Invent. math. 98, 425–444 (1989)

Inventiones mathematicae

Self maps of homogeneous spaces

K.H. Paranjape and V. Srinivas
Tata Institute of Fundamental Research, School of Mathematics, Homi Bhabha Road, Bombay 400005, India

Introduction

This paper arose out of an attempt to understand the following problem of Lazarsfeld [L]:

Problem 1. *Suppose G is a semi-simple algebraic group over $\mathbf{C}$, $P \subset G$ a maximal parabolic subgroup, $Y = G/P$. Let $f\colon Y \to X$ be a finite, surjective morphism of degree > 1 to a smooth variety X; then is $X \cong \mathbf{P}^n$? ($n = \dim X = \dim Y$)*

Lazarsfeld (*loc. cit.*) answers this in the affirmative when $Y = \mathbf{P}^n$, using the proof by S. Mori [M] of Hartshorne's conjecture. The general case seems to be open even for Grassmann varieties.

In this paper, we show (see Proposition 2): *if $Y = G/P$ is as above and $f\colon Y \to Y$ is a finite self map of degree > 1, then $Y \cong \mathbf{P}^n$.*

More generally, we prove the following:

Theorem. *Let G be a simply connected, semi-simple algebraic group over $\mathbf{C}$. Let $P \subset G$ be a parabolic subgroup, and let $Y = G/P$ be the homogeneous space. Let $f\colon Y \to Y$ be a generically finite morphism. Then there exist parabolic subgroups $P_0, P_1, \ldots, P_m$ of G containing P, and a permutation σ of $\{1, 2, \ldots, m\}$ such that:*

(i) *there are isomorphisms $G/P_i \cong \mathbf{P}^{n_i}$ for $i \geqq 1$, for some integers $n_i > 0$, such that $n_{\sigma(i)} = n_i$ for all i.*
(ii) *there is a finite morphism $\pi_i\colon \mathbf{P}^{n_i} \to \mathbf{P}^{n_i}$ for each $i > 0$.*
(iii) *the natural morphism*

$$Y \to G/P_0 \times G/P_1 \times \ldots \times G/P_m$$

is an isomorphism, under which $f\colon Y \to Y$ corresponds to the product $f_0 \times f_1 \times \ldots \times f_m$, where $f_0\colon G/P_0 \to G/P_0$ is an isomorphism and f_i the composite

$$G/P_{\sigma(i)} \cong \mathbf{P}^{n_i} \xrightarrow{\pi_i} \mathbf{P}^{n_i} \cong G/P_i\,.$$

We also show that Problem 1 has an affirmative answer if Y is a smooth quadric hypersurface of dimension $\geqq 3$ (Proposition 8); this includes the case of Grassmannian $Y = \mathbf{G}(2, 4)$. We also show:

Proposition 6. *Let $k \leqq n, 2 \leqq l \leqq m$ be integers, such that there exists a finite surjective morphism between Grassmann varieties*

$$f: \mathbf{G}(k, k+n) \rightarrow \mathbf{G}(l, l+m)$$

Then $k = l, m = n$ and f is an isomorphism.

In the spirit of Lazarsfeld's problem, we pose the following:

Problem 2. *Let $f: A \rightarrow X$ be a finite surjective morphism from a simple abelian variety A over $\mathbf{C}$ to a smooth variety X. Suppose that f is not étale. Then is $X \cong \mathbf{P}^n$? ($n = \dim A = \dim X$).*

This is easily proved for $\dim A \leqq 2$.

We wish to thank M.V. Nori, V.B. Mehta and S. Subramaniam for stimulating discussions.

1. Proof of the theorem

One of the tools used in the proof is the following "Bertini theorem" due to M.V. Nori.

Proposition 1. *Let $Y = G/P$ be a homogeneous space for a simply connected algebraic group over $\mathbf{C}$, and let $\pi: \tilde{Y} \rightarrow Y$ be a finite, surjective morphism with branch divisor $B \subset Y$. Let Z be an irreducible variety, $f: Z \rightarrow Y$ a non-constant morphism. Then either*

(i) *for a non-empty Zariski open subset $U \subset G$,*

$$Z_g = (Z \times_{g(f)} \tilde{Y})_{\text{red}}$$

is irreducible for all $g \in U$ (here $g(f): Z \rightarrow Y$ is obtained from f by pointwise translation by g on Y), or

(ii) *there exists a closed subgroup P' of G, containing P, such that if $h: Y \rightarrow G/P'$ is the quotient map, then the composite $h \circ f: Z \rightarrow G/P'$ is constant, and for some non-zero effective divisor $D \subset G/P'$, $h^{-1}(D)$ is a component of B.*

Proof. This result is implicit in [N]. We only need the special case when P is a maximal parabolic subgroup of a simply connected semi-simple group G, so that only the possibility (i) can occur. We give the proof in this case, leaving the general case to the reader.

Replacing Z by its normalization, we are reduced to the case when Z is normal. Let

$$B_g = g(f)^{-1}(B) = f^{-1}(g^{-1}B) \subset Z$$

where $g^{-1}B$ is the translate of B by g^{-1}. If $p: Z_g \rightarrow Z$ is the natural map, then $Z_g - p^{-1}(B_g)$ is Zariski dense in Z_g, when it is non-empty (which it is for all g lying in some non-empty Zariski open subset of G). Since

$$Z_g - p^{-1}(B_g) \rightarrow Z - B_g$$

is an étale covering space, $Z_g - p^{-1}(B_g)$ is normal; hence to prove that it is irreducible, it suffices to prove that it is connected. To prove connectedness, it suffices to prove that the map on fundamental groups

$$g(f)_*: \pi_1(Z - B_g) \to \pi_1(Y - B)$$

is surjective (we omit the base points in the notation).

Consider the surjective morphism

$$\mu: G \times Z \to Y, \quad \mu(g, z) = g(f)(z) .$$

The variety $G \times Z$ is connected, and the fibres of μ are principal P-bundles over Z, and in particular are connected. Hence, if $W = (G \times Z) - \mu^{-1}(B)$ then

$$\mu_*: \pi_1(W) \to \pi_1(Y - B)$$

is surjective. If $\varphi: W \to G$ is induced by the projection, then the scheme theoretic fibre

$$\varphi^{-1}(g) = \{g\} \times (Z - B_g) .$$

Hence it suffices to prove that for a non-empty Zariski open set $U \subset G$, the inclusion of the fibre of φ induces a surjection

$$\pi_1(\varphi^{-1}(g)) \to \pi_1(W), \quad \forall g \in U .$$

By lemma (1.5) of [N], there is a non-empty Zariski open subset $U \subset G$ such that $\varphi^{-1}(U) \to U$ is a fibre bundle for the complex ("classical") topology; further, for all $g \in U$, there is an exact sequence

$$\pi_1(\varphi^{-1}(g)) \to \pi_1(W) \to \pi_1(G) \to 0$$

under the following additional hypothesis: there is a codimension 2 subvariety $L \subset G$ such that $U \subset (G - L)$, and for each $g \in (G - L)$, the scheme theoretic fibre $\varphi^{-1}(g)$ is non-empty and has a smooth point. Since G is simply connected

$$\pi_1(\varphi^{-1}(g)) \to \pi_1(W)$$

is surjective if

$$T = \{g \in G \mid \varphi^{-1}(g) \text{ is empty}\}$$

has codimension $\geqq 2$ in G (since $\varphi^{-1}(g)$ is an open subset of the normal variety Z, it has smooth points if it is non-empty).

