INVARIANT FORMS FOR REPRESENTATIONS OF GL_2 OVER A LOCAL FIELD

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1. Introduction. Let $\mathbb{K}$ be a commutative separable cubic algebra over a nonarchimedean local field k. The aim of this paper is to study conditions under which irreducible, admissible representations of $GL_2(\mathbb{K})$ have $GL_2(k)$-invariant linear forms. This work is an extension of author's previous work [P] in which he studied this question for the case $\mathbb{K} = k \oplus k \oplus k$. In this paper we will only consider the case when $\mathbb{K}$ is either of the form $K \oplus k$ where K is a quadratic field extension of k, or is a cubic field extension of k.

Let D_k be the unique quaternion division algebra over k, and let $D_{\mathbb{K}} = D_k \otimes_k \mathbb{K}$. We recall that to a discrete series representation of $GL_2(k)$ (by which we will always mean an irreducible representation which has a twist whose matrix coefficients are square integrable modulo centre) there is a finite dimensional irreducible representation V' of $D_{\mathbb{K}}^*$ associated by Jacquet-Langlands which satisfies the character identity $\mathrm{ch}(V)(x) = -\mathrm{ch}(V')(x)$ at all the regular elliptic elements x of $GL_2(k)$. We extend this correspondence to one between representations of $GL_2(\mathbb{K})$ and representations of $D_{\mathbb{K}}^*$ as follows. If $\mathbb{K} = K \oplus k$, where K is a quadratic field extension of k, then $GL_2(\mathbb{K}) = GL_2(K) \times GL_2(k)$ and a representation V of $GL_2(\mathbb{K})$ is the tensor product $V_1 \otimes V_2$ of a representation V_1 of $GL_2(K)$ and a representation V_2 of $GL_2(k)$. In this case $D_{\mathbb{K}}^* = GL_2(K) \times D_{\mathbb{K}}^*$. We define the representation V' of $D_{\mathbb{K}}^*$ to be $V_1 \otimes V_2'$ if the representation V_2 of $GL_2(k)$ is a discrete series representation, and to be the zero representation if V_2 is not a discrete series representation. If $\mathbb{K}$ is a cubic field extension of k then $D_{\mathbb{K}}$ is the unique quaternion division algebra over the field $\mathbb{K}$ and we let V' be the representation of $D_{\mathbb{K}}^*$ associated to the representation V of $GL_2(\mathbb{K})$ (by the Jacquet-Langlands correspondence) if it is a discrete series representation, and zero otherwise.

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In this paper we will be dealing mainly with *generic* representations of $GL_2(\mathbb{K})$ by which we will mean any infinite dimensional representation of $GL_2(\mathbb{K})$ if $\mathbb{K}$ is a cubic field extension of k, and if $\mathbb{K} = K \oplus k$ to be the tensor product of an infinite dimensional representation of $GL_2(K)$ with an infinite dimensional representation of $GL_2(k)$.

We now state the principal theorems proved in this paper.

THEOREM A. *For an irreducible, admissible representation V of $GL_2(\mathbb{K})$, the space of $GL_2(k)$-invariant linear forms on V is at most one-dimensional.*

THEOREM B. *Let V be an irreducible, admissible, generic representation of $GL_2(\mathbb{K})$ such that the central character of V restricted to $k^* \subseteq \mathbb{K}^*$, is trivial. Then either there exists a $GL_2(k)$-invariant linear form on V which is unique up to scalars or the representation V' of $D^*_{\mathbb{K}}$ is a non-zero representation with a $D^*_{\mathbb{K}}$-invariant linear form which is also unique up to scalars. Moreover, only one of the two possibilities occurs.*

We will see in Section 4 that in the case when $\mathbb{K} = K \oplus k$ and the representation V of $GL_2(\mathbb{K})$ is $V_1 \otimes Sp_k$ where V_1 is a discrete series representation of $GL_2(\mathbb{K})$, and Sp_k is the special (or, Steinberg) representation of $GL_2(k)$, this theorem reduces to the following theorem about representations of $GL_2(K)$ which is the local analogue of a theorem of Jacquet and Lai [J-Lai] (but does not follow from their global theorem). This theorem was independently and simultaneously obtained by Jeff Hakim in his doctoral dissertation and appears as Theorem 2 in his announcement in [Ha], and also by Y. Flicker.

THEOREM C. *For K a quadratic extension of the local field k, and V a discrete series representation of $GL_2(K)$, V has a $GL_2(k)$-invariant linear form iff V has a D^*_k-invariant linear form.*

In Section 8 we construct an 8-dimensional representation, to be denoted by $M^k_{\mathbb{K}}\sigma_\pi$, of the Deligne-Weil group of k associated to an irreducible admissible representation π of $GL_2(\mathbb{K})$ and prove the following theorem about the epsilon factor associated to $M^k_{\mathbb{K}}\sigma_\pi$. In this theorem and in the rest of the paper we will use $\omega_{\mathbb{K}/k}$ for the following quadratic character of k^*.

(a) If $\mathbb{K} = K \oplus k$ with K a quadratic field extension of k then $\omega_{\mathbb{K}/k} = \omega_{K/k}$ where $\omega_{K/k}$ is the quadratic character of k^* associated by local class field theory to K.

(b) If $\mathbb{K}$ is a cubic field extension of k then $\omega_{\mathbb{K}/,k}$ will be the trivial character of k^* if $\mathbb{K}$ is Galois over k, and will be the quadratic character $\omega_{L/k}$ of k^* if $\mathbb{K}$ is not Galois over k and L is the unique quadratic extension of k contained in the Galois closure of $\mathbb{K}$.

THEOREM D. *For π an irreducible, admissible, generic representation of $GL_2(\mathbb{K})$ which is not supercuspidal, ψ a nontrivial additive character of k, $\epsilon(M^k_{\mathbb{K}}\sigma_\pi, \psi) \cdot \omega_{\mathbb{K}/k}(-1)$ is independent of ψ and takes the value $+1$ iff the representation π of $GL_2(\mathbb{K})$ has a $GL_2(k)$-invariant linear form, and takes the value -1 otherwise.*

Remark. In the case $\mathbb{K} = k \oplus k \oplus k$, this result about epsilon factors was suggested by M. Harris and proved in [P] for all generic representations when the residue characteristic of k is not equal to 2. Needless to say, we expect Theorem D to be true for all generic representations.

If $\mathbb{K} = K$ is a cyclic cubic field extension of k then as we will see in Section 8, the statement of Theorem D is equivalent to the following theorem (which has no reference to epsilon factors).

THEOREM E. *Let K/k be a cyclic cubic extension and π an irreducible admissible generic representation of $GL_2(K)$ whose central character restricted to k^* is trivial. Then π has a $GL_2(k)$-invariant linear form iff $\pi \otimes \pi' \otimes \pi''$ has a $GL_2(K)$-invariant linear form where π' and π'' are the representations of $GL_2(K)$ obtained by using the two nontrivial automorphisms of K/k.*

Because of Theorem B, it suffices to prove Theorem E in the analogous situation of quaternion division algebras. The proof of this statement about finite dimensional representations of quaternion division algebras has been obtained in odd residue characteristic by explicit character formulae. We intend to take this up elsewhere.

We now give a brief outline of the contents of the paper. In Section 2 we fix the notation and other preliminaries that we will be using throughout the paper. Large parts of this section are devoted to recalling the realization of a supercuspidal representation of $GL_2(k)$ as an induced representation from a maximal compact-modulo-centre subgroup, and the identities relating the characters of these representations and the character of the corresponding representation of the division algebra.

In Section 3 we prove that for a representation $V_1 \otimes V_2$ of $GL_2(K) \times GL_2(k)$, the space of $GL_2(k)$-invariant linear forms on $V_1 \otimes V_2$ is at most one-dimensional. The proof follows the method of Gelfand-Kazhdan. In Section 4 we prove Theorem B for $GL_2(K) \times GL_2(k)$. The proof is divided into several cases. In the case when the representation $V_1 \otimes V_2$ of $GL_2(K) \times GL_2(k)$ has V_2 supercuspidal, we compare (using the well-known results on orbital integrals of matrix-coefficients of supercuspidal representations) the representation $V_1 \otimes V_2$ of $GL_2(K) \times GL_2(k)$ to a representation $P \otimes V_2$ of $GL_2(K) \times GL_2(k)$ where P is a principal series representation of $GL_2(K)$ with the same central character as V_1 for which Theorem B was one of the previous case. This comparison has been possible because of a theorem of Casselman (and Silberger in odd residue characteristic) that the representations V_1 and P of $GL_2(K)$ are isomorphic on a compact-modulo-centre subgroup of $GL_2(K)$ except for finite dimensional parts. Theorem B for the representation $V \otimes Sp_k$ where V is a supercuspidal representation of $GL_2(K)$ is reduced to Theorem C in Section 4, and Theorem C is proved in Section 5. For the proof of Theorem C also, the comparison of the representation V of $GL_2(K)$ to a principal series representation has been one of the important steps.

In Section 6 we prove Theorem B for GL_2 over a cubic extension K/k. The proof is divided into several cases and again the only case which gives any trouble is the supercuspidal case. The idea of the proof in the supercuspidal case is as follows (actually one simple case has to be treated differently). Suppose that the representation V of $GL_2(K)$ is induced from a finite dimensional representation W of a maximal compact-modulo-centre subgroup $\mathcal{K}$ of $GL_2(K)$. By Mackey theory, the representation V restricted to $GL_2(k)$ is a sum of representations of $GL_2(k)$, one of which is induced from the representation W restricted to $\mathcal{K} \cap GL_2(k)$, to $GL_2(k)$. We prove that W has a $\mathcal{K} \cap GL_2(k)$-invariant linear form iff the representation V' of D_K^* does not have a D_k^*-invariant linear form. This turns out to be quite easy, and is in fact omitted as the proof is similar to and even simpler than the proof of Theorem C given in Section 5. Most of the work in this section is in proving that the other constituents of the representation V restricted to $GL_2(k)$ coming from the Mackey theory, do not have a $GL_2(k)$-invariant linear form.

In Section 7 we introduce the concept of multiplicative induction and state without proof some of its basic properties which are used in this paper.

In Section 8 we prove Theorem D. The proofs here either follow from the basic properties of the epsilon factor, or are reduced to a theorem of Tunnell which tells, in terms of epsilon factors, when a character of a quadratic field extension of k appears in an irreducible representation of $GL_2(k)$. In one case we have to appeal to a theorem of Y. Flicker and H. Jacquet according to which, for K a quadratic field extension of k, a representation of $GL_2(K)$ with trivial central character has a $GL_2(k)$-invariant linear form iff it is the base change of a representation of $GL_2(k)$ with non-trivial central character.

In Section 9 we give an application of the Theorem D on epsilon factors for $\mathbb{K} = k \oplus k \oplus k$ (proved in [P] for the odd residue characteristic). We also deduce the equivalence of Theorem D with Theorem E in the case of cyclic cubic extension.

We end this introduction by mentioning that the results of this paper should have applications to the central critical value of the Rankin triple product L-function, cf. [Ps-R], of an automorphic representation of a cubic algebra over a number field just as in the case $k \oplus k \oplus k$, treated by M. Harris and S. Kudla in [H-K].

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2. Notation and other preliminaries. In this paper k will always denote a fixed nonarchimedean local field with $\mathcal{O}_k$ as the ring of integers, π_k as a uniformizing parameter, v_k as the valuation, and q the cardinality of the residue field. The absolute value on k will be $\|x\|_k = q^{-v_k(x)}$. When there is no danger of confusion, we will omit the subscript k. The corresponding objects for any extension K of k will be denoted by the subscript K.

In this paper we will always use normalized induction. For a character χ of $B(K)$, the group of upper triangular matrices, $V_\chi = \mathrm{Ind}_{B(K)}^{GL_2(K)}\chi$ is the space of functions f on $GL_2(K)$ such that $f(bg) = \delta_K^{1/2}(b)\chi(b)$, $b \in B(K)$ and δ_K is the character of $B(K)$ given by

$$\delta_K\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \frac{\|\alpha\|_K}{\|\delta\|_K}.$$

For a subgroup H of G and a representation V of H, $\mathrm{ind}_H^G V$ will denote the compact induction.

For a representation V of a group G, V^* will denote the space of linear functionals on V, and $\tilde{V}$ will denote the subspace of those elements of V^* which are left invariant by a compact open subgroup of G.

The results about supercuspidal representations in this paper are proved via their realization as induced representations from a finite dimensional representation of a maximal compact-modulo-centre subgroup and certain character identities. We recall that there are two conjugacy classes of maximal compact-modulo-centre subgroups of $GL_2(k)$ (these correspond to conjugacy classes of maximal compact subgroups of $PGL_2(k)$); one of the conjugacy classes is represented by $k^* \cdot GL_2(\mathcal{O}_k)$ and the other one by

$$J_k = k^* \cdot \Gamma_0(\pi) \cup k^* \cdot \begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix} \Gamma_0(\pi),$$

where for $n \geq 0$, $\Gamma_0(\pi^n)$ denotes the group

$$\Gamma_0(\pi^n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_k) : c \equiv 0 \pmod{\pi^n} \right\}.$$

Define a decreasing filtration $GL_2(\mathcal{O}_k)(n)$, for $n \geq 1$, on $k^* \cdot GL_2(\mathcal{O}_k)$ by

$$GL_2(\mathcal{O}_k)(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_k) \,\middle|\, \begin{array}{l} a, d \equiv 1 \pmod{\pi^n} \text{ and} \\ b, c \equiv 0 \pmod{\pi^n} \end{array} \right\},$$

and a decreasing filtration $J_k(n)$, for $n \geq 1$, on J_k by

$$J_k(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_k) \,\middle|\, \begin{array}{l} a, d \equiv 1 \pmod{\pi^n},\ b \equiv 0 \pmod{\pi^n} \\ \text{and } c \equiv 0 \pmod{\pi^{n+1}} \end{array} \right\}.$$

We will use $\mathcal{K}$ to denote either of the conjugacy class of maximal compact-modulo-centre subgroup, and $\mathcal{K}(n)$ to denote the filtration defined above on $\mathcal{K}$.

Definition. A finite-dimensional irreducible representation W of $\mathcal{K}/\mathcal{K}(n)$, for $n \geq 1$, is called *very cuspidal* of *level n* if the representation W does not contain the trivial character of the subgroup

$$\begin{pmatrix} 1 & \pi^{n-1}\mathcal{O}_k \\ 0 & 1 \end{pmatrix} \subset \mathcal{K}/\mathcal{K}(n).$$

We now recall the definition of the *conductor* of a representation of $GL_2(k)$. The conductor of a representation Π of $GL_2(k)$, with central character ω, is the smallest integer n such that

$$\Pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} v = \omega(a)v$$

for some $v \neq 0$ in the representation space of Π and all matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\pi^n).$$

The conductor of the principal series $V_{(\psi_1,\psi_2)}$ is cond ψ_1 + cond ψ_2.

