SINGULAR SOLUTIONS OF SIMULTANEOUS ORDINARY DIFFERENTIAL EQUATIONS.

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[§ 1]. *Introduction.*

In considering a system of ordinary simultaneous differential equations, it is sufficient to regard the equations as involving differential coefficients of the first order only; for, equations involving differential coefficients of higher orders can be easily reduced to this canonical form by introducing more variables. We shall mostly confine ourselves to equations involving three variables only, so that the equations may be taken in the form

$$\phi_r\,(x, y, z, y', z') = 0,\ (r = 1, 2) \qquad \ldots \qquad (1)$$

where $y' = \dfrac{dy}{dx}$, $z' = \dfrac{dz}{dx}$.

That such a pair of equations can possess singular integrals* is well known, and when ϕ_1 and ϕ_2 are taken in certain particular forms, methods of obtaining the singular solutions are known. A list of chief references is given at the end. The subject will be studied in this paper by adopting a different method of procedure, which enables the singular solutions to be determined even when ϕ_1 and ϕ_2 involve elementary transcendental or irrational expressions.†

[§§ 2-4]. *General Solutions of the Equations* (1).

§ 2. Let the equations (1) be solved for y' and z', giving continuous functions of x, y and z possessing continuous first partial derivatives. [A]

* The definition of *Singular Solutions* adopted is that they are solutions not included in the general solutions. The latter should, however, be carefully defined by the *uniqueness* property in the fundamental existence theorems of Cauchy. *Vide* my paper in the *Journal of the Indian Mathematical Society* (Jubilee Volume, Vol. 20), also in the *Half-Yearly Journal of the Mysore University*, Vol. V.

† The only attempt in this direction so far is by M. J. M. Hill who in a paper (Ref. 1) on partial differential equations has discussed some examples of the type $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$. His methods are, however, very tentative. I have found, however, his examples very instructive and useful in the present method of procedure, and several of his examples have been used in this paper.

Let

$$y' = \psi_1 (x, y, z) \quad \ldots \quad (2)$$

$$z' = \psi_2 (x, y, z) \quad \ldots \quad (3)$$

Hence, $$y'' = \frac{\partial \psi_1}{\partial x} + y' \frac{\partial \psi_1}{\partial y} + \psi_2 \frac{\partial \psi_1}{\partial z} \quad \ldots \quad (4)$$

Let z be eliminated between (2) and (4), so as to result in a differential equation of the second order in x and y, in which the coefficients are all continuous functions. For this, (2) must be solved for z, and the value substituted in (4). [B]

Let the resulting equation be written

$$f(y'', y', y, x) = 0 \quad \ldots \quad (5)$$

The equation (5) possesses two independent first integrals of the type

$$F_r (x, y, y', c_r) = 0, \quad (r = 1, 2) \quad \ldots \quad (6)$$

Substituting $y' = \psi_1$, the general solutions‡ of (1) are obtained :

$$F_r (x, y, \psi_1, c_r) = 0, \quad (r = 1, 2) \quad \ldots \quad (7)$$

Finally, (7) may be put into the form

$$u (x, y, z) = c_1 ; \quad v (x, y, z) = c_2 \quad \ldots \quad (8)$$

§ 3. [The above general method of solution has one important defect—a defect which cannot be removed. We are led to believe that the general solutions of any given equations (1) can be *completely* represented in the form $u = c_1$; $v = c_2$, or in the rationalised form $F_r (x, y, z, c_r) = 0$, $(r = 1, 2)$, each of the F's involving *only one constant*. This is not necessarily the fact. Consider for example the generalised Clairaut-equations

$$\phi_r (y - xy', \ z - xz', \ y', z') = 0, \ (r = 1, 2).$$

Differentiating and solving, we get $y'' = 0 = z''$, showing the integral curves (other than singular solutions) must necessarily be straight lines. But the general method of § 2 will not give straight lines only. (5) should necessarily reduce to $y'' = 0$, whose first integrals are $y' = a$, and $y - xy' = b$.

Consider a concrete example :

$$y = xy' + y'^2 + z' ; \ z = xz' + y'z'.$$

Eliminating a and b from the known general solutions

$$y = ax + a^2 + b ; \ z = bx + ab,$$

we get

$$a (x+a)^2 - y (x+a) + z = 0 \quad \ldots \quad (9)$$

$$b^3 - b^2 y - bzx + z^2 = 0 \quad \ldots \quad (10)$$

The function $\psi_1 (x, y, z)$ of (2) is evidently the same as that obtained by solving (9) for a. Hence the equation $u = c_1$ in (8) gives (9) on rationalisation. Similarly $v = c_2$ leads to an equation of degree higher than the first.

‡ Forsyth, *Treatise on Differential Equations*, §176.

The general solutions (8) will thus represent curves of a degree higher than the first. They must necessarily be all degenerate, breaking up into straight lines which represent solutions of the given differential equations, and other curves which are not solutions.*

Assuming that there exist for the equations (1) *general solutions which are completely representable in the form* $F_r(x, y, z, a, b) = 0$, $r = 1, 2$,† *the solutions might only be partially represented when* $F_1 = 0$ *and* $F_2 = 0$ *are transformed into some other form.*

The explanation for the appearance of the extraneous curves seems to be as follows: In geometry, the curve given by the equations $\phi_r(x, y, z, a, b) = 0$, $r = 1, 2$ is known to form only a part of the curve common to the surfaces obtained by rationalising the equations $\psi_1 = a$ and $\psi_2 = b$ (continuing the notation of § 2). Similarly we must regard the equations (2) and (3) as representing something more than the original equations (1).

If, however, *we start* with the equations (2) and (3), (8) will completely represent their general solutions. Thus, when we deal with equations of the type $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, the general solutions are *completely* representable in the form $u = c_1$; $v = c_2$.

We shall hereafter be considering the general solutions of (1) in the form $u=c_1$; $v=c_2$ with the understanding that this *includes* the general solutions though not of necessity *completely* equivalent to them.]

§ 4. The equations (1) can have only one set of independent integrals. If $u=c_1$; $v=c_2$ be such a set, the general solutions may also be supposed to be *included* in general in $\theta_1(u, v) = c_1$; $\theta_2(u, v) = c_2$, where θ_1 and θ_2 may be any arbitrary functions, subject to suitable limitations.

These facts are easily proved, and are written down here only for the sake of reference.

[§§ 5-9]. *Equations leading to Singular Solutions.*

§ 5. Our object is to determine a pair of equations satisfying the given differential equations, which are not *included* in the general solutions (8) for any values of c_1 and c_2 whatever. At least one of the two equations to be determined must be functionally different from u and v. Only one of the equations may be different or both, and accordingly two classes of singular solutions might exist.

* A. Mayer (Ref. 4) expresses the solutions in the form $y = \Phi_1(x, a, b)$; $z = \Phi_2(x, a, b)$—a form possessing similar limitations. Also Ref. (7), p. 324.

† It is however possible that solutions may exist only in the form $x = \theta_1(t, a, b)$; $y = \theta_2(t, a, b)$; $z = \theta_3(t, a, b)$, t being a parameter.

Assuming the existence of such singular solutions, it follows that their existence must be accounted for by the fact that the general theory of §2 *can break down somewhere.* Now this may happen in either of the two places marked in §2 as [A] and [B].

In order that the theory may break down in the place marked [A], *i.e.*, in order that it may not be possible to obtain from (1), unique solutions $y' = \psi_1(x, y, z)$, $z' = \psi_2(x, y, z)$ in the form of continuous functions with continuous first partial derivatives, it is necessary that

either $\frac{\partial(\phi_1, \phi_2)}{\partial(y', z')} = 0,$ .. .. .. (11)

or ϕ_1 or ϕ_2 or any of their first partial derivatives with respect to x, y or z = infinite (or indeterminate)* .. .. .. (12)

Conversely, if a function $w_1(x, y, z) = 0$ is constructed by means of values of x, y and z satisfying (11) or (12), $w_1 = 0$ when taken along with another suitable equation, *may* represent a solution of the given differential equations and we might expect these solutions to be singular.

In order that the general theory may break down† in the place marked [B] in § 2, *i.e.*, in order that the result of eliminating z between (2) and (4) might involve coefficients or functions which can admit of discontinuous derivatives for finite values of the variables, either it must not be possible to solve (2) for z in the form of a unique continuous function of x, y, y' with continuous derivatives, or otherwise the first derivatives of ψ_2 may not be continuous for all finite values of x, y, z. We can write as necessary conditions,

either $\frac{\partial\psi_1}{\partial z} = 0,$ .. .. (13)

or ψ_1 or ψ_2 or any of their first partial derivatives = infinite (or indeterminate) .. .. (14)

The condition (14) however resolves itself into the conditions (11) and (12), so that (13) is the only new condition that presents itself. We shall however prove in § 9, that the equation $\frac{\partial\psi_1}{\partial z} = 0$, call it $w_2(x, y, z) = 0$, does not give rise to any true singular solutions, unless w_2 is included in w_1.