Now $\varphi^{-1}(g)$ is empty $\Leftrightarrow Z = B_g \Leftrightarrow g(f)(Z) \subset B$. Let $k: G \to G/P = Y$ be the quotient map, and let $k^{-1}(B) = B'$, $k^{-1}f(Z) = Z'$. Then

$$g(f)(Z) \subset B \Leftrightarrow gZ' \subset B' ,$$

so that

$$T = \{g \in G \mid gZ' \subset B'\}$$

where $B' \subset G$ is a divisor. Replacing Z', B' by suitable translates (which replaces T by an isomorphic subvariety of G), we may assume that the identity element e of G lies in $Z' \subset B'$. Then $T \subset B'$. If T_0 is an irreducible component of T which is a divisor in G, then

$$T_0 = T_0 e \subset T_0 Z' \subset B' ,$$

and the Zariski closure of $T_0 Z'$ in G is irreducible. Hence $T_0 = T_0 Z'$ is a divisor, and

$$Z' \subset \{g \in G \mid T_0 g = T_0\} = P' .$$

But P is a proper subset of Z' and hence also of P', and P' is a closed subgroup of G. Since P is a maximal parabolic subgroup of G, we have $P' = G$, a contradiction. Hence $\operatorname{codim}_G T \geqq 2$ and this completes the proof. □

To prove the Theorem, we first prove it in the following special case:

Proposition 2. *Let $P \subset G$ be a maximal parabolic subgroup, $Y = G/P$, and $f\colon Y \to Y$ a non-constant morphism. Then either f is an isomorphism, or $Y \cong \mathbf{P}^n$.*

Proof. Since Pic $Y = \mathbf{Z}$, generated by the class of a very ample divisor (see [B]), it follows that f is finite. To show that it is an isomorphism, it suffices to prove that it has degree 1. By the theorem of S. Mori [M], if the tangent sheaf T_Y is ample, then $Y \cong \mathbf{P}^n$. Using this, we show that if $Y \ncong \mathbf{P}^n$, then $\deg f = 1$.

If $y \in Y = G/P$, then $y = gP$ for some $g \in G$, and we may identify the tangent space at y,

$$T_{y,Y} = \operatorname{Lie} G / \operatorname{Ad}(g) \operatorname{Lie} P .$$

Thus we have a natural map $\operatorname{Lie} G \to H^0(Y, T_Y)$ whose image generates T_Y at every point, and we have a surjection of locally free sheaves

$$(\operatorname{Lie} G) \otimes_{\mathbf{C}} \mathcal{O}_Y \to T_Y .$$

This gives a closed embedding of Y-schemes

$$\mathbf{P}_Y(T_Y) \to \mathbf{P}_Y((\operatorname{Lie} G) \otimes_{\mathbf{C}} \mathcal{O}_Y) = Y \times \mathbf{P}(\operatorname{Lie} G) ,$$

giving rise to the diagram

$$\begin{array}{ccc} \mathbf{P}_Y(T_Y) & \xrightarrow{\alpha} & \mathbf{P}(\operatorname{Lie} G) \\ \beta \downarrow & & \\ Y & & \end{array}$$

(where α, β are induced by the projections on $Y \times \mathbf{P}(\operatorname{Lie} G)$). The morphism α restricts to a linear embedding on each fibre of β.

Lemma 3. *Let Y be a projective variety, $\mathcal{E}$ a locally free sheaf on Y, such that there is a surjection*

$$V \otimes_{\mathbf{C}} \mathcal{O}_Y \to \mathcal{E}$$

for a finite dimensional vector space V, giving rise to a diagram

$$\begin{array}{ccc} \mathbf{P}_Y(\mathcal{E}) & \xrightarrow{p} & \mathbf{P}(V) \\ q \downarrow & & \\ Y & & \end{array}$$

Let $Z \subset Y$ be an irreducible subvariety and $r \geqq 1$ an integer. Then the following are equivalent:

(i) *$\mathscr{E} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank r*
(ii) *there is a Zariski open set $U \subset Z$, such that for every irreducible curve $C \subset Z$ which meets U, $\mathscr{E} \otimes \mathcal{O}_C$ has a trivial direct summand of rank r*
(iii) *the linear subspace*

$$\bigcap_{y \in Z} p(q^{-1}(y)) = \bigcap_{y \in Z} p(\mathbf{P}(\mathscr{E}_y)) \subset \mathbf{P}(V)$$

has dimension $\geqq r-1$.

Proof. Clearly (iii) is equivalent to the existence of a surjection $\varphi: V \to L$, with $\dim L = r$, such that for each $y \in Z$, φ factors through the quotient $V \to \mathscr{E}_y$. This is equivalent to the existence of a factorization

$$\begin{array}{ccc} V \otimes \mathcal{O}_Z & \xrightarrow{\varphi \otimes 1} & L \otimes \mathcal{O}_Z \\ & \searrow \quad \nearrow & \\ & \mathscr{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z & \end{array}$$

i.e. $\mathscr{E}|_Z$ has a trivial quotient of rank r (since Z is irreducible projective, a surjection $V \otimes \mathcal{O}_Z \to L \otimes \mathcal{O}_Z$ must have the form $\varphi \otimes 1$ for some φ). Since $\mathscr{E}$ is generated by global sections, this is equivalent to $\mathscr{E}$ having a trivial direct summand of rank r. Thus (i) $\Leftrightarrow$ (iii), and (i) $\Rightarrow$ (ii). To prove that (ii) $\Rightarrow$ (iii), suppose that (iii) does not hold i.e. the intersection of linear spaces defined in (iii) has dimension $< r-1$ (if $r = 1$, we take this to mean it is empty). Then there is a finite set of points $y_1, \ldots, y_m \in Z$ such that

$$\dim\left(\bigcap_{i=1}^{m} \mathbf{P}(T_{y_i Y})\right) < r - 1 .$$

We can then find an irreducible curve $C \subset Z$ such that C meets U and contains all of the y_i. Then by (i) $\Leftrightarrow$ (iii) applied to C, we see that $\mathscr{E} \otimes \mathcal{O}_C$ does not have a trivial direct summand of rank r, so that (ii) does not hold for Z. This completes the proof of the lemma. □

Since

$$(\alpha, \beta): \mathbf{P}(T_Y) \to \mathbf{P}(\operatorname{Lie} G) \times Y$$

is an embedding, we see that for any $x \in \mathbf{P}(\operatorname{Lie} G)$, the map β induces an isomorphism of $\alpha^{-1}(x)$ onto its image in Y. In particular, if $D \subset \alpha^{-1}(x)$ is an irreducible curve, then $C = \beta(D)$ is an isomorphic curve, such that

$$x \in \bigcap_{y \in C} \mathbf{P}(T_{y,Y}) .$$

Thus $T_Y \otimes \mathcal{O}_C$ has a trivial direct summand. The tangent sheaf T_Y is not ample (since we have assumed that Y is not isomorphic to $\mathbf{P}^n$) and hence α is not finite. So there are curves C on Y as above.

On the other hand, the top Chern class $c_n(T_Y) \in H^{2n}(Y, \mathbf{Z}) \cong \mathbf{Z}$ equals the topological Euler characteristic of Y, which is non-zero (for example this follows from the fact that Y has a cell decomposition with even dimensional cells). Thus T_Y does not have any trivial direct summands.

Let

$$A = \{Z \subset Y | Z \text{ is irreducible, and } T_Y \otimes \mathcal{O}_Z \text{ has a trivial direct summand}\} .$$

Then by the discussion above, A contains some curves, while $Y \notin A$; further, by lemma 3,

$$Z \in A \Leftrightarrow \exists x \in \mathbf{P}(\text{Lie } G) \text{ such that } Z \subset \beta(\alpha^{-1}(x)) .$$

Let $m = \max\{\dim Z | Z \in A\}$, and let

$$S = \{Z \in A | \dim Z = m\} .$$

Then $1 \leqq m < n = \dim Y$, and each $Z \in S$ is an irreducible component of $\beta(\alpha^{-1}(x))$ for some $x \in \mathbf{P}(\text{Lie} G)$.