The *level* of a representation V of D_k^* is the minimum n such that the representation V is trivial on $D_k^*(n)$, where for $n = 0$, $D_k^*(n) = \mathcal{O}_{D_k}^*$ with $\mathcal{O}_{D_k}$ the ring of integers in D_k, and for $n > 0$, $D_k^*(n) = \{x \in \mathcal{O}_{D_k} | \pi_{D_k}^n \text{ divides } x - 1\}$ with π_{D_k} a uniformizing parameter of $\mathcal{O}_{D_k}$. The *conductor* of an irreducible representation V of D_k^* with level n is defined to be $n + 1$.

For a representation V of $GL_2(k)$, and the representation V' of D_k^* associated to V by the Jacquet-Langlands correspondence, we have cond V = cond V'.

The *minimal conductor* of a representation V of $GL_2(k)$, or of D_k^*, is the minimum of the conductors of the representations $V \otimes \chi$, where χ runs over the characters of k^*. The representation V will be called *minimal* if cond $V \leq$ cond $V \otimes \chi$, for χ any character of k^*.

The following theorem, in this precise form, is due to Kutzko, Theorem 4.3 in [Ku1].

THEOREM 2.1. *There exists a bijective correspondence, obtained by compact induction, between very cuspidal representations of a set of conjugacy classes of maximal compact-modulo-centre subgroups of* $GL_2(k)$ *and minimal irreducible supercuspidal representations of* $GL_2(k)$. *A minimal representation of* $GL_2(k)$ *of even conductor* $2n$ *is compactly induced from a very cuspidal representation of* $k^* \cdot GL_2(\mathcal{O}_k)$ *of level* n, *and a minimal representation of odd conductor* $2n+1$ *is compactly induced from a very cuspidal representation of* J_k *of level* n. ■

For a function f on D_k^*, invariant under conjugation, we define a class function $\hat{f}$ on $GL_2(k)$ by defining the value of the function $\hat{f}$ on a regular elliptic conjugacy class in $GL_2(k)$ to be the value of f on the corresponding conjugacy class in D_k^*, and by defining $\hat{f}$ to be zero on all the other conjugacy classes of $GL_2(k)$.

We recall from [J-L] Theorem 7.7, that the character of an irreducible, admissible representation V of $GL_2(k)$, in the sense of distributions, is represented by a locally-L^1 function, locally constant on the set of regular semisimple elements of $GL_2(k)$. We let $\mathrm{ch}(V)$ denote this function on the regular semisimple elements of $GL_2(k)$, and undefined at the other conjugacy classes.

In the following lemma, the character of a supercuspidal representation of $GL_2(k)$ on regular elliptic elements is obtained from [J-L] Proposition 15.5, and on split elements, from Proposition 5.5 and Proposition 6.11 in [Ku2].

LEMMA 2.2. *For a minimal supercuspidal representation* V *of* $GL_2(k)$ *of conductor* $2n$ *or* $2n+1$, *and of central character* ω, *the distribution* $\mathrm{ch}(V) + \mathrm{ch}(V')$ *on* $GL_2(k)$ *is represented by the class function*

$$\mathrm{ch}(V) + \mathrm{ch}(V') = \omega(\alpha)\left[\frac{2\|\beta\|}{\|\alpha - \beta\|} - \dim(V')\right] \text{ at } \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

$$\text{for } v\left(\frac{\alpha}{\beta} - 1\right) \geq n,$$

$$= 0 \text{ at all the other conjugacy classes in } GL_2(k).$$

■

We now introduce the concept of a $\mathcal{K}$-*generic* element, due to [Ku2]. For $\mathcal{K} = k^* \cdot GL_2(\mathcal{O}_k)$, $K_u^* \subset k^* \cdot GL_2(\mathcal{O}_k)$, for K_u the quadratic unramified extension of k, and the embedding is unique up to conjugation by $k^* \cdot GL_2(\mathcal{O}_k)$. Any element of $k^* \cdot GL_2(\mathcal{O}_k)$, conjugate by an element of $k^* \cdot GL_2(\mathcal{O}_k)$ to an element of $K_u^* - k^*$, will be called $k^* \cdot GL_2(\mathcal{O}_k)$-generic. A $k^* \cdot GL_2(\mathcal{O}_k)$-generic element has the same k-valuation for the off-diagonal entries. Similarly, $K_r^* \subset J_k$, for K_r any separable quadratic ramified extension of k and the embedding is unique up to conjugation by J_k. Any element of J_k, conjugate by an element of J_k to an element of $K_r^* - k^*$, will be called J_k-generic. The k-valuations of the off-diagonal entries of a J_k-generic element differ by 1.

The following lemma, cf. [Ku2] Lemma 1.4, is the main reason why it is easy to deal with $\mathcal{K}$-generic elements.

LEMMA 2.3. *If an element of* $\mathcal{K}$ *is conjugate to a* $\mathcal{K}$-*generic element in* $\mathcal{K}$ *by* $g \in GL_2(k)$ *then* $g \in \mathcal{K}$. ∎

The following lemma is from [Ku2] Propositions 5.5 and 6.11.

LEMMA 2.4. *For a supercuspidal representation* V *of central character* ω *which is induced from a very cuspidal representation* W *of level* n, *of a maximal compact-modulo-centre subgroup* $\mathcal{K}$, *we have the following character identity*:

$$\mathrm{ch}(W)(x) = \mathrm{ch}(V)(x) \text{ if } x \notin k^*\mathcal{K}(n), \text{ but is } \mathcal{K}\text{-generic}$$

$$\mathrm{ch}(W)(\lambda) = \omega(\lambda)\cdot[q^n - q^{n-1}] \text{ if } \lambda \in k^*$$

$$\mathrm{ch}(W)\left[\lambda\begin{pmatrix}1 & \pi^{n-1}\\ 0 & 1\end{pmatrix}\right] = -\omega(\lambda)\cdot q^{n-1} \text{ if } \lambda \in k^*$$

$$\mathrm{ch}(W)(x) = 0 \text{ at the other conjugacy classes.}$$ ∎

We will also need to know the character of a principal series. The following lemma is Proposition 7.6 in [J-L].

LEMMA 2.5. *The character of the* (*not necessarily irreducible*) *principal series* $V_{(\psi_1,\psi_2)}$ *is concentrated on split elements, and its value at a split element* $x \in GL_2(k)$ *with eigenvalues* α, β *is given by*

$$\mathrm{ch}\, V_{(\psi_1,\psi_2)}(x) = [\psi_1(\alpha)\psi_2(\beta) + \psi_1(\beta)\psi_2(\alpha)]\,\frac{\|\alpha\beta\|^{1/2}}{\|\alpha - \beta\|}.$$ ∎

The following lemma was proved by Gelfand and Graev in odd residue characteristic, and can be deduced from Section IV of [Ho] in general (for a proof, see also [C] Proposition 6.5).

LEMMA 2.6. *The dimension of a finite-dimensional, irreducible representation of D_k^* depends only on the minimal conductor of the representation. If the minimal conductor is $2n + 1 > 1$, then the dimension of the representation is $q^{n-1}(q + 1)$, and if the minimal conductor is $2n > 0$, then the dimension of the representation is $2q^{n-1}$. For representations of minimal conductor 1, the dimension is 1.* ■

We will often need to use the following theorem of Casselman cf. [Ca], generalizing a theorem of Silberger in odd residue characteristic.

THEOREM 2.7. *Any two irreducible, admissible, infinite dimensional representation of $GL_2(k)$, which have the same central character are isomorphic except for finite dimensional parts when restricted to a compact-modulo-centre subgroup of $GL_2(k)$.* ■

Remark 2.8. Actually Casselman proved this theorem only for $GL_2(\mathcal{O}_k)$ but the result for $GL_2(\mathcal{O}_k)$ implies the result for the other maximal compact subgroup J_k as can be seen as follows.

The difference of the characters of the two representations in Theorem 2.7 is a locally L^1-function on $GL_2(k)$ on which the centre acts by a character. It is smooth on the set of regular semisimple elements and also in a neighborhood of the identity because of Casselman's theorem and is therefore smooth everywhere. This clearly proves Theorem 2.7 in general.

Remark 2.9. Let V be a discrete series representation of $GL_2(k)$ with central character ω, V' the corresponding representation of D_k^*, χ a character of K^* for K a (*not necessarily separable*) quadratic field extension of k which restricts to ω on k^*. Then it easily follows from Theorem 2.7 (by comparing V to a principal series representation with the same central character as V, and noting that the K-type information of a principal series is very easy to obtain) together with the character relation ch $V = -$ch V' at regular elliptic elements of K^* if K is separable field extension of k and by approximating an inseparable element of K^* by a separable elliptic element, that the character χ either appears in V with multiplicity one or appears in V' with multiplicity one and exactly one of the two possibilities holds.

3. Multiplicity one for $GL_2(K) \times GL_2(k)$. The aim of this section is to prove that for any irreducible representation V of $GL_2(K) \times GL_2(k)$ where K is a separable quadratic field extension of k, the space of $GL_2(k)$-invariant linear forms on V is at most one-dimensional. By the theorem of Gelfand and Kazhdan as proved in [P], Lemma 4.2, it suffices to construct an involution i on $GL_2(K) \times GL_2(k)$, i.e., $i(XY) = i(Y)i(X)$, $i(i(X)) = X$ for all X, Y belonging to $GL_2(K) \times GL_2(k)$ such that any distribution on $GL_2(K) \times GL_2(k)$ which is bi-invariant under the diagonally embedded $GL_2(k)$, is invariant under the involution i. As in Lemma 4.3 of [P], it suffices to prove the following proposition.

PROPOSITION 3.1. *Any distribution on $GL_2(K)$ which is invariant under the inner conjugation action of $GL_2(k)$ is invariant under the involution $i : X \mapsto i(X) = \operatorname{tr} X - X$.*

Proof. Let U be the two-dimensional vector space over k whose automorphism group is GL_2 over k. For a matrix X in $GL_2(K)$, let $\overline{X}$ denote the matrix obtained by applying the nontrivial element of the Galois group of K/k to the matrix entries of X. Define

$$\pi : GL_2(K) \to K \times K \times K$$

$$X \mapsto (\det X, \operatorname{tr} X, \operatorname{tr}(X\overline{X})).$$

It is clear that the mapping π commutes with the inner conjugation action of $GL_2(k)$, and also with i. By the Localization Principle of Bernstein cf. [Be] page 58, it suffices to prove that any distribution on a fibre of the mapping π which is $GL_2(k)$-invariant is also invariant under the involution i. For this we study the various fibres of the mapping π and in each case we check that a $GL_2(k)$-invariant distribution on that fibre is invariant under the involution. We will denote by F_X the fibre of the mapping π passing through X.

Case I. X and $\overline{X}$ cannot simultaneously be triangulated over the algebraic closure k^a of k. In this case one sees by an explicit calculation that for any other element Y of F_X, there exists g belonging to $GL_2(K)$ such that $gXg^{-1} = Y$ and $g\overline{X}g^{-1} = \overline{Y}$. Clearly, g is unique up to scalars and therefore g can be assumed to be in $GL_2(k)$. It follows that F_X is a single $GL_2(k)$-orbit. Therefore a $GL_2(k)$-invariant distribution on this fibre is invariant under the involution $X \mapsto i(X)$.

Case II. X and $\overline{X}$ can simultaneously be triangulated over the algebraic closure k^a of k but not over the separable closure k^s of k. In this case $\mathrm{ch}(k) = 2$, and the eigen values of X are equal. It follows that $\mathrm{tr}(X) = 0$, and $i(X) = X$, so any distribution on F_X is invariant under i.

Case III. X and $\overline{X}$ can simultaneously be triangulated over K with a simultaneous eigenvector not belonging to U.

Let $e \in U \otimes_k K$ be such that the line $\langle e \rangle$ is not defined over k, and such that e is a simultaneous eigenvector of X and $\overline{X}$. Let $Xe = \lambda e$ and $\overline{X}e = \mu e$. This implies that $\overline{X}\overline{e} = \overline{\lambda}\overline{e}$ and $X\overline{e} = \overline{\mu}\overline{e}$. Therefore the matrix of X with respect to the k-basis $e + \overline{e}$, $de + \overline{d}\overline{e}$ of U where

$$d \in K - k \text{ is } \begin{pmatrix} a & -d\overline{d}b \\ b & a + (d + \overline{d})b \end{pmatrix} \text{ for } a, b \in K.$$

This implies that X is $GL_2(k)$-conjugate to a matrix determined in terms of tr X and det X, i.e. the fibre of π passing through X consists of a single $GL_2(k)$-orbit, and therefore any $GL_2(k)$-invariant distribution on the fibre is invariant under the involution i.

Case IV. X and $\overline{X}$ can simultaneously be triangulated over k^s but not over K.

In this case there exists $e \in U \otimes_k k^s$ which is a simultaneous eigenvector of X and $\overline{X}$ and such that the line $\langle e \rangle$ is not defined over K. Assume that $Xe = \lambda e$ and $\overline{X}e = \mu e$. It follows that $(X + \overline{X})e = (\lambda + \mu)e$. As $(X + \overline{X})$ belongs to $M_2(k)$, $\langle e \rangle$ can be defined over a separable quadratic field extension L of k. Let $x \mapsto x^\sigma$ be an automorphism of k^s over k which acts nontrivially on L but trivially on K. As

$$\left\{\begin{aligned} Xe &= \lambda e \\ \overline{X}e &= \mu e \end{aligned}\right\} \Rightarrow \begin{aligned} Xe^\sigma &= \lambda^\sigma e^\sigma \\ \overline{X}e^\sigma &= \mu^\sigma e^\sigma, \end{aligned}$$

we conclude as in Case III by looking at the matrix of X with respect to the k-basis $e + e^\sigma$, $\tilde{d}e + \tilde{d}^\sigma e^\sigma$ with $\tilde{d} \in L - k$ that the fibre consists of a single $GL_2(k)$-orbit and any $GL_2(k)$-invariant distribution on the fibre is invariant under the involution i.

