

A geometric characterization of arithmetic varieties

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Abstract. A result of Belyi can be stated as follows. Every curve defined over a number field can be expressed as a cover of the projective line with branch locus contained in a rigid divisor. We define the notion of geometrically rigid divisors in surfaces and then show that every surface defined over a number field can be expressed as a cover of the projective plane with branch locus contained in a geometrically rigid divisor in the plane. The main result is the characterization of arithmetically defined divisors in the plane as geometrically rigid divisors in the plane.

Keywords. Equisingular; geometrically rigid.

1. Introduction

This paper is an attempt to generalize a result of Belyi (see [1]).

Theorem (Belyi). *Let C be a smooth projective curve over an algebraic number field and T a finite set of closed points in C . There is a finite morphism $f : C \rightarrow \mathbb{P}^1$ so that the image $f(T)$ and the branch locus of f are contained in the set of three points $\{0, 1, \infty\}$.*

We note that this gives a completely geometric characterization of algebraic curves over number fields, since any deformation of a triple of points in \mathbb{P}^1 is in fact trivialized by an automorphism of \mathbb{P}^1 .

A naive generalization of this could require a surface over a number field to be expressible as a cover of \mathbb{P}^2 that is étale outside four general lines; however, as Kollár pointed out, this fails since the fundamental group of the complement of four general lines in \mathbb{P}^2 is abelian, whereas many surfaces have non-abelian fundamental groups. Thus one needs to look at more general divisors in \mathbb{P}^2 . The problem is that these divisors have non-trivial flat deformations. We need to find an algebraic notion that restricts the possible deformations. Thus, in §1 we define the notion of *geometrically rigid* divisors on a surface.

Let C be any collection of 4 or less lines in general position in \mathbb{P}^2 . From the definitions in §1 it follows easily that, C is geometrically rigid. Moreover, it is equally clear that collections of five or more lines in general position in \mathbb{P}^2 are *not* geometrically rigid. Geometrically rigid divisors in \mathbb{P}^2 (and hence their singular loci) are defined over $\overline{\mathbb{Q}}$ (see Lemma 7):

Theorem 1. *Let C be any divisor in \mathbb{P}^2 defined over \mathbb{C} which is geometrically rigid. There is an automorphism g of \mathbb{P}^2 so that $g(C)$ is defined over $\overline{\mathbb{Q}}$.*

Now, if C is a curve of degree 1 or 2 in \mathbb{P}^2 , then C is geometrically rigid but a general curve of degree 3 or more is not. In spite of this we will see that there are many geometrically rigid divisors in \mathbb{P}^2 . In fact (see the end of §3),

Theorem 2. *Let C be any divisor in \mathbb{P}^2 defined over $\overline{\mathbb{Q}}$, and T be a finite set of points in \mathbb{P}^2 defined over $\overline{\mathbb{Q}}$. There is a geometrically rigid divisor D in \mathbb{P}^2 so that $C \subset D$ and T is contained in the singular locus of D .*

These results give a geometric characterization of reduced algebraic subschemes of \mathbb{P}^2 that are defined over $\overline{\mathbb{Q}}$. As an easy corollary we have a generalization of Belyi's characterization to the case of surfaces.

COROLLARY 3

Let S be a smooth projective surface, C a divisor in S and T a finite set of points in S .

Assume that S , C and T are defined over $\overline{\mathbb{Q}}$, then there is a geometrically rigid divisor D in \mathbb{P}^2 and a finite morphism $f : S \rightarrow \mathbb{P}^2$ so that the image of C and the branch locus of f are contained in D ; moreover, the image of T is contained in the singular locus of D .

Conversely, suppose there is a tuple (S, C, T) as above over \mathbb{C} and a finite morphism $f : S \rightarrow \mathbb{P}^2$ so that the image of C and the branch locus of f are sub-divisors of a geometrically rigid divisor D and the image of T is contained in the singular locus of D . Then the tuple (S, C, T) is isomorphic to (the base-change to \mathbb{C} of) a tuple (S_0, C_0, T_0) which is defined over $\overline{\mathbb{Q}}$.

It is reasonably clear that these results should be extendable *mutatis mutandi* to higher dimension.

2. Geometricrigidity

Throughout the paper we work with schemes of finite type over a field of characteristic zero.