Let $W = \alpha(\mathbf{P}(T_Y))$; then the morphism $\mathbf{P}(T_Y) \to W$ has a flattening stratification (see [Mu]), so that the set $\{\deg Z | Z \in S\}$ is finite. Hence from the theory of the Chow variety, we see that the non-empty set of cohomology classes

$$\text{Cl}[S] = \{[Z] \in H^{2n-2m}(Y, \mathbf{Z}) | Z \in S\}$$

is finite.

Lemma 4. *The map on cohomology groups*

$$f^*: H^{2n-2m}(Y, \mathbf{Z}) \to H^{2n-2m}(Y, \mathbf{Z})$$

maps Cl$[S]$ *into itself.*

Proof. If $Z \in S$, then any translate $gZ \in S$, and

$$[Z] = [gZ] \in H^{2n-2m}(Y, \mathbf{Z}) .$$

From Proposition 1, it follows that for each $c \in \text{Cl}[S]$, there exists $Z \in S$ with $[Z] = c$, such that $f^{-1}(Z)_{\text{red}} = Z'$ is irreducible, and represents the inverse image $f^*(Z)$ as a cycle (i.e. the scheme theoretic inverse image $f^{-1}(Z)$ is reduced at the generic point of Z'). We claim $Z' \in S$. Since $\dim Z' = m$, it suffices to prove that $Z' \in A$. Since Z is not contained in the branch locus of f, the map of locally free sheaves

$$T_Y \otimes \mathcal{O}_{Z'} \xrightarrow{df \otimes 1} f^* T_Y \otimes \mathcal{O}_Z ,$$

is an isomorphism at the generic point of Z'. As $Z \in S$, $f^* T_Y \otimes \mathcal{O}_Z$, has a trivial direct summand; hence there is a map $T_Y \otimes \mathcal{O}_{Z'} \to \mathcal{O}_{Z'}$ which is generically surjective, hence surjective (since $T_Y \otimes \mathcal{O}_{Z'}$ is generated by global sections). Hence $Z' \in S$.

But clearly

$$f^*(c) = f^*[Z] = [f^*Z] = [Z'] \in \text{Cl}[S] .$$

□

Since Cl$[S]$ is finite, it follows that some iterate $f^k = f \circ f \circ \ldots \circ f$ has the property that $(f^k)^*(c) = c$ for some non-zero $c \in \text{Cl}[S]$. Hence, in order to show that $\deg f = 1$, we may assume without loss of generality that $f^* c = c$.

Let $h \in H^2(Y, \mathbf{Z})$ be the Chern class c_1 of the ample generator of Pic $Y = \mathbf{Z}$. Then $H^2(Y, \mathbf{Z}) = \mathbf{Z} \cdot h$, and $H^{2n}(Y, \mathbf{Q}) = \mathbf{Q} \cdot h^n$. Also $f^* h^n = (\deg f) \cdot h^n$, so that $f^* h = (\deg f)^{1/n} \cdot h$. Now $c \cup h^m = [Z] \cup h^m = d \cdot h^n$, where $d = \deg Z / \deg Y$ is a positive rational. Thus

$$f^*(c \cup h^m) = f^*(d \cdot h^n) = (\deg f) d \cdot h^n .$$

On the other hand,

$$f^*(c \cup h^m) = f^* c \cup (f^* h)^m = c \cup (\deg f)^{m/n} \cdot h^m = (\deg f)^{m/n} d \cdot h^n .$$

Hence $\deg f = (\deg f)^{m/n}$, where $m = \dim Z < n$. Thus $\deg f = 1$, and this shows that f is an isomorphism. This completes the proof of Proposition 2. □

We now prove the Theorem in the general case, when $Y = G/P$, P is any parabolic subgroup, and $f\colon Y \to Y$ is a finite self-map. Let $P' \supset P$ be a parabolic subgroup, and let $\mathscr{L} \in \operatorname{Pic} Y$ be the pullback to Y of a very ample invertible sheaf $\mathscr{L}'$ on G/P', under the natural map

$$Y = G/P \to G/P' .$$

From the theory of dominant weights (see [B]), $H^0(Y, f^* \mathscr{L})$ gives a base-point free linear system on Y, such that for a unique parabolic subgroup $\tau(P')$ of G which contains P, the morphism

$$Y \to \mathbf{P}(H^0(Y, f^* \mathscr{L}))$$

is identified with the natural map

$$Y = G/P \to G/\tau(P')$$

composed with a projective embedding of the latter by a complete linear system. The map

$$f^*\colon H^0(Y, \mathscr{L}) \to H^0(Y, f^* \mathscr{L})$$

gives rise to a diagram with surjective arrows

$$\begin{array}{ccc} Y & \longrightarrow & G/\tau(P') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & G/P' \end{array}$$

(this diagram defines the map f'; the horizontal arrows are the natural ones).

Let $\mathfrak{p}$ be the set of parabolic subgroups of G containing P. Then $\mathfrak{p}$ is a finite set (see [B]), which is an ordered lattice with respect to the partial order given by inclusion. Fix a very ample $\mathscr{L}' \in \operatorname{Pic}(G/P')$ for each $P' \in \mathfrak{p}$. Then the above construction yields a map of sets $\tau\colon \mathfrak{p} \to \mathfrak{p}$.

Lemma 5. *$\tau\colon \mathfrak{p} \to \mathfrak{p}$ is an isomorphism of ordered lattices.*

Proof. We must show that τ is bijective (i.e. that it is injective, as $\mathfrak{p}$ is finite), and preserves the partial order i.e. $P' \subset P'' \Rightarrow \tau(P') \subset \tau(P'')$.

We first remark that for any $P' \in \mathfrak{p}$,

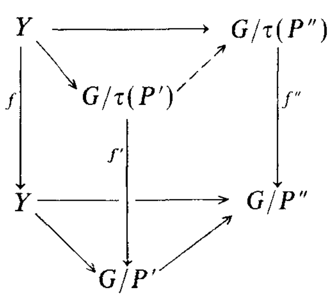
$$Y \to G/\tau(P') \xrightarrow{f'} G/P'$$

is the Stein factorization of the composite

$$Y \to Y \to G/P'$$

so that $\tau\colon \mathfrak{p} \to \mathfrak{p}$ is independent of the choices $\mathscr{L}' \in \operatorname{Pic}(G/P')$. Since the fibres of $Y \to G/\tau(P')$ are connected and $G/\tau(P')$ is smooth, this remark will follow if we prove that f' has finite fibres. If $x \in G/P'$, and $Z \subset Y$ is its inverse image under the natural map $Y = G/P \to G/P'$, then $\mathscr{L} \otimes \mathcal{O}_Z \cong \mathcal{O}_Z$, so that, $f^*\mathscr{L} \otimes \mathcal{O}_{f^{-1}(Z)} = \mathcal{O}_{f^{-1}(Z)}$. Hence each connected component of $f^{-1}(Z)$ is mapped to a point by the linear system associated to $f^*\mathscr{L}$, i.e. by the natural morphism $Y \to G/\tau(P')$. But $(f')^{-1}(x)$ consists of the finite set of images in $G/\tau(P')$ of connected components of $f^{-1}(Z)$, and thus is finite.

In particular, $\dim G/P' = \dim G/\tau(P')$, so that $\dim P' = \dim \tau(P')$. If $P' \subset P''$, then the natural map $Y \to G/P''$ factors through the natural map $Y \to G/P'$. By the functoriality of the Stein factorization, there is a unique map $G/\tau(P') \to G/\tau(P'')$ making the following diagram commute:



Since $Y \to G/\tau(P')$ and $Y \to G/\tau(P'')$ are the natural maps, this means $\tau(P'') \subset \tau(P'')$. Thus τ preserves the partial ordering on $\mathfrak{p}$.

