Geometry VI

6. Space – the Final Frontier

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In this concluding article of the series the author gives a brief overview of some modern aspects of geometry—Topology and Algebraic Geometry. The subject is now so vast that it is best to learn about it from the excellent books available—some of which are suggested at the end.

Holes in the Whole

The studies of Gauss and Riemann described in earlier articles can be classified as *local* geometry. Some geometrical attributes such as 'holes' or equivalently the notion of an object 'closing upon itself' are *global*. The first mathematical measure of this kind was described by Euler.

A (convex) *polyhedron* is a finite body bounded by planes so that any point lying between two points of the polyhedron also lies in the polyhedron. The boundary of a polyhedron then consists of a number F of (plane) polygonal faces which meet in edges (linear segments) that are E in number. The edges terminate in vertices (points) that are V in number. The numbers V, E, F differ according to the polyhedron we choose, but as Euler proved we always have the identity

$$V - E + F = 2$$

In order to understand the nature of this statement let us look at the similar statement for polygons (which are the two dimensional analogues of polyhedra). The boundary of a polygon has say E edges and V vertices where we have $V - E = 0$ (Exercise). Euler's proof is entirely analogous and quite elementary.

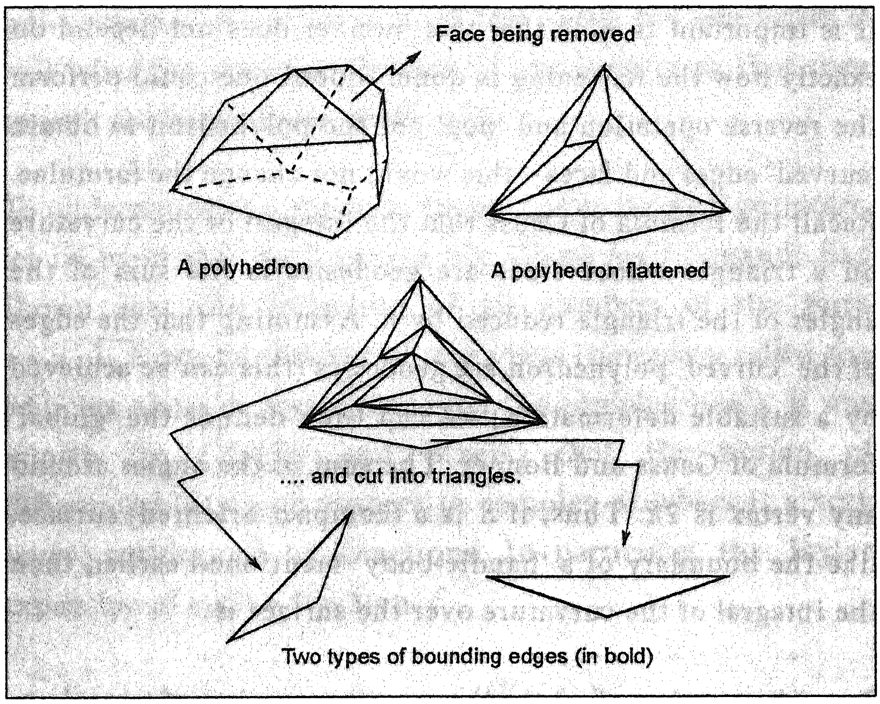

Figure 1 Euler's proof

We remove one face of the boundary of the polyhedron after which it can be 'flattened' onto the plane. We then cut each polygon into triangles as shown in *Figure 1*. This doesn't change the quantity $V - E + F$ (Exercise). Each bounding edge now occurs in one of the two configurations shown in *Figure 1*. Removing one edge in the first case and the two bounding edges and the included vertex in the other case makes exactly one face disappear. Thus this operation does not change $V - E + F$. By successively whittling down the figure in this manner we obtain a triangle. We then have 3 –3 +1 + (the one face removed originally)=2. Now we may take various common objects and flatten the round edges (in one's mind) and obtain a (finite) solid object bounded by polygonal faces. The only property of a polyhedron that is missing would be that of *convexity*. We could then calculate the *Euler characteristic* $V - E + F$ for the boundary of such an object and ask what we would obtain. The answer turns out to be an even number of the form $2(1- g)$ where g (called the *genus*) is the number of 'holes' that the body[1] has (Exercise: Convince yourself of this regarding various objects around you).