Let A be a smooth family of divisors in a smooth surface S ; in other words let $C \subset \mathcal{S} = A \times S$ be a divisor with A smooth. More generally, we can consider the case of non-constant ambient spaces by only assuming that $S \rightarrow A$ is a smooth projective morphism. We are interested in *topologically trivial* families $p : (S, C) \rightarrow A$. Over the field of complex numbers this can be characterized by saying that any point $a \in A$ has an analytic neighborhood U so that the pair $(U \times S, p^{-1}U)$ is homeomorphic over U to $U \times (S, p^{-1}(a))$. The geometric notion of *equisingular* families results in topologically trivial families.

Remark 1. The notion of equisingularity was first defined and studied by Zariski in a series of papers [2,3]. Theorem 7.4 in [3] proves the equivalence of his definition with that studied here. Alternatively, one can directly prove Lemmas 4, 6 and 7 using his definitions. We require a specialized application of Zariski's results which we develop in this section.

A special case is that of a *family of divisors with normal crossings* which is characterized by the following properties:

1. The divisor C is a divisor with normal crossings in \mathcal{S} .
2. Each component of C is smooth over A .
3. The critical locus of $C \rightarrow A$ is étale over A .

In particular, each component of the critical locus of $C \rightarrow A$ meets and is contained in exactly two components of C .

Now let $\mathcal{S}_n \rightarrow \cdots \rightarrow \mathcal{S}_0 = \mathcal{S}$ be a sequence of blow-ups with irreducible reduced centres $A_k \subset \mathcal{S}_k$ such that $A_k \rightarrow A$ is finite étale. Moreover, let \mathcal{C}_k denote the reduced union of the total transform of \mathcal{C} in \mathcal{S}_k and the exceptional locus of $\mathcal{S}_k \rightarrow \mathcal{S}$. We further assume that either,

1. A_k is contained in the critical locus of $\mathcal{C}_k \rightarrow A$ or,
2. A_k is contained in \mathcal{C}_k but misses the critical locus of $\mathcal{C}_k \rightarrow A$ entirely or,
3. A_k lies in the complement of \mathcal{C}_k in \mathcal{S}_k .

While the latter two are irrelevant to the desingularization it is useful to allow these to simplify the proofs. If \mathcal{C}_n is family of divisors with normal crossings, then we call such a sequence of blow-ups a *simultaneous desingularization* of the family of divisors $\mathcal{C} \rightarrow S$. If such a sequence of blow-ups exists then we say that the family is *simultaneously desingularizable* or *equisingular*. In order to understand how one arrives at this definition we state

Lemma 4. Fix a ground field k of characteristic zero. Let $\mathcal{S} \rightarrow A$ be a smooth family of projective surfaces of a reduced scheme A . Let $\mathcal{C} \subset \mathcal{S}$ be a reduced divisor. There is an open dense subset U of A over which \mathcal{C} is an equisingular family.

Proof. We can replace A by its smooth locus and further operate on each component individually; thus we can assume that A is smooth and irreducible. Now, consider the reduced critical locus of $\mathcal{C} \rightarrow A$. This is a closed subscheme B of \mathcal{C} which is generically finite over A . Thus the locus where $B \rightarrow A$ is not étale is a proper closed subscheme of A . We can replace A by the complement of this closed subscheme. Now we can take $A_1 = B$ and perform a blow-up of \mathcal{S} along A_1 to obtain \mathcal{S}_1 . Since A_1 is étale over A the resulting family $\mathcal{S}_1 \rightarrow A$ is smooth. Let \mathcal{C}_1 denote the (reduced) union of the strict transform of \mathcal{C} in \mathcal{S}_1 and the exceptional locus of the blow-up. We can now inductively construct the sequence \mathcal{S}_n as above. By the embedded desingularization of curves in characteristic zero, there is an n so that the generic fibre of $\mathcal{C}_n \rightarrow A$ is a divisor with normal crossings; i.e. each irreducible component (not geometrically irreducible component) of this generic fibre is smooth over the function field of A and at most two of them meet at any singular point (which is closed over the function field of A) and this meeting is transversal. Now replace A by the open subset where the critical locus of $\mathcal{C}_n \rightarrow A$ is étale and each component of $\mathcal{C}_n \rightarrow A$ is smooth. It follows that $\mathcal{C}_n \rightarrow A$ is a family of divisors with normal crossings in $\mathcal{S}_n \rightarrow A$. □

One point that is important from our perspective is the fact that U is defined over k since all schemes are of finite type over k . We also note the following lemma.