Next, if $P', P'' \in \mathfrak{p}$, the natural map $G/P \to G/P' \times G/P''$ factors into the natural maps $G/P \twoheadrightarrow G/P' \cap P'' \hookrightarrow G/P' \times G/P''$. Consider the diagram

$$\begin{array}{ccc} Y & \longrightarrow & G/\tau(P') \times G/\tau(P'') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f' \times f''} \\ Y & \longrightarrow & G/P' \times G/P'' \end{array}$$

which yields the diagram with surjective arrows

$$\begin{array}{ccc} Y & \longrightarrow & G/\tau(P') \cap \tau(P'') \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ Y & \longrightarrow & G/P' \cap P'' \end{array}$$

(this diagram defines $\bar{f}$). Note that $f' \times f''$, and hence $\bar{f}$, is finite. Since $\tau(P') \cap \tau(P'') \in \mathfrak{p}$ is connected, the fibres of $Y \to G/\tau(P') \cap \tau(P'')$ are connected. Thus we see that

$$\tau(P' \cap P'') = \tau(P') \cap \tau(P'')$$

by the uniqueness of the Stein factorization. Hence if

$$\tau(P') = \tau(P'') = \tau(P') \cap \tau(P'')$$

then $\dim P' = \dim P'' = \dim P' \cap P''$ since τ preserves dimensions. Since P', P'', $P' \cap P''$ are all connected they are equal. □

Let $\mathcal{M} \subset \mathfrak{p}$ be the subset of maximal parabolic subgroups $P' \subset G$ which contain P. From lemma 5, τ restricts to a bijection on $\mathcal{M}$. Let $\mathcal{M}_1 \subset \mathcal{M}$ be the subset consisting of parabolics P' such that for each $j > 0$, if $P'' = \tau^{j-1}(P')$, then

$$f'' \colon G/\tau(P'') \to G/P''$$

is an isomorphism. Clearly $\tau(\mathcal{M}_1) = \mathcal{M}_1$. If $P' \in \mathcal{M} - \mathcal{M}_1$, then for some $j > 0$, if $P'' = \tau^{j-1}(P')$,

$$f'' \colon G/\tau(P'') \to G/P''$$

has degree > 1. But τ is a bijection of a finite set, so that for some $n \geqq j$, $\tau^n(P') = P'$; thus the composite

$$G/P' = G/\tau^n(P') \to G/\tau^{n-1}(P') \to \ldots \to G/\tau(P') \to G/P'$$

is a finite self map of degree > 1. Hence, by proposition 2, we have $G/P' \cong \mathbf{P}^{n'}$ for some n'.

Let

$$P_0 = \bigcap_{P' \in \mathcal{M}_1} P', \quad X = G/P_0$$

and let $\mathcal{M} - \mathcal{M}_1 = \{P_1, \ldots, P_m\}$; then $\tau(P_i) = P_{\sigma(i)}$ for some permutation σ of $\{1, \ldots, m\}$, and $G/P_i \cong \mathbf{P}^{n_i}$ for some integer $n_i > 0$, for $i = 1, \ldots, m$. Since $\dim G/P' = \dim G/\tau(P')$, we have $n_i = n_{\sigma(i)}$ for all i. Since $\tau(\mathcal{M}_1) = \mathcal{M}_1$, we have $\tau(P_0) = P_0$ by lemma 5. Let

$$f_0 \colon X = G/P_0 = G/\tau(P_0) \to G/P_0 = X \ ,$$

and

$$f_i \colon G/P_{\sigma(i)} = G/\tau(P_i) \to G/P_i, \quad i > 0 \ ,$$

be the maps induced by f as constructed above. Then f_0 is an isomorphism by the choice of $\mathcal{M}_1$. For each $i > 0$, f_i can be written as a composite

$$G/P_{\sigma(i)} \cong \mathbf{P}^{n_i} \xrightarrow{\pi_i} \mathbf{P}^{n_i} \cong G/P_i$$

where π_i is a finite self map of the projective space. Then we have a commutative square

$$\begin{array}{ccc}
Y & \longrightarrow & X \times \prod_{i=1}^{m} G/P_i \\
\Big\downarrow f & & \Big\downarrow f_0 \times \prod_{i=1}^{m} f_i \\
Y & \longrightarrow & X \times \prod_{i=1}^{m} G/P_i
\end{array}$$

where the horizontal maps are closed embeddings, since (see [B])

$$P = \bigcap_{P' \in \mathcal{M}} P' .$$

Thus to finish the proof of the theorem, we only need to show that

$$\Psi : Y = G/P \to X \times \prod_{i=1}^{m} G/P_i$$

is surjective (and hence an isomorphism). Replacing f by an iterate does not change the subset $\mathcal{M}_1 \subset \mathcal{M}$; hence to show that Ψ is an isomorphism, we may replace f by an iterate so that, without loss of generality, we may assume that σ is the identity permutation. Thus f_i: $G/P_i \to G/P_i$ is a finite self map, which is an isomorphism for $i = 0$, and a map $\mathbf{P}^{n_i} \to \mathbf{P}^{n_i}$ of degree > 1 for $1 \leqq i \leqq m$.

Fix an integer $j \in \{1, \ldots, m\}$. If F is a fibre of the natural map

$$Y \xrightarrow{p} X \times \prod_{i \neq j} G/P_i$$

then $\dim F > 0$ and F maps isomorphically to its image $\bar{F}$ under the natural map

$$Y \xrightarrow{q} G/P_j = \mathbf{P}^{n_j} .$$

Further, p, q are G-equivariant (for the left G-action), so that the translate gF (which is another fibre of p) maps isomorphically under q to $g\bar{F}$. From proposition 1 applied to f_j: $\mathbf{P}^{n_j} \to \mathbf{P}^{n_j}$, $f_j^{-1}(g\bar{F})$ is irreducible and has multiplicity 1 as a cycle, for all g in a non-empty Zariski open set in G. If F' is an irreducible component of $f^{-1}(F)$, then from the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & X \times \prod_{i \neq j} G/P_i \\ \Big\downarrow f & & \Big\downarrow f_0 \times \prod_{i \neq j} f_i \\ Y & \xrightarrow{p} & X \times \prod_{i \neq j} G/P_i \end{array}$$

we see that F' is also a fibre of p (since $\dim F' = \dim F$, and the non-empty fibres of p are precisely the translates gF). Now $q(F') = \bar{F}' \subset \mathbf{P}^{n_j}$ is contained in the inverse image $f_j^{-1}(\bar{F})$. Thus, replacing F by a translate gF, so that $f_j^{-1}(\bar{F})$ is irreducible, we have $f_j^{-1}(\bar{F}) = \bar{F}'$ as a cycle. But $\bar{F}'$ is a translate of $\bar{F}$. Hence if $s = \dim F$,

$$[\bar{F}] = [\bar{F}'] = f_j^*[\bar{F}] \in H^{2n_j - 2s}(\mathbf{P}^{n_j}, \mathbf{Z}) .$$

Since $\deg f_j > 1$, this forces $s = n_j$ i.e. $\bar{F} = \mathbf{P}^{n_j}$. Hence $Y \subset X \times \prod_{i=1}^{m} \mathbf{P}^{n_i}$ is the inverse image of its projection to $X \times \prod_{i \neq j} \mathbf{P}^{n_i}$, for all $j \in \{1, \ldots, m\}$. Since $Y \to X$

induced by Ψ is just the natural surjection $G/P \to G/P_0$, we see that Ψ is a bijection. This completes the proof of the theorem. □

2. Maps between Grassmann varieties

Let $\mathbf{G}(k, N)$ denote the Grassmann variety of k-dimensional quotients of an N-dimensional vector space over $\mathbf{C}$.