Case V. X and $\overline{X}$ can simultaneously be triangulated over K with a simultaneous eigenvector belonging to U. In this case we can assume that

$$X = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

As we have already considered all the other possible cases, every matrix in the fibre of the map π, passing through X, also has a simultaneous eigenvector belonging to U. Therefore the fibre of the mapping π passing through X consists of $GL_2(k)$-orbits passing through the points

$$\begin{pmatrix} a & * \\ 0 & d \end{pmatrix}, \begin{pmatrix} d & * \\ 0 & a \end{pmatrix}$$

where $*$ takes arbitrary values from K. If $a = d$, then

$$i\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & a \end{pmatrix}$$

which is a conjugate of

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Therefore a $GL_2(k)$-invariant distribution on such a fibre is invariant under the involution i. If $a \neq d$, define the map

$$q : F_X \to \mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$$

$$A \mapsto [L_1], [L_2]$$

where L_1 and L_2 are the eigenvectors of $A \in F_X$ with eigenvalues a and d respectively. It is clear that the map q defines a $GL_2(k)$-equivariant isomorphism of F_X with $\mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$. The involution i becomes $(x, y) \mapsto (y, x)$ on $\mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$. We therefore need to prove that any $GL_2(k)$-invariant distribution on $\mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$ is invariant under the involution $(x, y) \mapsto (y, x)$. Observe now that $Y = (0, x) \cup (y, \infty) \cup (\infty, z) \cup (w, 0)$ where x, w take values from k^* and y, z from k, is a closed subset of $\mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$, and is invariant under the normaliser $N(T)$ of the diagonal torus T of $GL_2(k)$. Y is isomorphic to the disjoint union of the two coordinate axes in the (x, y)-plane. The element

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

permutes the two disjoint copies. Clearly, $GL_2(k) \times_{N(T)} Y \cong \mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$, and therefore from the lemma on page 60 of [Be], a $GL_2(k)$ invariant distribution on $\mathbf{P}^1_K \times \mathbf{P}^1_k \cup \mathbf{P}^1_k \times \mathbf{P}^1_K - \Delta\mathbf{P}^1_k$ can be identified to an $N(T)$-invariant distribution on Y. But an $N(T)$-invariant distribution on Y can be identified to a T-invariant distribution on the union of the two coordinate axes in the (x, y)-plane. It therefore suffices to prove the following lemma whose proof is easy and is omitted. ■

LEMMA 3.2. *A distribution on the union of the two coordinate axes in the (x, y)-plane, which is invariant under the action of k^* given by $\{t(x, y)\} \mapsto (tx, t^{-1}y)$, with $t \in k^*$ and $x, y \in k$, is invariant under the involution $(x, y) \mapsto (y, x)$.* ■

4. Theorem B for $GL_2(K) \times GL_2(k)$. The proof of Theorem B for $GL_2(K) \times GL_2(k($ will be divided into the following cases.

Case I. $V = V_1 \otimes V_2$, V_1 a principal series representation of $GL_2(K)$.

Case II. $V = V_1 \otimes V_2$, V_2 a principal series representation of $GL_2(k)$.

Case III. $V = V_1 \otimes V_2$, V_2 a supercuspidal representation of $GL_2(k)$.

Case IV. $V = V_1 \otimes V_2$, V_2 a special representation of $GL_2(k)$.

We begin with the general remark that for a character χ of K^*, the representation $(V_1 \otimes \chi) \otimes V_2$ of $GL_2(k)$ is the same as $V_1 \otimes (V_2 \otimes \chi|_{k^*})$.

Since any character of k^* can be extended to a character of K^*, and as the Jacquet-Langlands correspondence is equivariant under twisting, we can replace one of V_1 or V_2 by a twist.

4.1 *Proof of Theorem B in Case I.* Suppose that V_1 is the principal series $V\chi$ where

$$\chi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \alpha(a)\beta(d)$$

for some characters α, β of K^*. Let χ' be the character of K^* defined by $\chi'(x) = \alpha(x)\beta(\bar{x})$. As $\mathbf{P}^1_K$ has two orbits under the action of $GL_2(k)$, one closed, equal to $\mathbf{P}^1_k = GL_2(k)/B(k)$, and the other open, equal to $\mathbf{P}^1_K - \mathbf{P}^1_k = GL_2(k)/K^*$, we get an exact sequence of $GL_2(k)$-modules:

$$0 \to \mathrm{ind}_{K^*}^{GL_2(k)}\chi' \to V_1 \to \mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}) \to 0.$$

Applying the functor $\mathrm{Hom}_{GL_2(k)}(-, \tilde{V}_2)$, we get the exact sequence

$$0 \to \mathrm{Hom}_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}), \tilde{V}_2] \to \mathrm{Hom}_{GL_2(k)}[V_1, \tilde{V}_2] \to$$

$$\to \mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{K^*}^{GL_2(k)}\chi', \tilde{V}_2] \to \mathrm{Ext}^1_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}), \tilde{V}_2] \to \cdots .$$

From Corollary 5.9 of [P], $\mathrm{Hom}_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}), \tilde{V}_2]$ is nonzero if and only if $\mathrm{Ext}^1_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}), \tilde{V}_2]$ is nonzero. It follows that $\mathrm{Hom}_{GL_2(k)}[V_1, \tilde{V}_2]$ (which is the space of $GL_2(k)$-invariant linear forms on $V_1 \otimes V_2$) is zero iff both $\mathrm{Hom}_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}), \tilde{V}_2]$ and $\mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{K^*}^{GL_2(k)}\chi', \tilde{V}_2] \cong \mathrm{Hom}_{K^*}[\chi', \tilde{V}_2]$ are zero. As $K^*B(k) = GL_2(k)$, a principal series representation of $GL_2(k)$ contains all characters of K^* whose restriction to k^* is the central character. It follows that if V_2 is a principal series representation then $V_1 \otimes V_2$ has a $GL_2(k)$-invariant linear form. If V_2 is not a principal series, i.e. if it is a discrete series representation, let V_2' be the representation of D_k^* associated to the discrete series representation V_2 of $GL_2(k)$. As $\mathbf{P}^1_K \cong D_k^*/K^*$, the principal series representation V_1 is $\mathrm{ind}_{K^*}^{D_k^*}\chi'$ as a D_k^*-module. It follows that $V_1 \otimes V_2'$ has a D_k^*-invariant linear form if and only if $\mathrm{Hom}_{K^*}[\chi', \tilde{V}_2'] \neq 0$. From Remark 2.9, a character of K^*, whose restriction to k^* equals the central character of $\tilde{V}_2$, occurs in precisely one of the representations $\tilde{V}_2$ or $\tilde{V}_2'$. This completes the proof of Theorem B in Case I, except that we must check that there is no representation V_2 of $GL_2(k)$ such that $\mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2}), \tilde{V}_2]$ is nonzero but $\mathrm{Hom}_{K^*}[\chi', \tilde{V}_2]$ is zero, a possibility only for $\tilde{V}_2$ a special representation which we assume to be Sp_k (if this were possible, $V_1 \otimes V_2$ would have a $GL_2(k)$-invariant linear form and $V_1 \otimes V_2'$ a D_k^*-invariant linear form). As $\mathrm{ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k^{1/2})$, must have Sp_k as a quotient,

$$\alpha(x) = \|x\|_k^{-1}, \qquad \beta(x) = \|x\|_k \quad \text{for} \quad x \in k^*.$$

As $\mathrm{Hom}_{K^*}(\chi', \tilde{V}_2) = 0$, $\chi'(x) = \alpha(x)\cdot\beta(\bar{x})$ must be the trivial character of K^*. It follows that

$$\frac{\alpha(x)}{\beta(x)} = \frac{\alpha(x)\beta(\bar{x})}{\beta(x\bar{x})} = \frac{1}{\|x\cdot\bar{x}\|_k} = \frac{1}{\|x\|_K} \quad \text{for} \quad x \in K^*,$$

which is contradictory to the fact that $V_1 = V_{(\alpha,\beta)}$ is an irreducible principal series. ■

Remark 4.1.1. The proof above shows that $V_1 \otimes V_2$ where $V_1 = V_{(\alpha,\beta)}$ is a principal series representation of $GL_2(K)$, has a $GL_2(k)$-invariant linear form iff the character $(\alpha(x)\cdot\beta(\bar{x}))^{-1}$ of K^* appears in V_2.

Remark 4.1.2. By an argument similar to the one used to prove Theorem B in Case I, we can also prove Theorem B in case $V_1 = Sp_K$. In this case one finds that $Sp_K \otimes V_2$, has a $GL_2(k)$-invariant linear form if and only if either V_2-Sp_k, or the trivial character of K^* appears in V_2.

4.2 *Proof of Theorem B in Case II.* $V_1 \otimes V_2$ is a representation of $GL_2(K) \times GL_2(k)$ with $V_2 = \mathrm{Ind}_{B(k)}^{GL_2(k)}\chi$, a principal series. Now $V_1 \otimes V_2$ has a $GL_2(k)$-invariant linear form iff $\mathrm{Hom}_{GL_2(k)}[V_1, \tilde{V}_2]$ is nonzero. By Frobenius reciprocity,

$$(*) \qquad \mathrm{Hom}_{GL_2(k)}[V_1, \mathrm{Ind}_{B(k)}^{GL_2(k)}\chi^{-1}] = \mathrm{Hom}_{B(k)}[V_1|_{B(k)}, \chi^{-1}\cdot\delta_k^{1/2}].$$

If V_1 is either a principal series or is a special representation, Theorem B has already been proved. If V_1 is supercuspidal then by Kirillov theory, $V_1|_{B(k)} = \mathrm{ind}_{N(k)}^{B(k)}\psi$, where ψ is a nontrivial character of $N(k)$. From this and (*), Theorem B follows easily. ■

4.3 *Proof of Theorem B in Case III.* $V_1 \otimes V_2$ is a representation of $GL_2(K) \times GL_2(k)$ with V_2 supercuspidal.

We can assume, possibly after twisting by a character of k^*, that V_2 is a minimal representation. Let V_2 be $\mathrm{ind}_{\mathcal{K}}^{GL_2(k)} W_2$ where W_2 is a very cuspidal representation of level n of a maximal compact-modulo-centre subgroup $\mathcal{K}$ of $GL_2(k)$. Then $V_1 \otimes V_2$ considered as a representation of $GL_2(k)$ is $\mathrm{ind}_{\mathcal{K}}^{GL_2(k)}[V_1|_{\mathcal{K}} \otimes W_2]$. Therefore $V_1 \otimes V_2$ has a $GL_2(k)$-invariant linear form iff $V_1|_{\mathcal{K}} \otimes W_2$ has a $\mathcal{K}$-invariant linear form. Therefore Theorem B reduces to the following proposition.

PROPOSITION 4.3.1. *With the notation as above, $V_1|_{\mathcal{K}} \otimes W_2$ has a $\mathcal{K}$-invariant linear form iff the representation $V_1 \otimes V_2'$ of D_k^* does not have a D_k^*-invariant linear form.*

Proof. For any (virtual) representation V of $GL_2(K)$, let $m(V, W_2)$ denote the multiplicity of the trivial representation of $\mathcal{K}$ in $V \otimes W_2$, and let $m(V, V_2')$ denote the multiplicity of the trivial representation of D_k^* in $V \otimes V_2'$. With this notation we wish to prove that $m(V_1, W_2) + m(V_1, V_2') = 1$.

Let P be a principal series representation of $GL_2(K)$ with the same central character as V_1. From the proof of Theorem B in Case I, we know that $m(P, W_2) + m(P, V_2') = 1$. Therefore to prove that $m(V_1, W_2) + m(V_1, V_2') = 1$, it suffices to prove that $m(V_1 - P, W_2) + m(V_1 - P, V_2') = 0$. As $V_1 - P$ is a finite dimensional (virtual) representation of any compact modulo-centre subgroup of $GL_2(K)$ by Theorem 2.8, we can use character theory to prove this relation.

By the Weyl integration formula for a function f on $GL_2(k)/k^*$, cf. [J-L] formula 7.2.2, we have

$$\int_{GL_2(k)/k^*} f(y)dy = \sum_{K_i} \int_{K_i^*} \Delta(x)\left[\frac{1}{2}\int_{K_i^*\backslash GL_2(k)} f(\bar{g}^{-1}x\bar{g})d\bar{g}\right]dx$$

where K_i are the distinct conjugacy classes of maximal tori in $GL_2(k)$ and

$$\Delta(x) = \left\|\frac{(x_1 - x_2)^2}{x_1x_2}\right\|_k$$

where x_1 and x_2 are the eigenvalues of x. We will use this formula to integrate the function $f(x) = \mathrm{ch}(V_1 - P)\cdot \mathrm{ch}\, W_2(x)$ on $\mathcal{K}$ (extended to

$GL_2(k)/k^*$ by setting it zero outside $\mathcal{K}$). We will use the following well-known result, cf. Proposition 5, [*Cl*].

LEMMA 4.3.2. *Let $F(g) = (gv, v)$ be a matrix coefficient of a supercuspidal representation π of a reductive p-adic group G with centre Z. Then the orbital integrals of F vanish on regular non-elliptic elements, and is given by the following formula at a regular elliptic element x contained in a torus T,*

$$\int_{T\backslash G} F(\bar{g}^{-1}x\bar{g})d\bar{g} = \frac{(v, v)\cdot \mathrm{ch}\, \pi(x)}{d(\pi)\cdot \mathrm{vol}(T/Z)},$$

where $d(\pi)$ denotes the formal degree of the representation π. ■

Since a matrix coefficient of W_2 (extended to $GL_2(k)$ by setting it zero outside $\mathcal{K}$) is a matrix coefficient of V_2, it follows that

(1) for the choice of Haar measure on $GL_2(k)/k^*$ giving $\mathcal{K}/k^*$ measure 1, dim $W_2 = d(V_2)$, and

(2) for K_i a separable quadratic field extension of k,

$$\int_{K_i^*\backslash GL_2(k)} \mathrm{ch}\ W_2(\bar{g}^{-1}x\bar{g})d\bar{g} = \frac{\mathrm{ch}\ V_2(x)}{\mathrm{vol}(K_i^*/k^*)} \qquad \text{(from Lemma 4.3.2).}$$

It follows that

$$m(V_1 - P, W_2) = \frac{1}{\mathrm{vol}(\mathcal{K}/k^*)}\int_{\mathcal{K}/k^*} \mathrm{ch}(V_1 - P)\cdot \mathrm{ch}\ W_2(x)dx$$

$$= \sum_{K_i} \frac{1}{2\ \mathrm{vol}(K_i^*/k^*)}\int_{K_i^*/k^*} \Delta\cdot \mathrm{ch}(V_1 - P)\cdot \mathrm{ch}\ V_2(x)dx,$$

the sum running over all quadratic field extensions of k. Similarly we have (with the same function Δ as before),

$$m(V_1 - P, V_2') = \sum_{K_i} \frac{1}{2\ \mathrm{vol}(K_i^*/k^*)}\int_{K_i^*/k^*} \Delta\cdot \mathrm{ch}(V_1 - P)\cdot \mathrm{ch}\ V_2'(x)dx.$$

Since ch $V_2(x) = -\mathrm{ch}\ V_2'(x)$ at regular elliptic elements, it follows that $m(V_1 - P, W_2) + m(V_1 - P, V_2') = 0$, completing the proof of Theorem B in this case. ■

4.4 *Proof of Theorem B in Case IV.* $V_2 = Sp_k$.

As the Jacquet-Langlands dual of Sp_k is the one-dimensional trivial representation of D_k^*, we need to prove that $V_1 \otimes Sp_k$ has a $GL_2(k)$-invariant linear form iff V_1 does not have a D_k^*-invariant linear form. As we have already considered the cases when V_1 is either a principal series or is a special representation, we only need to consider the case when V_1 is supercuspidal. The following lemma reduces this to a question about $GL_2(K)$-representations.