Lemma 5. Let B_k be the image of the critical locus B_n of $\mathcal{C}_n \rightarrow A$ in \mathcal{S}_k for each k . Then $B_n \rightarrow B_k$ and $B_k \rightarrow A$ are étale. Any component of B_k that meets A_k is actually A_k . Let \mathcal{D}_k be a union of components of \mathcal{C}_k . If \mathcal{D}_k and a component of B_k meet then the latter is contained in the former. Finally, the critical locus of $\mathcal{D}_k \rightarrow A$ is a union of components of B_k .

Proof. We prove the statements by downward induction on k ; we start at $k = n$ where this is true by the definition of a family of divisors with normal crossings. Now suppose that the result is proved for B_{k+1} and for all divisors of the form \mathcal{D}_{k+1} . Let E_k be the exceptional

locus of $S_{k+1} \rightarrow S_k$. Then E_k is contained in C_{k+1} by the definition of C_{k+1} . The map $E_k \rightarrow A$ factors through $A_k \rightarrow A$ which is étale.

Let Y be the union of those connected components of B_k which meet A_k ; in particular, this includes those components which contain points where $B_{k+1} \rightarrow B_k$ is not an isomorphism. Let X be the inverse image of Y in B_{k+1} ; by the induction hypothesis $X \rightarrow A$ is étale. Moreover, each component of X meets E_k . By choosing $\mathcal{D}_{k+1} = E_k$ we see that X is contained in E_k by the induction hypothesis. Thus, the morphism $X \rightarrow A$ is étale and factors through $A_k \rightarrow A$. It follows that $Y = A_k$. Thus B_k is the disjoint union of A_k and components disjoint from A_k . The remaining components descend isomorphically from components of B_{k+1} and $B_{k+1} \rightarrow A$ is étale by induction. Hence $B_k \rightarrow A$ is étale.

Let \mathcal{D}_k be a union of irreducible components of C_k and suppose that \mathcal{D}_k meets A_k . Let \mathcal{D}_{k+1} be its strict transform in S_{k+1} . Then \mathcal{D}_{k+1} must meet E_k ; let Z be any component of $\mathcal{D}_{k+1} \cap E_k$. This is a divisor in E_k which is contained in the critical locus of $\mathcal{D}_{k+1} \cup E_k \rightarrow A$. By the induction hypothesis applied to $\mathcal{D}_{k+1} \cup E_k$ we see that Z is a component of B_{k+1} . Hence, $Z \rightarrow A$ is étale by induction, and the image of Z is A_k as above. Thus \mathcal{D}_k contains A_k .

Finally, any critical point p of $\mathcal{D}_k \rightarrow A$ which is not the image of a critical point of $\mathcal{D}_{k+1} \rightarrow A$, would have to lie in A_k . Either (a) there are two points q and q' that lie in $\mathcal{D}_{k+1} \cap E_k$ over p , or (b) there is a point q in $\mathcal{D}_{k+1} \cap E_k$ where this intersection is not transversal. In case (a), let Z and Z' be the components of $\mathcal{D}_{k+1} \cap E_k$ that contain q and q' respectively ($Z = Z'$ is a possibility). Then $Z \rightarrow A_k$ and $Z' \rightarrow A_k$ are étale as explained above. In particular, $\mathcal{D}_k \rightarrow A$ has critical points along A_k . In case (b), let Z be the component of $\mathcal{D}_{k+1} \cap E_k$ that contains q . The map $Z \rightarrow A_k$ is étale as above, hence Z is smooth. Thus the intersection of \mathcal{D}_{k+1} and E_k is non-transversal everywhere along Z . Thus, in this case A_k is contained in the critical locus of $\mathcal{D}_k \rightarrow A$ again. Any critical point of $\mathcal{D}_k \rightarrow A$ is thus either contained in A_k which is contained in this critical locus or contained in the image of the critical locus of $\mathcal{D}_{k+1} \rightarrow A$ which is a union of components of B_k . Since A_k is contained in B_k in both cases (a) and (b), it follows that the critical locus of $\mathcal{D}_k \rightarrow A$ is a union of components of B_k . □

In particular, note that this means that A_k is a connected component of the critical locus of $C_k \rightarrow A$ if it meets this locus; this strengthens condition (1) in the definition above. The fundamental lemma that we will use in our constructions is a corollary of the above lemma.

