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Lifting orthogonal representations to spin groups and local root numbers

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Abstract. Representations of D_k^*/k^* for a quaternion division algebra D_k over a local field k are orthogonal representations. In this note we investigate when these orthogonal representations can be lifted to the corresponding spin group. The results are expressed in terms of local root number of the representation.

Keywords. Orthogonal representations; spin groups; local root numbers.

Let D be a quaternion division algebra over a local field k. Then D_k^*/k^* is a compact topological group, and all its irreducible representations are finite dimensional. It can be seen that, in fact, all the irreducible representations are orthogonal, i.e. for any irreducible representation V of D_k^*/k^*, there exists a quadratic form q on V such that the representation takes values in $O(V)$. Using the natural embedding of $O(V)$ in $SO(V\oplus \mathbf{C})$ given by $g\mapsto(g,\det g)$, we get a homomorphism of D_k^*/k^* into $SO(V\oplus\mathbf{C})$. In this note we investigate when this can be lifted to the spin group of the quadratic space $V\oplus\mathbf{C}$. The results are expressed in terms of the local root number of the representation V, or of the corresponding two dimensional symplectic representation of the Weil–Deligne group. We recall that by a theorem of Deligne [D1] the local root number of an orthogonal representation of the Weil–Deligne group W_k' of a local field k is expressed in terms of the second Stiefel–Whitney number of the representation, or equivalently in terms of the obstruction to lifting the orthogonal representation to the spin group. In our case we have a symplectic two dimensional representation of the Weil–Deligne group and its root number is being related to the lifting problem for the orthogonal representation of the quaternion division algebra. The formulation of Deligne's theorem is very elegant and has important global consequences. We, however, have not succeeded in making such an elegant formulation of our results and have neither succeeded in any global application.

As the problem is trivial in the case of an archimedean field, we will confine ourselves to the non-archimedean case only. We have been able to treat the case of only those non-archimedean fields with odd residue characteristic; we will tacitly assume this to be the case all through, and let q denote the cardinality of the residue field of k, and ω the unique non-trivial quadratic character of $\mathbf{F}_q^*$.

Lemma 1. *Any finite dimensional irreducible representation of D_k^*/k^* is orthogonal.*

Proof. If $x\mapsto\bar{x}$ denote the canonical anti-automorphism of D_k^* such that $x\cdot\bar{x}=Nrd(x)$ where $Nrd(x)$ is the reduced norm of x, then as an element of D_k^*/k^*, $\bar{x}=x^{-1}$. By the

Skolem–Noether theorem, x and $\bar{x}$ are conjugate, and therefore x is conjugate to x^{-1} in D_k^*/k^*. By character theory, this implies that every representation of D_k^*/k^* is self-dual. Now it can be proved that for any irreducible representation V of D_k^*/k^*, there exists a quadratic extension L of k such that the trivial character of L^* appears in V; see Lemma 2 below for precise statement. Since every character of L^* appears with multiplicity $\leqslant 1$ in any irreducible representation of D_k^*, cf. Remark 3.5 in [P], the eigenspace corresponding to the trivial character of L^* is one-dimensional. The unique non-degenerate bilinear form on V must be non-zero on this one-dimensional subspace, and therefore the bilinear form must be symmetric.

The following Lemma follows easily from the construction of representations of D_k^*; it can also be proved using the theorem of Tunnell [Tu].

Lemma 2. *Let π be an irreducible representation of D_k^*/k^* associated to a character of a quadratic extension K of k. Let L be the quadratic unramified extension of k if K is ramified, and one of quadratic ramified extensions if K is unramified. Then the trivial representation of L^* appears in π. The trivial representation of K^* appears in π if and only if K/k is a ramified extension of k, and $q \equiv 3 \bmod (4)$.*

The proof of Lemma 1 shows more generally that a self-dual irreducible representation V of a group G must be orthogonal if we can find a subgroup H such that the restriction of V to H is completely reducible and contains the trivial representation of H with multiplicity one. From this remark, one gets the following Proposition.