4.4.1 LEMMA. *For V a supercuspidal representation of $GL_2(K)$, $V \otimes Sp_k$ has a $GL_2(k)$-invariant linear form iff V does not have a $GL_2(k)$-invariant linear form.*

Proof. The representation Sp_k sits in the exact sequence

$$0 \to \mathbf{C} \to \mathcal{S}(\mathbf{P}^1_k) \to Sp_k \to 0.$$

Therefore we have the exact sequence of $GL_2(k)$-modules

$$(*) \qquad 0 \to V \to V \otimes \mathcal{S}(\mathbf{P}^1_k) \to V \otimes Sp_k \to 0.$$

As $\mathcal{S}(\mathbf{P}^1_k) \cong \mathrm{ind}_{B(k)}^{GL_2(k)} \delta_k^{-1/2}$, it follows from the Kirillov model of the supercuspidal representation V that $V \otimes \mathcal{S}(\mathbf{P}^1_k)$ has a one-dimensional space of $GL_2(k)$-invariant linear forms. It therefore suffices to prove that the exact sequence (*) splits. Since V is supercuspidal, write V as $\mathrm{ind}_{\mathcal{K}}^{GL_2(k)} W$ where W is a finite dimensional representation of a compact-modulo-centre subgroup $\mathcal{K}$. It follows from the Mackey's theory that V restricted to $GL_2(k)$ is a sum of representations induced from finite dimensional representations of compact-modulo-centre subgroups of $GL_2(k)$, say

$$V|_{GL_2(k)} = \oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} W_i.$$

The exact sequence (*) therefore becomes

$$0 \to \oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} W_i \to [\oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} W_i] \otimes \mathcal{S}(\mathbf{P}^1_k) \to [\oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} W_i] \otimes Sp_k \to 0,$$

or,

$$0 \to \oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} W_i \to \oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} (W_i \otimes \mathcal{S}(\mathbf{P}^1_k)|_{\mathcal{K}_i}) \to [\oplus \mathrm{ind}_{\mathcal{K}_i}^{GL_2(k)} W_i] \otimes Sp_k \to 0.$$

Now $\mathcal{K}_i$ being compact-modulo-centre groups, the exact sequence of $\mathcal{K}_i$-modules

$$0 \to W_i \to W_i \otimes \mathcal{S}(\mathbf{P}^1_k) \to W_i \otimes Sp_k \to 0$$

splits. Therefore the exact sequence (*) also splits. ■

Remark 4.4.2. The same argument proves that more generally any exact sequence of modules for a locally compact group splits after tensoring with a representation which is induced from a compact-modulo-centre subgroup.

Theorem B therefore reduces to Theorem C in this case. We prove Theorem C in the next section.

5. Theorem C. The next theorem is Theorem C of the Introduction.

THEOREM 5.1. *Let V be a discrete series representation of $GL_2(K)$ where K is a quadratic extension of k. Then the representation V has a D_k^*-invariant linear form iff V has a $GL_2(k)$-invariant linear form.*

Proof. As the theorem becomes vacuous unless the central character of V restricted to k^* is trivial, we will assume this to be the case in what follows. If V is the special representation $Sp_K \otimes \chi$ where the character χ of K^* is trivial on k^{*2}, it is easy to see that V has a D_k^*-invariant linear form iff χ restricted to k^* is nontrivial but is trivial on the norm of elements from K^*. By an analysis similar to the one we do in Section 6.2, it follows that the same condition must hold for the existence of a $GL_2(k)$-invariant linear form on $Sp_k \otimes \chi$.

We now assume that V is supercuspidal and induced from a representation W of level n of a maximal compact-modulo-centre subgroup $\mathcal{K}$ of $GL_2(K)$. We will work out the details of the proof assuming that this maximal compact-modulo-centre subgroup of $GL_2(K)$ is $K^*GL_2(\mathcal{O}_k)$; the proof in the other case is similar. In the proof we will calculate the dimension of D_k^*-invariant linear forms on V by comparing V to a principal series representation $P = V_{(\alpha,\beta)}$ of $GL_2(K)$ with the same central character as V. As the central character of V restricted to k^* is trivial, we can assume that $P = V_{(\alpha,\beta)}$ is such that $\alpha(x)\beta(\bar{x}) = 1$ for $x \in K^*$ ($x \to \bar{x}$ is the non-trivial automorphism of K/k). By Frobenius reciprocity, P has a one-dimensional space of D_k^*-invariant linear forms. It therefore suffices to prove that the dimension of $GL_2(k)$-invariant linear forms on V is one more than the dimension of D_k^*-invariant linear forms in the finite dimensional (virtual representation) $V - P$ of D_k^*. The proof will be divided into two cases depending on whether K is unramified over k or not.

Case 1. K/k is unramified. From the Mackey theory,

$$V|_{GL_2(k)} = \mathrm{ind}_{k^*GL_2(\mathcal{O}_k)}^{GL_2(k)} W \oplus \text{other terms.}$$

By a calculation very similar to the one we do in Section 6.3, it follows that in this decomposition the "other terms" do not have a $GL_2(k)$-invariant linear form. We omit the details here. Therefore to prove Theorem C, it suffices to prove that the dimension of $k^*GL_2(\mathcal{O}_k)$-invariant vectors in W is one more than the dimension of D_k^*-invariant linear forms in the finite dimensional (virtual representation) $V - P$ of D_k^*. Since K/k is unramified, no element of $k^*GL_2(\mathcal{O}_k)$ is $\mathcal{K}$-generic. Therefore from Lemma 2.4, $\mathrm{ch}(W)$ is nonzero only on the conjugacy classes of

$$\begin{pmatrix} 1 & \pi^{n-1} \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with values $-q^{2(n-1)}$ and $q^{2n} - q^{2(n-1)}$ respectively. Therefore

$$\sum_{x \in \frac{GL_2(\mathcal{O}_k)}{GL_2(\mathcal{O}_k)(n)}} \mathrm{ch}_W(x) = -q^{2(n-1)}(q-1)(q+1) + q^{2n} - q^{2n-2}$$

$$= 0.$$

Now for the division algebra side. Elements of $K^* \hookrightarrow D_k^*$ become diagonalizable in $GL_2(K)$, and an element of E^* for a quadratic field $E \neq K$ becomes an element of a ramified quadratic extension of K. As $\alpha(x)\beta(\bar{x}) = 1$ for $x \in K^*$, it follows from Lemmas 2.2 and 2.5 that

$$\begin{aligned} \mathrm{ch}(V - P)(x) &= -\dim V' && \text{if } x \in D_k^*(2n-1) \\ &= \frac{2\|x\bar{x}\|_K^{1/2}}{\|x - \bar{x}\|_K} && \text{if } x \in K^* - K^*(n) \\ &= 0 && \text{if } x \in E^* - E(2n-1),\ E \neq K. \end{aligned}$$

By the Weyl integration formula,

$$\int_{D_k^*/k^*} \operatorname{ch}(V - P)(x)dx$$

$$= -\frac{\operatorname{vol}(D_k^*/k^*)}{2\operatorname{vol}(K^*/k^*)} \int_{\frac{K^*-K^*(n)}{k^*}} \left\| \frac{(x-\bar{x})^2}{x\bar{x}} \right\|_k \cdot \frac{2\|x\bar{x}\|_K^{1/2}}{\|x-\bar{x}\|_K} dx$$

$$-\dim V' \cdot \operatorname{vol}(D^*(2n-1)).$$

Since for an element a of k, $\|a\|_k^2 = \|a\|_K$, it follows that

$$\frac{1}{\operatorname{vol}(D_k^*/k^*)} \int_{D_k^*/k^*} \operatorname{ch}(V - P)(x)dx$$

$$= -\frac{(q+1)q^{n-1} - 1}{(q+1)q^{n-1}} - \frac{2q^{2(n-1)}}{2(q+1)q^{3(n-1)}}$$

$$= -1.$$

This completes the proof of the theorem in this case.

Case 2. K/k ramified. From the Mackey theory,

$$V|_{GL_2(k)} = \operatorname{ind}_{k^*GL_2(\mathcal{O}_k)}^{GL_2(k)} W \otimes \operatorname{ind}_{J_k}^{GL_2(k)} W' \oplus \text{other terms}.$$

The representation W' of J_k is obtained from W by the inclusion of J_k in $\mathcal{H}$ via the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & \pi_K b \\ \dfrac{c}{\pi_K} & d \end{pmatrix}.$$

By a calculation very similar to the one that we do in Section 6.3, it follows that in this decomposition the "other terms" do not have a $GL_2(k)$-invariant linear form. We omit the details here. Therefore to prove Theorem C, it suffices to prove that the sum of dimensions of $k^*GL_2(\mathcal{O}_k)$-invariant vectors in W and the space of J_k-invariant vectors in W' is one more than the dimension of D_k^*-invariant linear forms in the finite dimensional (virtual representation) $V - P$ of D_k^*.

The only quadratic field extensions of k which when composed with K give the unramified quadratic extension of K are the unramified quadratic extension K_u of k, and a unique ramified quadratic extension L of k. Therefore the elements of $GL_2(k)$ which have a conjugate which is $\mathcal{K}$-generic, is either from K_u, or from L. It is easy to see that the only elements of $k^*GL_2(\mathcal{O}_k)$ which are $\mathcal{K}$-generic are the $k^*GL_2(\mathcal{O}_k)$-generic elements. Similarly the only elements of J_k which become $\mathcal{K}$-generic under the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & \pi_K b \\ \pi_K^{-1} c & d \end{pmatrix},$$

are those J_k-generic elements which do not come from the field K.

Let $m(W)$ (resp. $m(W')$) denote the dimension of the space of $k^*GL_2(\mathcal{O}_k)$-invariant (resp. J_k-invariant) vectors in W (resp. in W'). We give the details of the proof assuming that n is even (the proof when n is odd is very similar). In this case W is a representation of $GL_2(\mathcal{O}_k)/GL_2(\mathcal{O}_k)(\frac{n}{2})$ and it is clear that no conjugate of

$$\begin{pmatrix} 1 & \pi_K^{n-1} \\ 0 & 1 \end{pmatrix} \in GL_2(\mathcal{O}_K)/GL_2(\mathcal{O}_K)(n)$$

comes from $GL_2(\mathcal{O}_k)/GL_2(\mathcal{O}_k)(\frac{n}{2})$. To simplify notation, we will use $\mathcal{G}$ for $k^*GL_2(\mathcal{O}_k)/k^*$, and $\mathcal{G}(n)$ for $k^*GL_2(\mathcal{O}_k)(n)/k^*$. We use the Weyl integration formula to calculate $m(W)$. By Lemmas 2.3 and 2.4, we have

$$\begin{aligned} m(W) &= \frac{1}{\operatorname{vol} \mathcal{G}} \int_{\mathcal{G}} \operatorname{ch} W(x)dx \\ &= \frac{1}{\operatorname{vol} \mathcal{G}} \left[\int_{\mathcal{G} - \mathcal{G}(n/2)} \operatorname{ch} W(x)dx + \int_{\mathcal{G}(n/2)} \operatorname{ch} W(x)dx \right] \\ &= \frac{1}{\operatorname{vol}(K_u^*/k^*)} \int_{\frac{[K_u^* - K_u^*(n/2)]}{k^*}} \Delta(x) \operatorname{ch} W(x)dx + \frac{\dim W}{[\mathcal{G} : \mathcal{G}(n/2)]} \end{aligned}$$

$$= \frac{1}{\operatorname{vol}(K_u^*/k^*)} \int_{\frac{[K_u^\bullet - K_u^\bullet(n/2)]}{k^*}} \Delta(x) \operatorname{ch} W(x) dx$$

$$+ \frac{q^{n-1}(q-1)}{q(q-1)(q+1)\cdot q^{3(n/2-1)}}.$$

Here and in what follows,

$$\Delta(x) = \left\| \frac{(x_1 - x_2)^2}{x_1 x_2} \right\|_k$$

where x_1 and x_2 are the eigenvalues of x.

We now do a similar calculation for $m(W')$. Let A be the subset of J_k which modulo $J_k(n/2)$ is either the trivial element, or is conjugate to the unipotent element

$$\begin{pmatrix} 1 & \pi_k^{(n/2)-1} \\ 0 & 1 \end{pmatrix}.$$

Since W' is a representation of $J_k/J_k(\frac{n}{2})$ and

$$\begin{pmatrix} 1 & \pi_k^{(n/2)-1} \\ 0 & 1 \end{pmatrix}$$

goes to a unipotent element

$$\begin{pmatrix} 1 & \pi_K^{n-1} u \\ 0 & 1 \end{pmatrix}$$

with u a unit in $\mathcal{O}_k$, we have (by the Weyl integration formula and Lemmas 2.3 and 2.4)

$$m(W') = \frac{1}{\operatorname{vol}(J_k/k^*)} \int_{J_k/k^*} \operatorname{ch} W'(x) dx$$

$$= \frac{1}{\operatorname{vol}(J_k/k^*)}\left[\int_{\frac{J_k - A}{k^*}} \operatorname{ch} W'(x)dx + \int_{A/k^*} \operatorname{ch} W'(x)dx\right]$$

$$= \frac{1}{\operatorname{vol}(L^*/k^*)}\int_{\frac{[L^* - L^*(n)]}{k^*}} \Delta(x)\operatorname{ch} W'(x)dx + \frac{(-q^{n-1})2(q-1)}{[J_k : J_k(n/2)]}$$

$$+ \frac{(q^n - q^{n-1})}{[J_k : J_k(n/2)]}$$

$$= \frac{1}{\operatorname{vol}(L^*/k^*)}\int_{\frac{[L^* - L^*(n)]}{k^*}} \Delta(x)\operatorname{ch} W'(x)dx + \frac{-q^n + q^{n-1}}{2(q-1)\cdot q^{(3n/2)-1}}.$$

Therefore,

$$m(W) + m(W') = \frac{1}{\operatorname{vol}(K_u^*/k^*)}\int_{\frac{[K_u^* - K_u^*(n/2)]}{k^*}} \Delta(x)\operatorname{ch} W(x)dx$$

$$+ \frac{1}{\operatorname{vol}(L^*/k^*)}\int_{\frac{[L^* - L^*(n)]}{k^*}} \Delta(x)\operatorname{ch} W'(x)dx$$

$$+ \frac{(q-1)}{2(q+1)q^{n/2}}.$$

We now do the calculation on the division algebra side. As $\alpha(x)\beta(\bar{x}) = 1$, it follows from Lemmas 2.2 and 2.5, that for $x \in K^* \subset D_k^*$ thought of as an element of $GL_2(K)$,

$$\operatorname{ch}(V - P)(x) = -\dim V' \qquad \text{if } v_K\left(\frac{x}{\bar{x}} - 1\right) \geq n,$$

$$= -\frac{2\|x\bar{x}\|_K^{1/2}}{\|x - \bar{x}\|_K} \qquad \text{otherwise.}$$

Also, for x elliptic and belonging to $D_k(n)$, $\operatorname{ch}(V - P)(x) = -\dim V'$. Elements of $K^* \hookrightarrow D_k^*$ are diagonalizable in $GL_2(K)$, and elements of a separable quadratic extension different from K remain elliptic.