Proposition 6. *Let $k \leqq n, 2 \leqq l \leqq m$ be integers, such that there exists a finite surjection morphism between Grassmann varieties*

$$f\colon \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m).$$

Then $k = l$, $m = n$ and f is an isomorphism.

Proof. Let $Z \subset \mathbf{G}(l, l+m)$ be an irreducible subvariety such that $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank r. Then an analogous statement holds for any translate of Z by $\mathrm{GL}_{l+m}(\mathbf{C})$ (regarding $\mathbf{G}(l, l+m)$ as a homogeneous space for $\mathrm{GL}_{l+m}(\mathbf{C})$). Replacing Z by a translate, we may assume that Z is not contained in the branch locus of f. Then if Z' is any irreducible component of $f^{-1}(Z)_{\mathrm{red}}$, we see that $f^* T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z'}$ has a trivial direct summand of rank r. Further, the natural map (induced by df)

$$T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z'} \to f^* T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z'}$$

is an injection of locally free sheaves which is an isomorphism at the generic point of Z'. Hence, $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z'}$ has a map to $\mathcal{O}_{Z'}^{\oplus r}$ which is generically surjective; since $T_{\mathbf{G}(k,k+n)}$ is generated by global sections, $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z'}$ has a trivial direct summand of rank r. As a consequence, if $\mathbf{G}(l, l+m)$ has a subvariety Z of dimension d such that $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z'}$ has a trivial direct summand of rank r, then $\mathbf{G}(k, k+n)$ also has such a subvariety of dimension d. We will use this to prove that $k = l, m = n$.

Let s, t be integers with $1 \leqq s \leqq l, 1 \leqq t \leqq m$. We have an embedding

$$X = \mathbf{G}(l-s, l+m-s-t) \to \mathbf{G}(l, l+m) = Y$$

which we may describe as follows: on X we have the universal quotient

$$\mathcal{O}_X^{\oplus l+m-s-t} \to \mathcal{Q}$$

where $\mathcal{Q}$ is locally free of rank $l-s$. This yields a quotient which is the composite

$$\mathcal{O}_X^{\oplus l+m} = \mathcal{O}_X^{\oplus l+m-s-t} \oplus \mathcal{O}_X^{\oplus s} \oplus \mathcal{O}_X^{\oplus t} \to \mathcal{Q} \oplus \mathcal{O}_X^{\oplus s} \oplus \mathcal{O}_X^{\oplus t} \to \mathcal{Q} \oplus \mathcal{O}_X^{\oplus s}.$$

By the universal property of $\mathbf{G}(l, l+m)$ this corresponds to the above morphism $X \to Y$. Let $Z_{s,t}$ denote the image. The universal exact sequence on $\mathbf{G}(l, l+m)$ restricted to $Z_{s,t}$ is the direct sum of the universal exact sequence on $\mathbf{G}(l-s, l+m-s-t)$ with the split sequence

$$0 \to \mathcal{O}_{Z_{s,t}}^{\oplus t} \to \mathcal{O}_{Z_{s,t}}^{\oplus s+t} \to \mathcal{O}_{Z_{s,t}}^{\oplus s} \to 0.$$

Thus, $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z_{s,t}}$ has a trivial direct summand of rank st. This is also true of any translate of $Z_{s,t}$ under $\mathrm{GL}_{l+m}(\mathbf{C})$.

Lemma 7. *Let Z be an irreducible subvariety of $\mathbf{G}(l, l+m)$. Suppose that $T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank r. Then there exists integers s, t with $1 \leqq s \leqq l$, $1 \leqq t \leqq m$ and $st \geqq r$, such that some $\mathrm{GL}_{l+m}(\mathbf{C})$ translate of $Z_{s,t}$ contains Z.*

Proof. Let $V = \mathbf{C}^{l+m}$, so that $\mathbf{G}(l, l+m)$ parametrizes l-dimensional quotients of V. If $x \in \mathbf{G}(l, l+m)$, then there is a corresponding l-dimensional quotient

$$V \to V/W_x$$

where W_x is of dimension m. Then the tangent space to $\mathbf{G}(l, l+m)$ at x is

$$T_x = \operatorname{Hom}(W_x, V/W_x) .$$

There is a surjection φ_x: $\operatorname{End}(V) \to T_x$ corresponding to a surjection of locally free sheaves

$$\mathcal{O}_{\mathbf{G}(l,l+m)} \otimes_{\mathbf{C}} \operatorname{End}(V) \to T_{\mathbf{G}(l,l+m)}$$

which gives rise to a morphism

$$\mathbf{P}(T_{\mathbf{G}(l,l+m)}) \to \mathbf{P}(\operatorname{End}(V)) ,$$

whose restriction to $\mathbf{P}(T_x)$ is induced by φ_x. Identifying $\operatorname{End}(V)$ with its dual space, we may identify the projective space $\mathbf{P}(\operatorname{End}(V))$ with the space of lines in $\operatorname{End}(V)$; then the subspace $\mathbf{P}(T_x)$ is the space of lines in $\operatorname{Hom}(V/W_x, W_x)$. Here we identify $\operatorname{Hom}(V/W_x, W_x)$ with

$$\{A \in \operatorname{End}(V) \mid \operatorname{im} A \subset W_x \subset \ker A\} .$$

Now if $Z \subset \mathbf{G}(l, l+m)$ is a subvariety, then by lemma 3,

$$T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_Z \text{ has a trivial direct summand of rank } r$$

$$\Leftrightarrow \dim\left(\bigcap_{x \in Z} \operatorname{Hom}(V/W_x, W_x) \right) \geqq r ,$$

where the intersection is taken in $\operatorname{End}(V)$. But if

$$V_1 = \bigcap_{x \in Z} W_x, \quad V_2 = \sum_{x \in Z} W_x ,$$

then $0 \subset V_1 \subset V_2 \subset V$, and

$$\bigcap_{x \in Z} \operatorname{Hom}(V/W_x, W_x) = \{A \in \operatorname{End}(V) \mid \operatorname{im} A \subset V_1 \subset V_2 \subset \ker A\}$$
$$= \operatorname{Hom}(V/V_2, V_1) .$$

If $t = \dim V_1$, $s = \dim V/V_2$, then

$$Z' = \{x \in \mathbf{G}(l, l+m) \mid V_1 \subset W_x \subset V_2\}$$

is a translate of $Z_{s,t}$ which clearly contains Z; also

$$\bigcap_{x \in Z'} \operatorname{Hom}(V/W_x, W_x) = \bigcap_{x \in Z} \operatorname{Hom}(V/W_x, W_x) = \operatorname{Hom}(V/V_2, V_1)$$

which has dimension st. Since the middle term has dimension $\geqq r$, we get $st \geqq r$. □