[1]For example, the doughnut has one hole and the sphere none. One can also consider more complicated handle-bodies i.e. a sphere with handles attached. A doughnut can be thought of as a sphere with one handle attached – why?

It is important to note that this number does not depend on exactly how the flattening is done. Indeed one could perform the reverse operation and 'pop' out the polyhedron to obtain 'curved' edges and faces – this would not change the formulae. Recall the formula of Gauss that the integral of the curvature on a triangle whose sides are geodesics is the sum of the angles of the triangle reduced by π. Assuming that the edges of the 'curved' polyhedron are geodesics (this can be achieved by a suitable deformation) we can then deduce the 'global' formula of Gauss and Bonnet. The sum of the angles around any vertex is 2π. Thus, if S is a (compact oriented) surface, like the boundary of a 'handle-body' mentioned earlier, then the integral of the curvature over the surface is

$$2\pi \times (\text{the number of vertices}) - \pi \times (\text{the number of triangles}).$$

Now each triangle has three edges and one face and each edge lies on two triangles; thus we have $2E=3F$ so that $V-E+F=V-F/2$ in this situation. Putting it all together we obtain

$$\int_S (\text{curvature}) = 2\pi \times (\text{Euler characteristic})$$

The Euler characteristic is only the first in a series of numbers and invariants that are defined for higher dimensional objects. The subject that studies such invariants is called *algebraic topology* and grew out of the works of Poincaré, Betti, Emmy Noether, Alexandroff and Lefschetz.

The Simplicity of Complex Objects

In addition to the paper of Riemann discussed in the earlier article, one other paper by him contributed tremendously to geometry. This was his paper on the geometric theory of functions of one complex variable. This paper gave rise to many beautiful geometrical ideas, some of which are briefly sketched below. This concluding section is not as self–contained as earlier parts of this series, but one hopes it will whet the appetite of some of our readers so that they consult the books listed at the end.

To understand what function theory has to do with geometry, let us wind the clock back a bit. Gauss (and Argand) had shown that the collection of all numbers of the form $a+\sqrt{-1}\,.\,b$ can be thought of as a plane (nowadays called the complex plane or sometimes even the complex line!). It was known (to Cauchy and others) that the notion of differentiability with respect to complex numbers is a very severe restriction[2] on functions. In particular, the Taylor expansion of such a function f

[2] Contrary to a common misconception among beginning calculus students, the Taylor series does *not* converge for 'most' functions.

$$f(z) = f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\cdots$$

converges absolutely for all z in the complex plane that lies within a certain radius of z_0. (Exercise: To see why this is a restriction, the reader is encouraged to compute the derivatives of all orders of the function exp $+(-1/x^2)$ at $x=0$). These properties imply (as Riemann pointed out), that any differentiable function of one complex variable is naturally defined on a 'generalised' planar region that consists of a number of 'sheets' lying over the complex plane glued together in a manner depending on the function being considered; this is today called the *Riemann surface* of the function. One example is the function $w=\sqrt{z}$; the Riemann surface consists of a complex plane with w as the coordinate variable. This lies over the z-plane in two sheets by sending the point with coordinate w to the point with coordinate $z = w^2$. A more interesting example is the function $w = \sqrt{z^3-z}$. It is clear that but for the points $z=0, \pm 1$ there are exactly two values of w, so that we again have a Riemann surface with two sheets. However, in this case there is *no global coordinate* on the Riemann surface[3].

[3] The uniformisation theorem formulated by Riemann can be thought of as a way of salvaging this situation.