Therefore,

$$\frac{1}{\operatorname{vol}(D_k^*/k^*)} \int_{D_k^*/k^*} \operatorname{ch}(V - P)(x)dx$$

$$= \frac{1}{\operatorname{vol}(D_k^*/k^*)} \left[\int_{\frac{D_k^* - k^* D_k^*(n)}{k^*}} \operatorname{ch}(V - P)(x)dx \right.$$

$$\left. + \int_{k^* D_k^*(n)/k^*} \operatorname{ch}(V - P)(x)dx \right]$$

$$= \frac{1}{\operatorname{vol}(K_u^*/k^*)} \int_{\frac{K_u^* - K_u^*(n/2)}{k^*}} \Delta(x) \operatorname{ch} V(x)dx$$

$$+ \frac{1}{\operatorname{vol}(L^*/k^*)} \int_{\frac{L^* - L^*(n)}{k^*}} \Delta(x) \operatorname{ch} V(x)dx$$

$$- \frac{1}{2\operatorname{vol}(K^*/k^*)} \int_{\frac{K^* - K^*(n)}{k^*}} 2dx - \frac{\dim V'}{[D_k^* : k^* D_k^*(n)]}$$

$$= \frac{1}{\operatorname{vol}(K_u^*/k^*)} \int_{\frac{K_u^* - K_u^*(n/2)}{k^*}} \Delta(x) \operatorname{ch} V(x)dx$$

$$+ \frac{1}{\operatorname{vol}(L^*/k^*)} \int_{\frac{L^* - L^*(n)}{k^*}} \Delta(x) \operatorname{ch} V(x)dx$$

$$- \frac{2q^{n/2} - 1}{2q^{n/2}} - \frac{2q^{n-1}}{2(q + 1)q^{3n/2-1}}$$

$$= \frac{1}{\operatorname{vol}(K_u^*/k^*)} \int_{\frac{K_u^* - K_u^*(n/2)}{k^*}} \Delta(x) \operatorname{ch} V(x)dx$$

$$+ \frac{1}{\operatorname{vol}(L^*/k^*)} \int_{\frac{L^* - L^*(n)}{k^*}} \Delta(x) \operatorname{ch} V(x)dx$$

$$- 1 + \frac{q - 1}{2(q + 1)q^{n/2}}.$$

Comparing the results obtained above for $GL_2(k)$ and D_k^*, it follows that the dimension of D_k^*-invariant linear forms on $V - P$ is one less than $m(W) + m(W')$, completing the proof of Theorem C.

6. $\mathbb{K}$, a cubic field. The proof of Theorem B when $\mathbb{K} = K$ is a cubic field extension of k will be divided into the following three cases. Our explicit proofs also give that the dimension of $GL_2(k)$-invariant forms is at most 1.

Case I. V a principal series representation.

Case II. V a special representation.

Case III. V a supercuspidal representation.

6.1 *Proof of Theorem B in Case I.* Let $V = \mathrm{Ind}_{B(K)}^{GL_2(K)}\chi$. The action of $GL_2(k)$ on $\mathbf{P}^1(K)$ has two orbits; one open equal to $\mathbf{P}^1(K) - \mathbf{P}^1(k) = GL_2(k)/Z(k)$ and the other closed, equal to $\mathbf{P}^1(k) = GL_2(k)/B(k)$. Therefore we have an exact sequence of $GL_2(k)$-modules:

$$0 \to \mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C} \to V \to \mathrm{Ind}_{B(k)}^{GL_2(k)}(\chi\cdot\delta_k) \to 0.$$

Applying the functor $\mathrm{Hom}_{GL_2(k)}[-, \mathbf{C}]$, where $\mathbf{C}$ is the one-dimensional module for $GL_2(k)$ with trivial action, we get:

$$0 \to \mathrm{Hom}_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}\chi\cdot\delta_k, \mathbf{C}] \to \mathrm{Hom}_{GL_2(k)}[V, \mathbf{C}] \to$$

$$\to \mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C}, \mathbf{C}] \to \mathrm{Ext}^1_{GL_2(k)}[\mathrm{Ind}_{B(k)}^{GL_2(k)}\chi\cdot\delta_k, \mathbf{C}] \to \cdots .$$

From [P], Corollary 5.9, $\mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{B(k)}^{GL_2(k)}\chi\cdot\delta_k, \mathbf{C}]$ is nonzero iff $\mathrm{Ext}^1_{GL_2(k)}[\mathrm{ind}_{B(k)}^{GL_2(k)}\chi\cdot\delta_k, \mathbf{C}]$ is nonzero. As clearly $\mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C}, \mathbf{C}]$ is nonzero (integration against a Haar measure on $GL_2(k)/Z(k)$!), we find that $\mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{B(K)}^{GL_2(K)}\chi, \mathbf{C}]$ is always nonzero as required by Theorem B. To prove that up to scalars there is only one nonzero $GL_2(k)$-invariant linear form on V, we need to prove that if $\mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{B(k)}^{GL_2(k)}\chi\cdot\delta_k, \mathbf{C}]$ is nonzero then any $GL_2(k)$-invariant linear form on V is zero on $\mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C}$. This is proved as in the Lemma 6.2.1 below. We omit the details here. ∎

6.2 *Proof of Theorem B in Case II.* $V = Sp_K \otimes \chi$, where χ is a character of K^*, trivial on k^{*2}. The representation of D_k^* associated to V by the Jacquet-Langlands correspondence is $\chi \circ \mathrm{Nrd}(x)$ where $\mathrm{Nrd}(x)$

denotes the reduced norm of x. Since the reduced norm map of the division algebra D_k^* surjects onto k^*, we need to prove that $V = Sp_K \otimes \chi$, with χ trivial on k^{*2}, has a $GL_2(k)$-invariant linear form iff χ is nontrivial on k^*.

We have the exact sequence of $GL_2(k)$-modules,

$$(*) \qquad 0 \to S(\mathbf{P}_K^1 - \mathbf{P}_k^1) = \mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C} \to S(\mathbf{P}_K^1) \to S(\mathbf{P}_k^1) \to 0.$$

Therefore going modulo the constant functions,

$$0 \to \mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C} \to Sp_K \to Sp_k \to 0.$$

Tensoring this exact sequence with the character χ of k^* (trivial on k^{*2}), we get the exact sequence of $GL_2(k)$-modules

$$0 \to \mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C} \to Sp_K \otimes \chi \to Sp_k \otimes \chi \to 0.$$

Therefore

$$0 \to \mathrm{Hom}_{GL_2(k)}[Sp_k \otimes \chi, \mathbf{C}] \to \mathrm{Hom}_{GL_2(k)}[Sp_K \otimes \chi, \mathbf{C}] \to$$

$$\to \mathrm{Hom}_{GL_2(k)}[\mathrm{ind}_{Z(k)}^{GL_2(k)}\mathbf{C}, \mathbf{C}] \to \mathrm{Ext}^1_{GL_2(k)}[Sp_k \otimes \chi, \mathbf{C}] \to \cdots .$$

Clearly $\mathrm{Hom}_{GL_2(k)}[Sp_k \otimes \chi, \mathbf{C}] = 0$ and since for a nontrivial character χ of k^*, $\mathrm{Hom}_{GL_2(k)}[\chi \circ \det, \mathbf{C}] = 0$, it follows from Corollary 5.9 of [P] that $\mathrm{Ext}^1_{GL_2(k)}[Sp_k \otimes \chi, \mathbf{C}] = 0$ if χ restricted to k^* is nontrivial. Therefore $\mathrm{Hom}_{GL_2(k)}[Sp_K \otimes \chi, \mathbf{C}] \cong \mathbf{C}$ if χ is nontrivial on k^* but trivial on k^{*2}. We prove that Sp_K has no $GL_2(k)$-invariant linear forms in the next lemma.

LEMMA 6.2.1. *For K a cubic field extension of k, the special representation Sp_K has no $GL_2(k)$-invariant linear form.*

Proof. A $GL_2(k)$-invariant linear form on Sp_K is equivalent to a $GL_2(k)$-invariant linear form on functions on $\mathbf{P}_K^1$ which is trivial on constant functions. Let ℓ be such a linear form which by the exact sequence (*) above can be assumed to be the integration of compactly supported functions on $\mathbf{P}_K^1 - \mathbf{P}_k^1 = GL_2(k)/Z(k)$ with respect to a Haar measure on $GL_2(k)/Z(k)$. For a compact open subset X on $\mathbf{P}_K^1$, let χ_X denote the characteristic function of X. Look at the action of

$$\begin{pmatrix} 1 & \alpha \\ 0 & \pi_k \end{pmatrix}$$

on the functions on $\mathbf{P}^1_K$, where α belongs to $\mathcal{O}_k$. As $(x + \alpha)/\pi_k$ is integral only if x is integral, and in which case it is integral for at most one value of α modulo π_k, it follows that

$$\sum_{\alpha \in \mathcal{O}_k/\pi_k} \begin{pmatrix} 1 & \alpha \\ 0 & \pi_k \end{pmatrix} \chi_{\mathcal{O}_K} - \chi_{\mathcal{O}_K}$$

is a negative function supported on $\mathbf{P}^1_K - \mathbf{P}^1_k$. Therefore

$$\ell\left(\sum_{\alpha \in \pi_k/\mathcal{O}_k} \begin{pmatrix} 1 & \alpha \\ 0 & \pi_k \end{pmatrix} \chi_{\mathcal{O}_K} - \chi_{\mathcal{O}_K} \right)$$

is nonzero. By using the invariance of ℓ with respect to the $GL_2(k)$ action, we find that $(q - 1)\ell(\chi_{\mathcal{O}_K}) \neq 0$, or $\ell(\chi_{\mathcal{O}_K}) \neq 0$.
Let

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since $w\chi_{\mathcal{O}_K} + \chi_{\mathcal{O}_K}$ is the sum of the constant function $\mathbf{1}$ on $\mathbf{P}^1_K$ and the characteristic function of $\mathcal{O}^*_K$, and as ℓ takes the value zero on the constant functions, it follows that

$$2\ell(\chi_{\mathcal{O}_K}) = \ell(w\chi_{\mathcal{O}_K} + \chi_{\mathcal{O}_K}) = \ell(\chi_{\mathcal{O}^*_K}) \neq 0.$$

We now split the proof (of the nonexistence of $GL_2(k)$-invariant linear form on Sp_K) into two cases depending on whether K/k is unramified or not.

If K/k is unramified then $\mathcal{O}^*_K = \mathcal{O}_K - \pi_k\mathcal{O}_K$, and therefore $\ell(\chi_{\mathcal{O}^*_K}) = \ell(\chi_{\mathcal{O}_K}) - \ell(\chi_{\pi_k\mathcal{O}_K}) = 0$, a contradiction.

If K/k is ramified then

$$\bigcup_{\alpha \in \mathcal{O}_k/\pi_k} [\alpha + \pi_K\mathcal{O}_K] = \mathcal{O}_K,$$

where the union is of disjoint sets. Again by invariance of ℓ, $q\ell(\chi_{\pi_K\mathcal{O}_K}) = \ell(\chi_{\mathcal{O}_K})$. But previously we have obtained $2\ell(\chi_{\mathcal{O}_K}) = \ell(\chi_{\mathcal{O}_K^*})$. As $\chi_{\mathcal{O}_K} = \chi_{\pi_K\mathcal{O}_K} + \chi_{\mathcal{O}_K^*}$, we obtain the system of equations

$$(q - 1)\ell(\chi_{\mathcal{O}_K}) = q\ell(\chi_{\mathcal{O}_K^*})$$

$$2\ell(\chi_{\mathcal{O}_K}) = \ell(\chi_{\mathcal{O}_K^*}),$$

with $\ell(\chi_{\mathcal{O}_K}) \neq 0$, which leads to a contradiction. ■

Though the author has been unable to formulate a more general result which would contain the above lemma, it seems likely that there should be one. We pose the following question.

Question. Suppose that the k-rational points $G(k)$ of a reductive algebraic group over a nonarchimedean local field k acts on the k-rational points $X(k)$ of an algebraic variety over k with two orbits, one closed and the other one open but not closed. Then

(a) If $X(k)$ is compact and the closed orbit does not have a $G(k)$-invariant measure, find conditions under which $X(k)$ does not have a $G(k)$-invariant distribution, and also find conditions under which $X(k)$ does not have a $G(k)$-invariant distribution which is zero on the constant functions.

(b) If the closed orbit has a $G(k)$-invariant measure, is it true that there is up to scalars, a unique $G(k)$-invariant distribution on $X(k)$.

Remark. We give a simple example to illustrate question (a). For a quadratic field extension K of k, both $\mathbf{P}^1_k \times \mathbf{P}^1_k$ and $\mathbf{P}^1_K$ satisfy the conditions for question (a). $\mathbf{P}^1_k \times \mathbf{P}^1_k$ has a $GL_2(k)$-invariant distribution which is zero on the constant functions but $\mathbf{P}^1_K$ does not have (though it does have a $GL_2(k)$-invariant distribution).

6.3 *Proof of Theorem B in Case III.* V is a supercuspidal representation of $GL_2(K)$.

Assume that V is induced from a very cuspidal representation W of level n of a maximal compact-modulo-centre subgroup $\mathcal{K}$ of $GL_2(K)$. We are interested in the restriction of $\operatorname{ind}_{\mathcal{K}}^{GL_2(K)} W$ to $GL_2(k)$, which by Mackey's theory is equal to

$$\bigoplus_{g \in GL_2(k) \backslash GL_2(K)/\mathcal{K}} \operatorname{ind}_{GL_2(k) \cap g\mathcal{K}g^{-1}}^{GL_2(k)} W^g,$$

where W^g is the representation of $GL_2(k) \cap g\mathcal{K}g^{-1}$ obtained by the composition

$$GL_2(k) \cap g\mathcal{K}g^{-1} \to \mathcal{K} \to \text{Aut } W$$

$$X \mapsto g^{-1}Xg.$$

We will give the details of the proof assuming that $\mathcal{K} = K^*GL_2(\mathcal{O}_K)$. Here also there are two cases to consider.

Case A. $\mathcal{K} = K^*GL_2(\mathcal{O}_K)$ where K is a ramified extension of k and the representation W of $\mathcal{K}$ is of level $n = 3m + 1$.