As a corollary, we observe that if $Z \subset \mathbf{G}(l, l+m)$ is such that $T_{\mathbf{G}(l, l+m)} \otimes \mathcal{O}_Z$ has a trivial direct summand, then

$$\dim Z \leqq \dim Z_{1,1} = \dim \mathbf{G}(l-1, l+m-2) = (l-1)(m-1) .$$

Hence if

$$f: \mathbf{G}(k, k+n) \to \mathbf{G}(l, l+m), \quad k \leqq n, 2 \leqq l \leqq m ,$$

is a finite morphism, $f^*(Z_{1,1})$ has an irreducible component $Z \subset \mathbf{G}(k, k+n)$ of dimension $(l-1)(m-1)$, such that $T_{\mathbf{G}(k, k+n)} \otimes \mathcal{O}_Z$ has a trivial direct summand. Applying lemma 7 to $\mathbf{G}(k, k+n)$, we see that

$$(k-1)(n-1) \geqq (l-1)(m-1) , \tag{1}$$

Since f is finite, we have $kn = lm$. Hence this implies

$$l + m \geqq k + n . \tag{2}$$

If equality holds, then since $kn = lm$, we must have $k = l$, $m = n$ and hence from the Theorem, f is an isomorphism. Hence it suffices to prove that strict inequality in (2) leads to a contradiction. Now

$$l + m > k + n \tag{3}$$

$$\Rightarrow (l+m)^2 > (k+n)^2$$

$$\Rightarrow (m-l)^2 > (n-k)^2; \quad \text{using } lm = kn$$

$$\Rightarrow m - l > n - k . \tag{4}$$

From (3) and (4),

$$m > n \geqq k > l \geqq 2 . \tag{5}$$

Let t_0 be the positive integer such that $t_0 l \geqq k$, while $(t_0 - 1)l < k$; then $t_0 \geqq 2$, as $k > l$. Let s be any integer satisfying

$$\frac{(k - t_0 - 1)m + 1}{k} \leqq s \leqq \frac{(k - t_0 + 1)m - 1}{k} . \tag{6}$$

Since $m > k$, there are at least two integers s satisfying (6). Then $(t_0 - 1)l \leqq (k-1)$ implies that

$$k - t_0 - 1 \geqq (l-1)t_0 - l \geqq 2(l-1) - l = l - 2 \geqq 0 ;$$

also, $(k - t_0 - 1) \leqq (k-1)$, so that

$$ks \leqq (k-1)m - 1 < km .$$

Hence for any s satisfying (6),

$$1 \leqq s \leqq (m-1) . \tag{7}$$

Consider the subvariety $Z_{1, m-s} \subset \mathbf{G}(l, l+m)$, for s satisfying (6). The subvariety is isomorphic to $\mathbf{G}(l-1, l-1+s)$, so it has dimension $(l-1)s$; also

$T_{\mathbf{G}(l,l+m)} \otimes \mathcal{O}_{Z_{1,m-s}}$ has a trivial direct summand of rank $m-s$. Hence there exists an irreducible component $Z \subset f^{-1}(Z_{1,m-s}) \subset \mathbf{G}(k, k+n)$ such that $\dim Z = (l-1)s$, and $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_Z$ has a trivial direct summand of rank $m-s$. By lemma 7 applied to $\mathbf{G}(k, k+n)$, there exist integers r, t with $1 \leqq t \leqq k-1$, $1 \leqq r \leqq n-1$ such that $Z \subset Z_{t,n-r} \subset \mathbf{G}(k, k+n)$. Now $Z_{t,n-r}$ has dimension $(k-t)r$ and $T_{\mathbf{G}(k,k+n)} \otimes \mathcal{O}_{Z_{t,n-r}}$ has a trivial direct summand of rank $t(n-r)$. Thus, t, r satisfy the system of inequalities:

$$r(k-t) \geqq s(l-1) \tag{8}$$

$$t(n-r) \geqq m-s\,. \tag{9}$$

Now $k-t \geqq 1$, so that (8) implies that

$$r \geqq \frac{s(l-1)}{k-t}\,, \tag{10}$$

while (9) implies that

$$r \leqq \frac{tn-m+s}{t}\,. \tag{11}$$

Combining (10) and (11), we obtain

$$(k-t)(tn-m+s) \geqq (l-1)ts\,.$$

Substituting $n = lm/k$, we get

$$(k-t)\left(\frac{tlm}{k} - m + s\right) \geqq (l-1)ts$$

$$\Rightarrow (k-t)\left(\frac{tlm}{k} - m\right) \geqq (tl-k)s$$

$$\Rightarrow ((k-t)m - ks)(tl-k) \geqq 0\,. \tag{12}$$

If $t > t_0$, then $tl - k > 0$, so that (12) yields $(k-t)m \geqq ks$. From (6), we get

$$(k-t)m \geqq (k-t_0-1)m + 1$$

$$\Rightarrow (t_0 - t + 1)m \geqq 1$$

$$\Rightarrow (t_0 + 1 - t) > 0$$

contradicting $t > t_0$.

If $t < t_0$, then $tl - k < 0$, and so (12) yields $(k-t)m \leqq ks$. From (6), we get

$$(k-t)m \leqq (k-t_0+1)m - 1$$

$$\Rightarrow (t - t_0 + 1)m \geqq 1$$

$$\Rightarrow (t + 1 - t_0) > 0$$

contradicting $t < t_0$.

Hence we must have $t = t_0$. Now if $t_0 l - k > 0$, then (12) again gives

$$(k-t_0)m \geqq ks\,. \tag{13}$$

But the interval

$$\left(\frac{(k-t_0)m}{k}, \frac{(k-t_0+1)m-1}{k}\right]$$

contains an integer s (which then satisfies (6)), since

$$(k-t_0+1)m-1-(k-t_0)m \geqq k ,$$

and this contradicts (13).

Hence, we are forced to choose $t = t_0$, where $t_0 l = k$. In this case, the inequalities (8) and (9) become (using $lm = kn = t_0 ln$, so that $m = t_0 n$)

$$r(t_0 l - t_0) \geqq s(l-1)$$

$$t_0(n-r) \geqq t_0 n - s$$

which yield the pair of inequalities

$$(rt_0 - s)(l-1) \geqq 0$$

$$0 \geqq rt_0 - s .$$

Since $l \geqq 2$, this forces $rt_0 = s$, so that t_0 divides s; also $t_0 \geqq 2$. But there are atleast two consecutive integers s satisfying (6); so we may choose s satisfying (6) but with $t_0 \nmid s$. Hence, in all cases, for some value of s satisfying (6), it is impossible to find any r, t with $1 \leqq t \leqq k-1$, $1 \leqq r \leqq n-1$ such that (8) and (9) hold. This proves Proposition 8. □

3. Remarks on Lazarsfeld's problem

In this section we show:

Proposition 8. *Let Y be a smooth quadric hypersurface,* $\dim Y = n \geqq 3$, *and $f\colon Y \to X$ be a finite surjective morphism of degree > 1 to a smooth variety X; then X is isomorphic to $\mathbf{P}^n$.*

Proof. We begin by reviewing the results of Mori [M]. He proves (Theorem 6 of [M]) that if X is a smooth, projective variety of dimension n such that the inverse of the canonical sheaf K_X^{-1} is ample, then for each $P \in X$, there is a non-constant morphism $u\colon \mathbf{P}^1 \to X$ with $\deg u^*(K_X^{-1}) \leqq n+1$, such that $P \in u(\mathbf{P}^1)$.

Now fix $P \in X$, and let $* \in \mathbf{P}^1$ be a fixed point. Assume that K_X^{-1} is ample, and let

$$d = \min\{\deg u^*(K_X^{-1}) | u\colon (\mathbf{P}^1, *) \to (X, P);\ u \text{ is non-constant}\} .$$

Then $d \leqq n+1$, and any $u\colon \mathbf{P}^1 \to X$ achieving this minimal degree is birational to its image. Let V be a connected component of $\underline{\mathrm{Hom}}^d((\mathbf{P}^1, *), (X, P))$, the scheme of morphisms $u\colon \mathbf{P}^1 \to X$ of degree d. Then $G = \mathrm{Aut}(\mathbf{P}^1, *)$ acts on V. Let $\tilde{V}$ be an irreducible component of the normalization of V. Then the G action on $\tilde{V}$ is proper and free, with a geometric quotient $\gamma\colon \tilde{V} \to W$, where W is a normal projective variety and γ is a principal G-bundle.