PROPOSITION 1

Every irreducible, admissible, self-dual, generic representation V of $GL(n,k)$, k non-archimedean, is orthogonal for any $n \geqslant 1$.

Indeed, the theory of new vectors for generic representations of $GL(n,k)$ (cf. [J–PS–S]) gives the existence of an open compact subgroup C such that the space of C-invariant vectors in V is one-dimensional.

According to a program begun by Carayol in [C] for the $GL(2)$ case, representations of D^* where D is a division algebra over a non-archimedean field, together with corresponding representations of $GL(n)$ (assumed to be supercuspidal) and W_k are expected to appear in the middle dimension cohomology (H^{n-1}) of a certain rigid analytic space. Considerations with Poincare duality suggest the following conjecture generalising lemma 1.

Conjecture. Let D^* be the multiplicative group of a division algebra central over a non-archimedean local field k. Let σ_π be the representation of W_k' associated by the local Langlands correspondence to π. Then whenever σ_π is self-dual, symplectic, and trivial on the $SL(2,\mathbf{C})$ factor of W_k', π is orthogonal.

The following Proposition calculates the determinant of a representation of D_k^*/k^*, and implies in particular that the determinant is never trivial; this was the reason why we have to consider the representation $V \oplus \mathbf{C}$ of D_k^*/k^* instead of just V.

PROPOSITION 2

Let π be an irreducible representation of D_k^*/k^* associated to a character of a quadratic *extension K of k. Then*

$$\det(\pi) = \omega_{L/k} \circ Nrd,$$

where $L = K$ if K is the quadratic unramified extension of k or if K is ramified with $q \equiv 1 \bmod (4)$; if K is ramified with $q \equiv 3 \bmod (4)$, then L is the other ramified quadratic extension.

Proof. Since the kernel of the reduced norm map is the commutator subgroup of D_k^*, we can write $\det(\pi)$ as $\mu \circ Nrd$ for a character μ of k^*. As π is self-dual, its determinant is of order $\leqslant 2$, and by class field theory, μ is either trivial or is $\omega_{E/k}$, for a quadratic extension E of k. For any quadratic extension M of k, write the decomposition of π as M^*-module as

$$\pi = \sum_{\mu \in X} \mu \oplus \sum_{\mu \in X} \mu^{-1} \oplus a \cdot 1 \oplus b \cdot \nu \qquad \text{(i)}$$

where a and b are integers $0 \leqslant a, b \leqslant 1$, ν is the unique character of M^*/k^* of order 2, and X is a finite set of characters of M^*/k^* of order $\geqslant 3$. Since the dimension of π is known to be even, $a = b$.

It follows that the determinant of π restricted to M^*/k^* is trivial if and only if the trivial representation of M^* does not appear in π in which case μ is trivial on the norm subgroup $Nrd(M^*)$. Lemma 2 now easily completes the proof.

Remark 1. It should be noted that self-dual representations π of D^* not factoring through D^*/k^* need not be orthogonal. For instance, for $k = \mathbf{R}$, $\pi = \rho \otimes \det(\rho)^{-1/2}$, where ρ is the standard two-dimensional representation of D^*, is a symplectic representation of D^*. It will be interesting to characterize self-dual representations of D_k^* which are orthogonal.