In this case the representation V' of D_K^* factors through $D_K^*/D_K^*(6m + 1)$ (but not through $D_K^*/D_K^*(6m)$). As the representation V' is irreducible, $D_K^*(6m)/D_K^*(6m + 1)$ has no invariant vectors in V'. If L is the quadratic unramified extension of k, then $L^*(m)$ surjects onto $D_K^*(6m)/D_K^*(6m + 1)$. Therefore V' has no D_k^*-invariant vector. As the unipotent subgroup

$$\begin{pmatrix} 1 & \pi_K^{3m}\mathcal{O}_K \\ 0 & 1 \end{pmatrix}$$

of $\mathcal{K}/\mathcal{K}(3m + 1)$ is in the image of $GL_2(\mathcal{O}_k)$, the representation W of $\mathcal{K}$ has no $GL_2(\mathcal{O}_k)$-invariant vector, by the definition of supercuspidality. For

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \pi_K \end{pmatrix}, \qquad g^{-1}GL_2(k)g \cap \mathcal{K}$$

consists of the matrices

$$\begin{pmatrix} a & \pi_K b \\ \pi_K^{-1}c & d \end{pmatrix}$$

where $a, d \in \mathcal{O}_k^*$ and $b, c \in \mathcal{O}_k$ with $\pi_k | c$. Clearly the group

$$\begin{pmatrix} a & \pi_K b \\ \pi_K^2 c & d \end{pmatrix}$$

with a, $d \in \mathcal{O}_k^*$ and b, $c \in \mathcal{O}_k$ has no $\mathcal{K}$-generic elements (as $\mathcal{K}$-generic elements have the same K-valuations for the off-diagonal elements) and also has no conjugate of

$$\begin{pmatrix} 1 & \pi_K^{3m} \\ 0 & 1 \end{pmatrix}$$

modulo $\mathcal{K}(3m + 1)$. It follows from Lemma 2.4 that the character of the representation W restricted to this subgroup is supported only on the trivial element. The dimension of W is $(q - 1)q^{3m}$ and the order of this subgroup is also $(q - 1)q^{3m}$, therefore the representation W has a one-dimensional space of invariant vectors for this subgroup. The proof of Case B will show that W^g has no $GL_2(k) \cap g\mathcal{K}g^{-1}$-invariant vector for g belonging to other double cosets. Therefore we are done in this case.

Case B. Cases not covered in Case A.

The proof of Theorem B in this case will be in two steps. The first step consists in proving that W has a $\mathcal{K}$-invariant linear form iff the representation V' of D_K^* does not have a D_k^*-invariant linear form. The second step consists in proving that for g belonging to a non-trivial double coset of $GL_2(k)\backslash GL_2(K)/\mathcal{K}$, the image of $GL_2(k) \cap g\mathcal{K}g^{-1}$ in $\mathcal{K}$, obtained by sending $X \in GL_2(k) \cap g\mathcal{K}g^{-1}$ to $g^{-1}Xg$, contains the unipotent matrices

$$\begin{pmatrix} 1 & 0 \\ \pi_K^{n-1}\mathcal{O}_K & 1 \end{pmatrix} \bmod \mathcal{K}(n),$$

and therefore by the definition of very cuspidality, the representation W^g does not have any $GL_2(k) \cap g\mathcal{K}g^{-1}$-invariant linear form. These two steps combined with Frobenius reciprocity will complete the proof of Theorem B.

The proof of the first step is similar to the proof of Theorem C given in Section 5, and in fact much simpler as the complications in that proof that an elliptic element of $GL_2(k)$ could split in $GL_2(K)$, or that an element of $GL_2(k)$ coming from a ramified field extension of k could become an element in $GL_2(K)$ which came from an unramified field extension of K, do not arise. We will omit the details of this step.

We now begin with the proof of the second step. Recall that $GL_2(K)/\mathcal{K}$ is the space of lattices up to homothesy in a two-dimensional vector space over K, say with basis e_1 and e_2. As $GL_2(k) \cap g\mathcal{K}g^{-1}$ is the stabilizer of the lattice $g(\mathcal{O}_K e_1 \oplus \mathcal{O}_K e_2)$ in $GL_2(k)$, we begin by writing lattices $\mathcal{L}$ in $Ke_1 \oplus Ke_2$ in a convenient form. By scaling, we can assume that $\mathcal{L}$ is contained in $\mathcal{O}_K e_1 + \mathcal{O}_K e_2$ but is not contained in $\pi_K[\mathcal{O}_K e_1 + \mathcal{O}_K e_2]$, i.e. there exists a vector e in $\mathcal{L}$ such that $e = \lambda_1 e_1 + \lambda_2 e_2$ with $\lambda_1, \lambda_2 \in \mathcal{O}_K$ and at least one of λ_1 or λ_2, say λ_1, is a unit. Choose f in $\mathcal{L}$ such that e and f form a basis of $\mathcal{L}$ over $\mathcal{O}_K$. Clearly we can assume that $e = e_1 + \lambda_2 \pi_K^j e_2$ and $f = \pi_K^i e_2$ with either $0 \leq j < i$ and λ_2 belonging to $\mathcal{O}_K^*$, or $\lambda_2 = 0$. For this lattice,

$$g = \begin{pmatrix} 1 & 0 \\ \lambda_2 \pi_K^j & \pi_K^i \end{pmatrix}.$$

Write an element X of $g^{-1}GL_2(k)g \cap \mathcal{K}$ as

$$X = g^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{with}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(k) \quad \text{and} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{K}.$$

Substituting

$$g = \begin{pmatrix} 1 & 0 \\ \lambda_2 \pi_K^j & \pi_K^i \end{pmatrix},$$

X can be written as

$$(*) \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a + \lambda_2 \pi_K^j b & \pi_K^i b \\ \pi_K^{-i}[c + \pi_K^j \lambda_2 (d - a) - b\lambda_2^2 \pi_K^{2j}] & d - b\lambda_2 \pi_K^j \end{pmatrix}.$$

Now we divide the proof into two parts depending on whether K/k is unramified or not.

Suppose K/k is unramified. In this case we can assume that $\pi_K = \pi_k$. Substituting a to be 1, b to be $\pi_k^{n+i-2j-1}b$, c to be $\pi_k^{n+i-1}c$ and d to be $1 + \pi_k^{n+i-j-1}d$, the matrix on the right hand side of the equation (*) becomes

$$(**) \qquad \begin{pmatrix} 1 + \lambda_2\pi_k^{n+i-j-1}b, & \pi_k^{n+2i-2j-1}b \\ \pi_k^{n-1}[c + \lambda_2 d - b\lambda_2^2], & 1 + \pi_k^{n+i-j-1}[d - b\lambda_2] \end{pmatrix}.$$

Since $i - j > 0$, it follows that the matrix in (**) modulo $\mathcal{K}(n)$ is

$$\begin{pmatrix} 1 & 0 \\ \pi_k^{n-1}[c + \lambda_2 d - b\lambda_2^2] & 1 \end{pmatrix}.$$

Clearly if the lattice $\mathcal{L}$ is not defined over k then we can, after adding an element of $\mathcal{O}_k$ to λ_2 (which amounts to replacing the lattice $\mathcal{L}$ by $g'\mathcal{L}$ where $g' \in GL_2(k)$), assume that λ_2 generates $\mathcal{O}_K/\pi_k$ over $\mathcal{O}_k/\pi_k$, and hence $[c + d\lambda_2 - b\lambda_2^2]$ takes arbitrary values from $\mathcal{O}_K/\pi_k$. From the definition of very cuspidality, it follows that W^g does not have any $GL_2(k) \cap g\mathcal{K}g^{-1}$-invariant linear form for g a nontrivial double coset in $GL_2(k)\backslash GL_2(K)/\mathcal{K}$. This completes the proof in the case K/k is unramified.

Assume now that K/k is a ramified extension. If j is not divisible by 3, the integer $n - 1$ is represented by one of the numbers $-i$, $-i + j$, $-i + 2j$ modulo 3 and it is clear that in each case, choosing a, b, c, $d \in k$ appropriately, the matrix

$$\begin{pmatrix} a + \lambda_2\pi_K^j b, & \pi_K^i b \\ \pi_K^{-i}[c + \pi_K^j\lambda_2(d - a) - b\lambda_2^2\pi_K^{2j}], & d - b\lambda_2\pi_K^j \end{pmatrix}$$

represents the unipotent subgroup

$$\begin{pmatrix} 1 & 0 \\ \pi_K^{n-1}\mathcal{O}_k & 1 \end{pmatrix}$$

modulo $\mathcal{K}(n)$. If j is divisible by 3 it is clear that we can change λ_2 by an element of $\mathcal{O}_k$ to reduce to a case when j is not divisible by 3. If the new j is less than i, we are done. If the new j is greater than or equal to i, the lattice $\mathcal{L}$ becomes equal to $\mathcal{O}_K e_1 + \pi_K^i \mathcal{O}_K e_2$. This is not defined over k iff 3 does not divide i. By scaling and using $GL_2(k)$, we can assume that $\mathcal{L} = \mathcal{O}_K e_1 + \pi_K^i \mathcal{O}_K e_2$. For

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \pi_K \end{pmatrix},$$

$g^{-1}GL_2(k)g \cap \mathcal{K}$ consists of the matrices

$$\begin{pmatrix} a & \pi_K b \\ \pi_K^{-1} c & d \end{pmatrix}$$

where $a, d \in \mathcal{O}_k^*$ and $b, c \in \mathcal{O}_k$ with $\pi_k | c$. As the level n is such that 3 does not divide $n - 1$, clearly

$$\begin{pmatrix} 1 & \pi_K^{n-1} \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 \\ \pi_K^{n-1} & 1 \end{pmatrix}$$

is in the stabilizer and hence has no invariant vectors. This completes the proof in the case when K/k is ramified. ∎

7. Multiplicative induction. In this section we introduce the concept of a multiplicative analogue of the usual process of induction to be called *multiplicative induction.*[1]

Let H be a subgroup of index m in a group G and suppose that V is an n-dimensional representation of H. Then we will construct a representation of G of dimension n^m, to be denoted $M_H^G V$ or MV if there is no cause of confusion, and called multiplicative induction of V (from H to G) in the following way.

Let $g_1, \ldots, g_m$ be a set of representatives for the left cosets of H in G. As a vector space $M_H^G V = V_1 \otimes \cdots \otimes V_m$ where each V_i is a copy

[1]It has been pointed out to me that this concept is known in the literature, cf. Curtis-Reiner, vol. 1, where it is called tensor induction but we have called it multiplicative induction as this name seems to be more suggestive and appropriate.

of V. We define the action of $g \in G$ on $v_1 \otimes \cdots \otimes v_m$ where $v_i \in V_i$ by

$$g(v_1 \otimes \cdots \otimes v_m) = w_1 \otimes \cdots \otimes w_m$$

where given $1 \le j \le m$, $w_j = h(g, i)v_i$, i being chosen such that $gg_i \in g_jH$, and $h(g, i)$ belonging to H being defined by $gg_i = g_jh(g, i)$.

It is easy to check that this defines a representation of G, and that changing the representatives $\{g_i\}$ of the left cosets of H in G produces an isomorphic representation.

The following lemma summarizes the properties of multiplicative induction that we will need. We will omit the proofs as they follow quite easily from the definition of multiplicative induction. In the lemma, H will always be a subgroup of finite index in a group G and we will be considering only finite dimensional representations.

LEMMA 7.1. (a) *For representations V_1 and V_2 of H,*

$$M_H^G(V_1 \otimes V_2) = M_H^G(V_1) \otimes M_H^G(V_1).$$

(b) *For a representation V of H,*

$$(M_H^G V)^* \cong M_H^G V^*.$$

(c) *For a one-dimensional representation χ of H, $M_H^G\chi$ is the one-dimensional representation of G obtained by composing χ with the transfer map from $G/[G, G]$ to $H/[H, H]$.*

(d) *If $[G : H] = 2$ and $V = V_1 \oplus V_2$ is the direct sum of two H-modules, then*

$$M_H^G(V_1 \oplus V_2) = M_H^G V_1 \oplus M_H^G V_2 \oplus \operatorname{Ind}_H^G(V_1 \otimes V_2^\sigma),$$

where V_2^σ is the representation of H obtained from V_2 by conjugating it by an element $\sigma \in G - H$.

(e) *If $[G : H] = 3$ and $V = V_1 \oplus V_2$ is the direct sum of two H-modules, there are two cases depending on whether H is normal in G or not.*

(i) *H is normal in G and σ is a generator of G/H. In this case*

$$M_H^G(V_1 \oplus V_2) = M_H^G V_1 \oplus M_H^G V_2 \oplus \operatorname{Ind}_H^G[V_1 \otimes V_1^\sigma \otimes V_2^{\sigma^2} \oplus V_1 \otimes V_2^\sigma \otimes V_2^{\sigma^2}].$$

(ii) *H is not normal in G. Let N be the subgroup of index 2 in H which is normal in G. Fix σ a nontrivial element of G/H and for a representation W of H, let W^σ denote the representation of N obtained by conjugating W restricted to N by σ (of course W^σ depends only on the left coset σH). Then*

$$M_H^G(V_1 \oplus V_2) = M_H^G V_1 \oplus M_H^G V_2 \oplus \mathrm{Ind}_H^G[V_1 \otimes M_N^H(V_2^\sigma) \oplus M_N^H(V_1^\sigma) \otimes V_2].$$

(f) *If $[G : H] = n$, then for a d-dimensional representation V of H,*

$$\det(M_H^G V)(g) = \chi(g)[M_H^G(\det V)]^d,$$

where $\chi(g)$ is the determinant of g acting on $V \otimes \cdots \otimes V$ (product over $[G : H]$-factors) by permuting the coordinates through the action of G on G/H. ∎

8. Epsilon factors I. In this section we prove Theorem D which characterizes those representations π of $GL_2(\mathbb{K})$ which have a $GL_2(k)$-invariant linear form in terms of epsilon factor of a certain representation of the Deligne-Weil group of k associated to π.

For a local field K, we let W_K denote the Weil group of K, and $W_K' = W_K \times SL_2(\mathbf{C})$ denote the Deligne-Weil group of K.

The representations of $W_K' = W_K \times SL_2(\mathbf{C})$ on which W_K acts trivially and $SL_2(\mathbf{C})$ acts by the unique irreducible representation of dimension n, will be denoted by $sp_K(n)$. The representation $sp_K(2)$ corresponds to the special representation Sp_K of $GL_2(K)$ via the local Langlands' correspondence.

For an arbitrary commutative semisimple algebra $A = K_1 \oplus \cdots \oplus K_r$, where K_i are local fields, define the Deligne-Weil group W_A' of A to be product $W_{K_1}' \times \cdots \times W_{K_r}'$ of the Deligne-Weil groups W_{K_i}' of K_i. With this notation, a representation π of $GL_2(A) = \Pi GL_2(K_i)$ being a tensor product of representations of $GL_2(K_i)$ naturally defines a representation σ_π of W_A' (via the local Langlands' correspondence). Observe that $W_A' = W_{K_1}' \times \cdots \times W_{K_r}'$ is contained in $W_k' \times \cdots \times W_k'$ (product over r-factors) as a subgroup of finite index. Therefore to this representation σ_π of W_A', we can perform multiplicative induction to construct a representation of $W_k' \times \cdots \times W_k'$. Restricting this representation of $W_k' \times \cdots \times W_k'$ to the diagonally embedded W_k', we obtain a representation of W_k' which by abuse of language will be denote by $M_A^k \sigma_\pi$.