Lemma 6. *Let $(S, C) \rightarrow A$ be an equisingular family of divisors in a family of smooth projective surfaces over a smooth variety A . Let $\mathcal{D} \subset C$ be a union of components of C , then $(S, \mathcal{D}) \rightarrow A$ is an equisingular family of divisors.*

Proof. Let $S_n \rightarrow \dots \rightarrow S_0 = S$ be a simultaneous desingularization of C as above. Let \mathcal{D}_k be the reduced total transform of \mathcal{D} in S_k . Since \mathcal{D}_n is a union of components of C_n , it too is a relative divisor with normal crossings over A . By the above lemma we see that whenever $\mathcal{D}_k \rightarrow A$ has a critical point on A_k , then A_k is contained in this critical locus. Moreover, if \mathcal{D}_k meets A_k then it contains it. Thus the given sequence of blow-ups is a simultaneous desingularization of \mathcal{D}_k . □

Let $C \subset S$ be a divisor. Let G be an algebraic group of automorphisms of S . Given a morphism $A \rightarrow G$, we can construct an equisingular family containing C as follows. Let $m : A \times S \rightarrow S$ denote the action of A on S and let $\mathcal{C} = m^{-1}(C)$. More generally, we say that a family $C \subset A \times S$ is G *iso-trivial*, if it is associated with a G -torsor on S . In

other words, each point $a \in A$ has an étale neighborhood $B \rightarrow A$ so that $\mathcal{C}_B = \mathcal{C} \times_A B$ is isomorphic over B to $m_B^{-1}(C_a)$ for some morphism $m_B : B \rightarrow G$. Any iso-trivial family is clearly equisingular.

We now define C to be a *geometrically rigid* divisor in S if this is the only way to construct equisingular deformations of C ; i.e. for any equisingular family $\mathcal{C} \subset A \times S$ parametrized by a smooth connected variety A so that C is the fibre $p^{-1}(a)$ for some point a in A , there is an algebraic group G of automorphisms of S so that the family $\mathcal{C} \rightarrow A$ is G iso-trivial.

The following lemma follows easily from the construction of universal deformations of divisors and the flattening stratification.

Lemma 7. Let S be smooth surface over an algebraically closed field k and C be a geometrically rigid divisor in S defined over an algebraically closed extension K of k . Then there is an automorphism g of S over K , so that $g(C)$ is the base change to K of a curve C_0 in S which is defined over k .

As a consequence, geometric rigidity is a sufficient criterion to reduce the field of definition.

Proof. Let H be the Hilbert scheme of divisors in S over k . Let A be the closure of the (non-closed) point of H which corresponds to C . Then A is a scheme of finite type over k to which we can apply Lemma 4 above. Thus replacing A by an open subscheme U defined over k we have an equisingular family $\mathcal{C} \rightarrow A$ in $S \times A$ with generic fibre isomorphic to the given C .

By the geometric rigidity of S it follows that this family is isotrivial for some algebraic group G of automorphisms of S . Thus there is a finite étale cover $A' \rightarrow A$ so that the family is group-theoretically trivial over A' . Since k is algebraically closed there is a k -valued point of A' . The fibre of \mathcal{C} at this point is then a ‘model’ of (S, C) which is defined over k . □

In particular, we note that Theorem 1 follows.

3. Constructions

We now give inductive constructions of geometrically rigid divisors to prove Theorem 2.

Lemma 8. Let D be a geometrically rigid divisor in \mathbb{P}^2 and let p, q be singular points of D . The divisor $D \cup \overline{pq}$ is geometrically rigid, where \overline{pq} is the line joining the points p and q .

Proof. Let $\mathcal{C} \rightarrow A$ be an equisingular deformation of $D \cup \overline{pq}$. We wish to construct a group-theoretic trivialization of this deformation over a finite étale cover of A .

