SOME PROPERTIES OF RECTILINEAR CONGRUENCES

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1. Among the ruled surfaces passing through a given ray of a rectilinear congruence, there are two ruled surfaces characterized by the property that the parameter of distribution is a maximum or a minimum. These surfaces have been called *mean ruled surfaces* ('surfaces of distribution' by some writers). The central point of the ray with respect to either of these surfaces coincides with the middle point of the ray, the corresponding central planes are at right angles, being in fact the bisectors of the angles between the principal planes, and the sum of the parameters of distribution for the two mean surfaces is twice the parameter of distribution for either of the principal surfaces.[1]

The object of this paper is to obtain a neat generalization of these and of some other results.

Taking any direction dv/du on the director surface, the distance from the director surface of the central point of the surface corresponding to the direction chosen is given by

$$r = -\frac{a\,du^2 + (b+b')\,du\,dv + c\,dv^2}{e\,du^2 + 2f\,du\,dv + g\,dv^2}, \tag{1}$$

the notation being the same as in Weatherburn. If we choose the middle surface of the congruence as the director surface, and if the parameters are chosen so that the principal surfaces correspond to the parametric curves, we have

$$f = 0, \quad b + b' = 0, \quad ag + ce = 0. \tag{2}$$

The equation (1) can now be written

$$(a + er)\,du^2 + (c + gr)\,dv^2 = 0. \tag{3}$$

[1] *Vide* Weatherburn, *Differential Geometry*, Vol. I, Art. 99, where these and other properties referred to in this paper are proved.

The parameter of distribution corresponding to a given dv/du, referred to the above choice of parameters, is given by

$$p=\frac{b}{(eg)^{\frac{1}{2}}}+2a\left(\frac{g}{e}\right)^{\frac{1}{2}}\frac{m}{e+gm^2}, \tag{4}$$

where $\quad m=dv/du.$

For a given value of r on a given ray, equation (3) gives two values of m, say m_1 and m_2. Let the corresponding parameters of distribution be called p_1 and p_2. Then we have,

$$p_1+p_2-\frac{2b}{(eg)^{\frac{1}{2}}}=2a\left(\frac{g}{e}\right)^{\frac{1}{2}}\left(\frac{m_1}{e+gm_1^2}+\frac{m_2}{e+gm_2^2}\right)$$

$$=0, \quad \text{since } m_1+m_2=0.$$

Hence, $\quad p_1+p_2=\frac{2b}{(eg)^{\frac{1}{2}}}=\text{constant}.$

If ϕ is the angle between the central plane for the surface corresponding to a given dv/du and the central plane for one of the mean surfaces, the parameter of distribution p for the surface is

$$p=\beta_1\cos^2\phi+\beta_2\sin^2\phi \tag{5}$$

where β_1 and β_2 are the parameters of distribution for the mean surfaces. Hence if

$$p_1=\beta_1\cos^2\phi+\beta_2\sin^2\phi,$$

we get

$$p_2=\beta_1\sin^2\phi+\beta_2\cos^2\phi.$$

Hence if ϕ is the angle corresponding to the value p_1, the angle corresponding to p_2 must be $90^\circ\pm\phi$. But we can discriminate between these values and reject $90^\circ+\phi$, by using the result[2]

$$r=\frac{\beta_2-\beta_1}{2}\sin 2\phi, \tag{6}$$

for if we write $90^\circ+\phi$ in place of ϕ, the sign of r is reversed, *i.e.*, we get a point on the other side of the middle point at the same distance.

We collect together these results as follows:

THEOREM.[3] *There are two surfaces through a given ray which have their central point on the ray at a given point. The sum of the parameters of*

[2] *Vide* Blaschke, *Differentialgeometrie*, Band 1, p. 208.

[3] The first part of the theorem was first proved by Delgleize (*Memoires de la Societe Royale des Sciences de Liege*, 1931, **16**, 1-9), and Ram Behari has proved the same by a somewhat different procedure, *Journ. Indian Math. Soc.*, (2), 1936, **2**, 92.

distribution for these two surfaces is a constant, as the point is varied. If the central plane for one of the mean surfaces makes an angle ϕ with the central plane for one of these two surfaces, it makes the angle $90° - \phi$ with the central plane for the other.