We will refer to Tate's article [Ta] as the general reference for epsilon factors. The epsilon factor used in this paper is in Tate's notation in [Ta] $\epsilon_L(\sigma, \psi) = \epsilon_D(\sigma \cdot \| \ \|^{1/2}, \psi, dx)$ where dx is the Haar measure on k, self-dual for Fourier transforms with respect to ψ. Here we only note the following properties which we will repeatedly use.

8.0.1. $\epsilon(\sigma_1 \oplus \sigma_2, \psi) = \epsilon(\sigma_1, \psi) \cdot \epsilon(\sigma_2, \psi)$.

8.0.2. $\epsilon(\sigma, \psi) \cdot \epsilon(\sigma^*, \psi) = \det \sigma(-1)$ where σ^* is the contragredient of σ.

8.0.3. $\epsilon(\sigma, \psi_a) = (\det)^{\dim \sigma}(a)\epsilon(\sigma, \psi)$ where $\psi_a(x) = \psi(ax)$.

8.0.4. $\epsilon(\mathrm{Ind}_K^k \sigma, \psi) = (\sigma, \psi_K)$ where σ is a (virtual) representation of W_K of dimension 0, and ψ_K is the additive character of K obtained by composing the trace map from K to k with the character ψ.

8.0.5. $\epsilon(sp_K(n) \otimes \rho, \psi) = \epsilon(\rho, \psi)^n \cdot \det(-F, \rho^I)^{n-1}$, where ρ is a representation of W_k, F is a geometric Frobenius of W_k, and I is the inertia subgroup of W_k.

From 8.0.2 and 8.0.4, it is easy to see that:

8.0.6. $\epsilon(\mathrm{Ind}_K^k \sigma, \psi) = \epsilon(\sigma, \psi_K)$ if $4 \mid \dim \sigma$ and where ψ_K is as in 8.0.4.

The following theorem of Tunnell will be basic to our calculations.

THEOREM 8.0.7. *Let π be an infinite dimensional irreducible admissible representation of $GL_2(k)$ with central character ω_π. For a separable quadratic extension K of k, let χ be a character of K^* which restricts to ω_π on k^* and let ψ be a non-trivial character of k. Then $\epsilon(\sigma(\pi)|_K \otimes \chi^{-1}, \psi_K)$ is independent of ψ and χ appears in π if and only if $\epsilon(\sigma(\pi)|_K \otimes \chi^{-1}, \psi_K)\omega_\pi(-1) = 1$, or equivalently if and only if $\epsilon(\sigma(\pi) \otimes \mathrm{Ind}_K^k \chi^{-1})\omega_\pi(-1)\omega_{K/k}(-1) = 1$.*

For a representation π of $GL_2(\mathbb{K})$ whose central character restricted to k^* is trivial, the representation $M_{\mathbb{K}}^k \sigma_\pi$ of W_k' by Lemma 7.1(f) has trivial determinant (if $\mathbb{K}$ is a cubic field extension of k then since the determinant of the automorphism of $\sigma_\pi \otimes \sigma_\pi \otimes \sigma_\pi$ sending $v_1 \otimes v_2 \otimes v_3$ to $v_2 \otimes v_1 \otimes v_3$ where $v_i \in \sigma_\pi$ is one, $\chi \equiv 1$). Therefore by 8.0.3 $\epsilon(M_{\mathbb{K}}^k \sigma_\pi, \psi)$, where ψ is a nontrivial character of k is independent of ψ, and we will omit ψ from the notation of epsilon factor and simply write $\epsilon(M_{\mathbb{K}}^k \sigma_\pi)$.

We will often use the following lemma from class field theory.

LEMMA 8.1. *For K a finite field extension of a local field k, the transfer map from $W_k/[W_k, W_k]$ to $W_K/[W_K, W_K]$ is identified to the in-*

clusion of k^* in K^* via the identification of $W_k/[W_k, W_k]$ (resp. $W_K/[W_K, W_K]$) to k^* (resp. K^*). ∎

Let $I\!K = K \oplus k$ be a cubic algebra over k and suppose that σ_{π_1} (resp. σ_{π_2}) is a two-dimensional representation of W_K (resp. W_k) corresponding to an irreducible, admissible, generic representation π_1 (resp. π_2) of $GL_2(K)$ (resp. $GL_2(k)$). We now begin with the proof of Theorem D. The proof will be done by a case-by-case analysis.

PROPOSITION 8.2. *With the notation as above, assume that π_1 is a principal series representation of $GL_2(K)$ (and therefore σ_{π_1} is a sum of two characters of K^*). Then $\epsilon(M_K^k\sigma_{\pi_1} \otimes \sigma_{\pi_2})\cdot\omega_{K/k}(-1) = 1$ iff the representation $\pi_1 \otimes \pi_2$ of $GL_2(K)$ has a $GL_2(k)$-invariant linear form.*

Proof. Let σ_{π_1} be the sum of two characters α and β of K^*. By Lemma 7.1(d),

$$M_K^k(\alpha \oplus \beta) = M_K^k\alpha \oplus M_K^k\beta \oplus \mathrm{Ind}_K^k(\alpha\beta')$$

where β' is the character of K^* given by $\beta'(x) = \beta(\bar{x})$. Therefore,

$$\begin{aligned}\epsilon[M_K^k\sigma_{\pi_1} \otimes \sigma_{\pi_2}] &= \epsilon[(M_K^k\alpha \oplus M_K^k\beta \oplus \mathrm{Ind}_K^k(\alpha\beta')) \otimes \sigma_{\pi_2}] \\ &= \epsilon[(\alpha\,|_{k^*} \oplus \beta|_{k^*} \oplus \mathrm{Ind}_K^k(\alpha\beta')) \otimes \sigma_{\pi_2}].\end{aligned}$$

By our condition on central characters, the dual of $\alpha\,|_{k^*} \otimes \sigma_{\pi_2}$ is $\beta\,|_{k^*} \otimes \sigma_{\pi_2}$. Therefore by 8.0.2,

$$\begin{aligned}\epsilon[M_K^k\sigma_{\pi_1} \otimes \sigma_{\pi_2}] &= \det(\alpha \otimes \sigma_{\pi_2})(-1)\cdot\epsilon[\mathrm{Ind}_K^k(\alpha\beta') \otimes \sigma_{\pi_2}] \\ &= (\alpha^2 \det \sigma_{\pi_2})(-1)\cdot\epsilon[\mathrm{Ind}_K^k(\alpha\beta') \otimes \sigma_{\pi_2}] \\ &= \det \sigma_{\pi_2}(-1)\cdot\epsilon[\mathrm{Ind}_K^k(\alpha\beta') \otimes \sigma_{\pi_2}].\end{aligned}$$

By Theorem 8.0.7, $\epsilon[\mathrm{Ind}_K^k(\alpha\beta') \otimes \sigma_{\pi_2}]\cdot(\det \sigma_{\pi_2}\cdot\omega_{K/k})(-1) = 1$ iff the character $(\alpha\beta')^{-1}$ of K^* appears in π_2. By Remark 4.1.1 we know that the character $(\alpha\beta')^{-1}$ of K^* appears in π_2 iff $\pi_1 \otimes \pi_2$ has a $GL_2(k)$-invariant linear form. ∎

PROPOSITION 8.3. *With the notation as before Proposition* 8.2, *assume now that* π_2 *is a principal series* (*and therefore* σ_{π_2} *is a sum of two characters*). *Then* $\epsilon[M_K^k(\sigma_{\pi_1}) \otimes \sigma_{\pi_2}]\omega_{K/k}(-1) = 1$.

Proof. Suppose that $\sigma_{\pi_2} = \mu \oplus \nu$. By the condition on central characters, the dual of $M_K^k(\sigma_{\pi_1}) \otimes \mu$ is $M_K^k(\sigma_{\pi_1}) \otimes \nu$. Therefore

$$\epsilon[M_K^k(\sigma_{\pi_1}) \otimes \sigma_{\pi_2}] = \det[M_K^k(\sigma_{\pi_1}) \otimes \mu](-1) \qquad \text{(by 8.0.2)},$$

$$= \omega_{K/k}(-1) \qquad \text{(by 7.1(f))}. \quad \blacksquare$$

We next consider representations of $GL_2(K) \times GL_2(k)$ of the type $\pi_1 \otimes \pi_2$ where π_2 is a special representation which by Lemmas 7.1(a) and 7.1(c) can be assumed to be Sp_k, and therefore the central character of π_1 restricted to k^* is trivial. From Lemmas 7.1(b) and 7.1(a) (and the assumption that the central character of π_1 restricted to k^* is trivial), we find that $M_K^k(\sigma_{\pi_1})$ is self dual. Therefore,

$$\epsilon[M_K^k\sigma_{\pi_1} \otimes sp(2)]$$

$$= \epsilon[M_K^k\sigma_{\pi_1}]^2 \cdot \det(-F, (M_K^k\sigma_{\pi_1})^I) \qquad \text{(by 8.0.5)}$$

$$= [\det M_K^k(\sigma_{\pi_1})](-1) \cdot \det(-F, (M_K^k\sigma_{\pi_1})^I) \qquad \text{(by 8.0.2)}$$

$$= \omega_{K/k}(-1) \cdot \det(-F, (M_K^k\sigma_{\pi_1})^I) \qquad \text{(by 7.1(f))}.$$

Since from Theorem B, $\pi_1 \otimes Sp_k$ has a $GL_2(k)$-invariant linear form iff π_1 does not have a $GL_2(k)$-invariant form, Theorem D reduces to the following proposition.

PROPOSITION 8.4. *For an irreducible, admissible, generic representation* π *of* $GL_2(K)$, $\det(-F, (M_K^k\sigma_\pi)^I)$ *is equal to one iff* π *has a* $GL_2(k)$-*invariant linear form.*

We will split the proof of this proposition into two cases depending on whether σ_π is an irreducible representation or not.

PROPOSITION 8.4.1. *For characters* α, β *of* K^* *such that* $(\alpha\beta)(x) = 1$ *for* x *belonging to* k^*, $\det(-F, [M_K^k(\alpha + \beta)]^I) = -1$ *if the character* $\alpha\beta'(x) = \alpha(x)\beta(\bar{x})$ *of* K^* *is trivial, and equals* 1 *otherwise.*

Proof. As $M_K^k(\alpha + \beta) = \alpha\,|_{k^*} \oplus \beta\,|_{k^*} \oplus \mathrm{Ind}_K^k(\alpha\beta')$ and $\alpha\beta$ equals 1 on k^*, either both the characters α and β are unramified characters of k^* or both are ramified, and $\det(-F, [M_K^k(\alpha + \beta)]^I) = \det(-F, [\mathrm{Ind}_K^k(\alpha\beta')]^I)$. If $\alpha\beta'$ does not factor through the norm mapping of $K^* \to k^*$, then the representation $\mathrm{Ind}_K^k(\alpha\beta')$ is irreducible and since I is a normal group with abelian quotient ($\cong \mathbf{Z}$), $\mathrm{Ind}_K^k(\alpha\beta')$ will have no I-invariant vectors and we will have $\det(-F, [M_K^k(\alpha + \beta)]^I) = 1$. If $\alpha\beta'$ does factor through a character χ of k^*, i.e. $\alpha\beta'(x) = \chi(N_K^k x)$ then $\mathrm{Ind}_K^k(\alpha\beta') = \chi \oplus \omega_{K/k}\chi$, and it is easy to see that $\mathrm{Ind}_K^k(\alpha\beta')$ has I-invariant vector iff $\alpha\beta'$ is trivial on I_K. If $\alpha\beta'$ is trivial on I_K, then if K/k is unramified (resp. ramified) then $[\mathrm{Ind}_K^k(\alpha\beta')]^I$ is two-dimensional (resp. one-dimensional) and $\det(-F, \mathrm{Ind}_K^k(\alpha\beta')^I)$ is $-(\alpha\beta')(\pi_K)$. It follows that $\epsilon(M_K^k(\alpha \oplus \beta) \otimes sp_k(2))$ is -1 iff the character $\alpha\beta'$ of K^* is trivial on I_K, and $\alpha\beta'(\pi_K) = 1$, i.e., the character $\alpha\beta'$ of K^* is trivial. ∎

This proposition together with Remark 4.1.1 completes the proof of Theorem D in this case.

We now turn to the case when σ_π is an irreducible representation.

PROPOSITION 8.4.2. *Let K/k be a quadratic field extension. For a 2-dimensional irreducible representation U of W_K whose determinant restricted to k^* is trivial, let MU denote the 4-dimensional representation of W_k, obtained by multiplicative induction from W_K to W_k. Then MU has a W_k-invariant vector iff* $\det(-F, (MU)^I) = -1$.

Proof. We will split the proof into several steps:

(1) If $(MU)^I = \{0\}$, then $\det(-F, (MU)^I) = 1$ and MU has no W_k-invariant vector.

(2) If $(MU)^I$ is one-dimensional, then $\det(-F, (MU)^I) = -1$ iff MU has a W_k-invariant vector.

(3) If $(MU)^I$ is two-dimensional, then since W_k/I is a cyclic group generated by F, MU contains two one-dimensional representations of W_k. Call them χ_1, χ_2. These are unramified characters of k^*. As MU restricted to W_K is $U \otimes U^\sigma$ for σ the nontrivial element of W_k/W_K, MU contains at most two one-dimensional (and therefore exactly two one-dimensional) representations of W_K (this also implies that $(MU)^I$ can't be more than two dimensional) and these must be distinct. In particular, χ_1 and χ_2 are distinct characters. Since MU is self-dual, if the character χ_1 were of order ≥ 3, χ_2 would be χ_1^{-1} and $\det(-F, (MU)^I)$ would be 1;

and of course MU would have no W_k-invariant vector. If both χ_1 and χ_2 were of order ≤ 2, then since they are distinct, we can assume that one of them is the trivial character and the other one is the unramified character of order 2. In this case MU has a W_k-invariant vector and $\det(-F, (MU)^I) = -1$. ■

The next proposition gives a criterion as to when $M_K^k U$ has a W_k invariant vector in terms of the representation U of W_K. For simplicity we will assume that U has trivial determinant.