Thus a geometric object can be associated with every differentiable function of one complex variable. Obviously, the study of this geometric object will provide information about the function. One important case is when the number of sheets is finite. Riemann showed that this is precisely the case of an algebraic function $w = f(z)$; i.e. one which satisfies an equation of the type

$$a_n(z)w^n + a_{n-1}(z)w^{n-1} + \cdots + a_0(z) = 0$$

where $a_k(z)$ are polynomial functions of z. Riemann showed that the surface can be identified with the boundary of a handle-body in this case; for example, the second Riemann surface considered above has genus 1 (*Figure 2*).

The equation above can be thought of as defining a curve in the plane (z, w) except that these variables are allowed to take complex values; for this reason we also refer to it as a complex algebraic curve. The relation between the geometry of a complex algebraic curve and its function theory was extensively studied by Brill and Max Noether[4]. The latter also extended this to the theory of complex algebraic surfaces; the famous Italian school of geometry grew out of his work. Following this, the purely algebraic approach led Chevalley, Weil and Zariski to formulate the theory of algebraic varieties. Algebraic geometry grew out of these seminal works.

[4] Emmy Noether was the daughter of Max Noether and developed much of the algebraic framework needed to give concrete and rigorous support to his results.

Summary

Differential geometry (which was introduced in the earlier article), topology and algebraic geometry form three major streams in the study of geometry today. Euclidean geometry survives as the study of linear spaces which is encompassed by linear algebra and lies at the basis of all geometrical research. The reader is advised to study some linear algebra from a good book on algebra such as the one by M Artin.

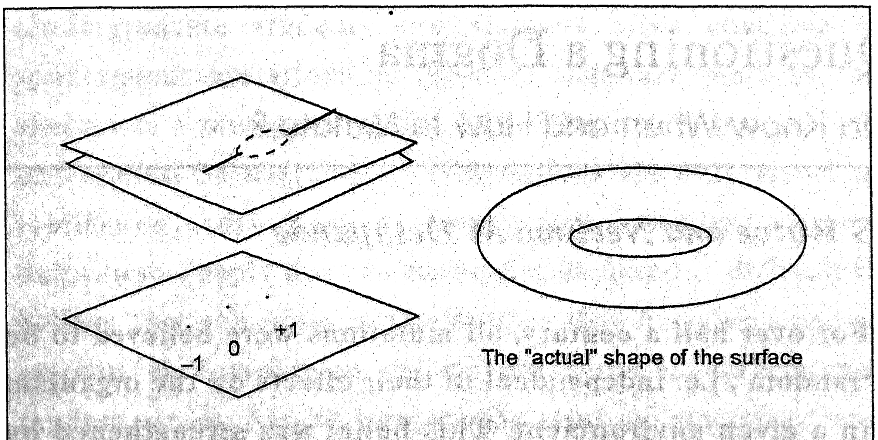

Figure 2 The Riemann surface of $w^2 = z^3 - z$

There are more things left unsaid in this short series than those that have been said. Each geometer without doubt would have her/his own list of beautiful items that have been ignored by the author. It can only be hoped that the reader has acquired an eagerness to explore these unexplored ideas. The author hopes he has demonstrated that this is a far more fruitful exercise than spending time squaring the circle or trisecting the angle.

I would like to thank V Pati for patiently hearing out my ideas and especially for pointing out flaws. Most importantly I am very grateful to A Sitaram for encouraging me to write this series and bearing with my occasionally boring re-working of old themes. I learnt most of the geometry I know from my teacher S Ramanan who turns sixty this year; I dedicate this series to him on this occasion and hope it meets his exacting standards.

Suggested Reading

S Lefschetz. *Topology*. AMS Colloquium Publ. Vol.12. 1930

H Weyl. *The concept of a Riemann surface*. Addison -Wesley. Princeton. 1964.

M Artin. *Algebra*. Prentice Hall India. 1994.

D Mumford. *The red book of varieties and schemes*. Narosa Publishing House. 1995.

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