* This follows from the theory of implicit functions. *Vide* Goursat-Hedrick, *Math. Analysis*, Vol. I, Chapter II. The remarks of Hedrick in the footnotes deserve attention.

† The break-down of the theory means that the conditions for Cauchy's existence theorem are not satisfied for (5). The existence of two and only two independent first integrals of this equation is an immediate corollary of the uniqueness of the general primitive as given by Cauchy's theorem. The possibility of existence of any "singular first integrals" depends upon the break-down of Cauchy's theorem as regards uniqueness.

§ 6. Let $w_3(x, y, z) = 0$ denote any one of the functions in virtue of which u, or v, or a first partial derivative of u or v becomes infinite.

§ 7. From the general solutions $u = c_1$, $v = c_2$, we obtain

$$y' = \psi_1(x, y, z) = \frac{\partial(u, v)}{\partial(z, x)} : \frac{\partial(u, v)}{\partial(y, z)}$$

$$z' = \psi_2(x, y, z) = \frac{\partial(u, v)}{\partial(x, y)} : \frac{\partial(u, v)}{\partial(y, z)}$$

The function w_1 may therefore arise out of any of the following causes :

(*i*) a first derivative of ψ_1 or ψ_2 may become infinite when a first derivative of u or v becomes infinite, *i.e.*, $w_1 = 0$ may be included in $w_3 = 0$;

(*ii*) a first derivative of ψ_1 or ψ_2 may become infinite when a second derivative of u or v becomes infinite ;

(*iii*) ψ_1 or ψ_2 or a first derivative of either may become infinite in virtue of $\frac{\partial(u, v)}{\partial(y, z)} = 0$, *i.e.*, w_1 may be a factor of $\frac{\partial(u, v)}{\partial(y, z)}$;

(*iv*) ψ_1^{-1} or ψ_2^{-1} or a derivative of either may become infinite in virtue of $\frac{\partial(u, v)}{\partial(z, x)} = 0$ or $\frac{\partial(u, v)}{\partial(x, y)} = 0$.

We can however prove that unless (*i*) be the case, $w_1 = 0$ will not contribute towards any singular solutions. We shall first consider a lemma.

§ 8. Let $f(x, y, z) = 0$; $g(x, y, z) = 0$ denote given solutions of the differential equations whose general solutions are $u = c_1$; $v = c_2$. It is assumed that f and g are analytic for all finite values of the variables, and may be made to represent proper surfaces. It is also assumed that u and v are finite when $f = 0$ and $g = 0$. If either of them becomes infinite when $f = 0$ and $g = 0$, it must be replaced by its reciprocal. Then we have:

The necessary and sufficient conditions that $f = 0$; $g = 0$ should be included in the general solutions are that the limits of the expressions $\frac{\partial(u, f, g)}{\partial(x, y, z)}$, $\frac{\partial(v, f, g)}{\partial(x, y, z)}$ should be *both* equal to zero.

A proof is easily supplied by following the method adopted for the case of two variables. [*Proc. L.M.S.*, Vol. 17 (Series 2), pp. 159-161.] We shall express this result in a more convenient form.

$$\text{Let } \frac{\partial(u, f, g)}{\partial(x, y, z)} = \frac{\partial g}{\partial x} L_1 + \frac{\partial g}{\partial y} M_1 + \frac{\partial g}{\partial z} N_1 = A,$$

$$\text{and } \frac{\partial(v, f, g)}{\partial(x, y, z)} = \frac{\partial g}{\partial x} L_2 + \frac{\partial g}{\partial y} M_2 + \frac{\partial g}{\partial z} N_2 = B,$$

so that A and B tend to zero as $f \to 0$, $g \to 0$. Hence

$$\frac{\partial g}{\partial y}(L_2 M_1 - L_1 M_2) + \frac{\partial g}{\partial z}(L_2 N_1 - L_1 N_2) = AL_2 - BL_1 \to 0,$$

i.e., expanding,

$$\left(\frac{\partial g}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial g}{\partial z}\frac{\partial f}{\partial y}\right) \cdot \frac{\partial (u, v, f)}{\partial (x, y, z)} \to 0.$$

We get two similar equations by eliminating $\frac{\partial g}{\partial y}$ or $\frac{\partial g}{\partial z}$. Hence, f and g being independent, we must have $f \to 0$, Lt $g \to 0 \; \frac{\partial (u, v, f)}{\partial (x, y, z)} = 0$. Similarly $f \to 0$, Lt $g \to 0 \; \frac{\partial (u, v, g)}{\partial (x, y, z)} = 0$. The same method gives the former pair of equations, *viz.*, $\frac{\partial (u, f, g)}{\partial (x, y, z)} \to 0$, $\frac{\partial (v, f, g)}{\partial (x, y, z)} \to 0$, from the latter.

THEOREM 1. *The necessary and sufficient condition that $f = 0$, $g = 0$ should be singular solutions is that the limiting value of at least one of the expressions $\frac{\partial (u, v, f)}{\partial (x, y, z)}$, $\frac{\partial (u, v, g)}{\partial (x, y, z)}$ should be distinct from zero, when $f \to 0$, $g \to 0$.*

If only one of these limits be different from zero, the solutions are called *singular solutions of the first order*. Singular solutions of the *second order* also may exist, for which, however, the limits of all the four determinants $(u_x \, v_y \, f_z)$, $(u_x \, v_y \, g_z)$, $(u_x \, f_y \, g_z)$, $(v_x \, f_y \, g_z)$ should be distinct from zero.

The conditions that the solutions $f = 0$, $g = 0$ might be particular are that Lt* $\frac{\partial (u, v, f)}{\partial (x, y, z)} = 0$ *and* Lt $\frac{\partial (u, v, g)}{\partial (x, y, z)} = 0$ either identically, or in virtue of the equations $f = 0$, $g = 0$.

Illustrations:

EXAMPLE 1.

To test the nature of the solutions $z - x - y = 0$; $z - 2y = b$ of the differential equations

$$\frac{dx}{1 + (z - x - y)^{\frac{1}{2}}} = \frac{dy}{1} = \frac{dz}{2}.$$

Here, $u = y + 2\,(z - x - y)^{\frac{1}{2}}$; $v = z - 2y$.

Hence, taking $f \equiv z - x - y$,

$$\underset{f \to 0, \; g \to 0}{\text{Lt}} \frac{\partial (u, v, f)}{\partial (x, y, z)} = \frac{\partial (y, z - 2y, z - x - y)}{\partial (x, y, z)} \neq 0.$$

The solutions are therefore singular solutions of the first order.

* The possibility of some of the first derivatives of u or v becoming infinite in virtue of $f = 0$, $g = 0$ is allowed. Compare *Mysore University Journal*, Vol. V.

Example 2.

To test the nature of the solutions $z = x+y$; $x^2 - y^2 =$ constant, of the equations

$$\frac{dx}{x\,(z-x-y)^{\frac{1}{2}} - y} = \frac{dy}{y\,(z-x-y)^{\frac{1}{2}} - x} = \frac{dz}{2(z-x-y)^{\frac{1}{2}}(1-\overline{z-x-y}) + (x+y)(\overline{z-x-y}^{\frac{1}{2}}-1)}.$$

Here, $u \equiv x+y\,(z-x-y)^{\frac{1}{2}}$, $v \equiv y+x\,(z-x-y)^{\frac{1}{2}}$

Write $f \equiv z-x-y$; $g \equiv x^2-y^2-c$.

Then, $\underset{f\to 0,\; g\to 0}{\mathrm{Lt}} \dfrac{\partial\,(u, v, f)}{\partial\,(x, y, z)} = 1$; $\underset{f\to 0,\; g\to 0}{\mathrm{Lt}} \dfrac{\partial\,(u, v, g)}{\partial\,(x, y, z)} = 0$.

The solutions are thus singular and of the first order.

Example 3.

To test the nature of the solutions $f \equiv z-x-y = 0$; $g \equiv x-c = 0$ of the equations

$$\frac{dx}{y^2\,(z-x-y)^{\frac{1}{2}}} = \frac{dy}{y\,(y-2x)} = \frac{dz}{y^2-2xy+4\,(x-y)\,(z-x-y)+y^2\,(z-x-y)^{\frac{1}{2}}}$$

Here, $u \equiv x+y\,(z-x-y)^{\frac{1}{2}}$; $v \equiv x^2+y^2\,(z-x-y)^{\frac{1}{2}}$.

$$\underset{f\to 0,\; g\to 0}{\mathrm{Lt}} \frac{\partial\,(u, v, f)}{\partial\,(x, y, z)} = \frac{\partial\,(x, x^2, z-x-y)}{\partial\,(x, y, z)} = 0.$$

$$\underset{f\to 0,\; g\to 0}{\mathrm{Lt}} \frac{\partial\,(u, v, g)}{\partial\,(x, y, z)} = \frac{\partial\,(x, x^2, x-c)}{\partial\,(x, y, z)} = 0.$$

The given solutions should therefore be *particular*. This is easily verified. The general integrals are also given by $v-u^2 =$ constant; $u =$ constant. (§ 4). But $v-u^2$ contains $(z-x-y)^{\frac{1}{2}}$ as a factor. The solutions $f = 0$, $g = 0$ are thus *included* in the general integrals taken in the form $v-u^2 =$ constant, $u =$ constant.