Assume further that for all $u \in V$,

$$H^1(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0 .$$

Then from the Riemann-Roch theorem,

$$\dim H^0(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = d \, ,$$

and Mori's arguments show that V is smooth of dimension d; hence W is smooth (and projective) of dimension $d-2$. Further, if $u^* T_X$ is ample for all $u \in V$, then

$$d = n+1, \quad W \cong \mathbf{P}^{n-1}, \quad \text{and } X \cong \mathbf{P}^n \, .$$

We now specialize to the situation when there is a finite surjective morphism $f\colon Y \to X$, where Y is a smooth quadric of dimension $n \geqq 3$. Let $B \subset X$ be the branch locus, and $R \subset Y$ be the ramification locus so that

$$f^* K_X \otimes K_Y^{-1} \cong \mathcal{O}_Y(-R) \text{ and } B = f(R)$$

(since Y is simply connected and every automorphism of Y has fixed points by the Lefschetz fixed point formula, R and B are effective and non-zero). As Pic $Y = \mathbf{Z}$, $\mathcal{O}_Y(R)$ is ample, and K_Y^{-1} is ample since Y is a quadric, K_X^{-1} is ample, and Mori's results apply.

Let $U = f(Y-R)$, so that $X - U \subset B$. Let $P \in U$, and let $Q \in Y - R$ with $f(Q) = P$. Then if $u\colon (\mathbf{P}^1, *) \to (X, P)$ is a curve such that $d = \deg u^* K_X^{-1}$, and C is the normalization of any irreducible component of $f^{-1}(u(\mathbf{P}^1))$ which passes through Q, then we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \xrightarrow{u} & X \end{array}$$

There is a map of locally free sheaves

$$v^* T_Y \to h^* u^* T_X$$

which is an isomorphism at the generic point of C, as $C \not\subset R$. Now

$$u^* T_X \cong \mathcal{O}_{\mathbf{P}^1}(m_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(m_n) \, ,$$

with $m_1 \leqq \ldots \leqq m_n$; since T_Y is generated by global sections, we see that $h^* u^* T_X$ is generated at the generic point of C by its global sections, so that $m_i \geqq 0$ for all i. Further,

$$v^* T_Y \cong \mathcal{O}_C^{\oplus r} \oplus \mathcal{E}$$

where $\mathcal{E}$ is an ample locally free sheaf, and $r \geqq 0$. Hence, $m_i > 0$ for all $i > r$. Also, the inclusion of sheaves $T_{\mathbf{P}^1} \to u^* T_X$ shows that $m_n \geqq 2$. In any case, $H^1(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-1)) = 0$, so that V is smooth of dimension d and W is smooth of dimension $d-2$.

Lemma 9. *If Y is a smooth quadric of dimension $n \geqq 3$, and $v\colon C \to Y$ a non-constant morphism from an irreducible projective curve C, then either $v^* T_Y$ is ample, or $v(C)$ is a line contained in Y and in this case,*

$$v^*(T_Y) \cong \mathcal{O}_C \oplus \mathcal{E} \, ,$$

where $\mathcal{E}$ is ample.

Proof. We note that Y is the space of isotropic lines in a quadratic space. As in section 1, we have a natural morphism

$$\mathbf{P}_Y(T_Y) \to \mathbf{P}(\operatorname{Lie} G),$$

where G is the corresponding orthogonal group. We may then identify $\operatorname{Lie} G$ with the space of skew-symmetric matrices, and for any $p \in Y$ the tangent space $T_{p,Y} = \operatorname{Hom}(p, p^{\perp}/p)$. From this it follows easily that, for any $p, q \in Y$ the linear subspaces $\mathbf{P}(T_{p,Y})$ and $\mathbf{P}(T_{q,Y})$ of $\mathbf{P}(\operatorname{Lie} G)$ intersect if and only if the lines p and q are orthogonal. Thus, from lemma 3, if $v(C) \subset Y$ is an irreducible curve such that $v^*(T_Y)$ has a trivial direct summand, then $v(C)$ lies in the projective space of an isotropic subspace of $\mathbf{P}^{n+1}$, i.e. for some $t > 0$,

$$v(C) \subset \mathbf{P}^t \subset Y \subset \mathbf{P}^{n+1}.$$

For a linear subspace $\mathbf{P}^t \subset Y$, we have the diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 \to T_{\mathbf{P}^t} \to & & T_Y|_{\mathbf{P}^t} & \to & N & & \to 0 \\
\| & & \downarrow & & \downarrow & & \\
0 \to T_{\mathbf{P}^t} \to & & T_{\mathbf{P}^{n+1}}|_{\mathbf{P}^t} & \to & \mathcal{O}_{\mathbf{P}^t}(1)^{\oplus n+1-t} & & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_{\mathbf{P}^t}(2) & = & \mathcal{O}_{\mathbf{P}^t}(2) & & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}$$

The middle row is split exact since $H^1(\mathbf{P}^t, T_{\mathbf{P}^t}(-1)) = 0$, hence so is the top row. Tensoring the last column with $\mathcal{O}_{\mathbf{P}^t}(-1)$ we see that

$$N \cong \Omega^1_{\mathbf{P}^t}(2) \oplus \mathcal{O}_{\mathbf{P}^t}(1)^{\oplus(n-2t)}.$$

Hence,

$$T_Y|_{\mathbf{P}^t} \cong \Omega^1_{\mathbf{P}^t}(2) \oplus \mathscr{E}$$

where $\mathscr{E}$ is ample. Thus $v^*(T_Y)$ has a trivial direct summand if and only if $v^*(\Omega^1_{\mathbf{P}^t}(2))$ has one. So it suffices to show that this is possible only if $v(C)$ is a line in $\mathbf{P}^t$. Taking duals, if

$$v^*(T_{\mathbf{P}^t}(-1)) \cong v^*(\mathcal{O}_{\mathbf{P}^t}(1)) \oplus \mathscr{F},$$

then $\mathscr{F}$ is generated by global sections and has trivial determinant and is thus trivial. But this clearly implies that $v(C)$ is a line, in which case we may take $t = 1$, and this yields the second conclusion. □

If $C \subset Y$ is a line then $r = 1$ and $\deg v^* T_Y = n$. Hence, in any case, $m_1 \geqq 0$, $m_i \geqq 1$ for $i > 1$, and $m_n \geqq 2$, so that $d \geqq n$, and $d - 2 = \dim W > 0$. Thus there are infinitely many distinct rational curves through P with minimal degree d.

We now consider two cases.

Case 1. *For some* $P \in U$, $\{u \in V \mid u^* T_X$ is not ample$\}$ *consists of atmost finitely many* G *orbits.*

In this case, $d = n + 1$, and W is smooth and projective of dimension $n - 1$. If $u \in V$ such that $u^* T_X$ is ample, then

$$u^* T_X \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n-1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)\,.$$

Hence u is an immersion. On the other hand, if $u^* T_X$ is not ample, then we have a diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \xrightarrow{u} & X \end{array}$$

where v is the embedding of a line in Y through $Q \in (f^{-1}(P) - R)$. Then $C \to X$ is unramified at $v^{-1}(Q)$, and so u is unramified at $h(v^{-1}(Q))$. This is valid for each irreducible component C of $f^{-1}(u(\mathbf{P}^1))$ through Q, and so u is unramified at $* \in \mathbf{P}^1$.