Lemma 3. *Let $SO(2n+1, \mathbf{C})$ correspond to the quadratic form $q = x_1x_2 + \ldots + x_{2n-1}x_{2n} + x_{2n+1}^2$, and T the associated maximal torus. For characters $(\chi_1, \ldots, \chi_n)$ of an abelian group G, let π be the representation of G with values in $SO(2n+1, \mathbf{C})$ given by $x \mapsto (\chi_1(x), \chi_1^{-1}(x), \chi_2(x), \chi_2^{-1}(x), \ldots, \chi_n(x), \chi_n^{-1}(x), 1)$. Then the representation π of G lifts to Spin $(2n+1, \mathbf{C})$ if and only if $\prod_{i=1}^n \chi_i = \mu^2$ for some character μ of G, i.e. if and only if $\prod_{i=1}^n \chi_i$ is trivial on the subgroup $G[2] = \{g \in G \mid 2g = 1\}$.*

Proof. The proof is a trivial consequence of the fact that the spin covering of $SO(2n+1, \mathbf{C})$ when restricted to the maximal torus $T = \{(z_1, z_1^{-1}, z_2, z_2^{-1}, \ldots, z_n, z_n^{-1}, 1) \mid z_i \in \mathbf{C}^*\}$ is the two-fold cover of T obtained by attaching $\sqrt{\prod z_i}$.

Lemma 4. *A homomorphism $\pi: D_k^*/k^* \to SO(n)$ can be lifted to the corresponding spin group if and only if π restricted to K^*/k^* can be lifted for any quadratic extension K of k.*

Proof. As the two sheeted coverings of a group G are classified by $H^2(G, \mathbf{Z}/2)$, one needs to prove that an element of $H^2(D_k^*/k^*, \mathbf{Z}/2)$ is trivial if and only if its restriction to $H^2(K^*/k^*, \mathbf{Z}/2)$ is trivial for all quadratic extensions K of k. Let D_1^* be the image in D_k^*/k^* of the first congruence subgroup of D_k^* under the standard filtration. Then since the residue characteristic of k is odd, $H^i(D_1^*, \mathbf{Z}/2)=0$ if $i>0$. It follows that $H^2(D_k^*/k^*, \mathbf{Z}/2) = H^2(D_k^*/k^* D_1^*, \mathbf{Z}/2)$. Now $D_k^*/k^* D_1^*$ is the dihedral group:

$$0 \to \mathbf{F}_{q^2}^*/\mathbf{F}_q^* \to D_k^*/k^* D_1^* \to \mathbf{Z}/2 \to 0,$$

where $\mathbf{F}_q$ is the residue field of k. Dividing $D_k^*/k^* D_1^*$ by the maximal subgroup H' of odd order of $\mathbf{F}_{q^2}^*/\mathbf{F}_q^*$, we again get the dihedral group $D_r = D_k^*/k^* D_1^* H'$ with $H^2(D_k^*/k^*, \mathbf{Z}/2) \cong H^2(D_k^*/k^* D_1^* H', \mathbf{Z}/2)$:

$$0 \to \mathbf{Z}/2^r \to D_r \to \mathbf{Z}/2 \to 0.$$

Clearly $\mathbf{Z}/2 \oplus \mathbf{Z}/2 \subseteq D_r$, and it can be seen from the explicit description of cohomology of dihedral groups, cf. [Sn, page 24], that $H^2(D_r, \mathbf{Z}/2)$ injects into $H^2(\mathbf{Z}/2 \oplus \mathbf{Z}/2, \mathbf{Z}/2) \oplus H^2(\mathbf{Z}/2^r, \mathbf{Z}/2)$ under restriction. An element of $H^2(\mathbf{Z}/2 \oplus \mathbf{Z}/2, \mathbf{Z}/2)$ is zero if and only if its restriction to all the three $\mathbf{Z}/2$'s in $\mathbf{Z}/2 \oplus \mathbf{Z}/2$ is zero. These three $\mathbf{Z}/2$'s come from the three quadratic extensions; also, $\mathbf{Z}/2^r$ comes from the quadratic unramified extension, proving the proposition.

The following Lemma summarizes the information we need about the characters of irreducible representations π of D^*/k^*, for k non-archimedean, cf. [Si, pages 50–51] where he calculates the characters of representations of $PGL(2, k)$.