§ 9. It follows immediately from Theorem 1 that $w_1 = 0$ will not contribute to any singular solution unless it is included in $w_3 = 0$. For, the equations (2) and (3) are equivalent to

$$\frac{\partial u}{\partial x}\,dx+\frac{\partial u}{\partial y}\,dy+\frac{\partial u}{\partial z}\,dz = 0\,;\quad \frac{\partial v}{\partial x}\,dx+\frac{\partial v}{\partial y}\,dy+\frac{\partial v}{\partial z}\,dz = 0.$$

If $w_1 = 0$ contributes towards a singular solution,

$\dfrac{\partial w_1}{\partial x}\,dx + \dfrac{\partial w_1}{\partial y}\,dy + \dfrac{\partial w_1}{\partial z}\,dz = 0$ must be satisfied by the values of y', z' obtained from the above. The Jacobian $\dfrac{\partial\,(u, v, w)}{\partial\,(x, y, z)}$ cannot attain a non-zero value unless one or more of the derivatives of u or v becomes infinite in virtue of $w_1 = 0$.

A similar argument holds to show that if $w_2 = 0$ leads to a singular solution, w_2 is included in w_3, *i.e.*, $w_2 = 0$ makes one of the first derivatives of u or v infinite, and from the values of ψ_1 and ψ_2 in terms of u and v we can conclude that w_2 is included in w_1 as well.

We conclude that in order to obtain the singular solutions from the differential equations, it is enough if we consider the function defined by (11) or (12). In order to obtain them from the general integrals taken in the form $u = c_1$, $v = c_2$, we must consider the function w_3.

The equation $w_1 = 0$, or $w_3 = 0$, as the case may be, must be associated with a suitable other equation whose formation will be now studied.

[§§ *10-11*]. *Determination of the equation to be associated along with* $w_1 = 0$ *or* $w_3 = 0$.

§10. For a particular method of formation of $w_1 = 0$, the question will be found discussed in Forsyth. (Ref. 5, pp. 150-153; also Ref. 3.) The result will be quoted here, with the symbols altered:

" If values of y' and z' satisfy the equations

(i) $\phi_1 (x, y, z, y', z') = 0$

(ii) $\phi_2 (x, y, z, y', z') = 0$

(iii) $0 = \dfrac{\partial (\phi_1, \phi_2)}{\partial (x, y')} + y' \dfrac{\partial (\phi_1, \phi_2)}{\partial (y, y')} + z' \dfrac{\partial (\phi_1, \phi_2)}{\partial (z, y')}$

(iv) $0 = \dfrac{\partial (\phi_1, \phi_2)}{\partial (x, z')} + y' \dfrac{\partial (\phi_1, \phi_2)}{\partial (y, z')} + z' \dfrac{\partial (\phi_1, \phi_2)}{\partial (z, z')}$

(v) $0 = J = \dfrac{\partial (\phi_1, \phi_2)}{\partial (y', z')}$,

then the equations $\phi_1 = 0, \phi_2 = 0$ possess a singular solution involving an arbitrary constant; *this singular solution is constituted by the combination of the general integral of* $H (x, y, y') = 0$ *which is the eliminant in* z *and* z' *between* $\phi_1 = 0, \phi_2 = 0, J = 0$ *with* $w_1 (x, y, z) = 0$ *which is the eliminant in* y' *and* z' *of the same three equations.*

The theorem may be divided into two parts; the second part given by the italics denotes the method of obtaining the singular solutions (supposed to exist) from (i), (ii) and (v); the first part gives (iii) and (iv) as the conditions of sufficiency in order that the equations (i), (ii) and (v) might give a set of solutions (singular or otherwise). The first part is the analogue of the familiar theorem. " If a singular solution of the equation $\phi(x, y, p) = 0$ exists, it must satisfy the equations $\phi = 0$; $\dfrac{\partial \phi}{\partial p} = 0$; $\dfrac{\partial \phi}{\partial x} + p \dfrac{\partial \phi}{\partial y} = 0$ simultaneously." It is now known that this theorem is not necessarily true if ϕ is considered as involving irrational or transcendental expressions. The theorem should be recast as follows: "*If* the singular solution satisfies $\phi = 0$ *and* $\frac{\partial \phi}{\partial p} = 0$, it must also satisfy $\frac{\partial \phi}{\partial x} + p\, \frac{\partial \phi}{\partial y} = 0$." Similar remarks hold in the present connection.

§ 11. In the second part of the theorem, the equation to be taken along with $w_1 = 0$ so as to constitute the singular solutions is obtained by eliminating z and z' between $\phi_1 = 0$, $\phi_2 = 0$ and $J = 0$, or what is the same thing, between $\phi_1 = 0$, $\phi_2 = 0$ and $w = 0$. This latter form of expression is more advantageous, since this does not take cognisance of the particular method of formation of the equation $w = 0$; in other words, w may be w_1 or w_3. (We have of course seen that if w is w_1, it is also w_3, and *vice versa*, if $w = 0$ could lead to singular solutions.)

The following is another way of obtaining the second equation, and is of considerable significance, though it is theoretically equivalent to the above mode of getting $H(x, y, y') = 0$. From the equations (1), we get in general $y' = \psi_1(x, y, z)$; $z' = \psi_2(x, y, z)$. These latter equations *go together* when we are considering any solutions which are particular cases of the general integrals of the given equations. Let us now consider *one* of these equations along with $w = 0$, where w means either w_1 or w_3. Thus,

$$w = 0\,;\; y' = \psi_1(x, y, z) \quad \ldots \quad (15)$$

$$\text{or} \quad w = 0\,;\; z' = \psi_2(x, y, z) \quad \ldots \quad (16)$$

(15) or (16) may be considered as a new pair of simultaneous equations. If any solution of (15) also satisfies (16) or *vice versa*, it satisfies (2) and (3) as well. Since singular solutions of (2) and (3) can be generated only through the functions w_1 or w_3, it follows that *such singular solutions are all given by* (15) *or* (16). It is also evident that *singular solutions of* (2) *and* (3) *that are given by* (15) *are equivalent to those given by* (16), for any such solution should satisfy both (15) and (16).

Let (15) give on elimination of z and solution for y', $y' = \lambda(x, y)$. This process is equivalent to that of eliminating z between $\phi_1 = 0$, $\phi_2 = 0$, $w = 0$. In other words, the equation $y' = \lambda(x, y)$ is equivalent to $H(x, y, y') = 0$, except that w is taken in its more general form. Similarly let (16) give $z' = \mu(z, x)$.

The general integral of $y' = \lambda(x, y)$, or $z' = \mu(z, x)$ when taken along with $w = 0$ *may* constitute a set of singular solutions of the given equations. The singular solution* of $y' = \lambda$, or $z' = \mu$, if any exist, when combined

* The singular solution of $y' = \lambda$, if any exist, is given by $\frac{\partial \lambda}{\partial x}$ = infinite, or $\frac{\partial \lambda}{\partial y}$ = infinite. *Vide* my paper in *Tohoku Mathematical Journal*, Vol. 30.

with $w = 0$ also *may* constitute solutions of (1). The latter solutions, if not included in the general solutions or in the singular solutions of the first order, will be *singular solutions of the second order.*

[§ 12]. *Simplification in the case where the two systems of surfaces* u = a *and* v = b *do not possess a common envelope.*

§ 12. When the general integrals $u = a$, $v = b$ are known, we shall arrive at a rule which enables us to write down the solutions of (15) or (16) straight-away in most cases. In most cases, the equation $w_3 = 0$ makes the first partial derivatives of only one of the functions u or v infinite. Let us suppose that $w_3 = 0$ makes one or more of the first derivatives of u infinite. Also, u itself can be taken as not being infinite when $w_3 = 0$, for otherwise, u may be replaced by u^{-1}. Let $\Omega(x, y, z) =$ constant be the equation which when taken along with $w_3 = 0$, gives singular solutions *of the first order.* From § 8, it follows that $\underset{w_3 \to 0}{\mathrm{Lt}} \frac{\partial(u, v, w_3)}{\partial(x, y, z)} \neq 0$, but $\frac{\partial(u, v, \Omega)}{\partial(x, y, z)} = 0$, so that Ω is a function of u and v, say $\Omega = \Omega(u, v)$. Now, $w_3 = 0$ envelopes the system of surfaces $u =$ constant, and unless Ω is independent of u, also the system $\Omega(u, v) =$ constant.† If therefore Ω is not independent of u, the curves $w_3 = 0$; $\Omega =$ constant will be the characteristic curves on the envelope of $\Omega =$ constant. We shall now prove that such curves cannot represent singular solutions at all for the given differential equations.