Hence, if we fix a non-zero tangent vector $t \in T_{*,\mathbf{P}^1}$ then the assignment $u \mapsto du(t)$ gives a morphism

$$\eta\colon V \to (T_{P,X} - \{0\}) \cong \mathbf{A}^n - \{0\}$$

which yields a commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\eta} & \mathbf{A}^n - \{0\} \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow{\delta} & \mathbf{P}^{n-1} \end{array}$$

As in Mori's paper [M], we see that if $u \in V$ such that $u^* T_X$ is ample, then η is smooth along the G orbit of u in V, and so δ is étale at $\gamma(u) \in W$. By assumption, this means that δ is étale outside a finite set. Since $n \geqq 3$ this means that δ is étale, and hence an isomorphism. On the other hand, if $u \in V$ such that $u^* T_X$ is not ample, then

$$H^1(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-2)) \neq 0\,.$$

The Zariski tangent space to the fibre of η at u is

$$H^0(\mathbf{P}^1, u^* T_X \otimes \mathcal{O}_{\mathbf{P}^1}(-2))$$

which has dimension > 1; hence η is not smooth at u. But $\delta \circ \gamma$ is a principal G-bundle so that η is a principal G_1-bundle, where G_1 is the subgroup of G fixing the tangent vector t. Hence η is smooth, and so $u^* T_X$ is ample for all $u \in V$. As in [M], this implies that $X \cong \mathbf{P}^n$.

Case 2. *For each* $P \in U$, $\{u \in V \mid u^* T_X$ is not ample$\}$ *consists of infinitely many* G *orbits.*

Since there are only a finite number of lines in Y joining distinct points of $f^{-1}(P)$, we see that there exists $u \in V$ such that $u^* T_X$ is not ample, and in the diagram

$$\begin{array}{ccc} C & \xrightarrow{v} & Y \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbf{P}^1 & \xrightarrow{u} & X \end{array}$$

C is a line such that $C \cap f^{-1}(P) = \{Q\}$. Since f is unramified at $Q, f \circ v$ is birational, and h is an isomorphism. Thus

$$\begin{aligned} n + 1 \geqq \deg h^* u^* K_X^{-1} &= \deg v^* K_Y^{-1} + \deg v^* \mathcal{O}_Y(R) \\ &= n + \deg v^* \mathcal{O}_Y(R) \geqq n + 1 . \end{aligned}$$

Hence we must have $d = n + 1$, and $\mathcal{O}_Y(R) = \mathcal{O}_Y(1)$. In particular, R is a hyperplane section of Y, and is reduced and irreducible; so f is simply ramified (has ramification index two) at the generic point of R. Thus $\dim V = n + 1$, and $\dim W = n - 1$.

As in case 1, we see that, by fixing a non-zero tangent vector $t \in T_{*,\mathbf{P}^1}$, we obtain a diagram

$$\begin{array}{ccc} V & \xrightarrow{\eta} & \mathbf{A}^n - \{0\} \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \pi} \\ W & \xrightarrow{\delta} & \mathbf{P}^{n-1} \end{array}$$

where for $u \in V$ such that $u^* T_X$ is ample, δ is étale at $\gamma(u)$.

The cone of lines in Y through Q is parametrized by a smooth quadric hypersurface $Z \subset \mathbf{P}(T^*_{Q,Y}) \cong \mathbf{P}(T^*_{P,Y})$. Since for any line $C \subset Y, \deg f^* K_X^{-1} \otimes \mathcal{O}_C = n + 1$, we see that $f|_C$ is birational for any line C meeting $f^{-1}(U)$. Thus we obtain a morphism $\zeta : Z \to W$ such that the composite

$$\delta \circ \zeta : Z \to \mathbf{P}^{n-1} = \mathbf{P}(T^*_{P,X})$$

is the natural embedding. Clearly the non-étale locus of δ is contained in $\zeta(Z)$. Hence δ is a finite morphism between smooth varieties, and its non-étale locus is a divisor, which must equal $\zeta(Z)$ if δ is not an isomorphism. As in case 1, if δ is an isomorphism, then $u^* T_X$ is ample for all $u \in V$, contradicting the hypothesis of case 2. Hence for every line $C \subset Y$ through $Q, f^* T_X \otimes \mathcal{O}_C$ is not ample on C.

We claim that $U = X - B$ i.e. $f^{-1}(B) = R$. If not, we can find $P \in U$ with Q, Q' in $f^{-1}(P)$, where $Q \notin R$, and $Q' \in R$, such that P is a smooth point of B, and f is simply ramified at Q'. We can find a line $C \subset Y$ through Q such that $C_1 = f(C)$ is smooth at P and transverse to B at P. Then we can find another line $C' \subset Y$ through Q' which maps birationally to C_1, since every irreducible component of $f^{-1}(C_1)$ must be a line. However simple ramification at Q' implies that $df(T_{Q',Y}) \subset T_{P,B}$. Since C_1 is transverse to B at P, this is a contradiction.

Now R is a hyperplane section of the smooth quadric Y of dimension $\geqq 3$; hence $\pi_1(Y-R)=0$ (this is clear if R is singular as $Y-R\cong \mathbf{A}^n$; if R is smooth, this follows from the facts (i) $\pi_1(Y-R)$ is abelian, and (ii) $H_1(Y-R, \mathbf{Z})=0$). Thus $Y-R$ is the universal covering space of $X-B$. In particular there is a finite group H of automorphisms of Y, which acts freely on $Y-R$, such that $X=Y/H$ (the automorphisms in H of $Y-R$ extend to Y as Y is the normalization of X in $\mathbf{C}(Y)$). Since f is simply ramified at the generic point of R, the inertia group of the corresponding discrete valuation on $\mathbf{C}(Y)$ has order two. The involution σ generating this inertia group extends to the ambient projective space $\mathbf{P}^{n+1}$, fixes the hyperplane spanned by R and has no other fixed points on Y. Thus σ has one other isolated fixed point in $\mathbf{P}^{n+1}-Y$ and the quotient map $Y\to Y/\langle\sigma\rangle$ is induced by the projection from this fixed point. Thus we have a factorization

$$\begin{array}{ccc} Y & \longrightarrow & \mathbf{P}^n \\ & f\searrow \quad \swarrow & \\ & X & \end{array}$$

From the result of Lazarsfeld mentioned in the introduction (see [L]), we must have $X\cong \mathbf{P}^n$, so that T_X is ample on every curve in X, contradicting the hypothesis of case 2. □

References

[B] Borel, A.: Linear Algebraic Groups. New York, W.A. Benjamin, 1969

[L] Lazarsfeld, R.: Some applications of the theory of Positive vector bundles. In: Complete Intersections, C.I.M.E., Acireale 1983. Lect. Notes. Math. vol. 1092, Berlin, Heidelberg, New York: Springer, 1984

[M] Mori, S.: Projective manifolds with ample tangent bundles. Ann. Math. **110**, 593–606 (1979)

[Mu] Mumford, D.: Lectures on Curves on Algebraic Surfaces. Ann. Math. Stud. **59**, Princeton: Princeton Univ. Press, 1966

[N] Nori, M.V.: Zariski's Conjecture and related problems. Ann. Sci. Ec. Norm. Super. **16**, 305–344 (1983)

Oblatum 13-X-1988

Note added in proof

1. Problem 2 has been answered affirmatively by O. Debarre.
2. The following observation by P. Polo and M.S. Raghunathan can be used to strengthen the Theorem. *If G is any simple connected, semi-simple algebraic group over* $\mathbf{C}$ *such that $G/P\cong \Pi\, G/P_i$, where P, P_i's are parabolic subgroups; then $G=\Pi\, G_i$ and there are parabolic groups $Q_i\subset G_i$ such that $P_i=p_i^{-1}(Q_i)$, where $p_i\colon G\to G_i$ is the projection.*
3. The Theorem has the following corollary: *Let G be a semi-simple, simply connected algebraic group over an algebraically closed field k with* $\operatorname{char} k=p>0$, *and let X be a projective homogeneous variety for G. Suppose X lifts to a smooth and proper scheme $\chi\to$* Spec *$W(k)$ over the Witt vectors of k, such that the absolute Frobenius morphism of X lifts to a morphism of χ (covering the Frobenius on $W(k)$). Then $X\cong \Pi\mathbf{P}^{n_i}$.*