}_1}{p_1 \cdot q_1} + \frac{\mathbf{e}_1 (\mathbf{p}_1 - m)}{p_1 \cdot q_2} \mathbf{e}_2 \right] u_1. \tag{44}$$

Reducing still further by the method now familiar, we obtain

$$M_1 + M_{11} = -4\pi i e^3 v(Q) \tilde{u}_2 \gamma_t u_1 \frac{p_1 \cdot e_1}{p_1 \cdot q_1} \frac{p_1 \cdot e_2}{p_1 \cdot q_2}. \tag{45}$$

In a similar manner, we reduce the other two pairs of elements and obtain

$$M_1 = -4\pi i e^3 v(Q) \tilde{u}_2 \gamma_t u_1 \left(\frac{p_2 \cdot e_1}{p_2 \cdot q_1} - \frac{p_1 \cdot e_1}{p_1 \cdot q_1} \right) \left(\frac{p_2 \cdot e_2}{p_2 \cdot q_2} - \frac{p_1 \cdot e_2}{p_1 \cdot q_2} \right). \tag{46}$$

On making the spur calculation, the cross-section can be written as

$$d\sigma_i = \frac{E_1}{p_1} \left[\frac{2\pi e^2}{4E_1 E_2} |v(Q)|^2 \frac{E_2 p_2 d\Omega_2}{(2\pi)^3} 4 (m^2 + 2E_1 E_2 - p_1 \cdot p_2) \right]$$

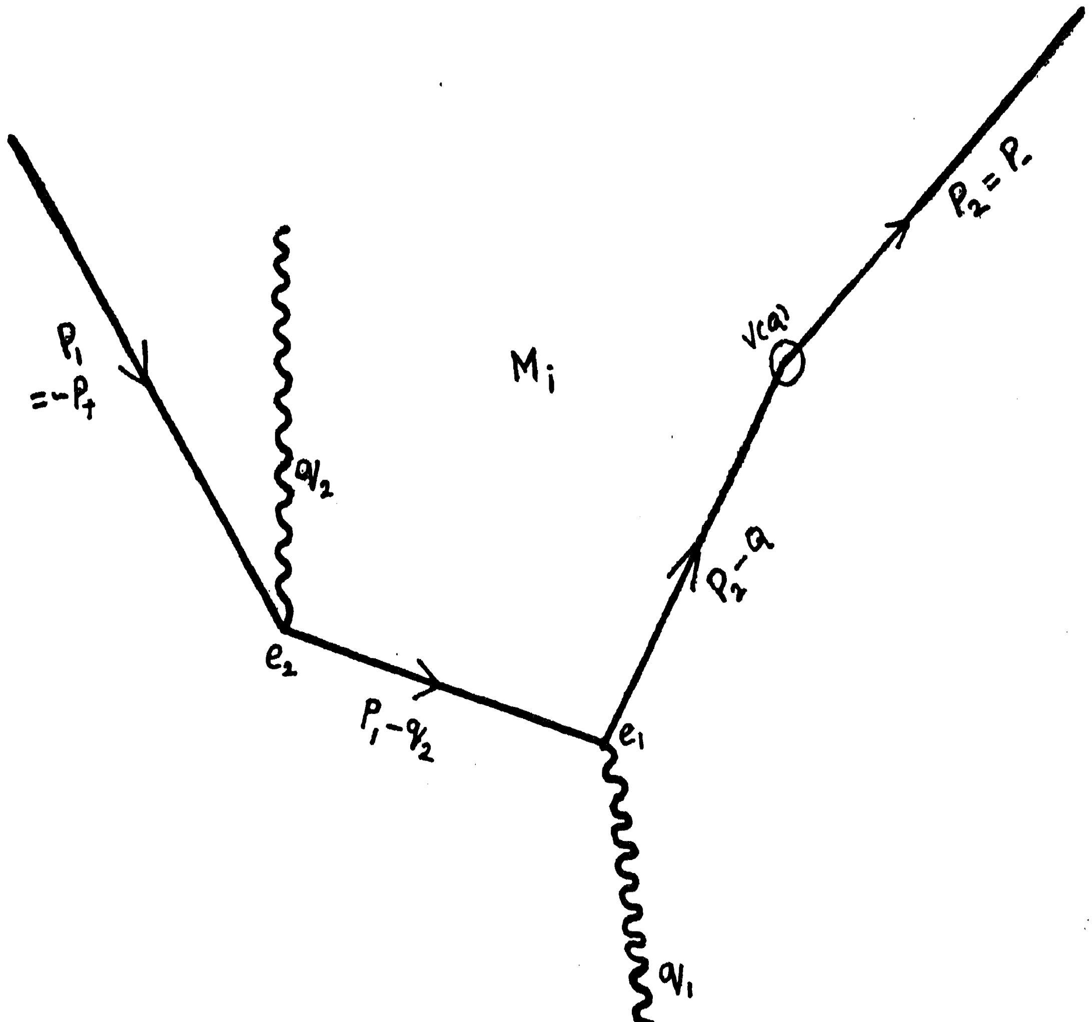
$$\times\left[\frac{e^2 d\omega_1 d\Omega\omega_1}{(2\pi)^2\omega_1}\left(\frac{p_2\cdot e_1}{p_2\cdot\dfrac{q_1}{\omega_1}}-\frac{p_1\cdot e_1}{p_1\cdot\dfrac{q_1}{\omega_1}}\right)^2\right]$$