Let $A_1 \rightarrow A$ (respectively $A_2 \rightarrow A$) be a component of the critical locus of $\mathcal{C} \rightarrow A$ which contains p (respectively contains q). These are étale covers of A by Lemma 5. Let $B \rightarrow A$ be a connected étale cover of A that dominates both covers; we have natural morphisms $P : B \rightarrow \mathbb{P}^2$ and $Q : B \rightarrow \mathbb{P}^2$ passing through p and q respectively. Let $\mathcal{L} \rightarrow B$ be the component of $\mathcal{C}_B = \mathcal{C} \times_A B$, that contains \overline{pq} . Then, the fibre of L over $b \in B$ consists of the line joining $P(b)$ and $Q(b)$. Let \mathcal{D}_B be the union of the remaining components of \mathcal{C}_B . By Lemma 6, the family $\mathcal{D}_B \rightarrow B$ is an equisingular deformation of D .

Now, by the geometric rigidity of D , we see that $\mathcal{D}_B \rightarrow B$ is iso-trivial. In particular, we take a further étale cover (which we also denote by B by abuse of notation) so that

the family \mathcal{D}_B is group-theoretic. Now, $P(B)$ and $Q(B)$ continue to be part of the critical locus of $\mathcal{D}_B \rightarrow B$, thus by the connectedness of B the trivialization of the family must take them to $B \times \{p\}$ and $B \times \{q\}$ respectively. But then the same trivialization also takes L to $B \times \overline{pq}$. Thus we have a group-theoretic trivialization of \mathcal{C}_B . \square

Starting with the geometrically rigid divisor Q of 4 lines in general position on \mathbb{P}^2 , we look at all the divisors obtained by iterated application of the above lemma. The usual constructions of projective geometry that give the field operations for points on a line give the following result.

PROPOSITION 9

Let T be any finite set of points in \mathbb{P}^2 defined over \mathbb{Q} . There is a geometrically rigid divisor D consisting of lines so that T is contained in the singular locus of D .

Proof. Fixing the reference quadrilateral Q consisting of four general lines in \mathbb{P}^2 also fixes a coordinate system so that the lines are given by $X = 0, Y = 0, Z = 0$ and $X + Y + Z = 0$. The singular points of the quadrilateral are $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1), (1 : -1 : 0), (1 : 0 : -1)$ and $(0 : 1 : -1)$.

For any $t \in \mathbb{P}^2(\mathbb{Q})$ and a geometrically rigid divisor C_0 containing Q we will construct a larger geometrically rigid divisor that contains t . We can then construct D by starting with Q and successively adding each point of the finite set T .

Thus we can assume that T consists of just one point. Since at least one coordinate of t is non-zero we can assume that it takes the form $(u : v : 1)$ in these coordinates for some rational numbers u and v .

Now, suppose that we can add to C_0 and produce a geometrically rigid divisor C so that the singular locus of C contains $(u : 0 : 1)$ and $(0 : v : 1)$. We can then add to C the line L joining $(u : 0 : 1)$ and $(0 : 1 : 0)$, and the line M joining $(0 : v : 1)$ and $(1 : 0 : 0)$, again producing a geometrically rigid divisor $C \cup L \cup M$ by Lemma 8. Now the point t is the intersection point of L and M so it is a singular point of this divisor as required.

Similarly, if we can add to C_0 to produce a geometrically rigid divisor C containing $(v : 0 : 1)$ in its singular locus then the divisor $C \cup L$ is also geometrically rigid, where L is the line joining $(v : 0 : 1)$ and $(1 : -1 : 0)$. The point $(0 : v : 1)$ which is the point of intersection of L and the line $X = 0$, is a singular point of this divisor. Thus to prove the result, it is enough to construct for each rational number u a divisor C_u containing C_0 so that the point $(u : 0 : 1)$ is in the singular locus of C_u .

We write $u = p/q$, where q is a positive integer and p is some integer. Suppose we can construct a divisor C containing C_0 so that $(0 : p : 1)$ and $(0 : -q : 1)$ are singular points of C . Let L be the line joining $(1 : 0 : -1)$ and $(0 : -q : 1)$; as before $C \cup L$ is a geometrically rigid divisor. Moreover, $(1 : -q : 0)$ is a singular point of this divisor as it lies on L and the line $Z = 0$. Let M be the line joining $(0 : p : 1)$ and $(1 : -q : 0)$; as before the divisor $C \cup L \cup M$ is geometrically rigid. The point $(p/q : 0 : 1)$ is a singular point of this divisor as it lies on M and the line $Y = 0$.