If the point is taken at either limit point, the two surfaces coincide, and hence the above-mentioned angle becomes 45° for a principal surface If the point is taken at either focal point, one of the surfaces is a developable, so that at such a point p_1 is zero while $p_2 = \beta_1 + \beta_2$.

2. The elimination of m between (3) and (4) leads after simplification, and the use of the equation $ag + ce = 0$, in (2), to

$$p - \frac{b}{(eg)^{\frac{1}{2}}} = \pm \left(\frac{a^2}{e^2} - r^2\right)^{\frac{1}{2}}.$$

We enunciate this as follows:

THEOREM. *If we take two points A and B equidistant from the middle point of a ray on either side, the parameters of distribution for the pair of surfaces which have their central point at A are equal to those for the surfaces which have their central point at B.*

3. *Some formulae.*

We easily obtain

$$\tan \phi = \frac{\sqrt{e} - \sqrt{g}\, m}{\sqrt{e} + \sqrt{g}\, m},$$

whence from (4), we get

$$p - \frac{b}{\sqrt{eg}} = \frac{a}{e} \cos 2\phi$$

i.e.,

$$p - \frac{\beta_1 + \beta_2}{2} = \frac{\beta_1 - \beta_2}{2} \cos 2\phi.$$

From this and from (6), we get

$$r = \pm \left(p - \frac{\beta_1 + \beta_2}{2}\right) \tan 2\phi. \qquad (7)$$

At a focal point, one of the values of p is zero, and let the corresponding ϕ be denoted by α. Then we know that

$$\tan \alpha = \pm \sqrt{\frac{-\beta_1}{\beta_2}}. \qquad (8)$$

Using (8) in either (7) or (6), *we get the distance of a focal point from the middle point as numerically equal to* $\sqrt{-\beta_1 \beta_2}$. (9)

The result given by (8) can be generalized in two ways.[4]

(*a*) We have

$$\frac{p_1}{p_2}=\frac{\beta_1 \cos^2\phi+\beta_2 \sin^2\phi}{\beta_1 \sin^2\phi+\beta_2 \cos^2\phi}.$$

Hence

$$\tan^2\phi=\frac{\beta_1 p_2-p_1\beta_2}{p_1\beta_1-p_2\beta_2}.$$

Using $\beta_2=p_1+p_2-\beta_1$, we obtain

$$\tan^2\phi=\frac{\beta_1-p_1}{\beta_1-p_2}=\frac{\beta_2-p_2}{\beta_2-p_1}. \tag{10}$$

The equation (8) is a particular case of (10) wherein $p_1=0$, $p_2=\beta_1+\beta_2$.

(*b*) Another generalization is the following equation, which is easily verified:

$$-\frac{p_1}{p_2}=\tan(\alpha-\phi)\tan(\alpha+\phi) \tag{11}$$

where α is given by (8).

Next, using (10) in (6), we obtain

$$\begin{aligned} r &= \pm\sqrt{(\beta_1-p_1)(\beta_1-p_2)} \\ &= \pm\sqrt{(\beta_2-p_1)(\beta_2-p_2)} \end{aligned} \tag{12}$$

which is a generalization of (9).

Also, (10) and (12) give

$$r\tan\phi=\beta_1-p_1=p_2-\beta_2. \tag{13}$$

Finally, if O, F, L denote the middle point, a focal point, and a limit point respectively, we have

$$\begin{aligned} \mathrm{OL}=\pm\frac{a}{e},\quad \mathrm{OF}^2 &= -\beta_1\beta_2 \\ &= \frac{a^2}{e^2}-\left(\frac{\beta_1+\beta_2}{2}\right)^2. \end{aligned}$$

Using these in the equation of § 2, we get

$$p=(\mathrm{OL}^2-\mathrm{OF}^2)^{\frac{1}{2}}\pm(\mathrm{OL}^2-\mathrm{OP}^2)^{\frac{1}{2}} \tag{14}$$

where P is any point on the ray, and where $(\mathrm{OL}^2-\mathrm{OF}^2)^{\frac{1}{2}}$ has a fixed sign, *viz.*, the sign of b.

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[4] One other generalization is given by K. Ogura, *vide* Ram Behari's paper in *Journ. Indian Math. Soc.*, 1938, **3**, 110. Ogura's paper is not accessible to me.