$$\times\left[\frac{e^2 d\omega_2 d\Omega\omega_2}{(2\pi)^2\omega_2}\left(\frac{p_2\cdot e_2}{p_2\cdot\dfrac{q_2}{\omega_2}}-\frac{p_1\cdot e_2}{p_1\cdot\dfrac{q_2}{\omega_1}}\right)^2\right] \tag{47}$$

so that the last two factors can be interpreted as the probability of emission of the weak photons, if there is elastic scattering from momentum p_1 to p_2 (*see* Feynman, 1953).

Scattering of a Photon with Pair Creation

In this process a photon of momentum q_1 and polarization e_1 is incident and after its collision with the nucleus, we obtain a photon of momentum q_2 and polarization e_2 besides a positron-electron pair of momentum p_+ and p_- respectively. The lowest order Feynman diagrams for this process are given below. As in the previous section, we have given only three diagrams since the other three can be obtained by the exchange of the positions of the photons.



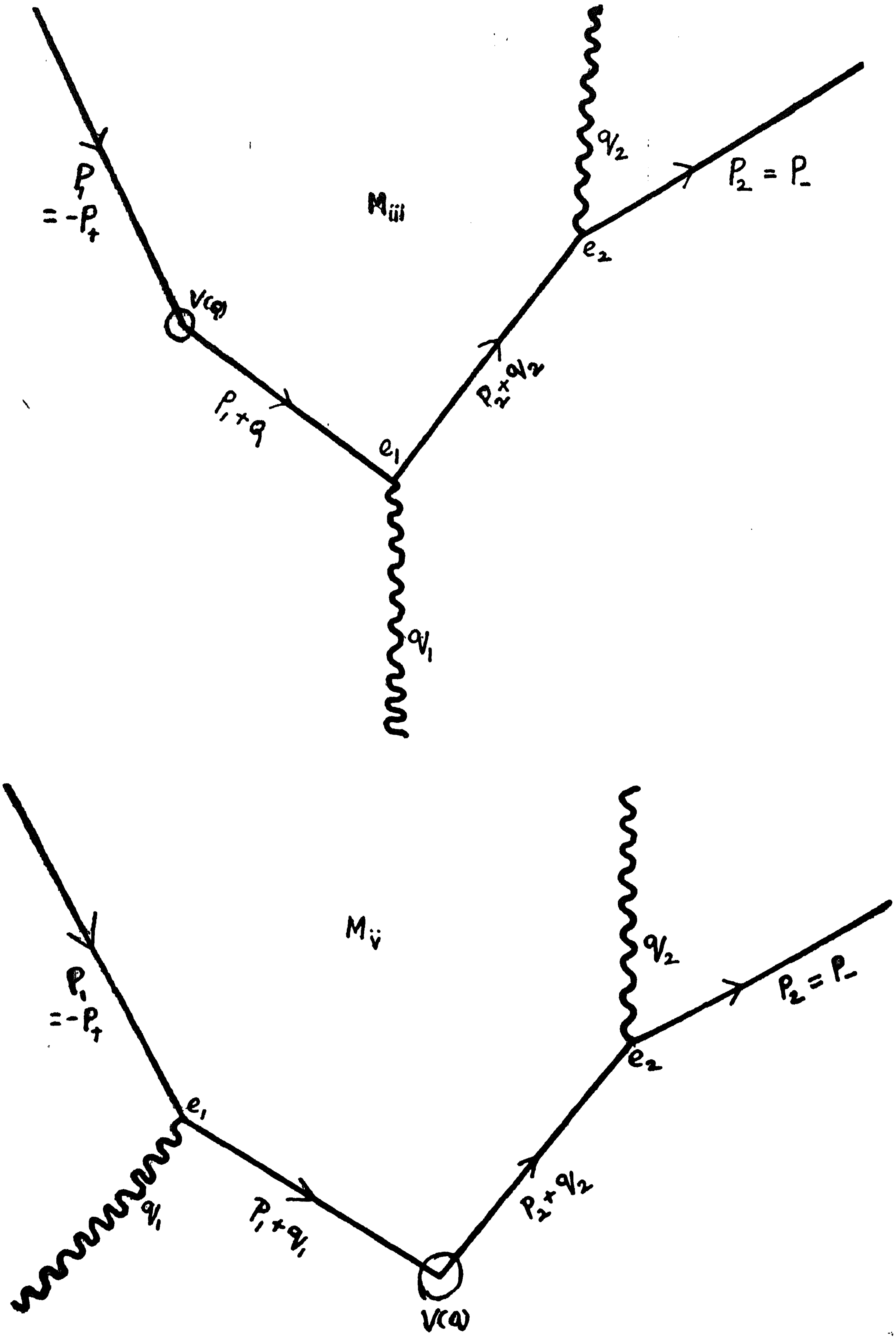


FIG. 2. Scattering of photon with a pair creation.

With respect to the directions of the arrows and without regard to the directions of increasing time, these diagrams look exactly like those for the process discussed in the previous section (*cf.* Fig. 1). However when the direction of time is taken into account, we notice that

(1) p_1 is the momentum of the electron travelling backward in time, *i.e.*, $p_1 = -p_+$.

(2) The photon q_1 is absorbed instead of being emitted.

(3) $p_2 = p_-$.