Thus we have finally reduced to the problem of constructing for each integer p a geometrically rigid divisor C_p containing C_0 for which $(0 : p : 1)$ is a singular point. We will do this by induction on the absolute value of p . Let L_1 be the line joining $(0 : 1 : 0)$ and $(-1 : 0 : 1)$, L_2 the line joining $(-1 : 1 : 0)$ and $(0 : 0 : 1)$. By Lemma 8 the divisor $Q \cup L_1 \cup L_2$ is geometrically rigid. The point $(-1 : 1 : 1)$ is the intersection point of L_1 and L_2 , hence it is a singular point of this divisor. Let M be the line joining this point to $(1 : 0 : 0)$. Then $Q \cup L_1 \cup L_2 \cup M$ is geometrically rigid. The point $(0 : 1 : 1)$ is the

intersection point of M and the line $X = 0$. Thus we have produced C_p for every p less than 1 in absolute value.

Now, suppose that we have constructed a divisor C containing C_0 which has $(0 : p : 1)$ and $(0 : q : 1)$ in its singular locus. Let L_1 be the line joining $(0 : p : 1)$ with $(1 : 0 : 0)$ and L_2 be the line joining $(-1 : 0 : 1)$ with $(0 : 1 : 0)$. The point $(-1 : p : 1)$ is a singular point of the geometrically rigid divisor $C \cup L_1 \cup L_2$. Let M_1 be the line joining $(0 : 0 : 1)$ and $(-1 : p : 1)$; the point $(-1 : p : 0)$ is a singular point of the geometrically rigid divisor $C \cup L_1 \cup L_2 \cup M_1$. Let M_2 be the line joining $(-1 : p : 0)$ and $(0 : q : 1)$; then $(-1 : p+q : 1)$ is a singular point of the geometrically rigid divisor $C \cup L_1 \cup L_2 \cup M_1 \cup M_2$. Finally, we add the line M_3 joining $(1 : 0 : 0)$ and $(-1 : p+q : 1)$ to obtain a geometrically rigid divisor which has $(0 : p + q : 1)$ as a singular point.

For every $p > 1$ we apply the latter construction to the divisor $C_{p-1} \cup C_1$ each of which we have already constructed by the induction hypothesis and has singular points at $(0 : p - 1 : 1)$ and $(0 : 1 : 1)$. For $p < 1$ we apply the latter construction to C_{p+1} which has $(0 : p + 1 : 1)$ and $(0 : -1 : 1)$ as singular points. This provides the required construction and hence the result is proved. \square

To construct points with coordinates in algebraic number fields we need to have curves of degree greater than one in geometrically rigid divisors.

Lemma 10. *Let D be a geometrically rigid divisor on a rational surface S and let T be a finite subset of the singular points of D . Let L be a divisor class on S so that the linear system $|L - T|$ has a unique element E . Then the divisor $D \cup E$ is geometrically rigid.*

Actually, we only need the regularity of S (i.e. $H^1(S, \mathcal{O}_S) = 0$) in the proof given below. Further generalizations even for irregular surfaces are possible.

Proof. Let $\mathcal{C} \rightarrow A$ be an equisingular deformation of the divisor $C = D \cup E$. Let B be the connected component of the critical locus of $\mathcal{C} \rightarrow A$ that contains T . This is finite étale over A by Lemma 5. By base change we may assume that $B \rightarrow A$ is an isomorphism. Thus, we can write $\mathcal{C} = \mathcal{D} \cup \mathcal{E}$ where \mathcal{D} is the union of irreducible components of \mathcal{C} that meet D and \mathcal{E} is the union of the irreducible components of \mathcal{C} that meet E . By Lemma 6, $\mathcal{D} \rightarrow A$ is an equisingular deformation of D . Thus by base change we have a group-theoretic trivialization of \mathcal{D} . Since B is contained in the critical locus of $\mathcal{D} \rightarrow A$, it is mapped into T by the trivialization. Thus, after applying this trivialization, $\mathcal{E} \rightarrow A$ becomes a family of divisors containing T .