To begin with, let $f(x, y, z, a) = 0$ be a system of surfaces where f involves no irrational or transcendental expressions, so that the equation of the envelope E is obtained in the usual way, by the equations $f = 0$, $\frac{\partial f}{\partial a} = 0$. It must be noted that *the direction cosines of the tangent at any point on a characteristic curve cannot be obtained directly from the equations* $f = 0$, $E = 0$; for the expressions $\frac{\partial(f, E)}{\partial(y, z)}$, $\frac{\partial(f, E)}{\partial(z, x)}$, $\frac{\partial(f, E)}{\partial(x, y)}$ are all equal to zero. The direction cosines will have to be obtained from the equations $f = 0$, $\frac{\partial f}{\partial a} = 0$, *i.e.*, they are proportional to $\frac{\partial\left(f, \frac{\partial f}{\partial a}\right)}{\partial(y, z)}$, etc., etc.

Now considering the form $u = a$, and its envelope $E = w_3 = 0$, the same thing happens. The expressions $\frac{\partial(u, w_3)}{\partial(y, z)}$, $\frac{\partial(u, w_3)}{\partial(z, x)}$, $\frac{\partial(u, w_3)}{\partial(x, y)}$ are

† For proof, see "Envelopes of Systems of Surfaces", *Tohoku Math. Journal*, Vol. 39.

either all equal to zero, or some of them are indeterminate. To obtain the directions of the tangent at a point on the characteristic, the equation $u = a$ must first of all be reduced or simplified by the help of $w_3 = 0$. The expression u (which is itself finite when $w_3 = 0$) involves a fractional power of w_3, or involves logarithmic or other transcendental functions of w_3. Let $u' = a$ be the equation obtained by simplifying $u = a$‡, with the help of $w_3 = 0$. (For instance, in Example 2, $w_3 \equiv z-x-y$; $u' \equiv x$.) Then $u' = a$, $w_3 = 0$ are the equations of the characteristic curve. If these equations should satisfy the differential equations, $\frac{\partial (u', u, v)}{\partial (x, y, z)} = 0$. But u' is a function of u and w_3, so that it follows that w_3 must be a function of u and v. $w_3 = 0$ cannot therefore give rise to singular solutions of the first order. In other words, the characteristics $u' = a$, $w_3 = 0$ can *at the most* give a set of solutions included in the general solutions. (See Example 11 for an illustration of this extreme case.) In no case can they give singular solutions.

We conclude that Ω is independent of u. *The singular solutions given by* $w_3 = 0$; $y' = \psi_1 (x, y, z)$ *are therefore equivalent to* $w_3 = 0$; $v (x, y, z) = b$. (We assume that $w_3 = 0$ is not included in $v = b$, for any value of b.)

A purely geometrical method of arriving at this result is considered in § 22.

We have assumed that $w_3 = 0$ makes the first derivatives of u only, infinite. If $w_3 = 0$ is also the envelope of the system $v = b$, the above arguments do not hold. In such cases, sometimes it may be possible to obtain a function $\psi (u, v)$, none of whose first partial derivatives becomes infinite when $w_3 = 0$. (*Vide* Example 9.) The singular solutions are then given by $w_3 = 0$; $\psi (u, v) = b$. If, however, it is not possible to find such a function, the actual solution of the differential equations $y' = \psi_1 (x, y, z)$; $w_3 = 0$ cannot be avoided.

[§ *13*]. *Remarks.*

§ 13. The methods of derivation of the singular solutions of the first order have been studied. So far, we have been dealing only with the necessary conditions for singular solutions. Any singular solutions that may exist will be derivable by the foregoing methods, but there is no guarantee that the equations derived from these methods do always actually constitute solutions, whether singular or particular. For the special case wherein the equation $w_1 = 0$ is formed by the elimination of y' and z' between $\phi_1 = 0$, $\phi_2 = 0$, $J = 0$, the third and fourth equations in the theorem quoted in § 10 give the conditions of sufficiency in order that the equations obtained by

‡ It is immaterial whether we use $u = a$, or $\Omega (u, v) =$ constant.

the process described in the theorem might constitute solutions (singular or particular). In other cases it will be perhaps simplest to make a direct verification of the fact whether the differential equations are satisfied. The theorem of § 8 may often be useful in deciding whether the solutions obtained are singular or are included in the general integrals.

[§§ *14-15*]. *On Singular Solutions of the Second Order.*

§ 14. We shall now examine the implication of the existence of singular solutions of the second order.

Consider one of the functions which goes by the name w_3. We shall assume that for this w_3, the equations (15) give rise to singular solutions of the *first* order. Eliminating z from (15), we obtain $y' = \lambda(x, y)$. (§ 11). The singular solution, if any, of this equation, taken along with $w_3 = 0$ will (in general) constitute singular solutions of the second order. If $w_3 = 0$ give on solution for z, $z = \theta(x, y)$, then

$$y' = \lambda(x, y) = \psi_1[x, y, \theta(x, y)].$$

The singular solution, if any exist, of this equation is given either by $\frac{\partial\lambda}{\partial x}$ = infinite, or by $\frac{\partial\lambda}{\partial y}$ = infinite. In most cases, $w_3 = 0$ will be an elementary function whose derivatives are finite for all finite values of the variables. The infinities of $\frac{\partial\lambda}{\partial x}$ and $\frac{\partial\lambda}{\partial y}$ can thus arise only

either (i) through the infinities of $\frac{\partial\psi_1}{\partial x}$, $\frac{\partial\psi_1}{\partial y}$, or $\frac{\partial\psi_1}{\partial z}$

or (ii) on account of the equation $\frac{\partial w_3}{\partial z} = 0$ [for this may cause the derivatives of $\theta(x, y)$ to become infinite].

Case (*i*). Let $w_1(x, y, z) = 0$ be the equation defined by $\frac{\partial\psi_1}{\partial x}$ or $\frac{\partial\psi_1}{\partial y}$ or $\frac{\partial\psi_1}{\partial z}$ = infinite (w_1 being supposed to be distinct from the particular w_3 that we are considering). The condition of sufficiency that $w_1 = 0$; $w_3 = 0$ may constitute singular solutions of the second order for the given equations is the same as the condition that $w_1(x, y, \theta) = 0$ may be a singular solution for the equation $y' = \lambda(x, y)$.* This requires that the equation $y' = \psi_1(x, y, \theta)$ should be satisfied by the value of y' given by

$$\frac{\partial w_1}{\partial x} + \frac{\partial w_1}{\partial y} y' + \frac{\partial w_1}{\partial \theta}\left(\frac{\partial\theta}{\partial x} + \frac{\partial\theta}{\partial y} y'\right) = 0 \text{ in virtue of } w_1(x, y, \theta) = 0.$$

* We suppose that the particular w_3 considered is not a function of u and v. If w_3 be a function of u and v, the above condition is not sufficient to give singular solutions of the second order. In fact, then, $w_3 = 0$; $y' = \psi_1$ do not give rise to *singular* solutions at all, while $w_1 = 0$, $w_3 = 0$ form singular solutions of the *first* order. (See Example 12.)

This is equivalent to the condition $\frac{\partial w_1}{\partial x} + \frac{\partial w_1}{\partial y} \psi_1 + \frac{\partial w_1}{\partial \theta} \left(\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} \psi_1 \right) = 0$ in virtue of $w_1 = 0$.

The equations $w_3 = 0$; $y' = \psi_1 (x, y, z)$ .. .. .. (15) have been supposed to give singular solutions of the first order. They may be replaced by $z = \theta (x, y)$; $y' = \psi_1$. But as the equations (15) and (16) are equivalent, it must be possible to derive the equation $z' = \psi_2 (x, y, z)$ from (15). Therefore $\frac{\partial \theta}{\partial x} + \frac{\partial \theta}{\partial y} \psi_1$ is equivalent to ψ_2. Hence $\frac{\partial w_1}{\partial x} + \frac{\partial w_1}{\partial y} \psi_1 + \frac{\partial w_1}{\partial z} \psi_2 = 0$ in virtue of the equations $w_1 (x, y, z) = 0$; $y' = \psi_1$. But, this is the condition that the equations $w_1 (x, y, z) = 0$; $y' = \psi_1$ may give rise to a set of singular* solutions of the first order.

Thus, in this case, there will be *two different* sets of singular solutions of the first order, one set given by $w_3 = 0$; $y' = \psi_1$, another given by $w_1 = 0$; $y' = \psi_1$. The singular solution of the second order is given by $w_1 = 0$, $w_3 = 0$.

