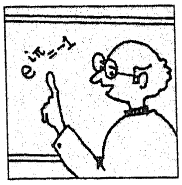
# Classroom



*In this section of* **Resonance**, *we invite readers to pose questions likely to be raised in a classroom situation. We may suggest strategies for dealing with them, or invite responses, or both. "Classroom" is equally a forum for raising broader issues and sharing personal experiences and viewpoints on matters related to teaching and learning science.*

## Are All Matrices Diagonalisable?

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The author's version of the answer is given below.

The author has frequently used the above question to gauge the solidity of the mathematical training of another person. The incorrect responses vary between:

An unhesitating "Yes".
An unhesitating "No" with no counter-example forthcoming.
A "Yes" after some thought.
"Yes, over the field of complex numbers".

The author holds the opinion that Linear Algebra forms the heart of modern mathematics. It is thus disappointing to see the number of students confused about such an important issue. This note is an attempt to show the ramifications of this question and provide a detailed answer. In what follows, $F$ is either the field of real numbers or the field of complex numbers (or more generally any perfect field, e.g. a finite field).

To begin with a matrix $A$ with entries from $F$ is said to be *diagonalisable* over $F$ if it is a square matrix and there is an invertible matrix $B$ with entries from $F$ such that $B^{-1}AB$ is a diagonal matrix. So we need to restrict ourselves only to square matrices for the purposes of our discussion. Now it is clear that if

$\{e_1, \ldots, e_n\}$ is the standard basis for vector space $F^n$ on which $A$ acts, then $\{Be_1, \ldots, Be_n\}$ is another basis with the property that each of the basis vectors is an *eigenvector* and the diagonal entries of $B^{-1}AB$ are just the *eigenvalues* of $A$. Thus an equivalent definition of diagonalisability is that there be a basisconsisting of eigenvectors.

If $A$ is a symmetric matrix over $\mathcal{R}$ then one can show that there is an orthogonal basis of eigenvectors and thus we obtain the diagonalisability of symmetric matrices. One can argue similarly for some other classes of matrices. This is perhaps what people who give the third answer are thinking of. However, there is an important class of matrices of which none except the zero matrix is diagonalisable (see below).

Now it is clear that a matrix like $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ cannot have eigenvalues over the field of real numbers; hence it is not diagonalisable over the real numbers. However it *is* diagonalisable over the field of complex numbers with eigenvectors $(1,0) \pm \sqrt{-1} \cdot (0, 1)$. One might be tempted to think that the problem dis-appears if we consider the field of complex numbers! However, this is not the case. From now on let us consider matrices only over the complex field. From the second definition of diagonalisability and the usual definition of eigenvectors we know that we should be looking at the roots of the characteristic polynomial $P_A(t) = \det(A - tI)$. Since we are working over the field of complex numbers we do have all roots of $P_A(t)$. Thus we have eigen-vectors corresponding to every eigenvalue. This is perhaps the reasoning of people who give the fourth answer. However, the problem is that if an eigenvalue occurs with multiplicity greater than one it is not clear that we can find two or more (linearly independent) eigenvectors.

Of course the correct answer is "No" because of the phenonomenon of nilpotent matrices. A matrix such as $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has 0 as its only eigenvalue but is not the zero matrix and thus it is not diagonalisable. It is clear that if $N$ is a nilpotent matrix (i.e. $N^k=0$ for some $k$) then it is diagonalisable if and only if $N=0$. In fact, the role of nilpotent matrices is made more precise by the *Jordan decomposition theorem* which says that any matrix $A$ over the real numbers (or more generally over a perfect field) can be written as $A=S+N$ where $S$ and $N$ commute, $N$ is nilpotent and $S$ is diagonalisable over the complex numbers (respectively over an algebraic extension of the ground field).

Now we can approach the question from a different viewpoint and ask for conditions which ensure that the nilpotent matrix $N$ is zero. One way to do this is to ensure that $P_A(t)$ has distinct roots. In this case we clearly have a basis of eigenvectors. Let $f(t)=t^n+c_1t^{n-1}+\ldots+c_n$ be any polynomial. There is a (universal) nonzero polynomial $D_n(C_1,\ldots,C_n)$ called the discriminant with the property that $D_n(c_1,\ldots,c_n)=0$ if and only if $f(t)$ has multiple roots (e.g. for $n=2$ we have $D_2=C_1^2-4C_2$). Now, the coefficients of the characteristic polynomial of a matrix $A$ are polynomials in the matrix entries of $A$.

Thus if $a_1, a_2, \ldots, a_n$ are the coefficients of the characteristic polynomial $P_A(t)$ of $A$, $P_A(t)$ has multiple roots if and only if $D_n(a_1,a_2,\ldots,a_n)=0$. Now, the zero set of a non-zero polynomial is always a 'thin set' in the sense of measure theory. In other words, if we bound the matrix entries of $A$ and choose them randomly i.e. uniformly (or in fact from any distribution with a continuous density) within these bounds then *with probability 1* we will find a diagonalisable matrix. Thus my own answer to the question posed above is two-fold:

- Every matrix is not diagonalisable. Take for example non- zero nilpotent matrices. The Jordan decomposition tells us how close a given matrix can come to diagonalisability.
- If we choose our matrix 'randomly' (in a uniform distribution) from within a bounded region, then it will turn out to be diagonalisable over the complex numbers with probability 1 (in other words 'almost always' in the sense of probability).