Now, the divisor class L has no deformation since S is rational. Thus, the divisor class of every fibre of $\mathcal{E} \rightarrow A$ is in the class L . By assumption, E is a unique class containing T , thus $\mathcal{E} \rightarrow A$ is the trivial family. Hence the trivialization for $\mathcal{D} \rightarrow A$ in fact gives a trivialization of \mathcal{E} and \mathcal{C} as well. \square

The above lemma allows us to apply the Lagrange interpolation formula to prove the following proposition.

PROPOSITION 11

Let T be a finite set of algebraic points on \mathbb{P}^2 , then there is a geometrically rigid divisor D so that T is contained in the singular locus of D .

Proof. As in the proof of Proposition 9, given a geometrically rigid divisor C_0 which contains the reference quadrilateral Q and a point $t \in \mathbb{P}^2(\mathbb{Q})$, we construct a larger divisor

$C \supset C_0$ so that t is in the singular locus of t . Since T is a finite set we can inductively add all the points $t \in T$ to obtain the required divisor D . Thus we can assume that T consists of one point t .

Again, as in the proof of Proposition 9 we can further reduce to the case where the point has the form $(u : 0 : 1)$ where u is an algebraic number. Let $f(T)$ be a monic polynomial with rational coefficients for which $f(u) = 0$; let n be the degree of f . Let F be the set of points $(k : f(k) : 1)$ for $k = 0, \dots, n^2$. The curve E defined by $YZ^{n-1} = f(X/Z)Z^n$ passes through these $n^2 + 1$ points. Thus it is the unique curve of degree n that passes through these points. Let C be a divisor (containing the quadrilateral Q) constructed using Proposition 9 which contains F in its singular locus. Lemma 10 then asserts that $D = C \cup E$ is geometrically rigid. The point $(u : 0 : 1)$ is a point of intersection of E and the line $Y = 0$ which lies in Q ; hence it is a singular point of D . \square

Finally, any curve of degree n defined over $\overline{\mathbb{Q}}$ is uniquely determined in its divisor class by $n^2 + 1$ distinct $\overline{\mathbb{Q}}$ -valued points on it.

Proof (of Theorem 2). Let C be any curve of degree n in \mathbb{P}^2 which is defined over $\overline{\mathbb{Q}}$. Let T be a collection of $n^2 + 1$ distinct points on this curve over $\overline{\mathbb{Q}}$. Let D be a geometrically rigid divisor in \mathbb{P}^2 that contains T in its singular locus. By Lemma 10 the divisor $D \cup C$ is geometrically rigid. Applying this argument to each component of a given divisor in \mathbb{P}^2 defined over $\overline{\mathbb{Q}}$, we have the result. \square

4. Remarks and open problems

A similar collection of arguments can be used to obtain geometrically rigid configurations in \mathbb{P}^n for $n \geq 3$. Projection arguments can be used to define the notion of equisingular deformations in higher (co-)dimensions. Arguments similar to the ones in the previous section can then be probably used to show:

Problem 1. For each k between 0 and $n - 1$, let T_k be a closed subscheme of \mathbb{P}^n of pure dimension k that is defined over $\overline{\mathbb{Q}}$. Then there is a geometrically rigid divisor S_{n-1} in \mathbb{P}^n so that if S_k is defined inductively as the singular locus of S_{k+1} , then S_k has pure dimension k and $T_k \subset S_k$.

Another possible generalization of Belyi's theorem is the following:

Problem 2. If C is a projective algebraic curve over a field of transcendence degree r there is a morphism $f : C \rightarrow \mathbb{P}^1$ for which the branch locus has cardinality less than or equal to $3 + r$.

Belyi's original arguments can be used to show that the branch locus can be assumed to be defined over the field of rational functions in r variables. However, there does not seem to be any obvious way to reduce the number of points to $3 + r$. The converse (that such a cover is defined over a field of transcendence degree at most r) follows from the fact that s -tuples of points in \mathbb{P}^1 have a moduli space of dimension $s - 3$.

Finally, it is clear from the above construction that the complexity of the configuration required to obtain rigidity is related to the height of the defining equation of a curve. Can this relation be explicitly used to define a notion of height?

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