[Conversely, *given* the existence of two different systems of singular solutions of the first order given by $w_3 = 0$; $y' = \psi_1$ and $w_1 = 0$; $y' = \psi_1$, we can usually infer the existence of a set of singular solutions of the second order, *viz.*, $w_3 = 0$, $w_1 = 0$, *but not always*. The equations $\frac{\partial w_3}{\partial x} + \frac{\partial w_3}{\partial y} \psi_1 + \frac{\partial w_3}{\partial z} \psi_2 = 0$, and $\frac{\partial w_1}{\partial x} + \frac{\partial w_1}{\partial y} \psi_1 + \frac{\partial w_1}{\partial z} \psi_2 = 0$ are satisfied in virtue of the equations $w_3 = 0$ and $w_1 = 0$ respectively. Hence the values of y' and z' for the curve of intersection of $w_3 = 0$ and $w_1 = 0$ satisfy the given differential equations. This is however provided that the y' and z' for this curve are *derivable from*

$$\frac{\partial w_3}{\partial x} + \frac{\partial w_3}{\partial y} y' + \frac{\partial w_3}{\partial z} z' = 0$$

$$\text{and } \frac{\partial w_1}{\partial x} + \frac{\partial w_1}{\partial y} y' + \frac{\partial w_1}{\partial z} z' = 0.$$

The exceptional cases are when the two surfaces $w_3 = 0$ and $w_1 = 0$ touch each other throughout their curve of intersection (vide *Example 7*), *or when the curve of intersection is a double line on one of the surfaces.*]

Case (*ii*). The equation $\frac{\partial w_3}{\partial z} = 0$ arises as the equation in virtue of which the solution $z = \theta (x, y)$ obtained by solving $w_3 = 0$ may not be in the

* That these solutions are singular does not necessarily follow from the proof. All that we can say from the arguments is that $w_1 = 0$; $y' = \psi_1$ constitute solutions of some kind. But if these solutions were particular, $w_1 = 0$ cannot associate itself with another equation to give singular solutions of the *second* order.

standard form (continuous with continuous first derivatives). Now, instead of starting with the equation (15), if we had started with (16), the exceptional case arises when $w_3 = 0$ cannot be solved for y in the prescribed form. The necessary condition therefore becomes

$$\text{either (iii)} \quad \frac{\partial \psi_2}{\partial x} \text{ or } \frac{\partial \psi_2}{\partial y} \text{ or } \frac{\partial \psi_2}{\partial z} = \text{infinite}$$

$$\text{or} \qquad \text{(iv)} \quad \frac{\partial w_3}{\partial y} = 0.$$

The condition (iii) leads to a second set of singular solutions of the first order, precisely as in case (i). Otherwise, we have $w_3 = 0$; $\frac{\partial w_3}{\partial z} = 0$; $\frac{\partial w_3}{\partial y} = 0$.

Again, any one of the three variables x, y, z may be taken as independent. Hence, starting with the combination $w_3 = 0$; $\frac{dy}{dz} = \frac{\psi_1}{\psi_2}$, we shall similarly obtain the condition $\frac{\partial w_3}{\partial x} = 0$.

In the absence of two different systems of singular solutions of the first order, we must have therefore $w_3 = 0$; $\frac{\partial w_3}{\partial x} = 0$; $\frac{\partial w_3}{\partial y} = 0$; $\frac{\partial w_3}{\partial z} = 0$; these being equivalent to two independent equations only.

We thus arrive at the following theorem:

THEOREM II. *Given the existence of one curve representing singular solutions of the second order for the given differential equations, it follows that either there will be two different systems of singular solutions of the first order, or otherwise the curve is a double line on the surface* $w_3(x, y, z) = 0$, *which leads to the singular solutions of the first order.*

The surface $w_3 = 0$ can be seen to form the focal surface or part of it, of the congruence of curves represented by the general solutions. When there is only one set of singular solutions of the first order, $w_3 = 0$ forms the entire focal surface. When there are two sets of singular solutions of the first order, the focal surface breaks up into two proper surfaces. The condition in the first footnote in this section implies that the focal surface w_3 is not itself a surface of the congruence.

§15. In connection with the above theorem, it will be proper here to refer to some results that have been obtained from entirely different considerations by A. C. Dixon in his paper (Ref. 2). Dixon confines himself to the Clairaut-equations $y - xy' = f_1(y', z')$; $z - xz' = f_2(y', z')$.

(1) The straight lines comprising the rectilinear congruence $y - ax = f_1(a, b)$; $z - bx = f_2(a, b)$ form bitangents to a surface, *viz.*, the focal surface. The *nodal curve* (but not the cuspidal curve) of this surface will in general

furnish a singular solution of the second order for the differential equations of the congruence.

(2) In special cases, the focal surface of the above rectilinear congruence may become a developable. The edge of regression (which is known to be a locus of unodes) furnishes a singular solution of the second order.

(3) A special case is considered wherein the Clairaut-equations represent the congruence comprising the inflexional tangents to a surface. This falls as a particular case of (1) when the focal points that lie on any ray of the congruence coincide. Dixon proves that if the surface contains a cuspidal curve (*i.e.* a locus of unodes), it furnishes a solution which, he says, is of the second order, and that the parabolic curve may also furnish a solution, if it happens to be a plane curve.

It may be mentioned that in this case, the asymptotic lines of the surface form singular solutions of the first order.* It is well known that the asymptotic lines do not in general possess an envelope, the exception being when the parabolic curve is a plane curve. When the parabolic curve is a plane curve it furnishes a solution of the differential equations. But this will not be a singular solution of the second order, for the parabolic curve is not a double line. In fact, since the parabolic curve in the case considered is the envelope of a singly infinite system of inflexional tangents properly selected out of the congruence, it is included in the singular solutions of the *first* order.

As regards the unodal locus, I do not agree with Prof. Dixon when he says that it represents a singular solution of the second order. A singular solution of the second order must necessarily represent the envelope of the curves given by the singular solutions of the first order, and the curve in question does not possess this property. In fact, in § 45 of Dixon's paper, it is proved that the tangent to the unodal locus at any point is a hyper-inflexional tangent thereat, *i.e.*, a tangent having contact of the third order. Now, when the equations of the congruence of the inflexional tangents are written down, any system or systems of hyper-inflexional tangents are

* The asymptotic lines may not probably exhaust, in every case, all the singular solutions of the first order. When a surface is of degree higher than four, it may happen that a straight line may be an inflexional tangent at one point P on the surface, and an ordinary tangent at another point Q. We might possibly have therefore a curve on the surface every tangent to which touches the surface inflexionally *elsewhere*. Such a curve evidently furnishes a singular solution of the first order (we must remember that the inflexional tangents satisfy the differential equations considered all along their length, and not merely at the points where they touch the surface inflexionally), but is not an asymptotic line.

usually included in them. Hence the unodal locus is the envelope of a particular ∞^1 of inflexional tangents included in the congruence. It provides therefore a singular solution of the *first* order only.

Consider Dixon's Example:

EXAMPLE 4.

The system of lines

$$y = 3a^2 b^4 x + \tfrac{1}{3} b^9 (1 - 6a^3)$$
$$z = b^3 (1 + 2a^3)\, x - \tfrac{3}{2} a^4 b^8$$

forms one system of inflexional tangents on a certain surface of degree twelve. The cuspidal curve is given by $x = 8\,t^5$; $y = \frac{160}{3}\,t^9$; $z = 40\,t^8$. This curve will be found to be the envelope of the ∞^1 lines obtained by putting $2a^3 = 1$ in the above equations, and these lines are the hyper-inflexional tangents at the unodes. The curve is thus included amongst the singular solutions of the *first* order.

[§ *16*]. *Summary of Results.*

§ 16. THEOREM 3. *Let the given differential equations be reduced to the form* $y' = \psi_1\ (x, y, z)$; $z' = \psi_2\ (x, y, z)$, *and let their general solutions be written* $u\ (x, y, z) = a$; $v\ (x, y, z) = b$. *Let* $w\ (x, y, z) = 0$ *denote an equation in virtue of which a first partial derivative of* ψ_1 *or* ψ_2 $\left(\text{or}\,\frac{1}{\psi_1}\ \text{or}\,\frac{1}{\psi_2}\right)$ *as well as a first partial derivative of u or v becomes infinite. The singular solutions (if any exist) of the given equations will be included in the equations*

$$w = 0;\ y' = \psi_1,\ \text{or}\ z' = \psi_2.$$

THEOREM 4. *If when any of the first partial derivatives of u becomes infinite, none of the first partial derivatives of v becomes infinite, and* vice versa, *we must search for singular solutions of the first order from the following equations:*

$$\frac{\partial u}{\partial x}\ \text{or}\ \frac{\partial u}{\partial y}\ \text{or}\ \frac{\partial u}{\partial z} = \text{infinite};\ v\ (x, y, z) = b \qquad \text{(i)}$$

$$\frac{\partial v}{\partial x}\ \text{or}\ \frac{\partial v}{\partial y}\ \text{or}\ \frac{\partial v}{\partial z} = \text{infinite};\ u\ (x, y, z) = a \qquad \text{(ii)}$$

Let the functions defined by the first halves in (i) *and* (ii) *be called* $w_{31} = 0$ *and* $w_{32} = 0$. *Then the singular solutions of the second order (if any exist) will be given by*

$$w_{31} = 0;\ w_{32} = 0 \qquad \text{(iii)}$$

$$\text{or by}\ w_{31} = 0;\ \frac{\partial w_{31}}{\partial x} = 0;\ \frac{\partial w_{31}}{\partial y} = 0;\ \frac{\partial w_{31}}{\partial z} = 0 \qquad \text{(iv)}$$

$$\text{or by}\ w_{32} = 0;\ \frac{\partial w_{32}}{\partial x} = 0;\ \frac{\partial w_{32}}{\partial y} = 0;\ \frac{\partial w_{32}}{\partial z} = 0 \qquad \text{(v)}$$

[§ 17]. *Generalisation.*

§ 17. The methods followed in the previous pages are obviously capable of being extended to equations involving more variables. Only one theorem will be written down here.