Proposition 8.4.3. *For a two-dimensional irreducible representation U of W_K with trivial determinant, $M_K^k U$ has a W_k-invariant vector iff U is the restriction of a representation of W_k whose determinant is $\omega_{K/k}$.*

Proof. We have the exact sequence

$$0 \to W_K \to W_k \to \mathbf{Z}/2 \to 0.$$

Let $A \in W_k$ go to the non-trivial element of $\mathbf{Z}/2$ and assume that $A^2 = m \in W_K$. The representation $M_K^k U$ of W_k is $U \otimes U$ as a vector space, the action of W_k being given by

$$n \cdot (u_1 \otimes u_2) = n \cdot u_1 \otimes A^{-1} n A \cdot u_2, \qquad n \in W_K$$

$$A \cdot (u_1 \otimes u_2) = m \cdot u_2 \otimes u_1.$$

Since U is 2 dimensional and the determinant of the representation of W_K on U trivial, U is isomorphic to its dual. Therefore $M_K^k U$ has a W_K-invariant vector if and only if $U \cong U^A$. As U is an irreducible W_K-module, $M_K^k U$ has at most one W_K-invariant vector. One can explicitly write this vector down when $U \cong U^A$ and calculate the action of A on it. By this simple calculation which we omit here, one concludes that A acts by $(-\det A)$. Therefore $M_K^k U$ has a W_k-invariant vector if and only if U is the restriction of a representation of W_k with determinant $\omega_{K/k}$. ■

The following theorem, cf. [F], now completes the proof.

Theorem 8.4.4. *A representation V of $GL_2(K)$ with trivial central character has a $GL_2(k)$-invariant linear form iff V is the base change of a representation of $GL_2(k)$ with central character equal to $\omega_{K/k}$.* ■

PROPOSITION 8.5. *For a two dimensional representation σ of W'_k, and a quadratic extension K of k, $\epsilon[M_K^k(sp_K(2)) \otimes \sigma]\omega_{K/k}(-1) = 1$ if and only if either $\sigma = sp_k(2)$ or the representation $\pi(\sigma)$ of $GL_2(k)$ contains the trivial representation of K^*.*

Proof. First assume that σ is a representation of W_k. Since $M_K^k(sp_K(2)) \cong sp_k(3) \oplus \omega_{K/k}$, it follows from 8.0.5 that

$$\epsilon[M_K^k(sp_K(2)) \otimes \sigma] = \epsilon(\sigma)^3\epsilon(\omega_{K/k} \otimes \sigma)\det(-F, \sigma^I)^2 = \epsilon(\sigma \oplus \omega_{K/k}\sigma).$$

Since $\mathrm{Ind}_K^k 1 = 1 \oplus \omega_{K/k}$, theorem 8.0.7 completes the proof of the proposition when σ is a representation of W_k. Next assume that $\sigma = sp_k(2) \otimes \chi$ where χ is a character of k^* of order 2. Then as $M_K^k(sp_K(2)) \cong sp_k(3) \oplus \omega_{K/k}$, $M_K^k(sp_K(2)) \otimes sp_k(2) \otimes \chi \cong sp_k(4) \otimes \chi \oplus sp_k(2) \otimes \chi(1 \oplus \omega_{K/k})$. From 8.0.5, $\epsilon(sp_k(4) \otimes \chi) = \epsilon(\chi)^4 \det(-F, \chi^I)^3$. Since χ is a character of order 2, $\det(-F, \chi^I) = -1$ or 1 depending on whether χ is trivial or not. Let us assume that χ is non-trivial. Then,

$$\epsilon[M_K^k(sp_K(2)) \otimes sp_k(2) \otimes \chi] = \epsilon[sp_k(2) \otimes \chi(1 \oplus \omega_{K/k})].$$

Again by theorem 8.0.7, $\epsilon[sp_k(2) \otimes \chi(1 \oplus \omega_{K/k})]\omega_{K/k}(-1) = 1$ if and only if the representation $Sp \otimes \chi$ of $GL_2(k)$ contains the character $\chi \cdot Nm\, x$ of K^*, completing the proof in the case χ is non-trivial. We omit the proof for trivial χ. ∎

Now we treat the case of a cubic field extension. Recall that for a cubic algebra $\mathbb{K}/k$, the quadratic character $\omega_{\mathbb{K}/k}$ of k was defined in the Introduction. We will need to use the following simple lemma.

LEMMA 8.6. *For a cubic field extension $\mathbb{K}$ of k and a character χ of $\mathbb{K}^*$, $\det(\mathrm{Ind}_{W_{\mathbb{K}}}^{W_k}\chi) = \omega_{\mathbb{K}/k} \cdot \chi\,|_{k^*}$.* ∎

PROPOSITION 8.7. *Let $\sigma_\pi = \alpha \oplus \beta$ be a reducible two-dimensional representation of $W_{\mathbb{K}}$ such that the character $\alpha\beta$ of $\mathbb{K}^*$ is trivial on k^*. Then $\epsilon(M_{\mathbb{K}}^k\sigma_\pi) = 1$.*

Proof. Case 1. $\mathbb{K}/k$ cyclic. By Lemma 7.1(e)(i),

$$\begin{aligned}\epsilon[M_{\mathbb{K}}^k(\alpha \oplus \beta)] &= \epsilon[\alpha\,|_{k^*} \oplus \beta\,|_{K^*} \oplus \mathrm{Ind}_{\mathbb{K}}^k(\alpha\alpha^\sigma\beta^{\sigma^2} \oplus \alpha\beta^\sigma\beta^{\sigma^2})] \\ &= \alpha(-1)\cdot\epsilon[\mathrm{Ind}_{\mathbb{K}}^k(\alpha\alpha^\sigma\beta^{\sigma^2}) \oplus \mathrm{Ind}_{\mathbb{K}}^k(\alpha\beta^\sigma\beta^{\sigma^2})].\end{aligned}$$

As the determinant of σ_π restricted to k^* is trivial, $\alpha\alpha^\sigma\alpha^{\sigma^2}\beta\beta^\sigma\beta^{\sigma^2} = 1$, and therefore

$$\alpha\alpha^\sigma\beta^{\sigma^2} = (\beta\beta^\sigma\alpha^{\sigma^2})^{-1}$$
$$= [(\alpha\beta^\sigma\beta^{\sigma^2})^{-1}]^{\sigma^2}.$$

This implies that $\mathrm{Ind}_{\mathbb{K}}^k(\alpha\alpha^\sigma\beta^{\sigma^2})$ and $\mathrm{Ind}_{\mathbb{K}}^k(\alpha\beta^\sigma\beta^{\sigma^2})$ are dual of each other. Therefore from 8.0.2 and Lemma 8.6,

$$\epsilon[M_{\mathbb{K}}^k(\alpha \oplus \beta)] = \alpha(-1)\det[\mathrm{Ind}_{\mathbb{K}}^k(\alpha\alpha^\sigma\beta^{\sigma^2})](-1)$$
$$= \alpha(-1)(\alpha\alpha^\sigma\beta^{\sigma^2})(-1) = 1.$$

Case 2. $\mathbb{K}/k$ non-Galais with L the quadratic extension of $\mathbb{K}$ such that L/k is Galois. By Lemma 7.1(e)(ii),

$$\epsilon[M_{\mathbb{K}}^k(\alpha\oplus\beta)] = \epsilon[\alpha|_{k^*}\oplus\beta|_{k^*}\oplus\mathrm{Ind}_{\mathbb{K}}^k(\alpha\otimes M_L^{\mathbb{K}}(\beta|_L^\sigma)\oplus\beta\otimes M_L^{\mathbb{K}}(\alpha|_L^\sigma))]$$
$$= \epsilon[\alpha|_{k^*}\oplus\beta|_{k^*}\oplus\mathrm{Ind}_{\mathbb{K}}^k(\alpha\otimes(\beta|_L)^\sigma|_{\mathbb{K}}\oplus\beta\otimes(\alpha|_L)^\sigma|_{\mathbb{K}})]$$

where σ is an element of $W_k - W_{\mathbb{K}}$, $\mathrm{Gal}(L/\mathbb{K}) = \{1, \tau\}$, and for the character β of $\mathbb{K}^*$, $(\beta|_L)(y) = \beta(y\cdot\tau y)$ for $y \in L^*$. Now $(\beta|_L)^\sigma|_{\mathbb{K}}$ is

$$(\beta|_L)^\sigma|_{\mathbb{K}}(x) = (\beta|_L)(\sigma x) = \beta(\sigma x\cdot\tau\sigma x) = \frac{\beta(N_{\mathbb{K}}^k x)}{\beta(x)} \quad \text{for } x \in \mathbb{K}^*.$$

Similarly for $(\alpha|_L)^\sigma|_{\mathbb{K}}$. It follows that $\alpha \otimes (\beta|_L)^\sigma|_{\mathbb{K}}$ and $\beta \otimes (\alpha|_L)^\sigma|_{\mathbb{K}}$ are dual of each other. Therefore from 8.0.2 and Lemma 8.6,

$$\epsilon[M_{\mathbb{K}}^k(\alpha + \beta)] = \alpha(-1)\det(\mathrm{Ind}_{\mathbb{K}}^k\alpha(\beta|_L)^\sigma|_{\mathbb{K}})(-1) = \omega_{\mathbb{K}/k}(-1). \quad \blacksquare$$

PROPOSITION 8.8. *For a character χ of $\mathbb{K}^*$ trivial on k^{*2}, $\epsilon[M_{\mathbb{K}}^k(sp_{\mathbb{K}}(2)\otimes\chi)]\cdot\omega_{\mathbb{K}/k}(-1) = 1$ iff χ is non-trivial on k^*.*

Proof. As a representation of $SL_2(\mathbf{C}) \times S_3$, where S_3 is the symmetric group on three symbols, we have

$$sp_{\mathbb{K}}(2) \otimes sp_{\mathbb{K}}(2) \otimes sp_{\mathbb{K}}(2) = sp_{\mathbb{K}}(4) \oplus [\rho \otimes sp_{\mathbb{K}}(2)]$$

where ρ is the unique two-dimensional irreducible representation of S_3. It follows that

$$M^k_{\mathbb{K}}(sp_{\mathbb{K}}(2) \otimes \chi) = [sp_{\mathbb{K}}(4) \oplus \rho_{\mathbb{K}/k} \otimes sp_{\mathbb{K}}(2)] \otimes \chi|_{k^*},$$

where $\rho_{\mathbb{K}/k}$ denotes the two-dimensional representation of W_k obtained by composing the representation ρ of S_3 with the action of W_k on $W_k/W_{\mathbb{K}}$. It is easy to see that $\det \rho_{\mathbb{K}/k} = \omega_{\mathbb{K}/k}$. Using 8.0.2 and 8.0.5 it follows that

$$(*) \quad \epsilon[M^k_{\mathbb{K}}(sp_{\mathbb{K}}(2) \otimes \chi)] \cdot \omega_{\mathbb{K}/k}(-1) = \det(-F, \chi^I) \cdot \det(-F, (\rho_{\mathbb{K}/k} \otimes \chi)^I).$$

If $\rho_{\mathbb{K}/k}$ is an irreducible representation of W_k then as W_k/I is a cyclic group, $(\rho_{\mathbb{K}/k} \otimes \chi)^I = \{0\}$. It follows from (*) that $\epsilon[M^k_{\mathbb{K}}(sp_{\mathbb{K}}(2) \otimes \chi)] \cdot \omega_{\mathbb{K}/k}(-1) = -1$ iff χ is the trivial character when restricted to k^*.

If $\rho_{\mathbb{K}/k}$ is a reducible representation of W_k then $\rho_{\mathbb{K}/k}$ will be the sum of two characters of order 3 of W_k, say ω and ω^2. As χ restricted to k^{*2} is trivial, the characters $\omega \otimes \chi$ and $\omega^2 \otimes \chi$ of k^* are dual of each other. It follows that if $\omega \otimes \chi$ is trivial on I, so is $\omega^2 \otimes \chi$, and $\det(-F, (\rho_{\mathbb{K}/k}\chi)^I) = 1$. Again it follows from (*) that $\epsilon[M^k_{\mathbb{K}}(sp_{\mathbb{K}}(2) \otimes \chi)] \cdot \omega_{\mathbb{K}/k}(-1) = -1$ iff χ is the trivial character when restricted to k^*. ■

9. Epsilon factors II.

PROPOSITION 9.1. *Let K/k be an odd degree Galois extension, and σ a self-dual representation of W_k with trivial determinant and such that* $4 \mid \dim \sigma$. *Then for an additive character ψ of k and ψ_K of K as in* 8.0.4,

$$\epsilon(\sigma, \psi) = \epsilon(\sigma|_K, \psi_K).$$

Proof. Observe that for the representation σ of W_k, we have

$$\mathrm{Ind}^k_K(\sigma|_K) = \sigma \otimes_{\mathbf{C}} \mathbf{C}[W_k/W_K],$$

where $\mathbf{C}[W_k/W_K]$ denotes the regular representation of W_k by left translation on W_k/W_K. By a result of Burnside, cf. Serre's: *Linear Representations of Finite Groups*, *Exercise* 13.9, the representation $\mathbf{C}[W_k/W_K]$ of W_k can be written as

$$\mathbf{C}[W_k/W_K] = \mathbf{1} \oplus W \oplus W^*$$

for a certain representation W of W_k/W_K. Therefore from 8.0.6 and 8.0.2,

$$\begin{aligned}\epsilon(\sigma|_K, \psi_K) &= \epsilon(\mathrm{Ind}_K^k(\sigma|_K), \psi) \\ &= \epsilon(\sigma \otimes_{\mathbf{C}} [\mathbf{1} \oplus W \oplus W^*], \psi) \\ &= \epsilon(\sigma, \psi) \cdot \det(\sigma \otimes W)(-1) \\ &= \epsilon(\sigma, \psi). \qquad \blacksquare\end{aligned}$$

Combining this proposition with Theorem D on epsilon factors for triple products (proved in [P] for odd residue characteristic), we obtain the following corollary.

COROLLARY 9.2. *Let π_1, π_2, and π_3 be irreducible, admissible, generic representations of $GL_2(k)$ such that the product of their central characters is trivial. Then for K/k, an odd degree Galois extension, the representation $\pi_1 \otimes \pi_2 \otimes \pi_3$ has a $GL_2(k)$-invariant form iff the base change of π_i's to K has a $GL_2(K)$-invariant trilinear form.* ■

We finally show the equivalence of Theorems D and E in the case when K is a cyclic extension of degree 3. Let π be an irreducible admissible generic representation of $GL_2(K)$ whose central character restricted to k^* is trivial. The restriction of the 8-dimensional representation $M_K^k \sigma_\pi$ to W_K is $\sigma_\pi \otimes \sigma_{\pi'} \otimes \sigma_{\pi''}$ where $\sigma_{\pi'}$ and $\sigma_{\pi''}$ are the representations of W_K associated to π' and π'' obtained by Galois conjugation of the representation π of $GL_2(K)$ by the nontrivial elements of the Galois group of K/k. Therefore from Proposition 9.1 and the Theorem D on epsilon factors for $\mathbb{K} = k \oplus k \oplus k$ (proved in [P]), we see that Theorem D in this situation is equivalent to Theorem E of the introduction.

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