Hence we can obtain the matrix elements corresponding to these diagrams from those of process (ii) if only we replace $\mathbf{p}_1$ by $-\mathbf{p}_+$, $\mathbf{q}_1$ by $-\mathbf{q}_1$ and $\mathbf{p}_2$ by $\mathbf{p}_-$. However the density of final states is different in this case since the particles in the final states are now a photon, an electron and a positron. Thus $d\sigma_i$ the differential cross-section for a polarization of *i*-th type is given by

$$d\sigma_i = \frac{2\pi}{2E_- 2\omega_1 2\omega_2 2E_+} (2\pi)^{-9} p_+{}^2 dp_+ d\Omega_+ p_-{}^2 dp_- d\Omega_- \omega_2{}^2 d\Omega\omega_2$$

$$\times 2\,[4\pi e^3 v\,(Q)]^2 \sum_{j=1}^{8} [l_{ij} m_{ij}\,\epsilon(j) + m^2 a_{ij}{}^2\,\epsilon'(j)]' \tag{48}$$

where the prime over the square bracket under summation sign indicates that we have to make the following substitutions

$-p_+$ for p_1; $\quad -E_+$ for E_1

p_- for p_2; $\quad E_-$ for E_2

and

$-\omega_1$ for ω_1.

If we are not interested in a particular state of polarization of the photons, we average over the polarization of the incoming photon and sum over polarization of the outgoing photon. This is done by summing over i and dividing by 2.

We hope that this process will be of considerable interest in high energy electron-photon cascades since it yields a photon, in addition to a pair. In a later contribution we hope to discuss these cross-sections under some useful approximations. We also propose to include these processes in the usual soft cascade theory and study the fluctuations in the number of electron pairs.

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