THEOREM 5. *Let the general solutions of the system of equations* $y_r' = \Psi_r(x, y_1, y_2, \cdots, y_n)$ *be written* $u_r(x, y_1, \cdots, y_n) = a_r$, $(r = 1, 2, \cdots, n)$. *Let* $w_r = 0$ *denote the equation in virtue of which any of the first partial derivatives of* u_r *becomes infinite. If each of the w's possesses this property with respect to one of the u's only, the singular solutions of the first order must be sought for from systems of equations containing one of the w's and its non-corresponding* $(n-1)$ *u's; the singular solutions of the second order may be obtained from systems of equations containing two of the w's and their non-corresponding* $(n-2)$ *u's, and so on; singular solutions of the* n^{th} *order may be given by all the w's taken together, if these are n in number.*

This process may not exhaust all the singular solutions of the second and higher orders, though those of the first order are obtained exhaustively. I shall not discuss this in detail here.

[§ 18]. *Examples.*

§ 18. EXAMPLE 5.

$$y = xy' + y'^2 + z'; \quad z = xz' + y'z'$$

The singular solutions of the first order, as obtained from the theorem quoted in § 10, are given by

$$\left.\begin{array}{l} w \equiv \{27(z-xy) - 2x(x^2-9y)\}^2 - 4(x^2+3y)^3 = 0 \\ \text{associated with} \quad y = -\tfrac{1}{4}(x+k)^2 + k^2 \end{array}\right\} \quad \ldots \quad (17)$$

while the singular solutions of the second order are given by the double line of $w = 0$, *viz.*, $x^2 + 3y = 0$; $x^3 + 27z = 0$.

The following remarks as regards the function w will be interesting. The general solutions of the given equations, *viz.*, $y = ax + a^2 + b$; $z = bx + ab$ are given as the *partial* intersections of the surfaces

$$a(x+a)^2 - y(x+a) + z = 0 \quad \ldots \quad (18)$$

$$b^3 - b^2y - bzx + z^2 = 0 \quad \ldots \quad (19)$$

Instead of solving for a and b, and finding when the derivatives of the functions $u(x, y, z)$, $v(x, y, z)$ become infinite, we can do the equivalent process of finding the a- and b-discriminants of the equations (18) and (19). *It will be found that the discriminants are identically the same,* both simplifying to the relation $w = 0$. This is not a mere accident, but is generally true whenever we are dealing with equations which are wholly algebraic and rational. It may be proved as follows:—Let a and b be eliminated in turn from the equations $f_r(x, y, z, a, b) = 0$, $(r = 1, 2)$, and let the eliminants be written after rationalisation in the form $u(x, y, z, a) = 0$, $v(x, y, z, b) = 0$. The a-discriminant of $u(x, y, z, a) = 0$ represents the condition that the solution for a MIGHT give a function one or more of whose first partial derivatives becomes infinite.* A similar statement holds for the b-discriminant of $v = 0$. If the result of eliminating a and b from $f_1 = 0$; $f_2 = 0$; $\frac{\partial(f_1, f_2)}{\partial(a, b)} = 0$ be written $w(x, y, z) = 0$, then $w = 0$ denotes the condition in virtue of which the solution for a or for b, or for both, from the equations $f_1 = 0$; $f_2 = 0$ might be a function one of whose first partial derivatives is infinite. Hence either of the following possibilities must be satisfied:

(a) The a-discriminant of u, and the b-discriminant of v must both be the same as w.

(b) Otherwise w can be broken up into two factors, *viz.*, the a-discriminant of u and the b-discriminant of v.†

The condition (b) can naturally be satisfied only in special cases. (An instance is provided by Example 7.) When $w = 0$ is non-degenerate, as in the present example (originally given by Serret), condition (a) must necessarily be satisfied, and there can exist only one set of singular solutions of the first order.

The singular solutions cannot be obtained by using Theorem 4. They have to be obtained by using Theorem 3.

[Forsyth (Ref. 5) writes the singular solutions of the first order in the form

$$y = -\tfrac{1}{4}(x + k)^2 + k^2; \; z = -\tfrac{1}{4}k(x - k)^2 \quad \ldots \quad \ldots \quad (20)$$

—a form simpler than that in (17). In fact, substituting $y = -\frac{1}{4}(x + k)^2 + k^2$ in $w = 0$ and solving for z, we obtain two values one of which is $z = -\frac{1}{4}k(x - k)^2$. The curves (17) are thus equivalent to the curves (20) together with the curves

$$y = -\tfrac{1}{4}(x + k)^2 + k^2; \; 108\, z = 27k^3 - 9\, kx^2 - 2x^3 \quad \ldots \quad (21)$$

---

* This is under the supposition that the system of surfaces $u = 0$ does not possess a surface-locus of double points. It is however a known fact that if $u = 0$, or $v = 0$ possesses such a surface-locus of double points, this surface is included as an extraneous factor in the focal surface $w = 0$ of the congruence $f_1 = 0$, $f_2 = 0$. We do not, again, take into consideration any particular surface of the system $u = 0$ that might appear in the a-discriminant.

† In both cases, it is clear that the focal surface of the given congruence $f_1 = 0$, $f_2 = 0$ is identical with that of the derived congruence $u = 0$, $v = 0$, although the "surfaces of the two congruences" are different.

The curves (21) are not at all solutions of the differential equations considered. Thus, when we take the three equations $\phi_1 = 0$; $\phi_2 = 0$; $\frac{\partial(\phi_1, \phi_2)}{\partial(y', z')} = 0$ (§ 10) only, omitting the two other equations in the theorem quoted in § 10, we obtain, in the present example, a singly infinite number of extraneous curves in addition to the singular solutions. The congruence does not possess any multiple points or tac-points. The occurrence of the extraneous system in spite of this fact is a noticeable thing. *The extraneous curves are those that satisfy* (15) *but not* (16).]

EXAMPLE 6.

To find the focal surface of the congruence

$$y = ax + a^2 + ab + b\,;\ z = bx + ab + b^2.$$

This example is taken here merely to illustrate the use of the remarks made in the previous example. The usual method, *viz.*, to eliminate a and b between the given equations and their Jacobian w.r.t. a and b may be replaced by the simpler but equivalent method of eliminating, say a between the given equations, and then finding the b-discriminant of the resulting equation.

EXAMPLE 7.

The congruence

$$y = ax + a^2 + ab + b\,;\ z = bx + b^2.$$

The focal surface $w = 0$ breaks up into two distinct surfaces

$$w_a \equiv 4x^3 + 16xy + 4x^2y + 16y^2 + 8yz + z^2 + 12xz - 16z = 0$$

and $w_b \equiv x^2 + 4z = 0.$

Eliminating b, the congruence may be considered as *included* in

$$z(a+1)^2 = (y - ax - a^2)\ (x + y - a^2)$$
$$z = bx + b^2.$$

The differential equations

$$y = xy' + y'^2 + y'z' + z'\,;\ z = xz' + z'^2$$

admit of two sets of singular solutions of the first order. One set is *included* in the equations $w_a = 0$; $z = bx + b^2$. The actual equations are $y = -\frac{1}{4}(x+b)^2 + b$; $z = bx + b^2$. We might expect the other set to be included in $w_b = 0$; $z(a+1)^2 = (y - ax - a^2)\ (x + y - a^2)$. *But this is not so.* The surface $w_b = 0$ is found to be the common envelope of the two systems $z = bx + b^2$, and $z(a+1)^2 = (y - ax - a^2)(x + y - a^2)$. The second set of singular solutions of the first order has therefore to be found out by actually solving the equations

$$x^2 + 4z = 0\,;\ y = \tfrac{1}{2}xy' + y'^2 - \frac{x}{2}.$$

$w_a = 0$ does not possess a double line, and since the equation $y = \frac{1}{2}xy' + y'^2 - \frac{x}{2}$ does not possess a singular solution, the given equations have no singular solutions of the second order. The reason why $w_a = 0$; $w_b = 0$ do not constitute solutions is that they touch each other along the entire curve $x^2+8x+16y = 0$; $x^2+4z = 0$. (*Vide* § 14.)

[These geometrical peculiarities are found to be generally true for any congruence of the type

$$y = ax + f_1(a, b) \qquad \ldots \qquad \ldots \qquad \ldots \quad (22)$$

$$z = bx + f_2(b) \qquad \ldots \qquad \ldots \qquad \ldots \quad (23)$$

There will be two distinct focal surfaces, one of which is the envelope of the system (23). Let (23) be solved for b in the form $v(x, y, z) = b$. Eliminating b between (22) and (23), we have

$$y - ax = f_1(a, v) \qquad \ldots \qquad \ldots \qquad \ldots \quad (24)$$

It now follows* that the envelope of the system (23), say E_1 is also an envelope of the system (24). The system (24) possesses another envelope, say E_2, which will constitute the other focal surface of the congruence.

Now E_1 touches the surface $y-ax = f_1(a, v)$ for any given value of a, all along the curve of intersection; so does E_2. Hence, at the point of intersection of the three surfaces E_1, E_2, and $y-ax = f_1(a, v)$, they have a common tangent plane. Therefore, *the two focal surfaces of the congruence* (22), (23) *touch each other all along their curve of intersection.*]

EXAMPLE 8.

$$\frac{dx}{3+2(z+x+2y)^{\frac{1}{2}}} = \frac{dy}{2(z-x-y)^{\frac{1}{2}}-2}$$

$$= \frac{dz}{1+2(z+x+2y)^{\frac{1}{2}}-4(z-x-y)^{\frac{1}{2}}-4(z-x-y)^{\frac{1}{2}}(z+x+2y)^{\frac{1}{2}}}$$

(M. J. M. Hill.)

The general solutions are

$$u \equiv x+(z-x-y)^{\frac{1}{2}} = a; \quad v \equiv y+(z+x+2y)^{\frac{1}{2}} = b.$$

There are two sets of singular solutions of the first order, *viz.*, $z = x+y$; $v = b$, and $z+x+2y = 0$; $u = a$. The singular solutions of the second order are given by $z = x+y$; $z+x+2y = 0$.

EXAMPLE 9.

Consider the equations of Example 2. Theorem 4 cannot be used, since $z = x+y$ makes the first derivatives of both u and v infinite. Theorem 3 readily gives the singular solutions of the first order in the form $z = x+y$; $x^2-y^2 = c$.

In this example, however, it is possible to find a function $\theta(u, v)$ which is free from irrationals. If we take the general solutions as included in

$$u = a; \quad v' \equiv u^2-v^2 = (x^2-y^2)(1-\overline{z-x-y}) = c,$$

* *Vide Tohoku Mathematical Journal*, **39**, 90-91.

the singular solutions can be obtained by Theorem 4 as included in $z = x+y$; $v' = c$.

EXAMPLE 10.

Consider the differential equations having for their general solutions

$$u \equiv x+y\,[\,\log (z-x-y)\,]^{-1} = a$$

$$v \equiv y+x\,[\,\log (z-x-y)\,]^{-1} = b \qquad \text{(M. J. M. Hill.)}$$

The singular solutions of the first order, *viz.*, $z = x+y$; $x^2-y^2 = c$ may be obtained by using Theorem 3. In this case, $\theta(u, v)$ cannot be found so as to be free from the transcendental expression.

EXAMPLE 11.

In example 3, we obtain the solutions $z-x-y = 0$; $x =$ constant, by using Theorem 3. We have seen that these are not singular solutions. They form the characteristic curves on the envelope of either of the systems $u = a$, and $v = b$. (See § 12.)

EXAMPLE 12.

$$\frac{y'^3}{3} + \frac{y'^2 x}{2} - (y'+1)\,y - z = 0\,; \quad y' + \frac{y'^2}{2} + z' = 0.$$

This example, given by Mayer (Ref. 4) has the peculiarity that it possesses a single singular solution of the first order, *viz.*, $y = -\frac{x^2}{4}$; $z = \frac{x^2}{4} - \frac{x^3}{24}$. The reason is that the focal surface of the congruence defined by the general primitive happens to be a particular surface of the congruence. The singular solution represents a double line on the focal surface. The double line would have given a singular solution of the *second* order, but for the above peculiarity about the focal surface.

[§§ *19-24*]. *The Geometry of the Singular Solutions.*

§ 19. Let us first consider the case where the two systems $u = a$ and $v = b$ do not possess a common envelope.

A necessary condition that the surface $w(x, y, z) = 0$ should be an envelope of the system $u(x, y, z) = a$ is that one of the first partial derivatives of u becomes infinite when $w = 0$. In fact, this condition is in most cases sufficient.* We shall however precisely write down the conditions of sufficiency in the form

$$\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial z}} = \underset{w \to 0}{\text{Lt}} \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial z}} \text{ and } \frac{\frac{\partial w}{\partial y}}{\frac{\partial w}{\partial z}} = \underset{w \to 0}{\text{Lt}} \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial z}} \qquad \ldots \qquad \ldots \quad (25)$$

* *Vide Tohoku Math. Journal*, Vol. 39.

Sufficient conditions that $w = 0$; $v = b$ may provide solutions of the given differential equations are that

$$\frac{\dfrac{\partial (w, v)}{\partial (z, x)}}{\dfrac{\partial (w, v)}{\partial (y, z)}} = \underset{w \to 0}{\text{Lt}} \left\{ \frac{\dfrac{\partial (u, v)}{\partial (z, x)}}{\dfrac{\partial (u, v)}{\partial (y, z)}} \right\} \quad \text{and} \quad \frac{\dfrac{\partial (w, v)}{\partial (x, y)}}{\dfrac{\partial (w, v)}{\partial (y, z)}} = \underset{w \to 0}{\text{Lt}} \left\{ \frac{\dfrac{\partial (u, v)}{\partial (x, y)}}{\dfrac{\partial (u, v)}{\partial (y, z)}} \right\} \qquad (26)$$

The two sets of conditions (25) and (26) are equivalent.

Thus,

$$\underset{w \to 0}{\text{Lt}} \frac{\dfrac{\partial (u, v)}{\partial (z, x)}}{\dfrac{\partial u}{\partial z} \dfrac{\partial v}{\partial z}} = \frac{\dfrac{\partial v}{\partial x}}{\dfrac{\partial v}{\partial z}} - \underset{w \to 0}{\text{Lt}} \frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial z}} = \frac{\dfrac{\partial (w, v)}{\partial (z, x)}}{\dfrac{\partial w}{\partial z} \dfrac{\partial v}{\partial z}}, \text{ using (25).}$$

Similarly,

$$\underset{w \to 0}{\text{Lt}} \frac{\dfrac{\partial (u, v)}{\partial (y, z)}}{\dfrac{\partial u}{\partial z} \dfrac{\partial v}{\partial z}} = \frac{\dfrac{\partial (w, v)}{\partial (y, z)}}{\dfrac{\partial w}{\partial z} \dfrac{\partial v}{\partial z}}$$

Hence, by division, we obtain one of the conditions in (26). The other condition is similarly obtained. The algebra may be reversed with slight manipulations.

The conditions of sufficiency that $w = 0$ might envelope the system $u = a$ are thus the same as the conditions of sufficiency that $w = 0$; $v = b$ might be solutions of the given differential equations, *w being supposed to be not a function of u and v.*

When the singular solutions of the given differential equations are obtainable from Theorem 4, *i.e.*, when the two systems $u = a$ and $v = b$ do not have a common envelope, we have therefore the following theorem:

THEOREM 6. *The singular solutions of the first order will be given either as the curves of intersections of the surfaces $v = b$ with the envelope of the system $u = a$, or as the curves of intersection of the surfaces $u = a$ with the envelope of the system $v = b$.*

The intersection of the two envelopes usually provides a singular solution of the second order. (*Vide* § 14 for exceptions.) We have assumed that the envelopes are not included in the surfaces of the congruence.

§ 20. The following property of the focal surface will be discussed in a separate note elsewhere.

For the congruence $F_r (x, y, z, a, b) = 0$, $(r = 1, 2)$, there exists in general a ∞^1 surfaces of the congruence which do not necessarily touch the focal surface. These surfaces are determined by the equation $\lambda \left(a, b, \dfrac{db}{da}\right) = 0$, which is the result of eliminating x, y, z from

$$F_1 = 0 \,;\, F_2 = 0 \,;\, \frac{\partial F_1}{\partial a} + \frac{\partial F_1}{\partial b} \frac{db}{da} = 0 \,;\, \frac{\partial F_2}{\partial a} + \frac{\partial F_2}{\partial b} \frac{db}{da} = 0.$$

For the congruence $u = a$, $v = b$ where the two systems $u = a$ and $v = b$ do not possess a common envelope, the surfaces $v = b$ are to be regarded as surfaces of the congruence which do not touch the focal surface defined by the envelope of $u = a$, except perhaps for special values of b.

If, however, the systems $u = a$ and $v = b$ possess a common envelope, this envelope touches in general (excepting perhaps a finite number of surfaces) all surfaces of the congruence.

For the case considered in § 19, we have therefore the following theorem :

THEOREM 7. *The singular solutions of the first order are the curves of intersection of the focal surface with those surfaces of the congruence which do not touch the focal surface.* When there are two or more focal surfaces, each of these must be considered in turn.

§ 21. Let us now *start* with a given congruence of curves $F_r(x, y, z, a, b) = 0$, $(r = 1, 2)$ where the F's are algebraic and rational, and consider the problem of obtaining *directly* the singular solutions of the corresponding differential equations. We cannot reduce this problem so as to be able to use Theorem 4, for if $F_1 = 0$ and $F_2 = 0$ are put in the form $u = a$ and $v = b$, we have seen while discussing Example 5, that as a rule, the two systems $u = a$ and $v = b$ will have a common envelope.

The following will be a geometrical method from first principles:

Let P be any point on the curve of intersection of the surface $F_1[(x, y, z, a, \phi(a)] = 0$ for an assigned value of a with the surface E_1 enveloping this system, [$b = \phi(a)$ being a pre-assigned relation between b and a]. These two surfaces will have the same normal PN_1 at P. If P also happens to lie on the curve of intersection of the surface $F_2[x, y, z, a, \phi(a)] = 0$ *for the same value of a* with the envelope E_2 of this system, and if PN_2 be the common normal at P, then PN_1 and PN_2 may be regarded as normals either to E_1 and E_2, or to the surfaces $F_1 = 0$, $F_2 = 0$. But the intersection of the latter surfaces is a particular curve of the congruence, and satisfies the differential equations of the congruence. Hence the curve (E_1, E_2) also satisfies these equations at P. The point P being common to four surfaces should satisfy the four equations

$$\left.\begin{aligned} F_1(x, y, z, a, b) = 0\,; \quad \frac{\partial F_1}{\partial a} + \frac{\partial F_1}{\partial b}\frac{db}{da} = 0 \\ F_2(x, y, z, a, b) = 0\,; \quad \frac{\partial F_2}{\partial a} + \frac{\partial F_2}{\partial b}\frac{db}{da} = 0 \end{aligned}\right\} \quad \cdots \quad \cdots \quad (27)$$

The curve (E_1, E_2) is the same as the locus of P. Eliminating x, y and z from (27), we obtain a relation $\theta\left(a, b, \frac{db}{da}\right) = 0$. The relation $b = \phi(a)$

should be taken so as to satisfy this equation. Let the general solution of $\theta = 0$ be $b = \lambda(a, k)$ where k is an arbitrary constant. Substituting and eliminating a, we obtain the equations of the curve (E_1, E_2). The equations involve an arbitrary constant k, and give the singular solutions of the first order. *Extraneous loci may however also occur.* If $\theta = 0$ admits of a singular solution, this will lead to the singular solutions of the second order.

The equations (27) have been deduced from analytical considerations in Forsyth (Ref. 5) and in Dixon (Ref. 2). Another method is given in the next article.

§ 22. The four equations (27) are the same as those that determine the surfaces of the congruence which may not touch the focal surface. (§ 20.) This proves that *Theorem 7 continues to hold good, in the main, for congruences of the type considered in* § 21.

There is another theorem which holds good for both the cases of §§ 19 and 21, *viz.*:

THEOREM 8. We can select singly infinite systems of curves belonging to the congruence, so as to possess an enveloping curve. *Any such enveloping curve will give a singular (in general) solution of the first order. If the curves representing singular solutions of the first order possess an envelope of their own, this envelope will represent in general a singular solution of the second order.*

This theorem is well known and will be found mentioned in the works of Goursat and Dixon. When applied to the congruence of § 19, it gives an alternate proof of Theorem 4, and when applied to the congruence of § 21, it at once gives the equations (27).

§ 23. Although Theorems 7 and 8 hold good for the congruences of §§ 19 and 21, *they do not remain true for the congruence $u=a$, $v=b$ wherein the two systems $u=a$ and $v=b$ possess a common envelope.* In this case, the so-called focal points on any curve, *i.e.*, the points of intersection of the three surfaces $u=a$, $v=b$, and the focal surface are points where the two surfaces $u=a$ and $v=b$ touch each other. They are therefore double points on the curve. Thus in Examples 9 and 10, the singular solutions $z=x+y$; $x^2-y^2=c$ are to be regarded as the loci of double points of singly infinite systems of curves of the congruence, rather than as their envelopes.

[The congruence $F_r(x, y, z, a, b)=0$, $r=1, 2$ can in general be considered as forming *part* of the congruence $u=a$, $v=b$ (obtained by solving for a and b) where the systems $u = a$ and $v = b$ have a common envelope. What is stated here is that if the general solutions of the differential equations are *completely* represented by $u = a$, $v = b$ where the two systems of surfaces have the same envelope, then Theorem 7 and the first part of Theorem 8 are not true.]

§ 24. Theorem 2 taken with the second part of Theorem 8 gives the following geometrical property :

If the congruence $F_r(x, y, z, a, b) = 0$, $(r = 1, 2)$ *possess a non-degenerate focal surface, and if the envelopes of singly infinite systems of curves selected out of the congruence possess an envelope of their own, the latter will be a double line on the focal surface.*

[§§ *25-26*]. *The Nodal Locus.*

§ 25. It was mentioned in § 21 that extraneous loci may sometimes occur, which do not constitute singular solutions. We shall not attempt here a study into the nature of the possible extraneous loci. Forsyth seems to believe that the nodal locus is always one such factor, for he writes (Ref. 5, p. 180), "the equation $J \equiv \frac{\partial (F_1, F_2)}{\partial (a, b)} = 0$ is satisfied at every node." This is not always true. If we consider a non-degenerate skew curve which is the *complete* intersection of two proper surfaces, there are two ways in which a node can arise. A double point on either surface is also a double point on the curve of intersection. A second type of node is defined by the theorem: "If two surfaces touch, the point of contact is a double point on their curve of intersection."* If now we consider the congruence $F_r(x, y, z, a, b) = 0$, $(r = 1, 2)$, it is easy to see, by constructing examples, that a node of either of the types on any curve of the congruence does not necessarily satisfy $\frac{\partial (F_1, F_2)}{\partial (a, b)} = 0$.

§ 26. The following example (Ref. 5, Art. 208) is instructive and illustrates a very different aspect.

Example 13.

$$F_1 \equiv x^2 + y^2 + z^2 - a^2 = 0\,; \quad F_2 \equiv ax^2 + by^2 + 2abxy - a^3 = 0.$$

The equations (27) give the extraneous locus $y = 0$, $z = 0$.

This is a nodal locus of the curves, but not of either of the types described above. When we put $b = 0$, the curves of the congruence reduce to $x^2 - a^2 = 0$; $y^2 + z^2 = 0$. The curve $x = a$; $y^2 + z^2 = 0$ consists of two lines meeting at $(a, 0, 0)$. The line $y = 0$, $z = 0$ is thus obtained as the locus of the nodes of degenerate curves of the system. The appearance of this type of a nodal locus as an extraneous factor while dealing with the equations (27) requires explanation. My explanation is that its occurrence must be accounted for by the fact that the line $y = 0$, $z = 0$ meets every curve of the congruence.

* Salmon, *Analytical Geometry of Three Dimensions*, § 203.

Through any point in space there pass in general a finite number of curves of the congruence $F_1 = 0$, $F_2 = 0$. If, however, (x_1, y_1, z_1) be a point through which pass an infinite number of curves of the congruence, then the equations $F_1 = 0$, $F_2 = 0$ cannot give definite solutions for a and b when $x = x_1$ $y = y_1$, $z = z_1$. Hence, by the theory of implicit functions, we must have $\frac{\partial (F_1, F_2)}{\partial (a, b)} = 0$ at (x_1, y_1, z_1). At such a point, the four equations (27) reduce to three only. If there be a locus of such points, this locus will figure in the process described in § 21.*

A curve which meets every curve of the congruence constitutes as a rule a singular solution of the first order for the differential equations of the congruence. For through every point on the curve there pass a ∞^1 curves of the congruence, and there will usually be one curve of the congruence which touches the given curve at the point. Hence the given curve is the envelope of a properly chosen ∞^1 of curves of the congruence. It provides therefore a singular solution of the first order.

Example 13 presents an exception to this general statement. Consider any point $(a, 0, 0)$. The surface $F_1 = 0$ is independent of b and the line $y = 0$, $z = 0$ does not happen to lie in the tangent plane to $F_1 = 0$ at $(a, 0, 0)$. It is therefore impossible to find out the value of b such that the corresponding curve of the congruence has $y = 0$, $z = 0$ as the tangent at $(a, 0, 0)$.

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* This is approximately equivalent to the following well-known result :—" If a given curve meets every curve of a congruence, the given curve lies on the focal surface." *This theorem is however not necessarily true*, unless the focal surface is blindly defined so as to include *any* surface obtained by eliminating a and b from $F_1 = 0$, $F_2 = 0$, $\frac{\partial (F_1, F_2)}{\partial (a, b)} = 0$, irrespective of any geometrical properties that the resulting surface may or may not possess. For example, the curve $r_1 = 0$, $r_2 = 0$ meets every curve of the congruence $u \equiv \frac{r_1}{r_2} = a$; $v = b$ where r_1 and r_2 are rational integral functions of x, y, z. The envelope of the system $v = b$ constitutes the sole focal surface, but this does not necessarily contain the curve $r_1 = 0$, $r_2 = 0$.