Lemma 5. *For K a quadratic extension of k, let $\pi = \pi_\chi$ be the representation of D^*/k^* attached to a character χ of K^*. Then we have the following table*

K/k	cond(χ)	dim(π)	cond(π)
unramified	f	$2q^{f-1}$	$2f$
ramified	$2f$	$(q+1)q^{f-1}$	$2f+1$

Let L be any quadratic extension of k, and x_0 the unique element of L^*/k^* of order 2. Denote by Θ_π the character of π. Then we have:

1. *If $L \neq K$, $\Theta_\pi(x_0)=0$.*
2. *If $L=K$ and K/k unramified, $\Theta_\pi(x_0) = (-1)^{f+1} 2\chi(x_0)$*
3. *If $L=K$ and K/k ramified,*

$$\Theta_\pi(x_0) = -2G_\chi \omega(2)\omega(-1)^{f-1}\chi(x_0),$$

where

$$G_\chi = \frac{1}{\sqrt{q}} \sum_{x \in (\mathcal{O}_K/\pi_K)^*} \chi(1+\pi_K^{2f-1}x)\omega(x).$$

We now begin analysing the lifting of orthogonal representations of D_k^*/k^* to spin groups.

PROPOSITION 3

Let π be an irreducible representation of D_k^*/k^* with values in $O(V)$ associated to a quadratic extension K of k. Then the associated representation with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group, $Spin(V \oplus \mathbf{C})$, when restricted to L^*/k^* for L a quadratic extension of k different from K if and only if $\omega(-2) = -1$ if K is a ramified extension, *and* $\omega(-1)^{f-1} = -1$ *if* K *is the unramified extension where* $2f$ *is the conductor of the representation* π. *(We recall that* ω *is the unique non-trivial quadratic character of* $\mathbf{F}_q^*$.)

Proof. Let $L = k(x_0)$ with $x_0^2 \in k^*$. Clearly x_0 is the unique element of L^*/k^* of order 2. As π is self-dual, whenever a character μ of L^* appears in π, so does μ^{-1}. Let us now write the decomposition of π as L^*-module as

$$\pi = \sum_{\mu \in X} \mu \oplus \sum_{\mu \in X} \mu^{-1} + a \cdot 1 + b \cdot \nu \tag{i}$$

where a and b are integers $0 \leqslant a, b \leqslant 1$, ν is the unique character of L^*/k^* of order 2, and X is a finite set of characters of L^*/k^* of order $\geqslant 3$. Since the dimension of π is even, $a = b$. Note that $\nu(x_0) = -1$ except in the case when L is a quadratic unramified extension of k with $q \equiv 3 \bmod (4)$ in which case $\nu(x_0) = 1$.

By Lemma 3, the representation π of L^*/k^* with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group, Spin $(V \oplus \mathbf{C})$, if and only if

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(x_0) = 1.$$

As x_0 has order 2 in L^*/k^*, all the characters of L^*/k^* take the value ± 1 on x_0. Let r be the number of characters μ from X such that $\mu(x_0) = 1$, and let s be the number of characters μ from X such that $\mu(x_0) = -1$. From Lemma 5, the character of π at x_0 is zero. Assuming that L is not the quadratic unramified extension with $q \equiv 3 \bmod (4)$, so that $\nu(x_0) = -1$, we have from the decomposition of π as in (i)

$$\dim(\pi) = 2(r+s) + 2a \tag{ii}$$

$$\Theta_\pi(x_0) = 2(r-s) = 0, \tag{iii}$$

From (ii) and (iii),

$$\dim(\pi) = 4s + 2a. \tag{iv}$$

Also,

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(x_0) = (-1)^{s+a}. \tag{v}$$

From (iv) and (v), and using Lemma 5 for the dimension of π, it follows that if K is a ramified extension of k, and L is not the quadratic unramified extension of k with $q \equiv 3 \bmod (4)$, the representation π restricted to L^*/k^* lifts to the spin group if and only if $q \equiv 5 \bmod (8)$ or $q \equiv 7 \bmod (8)$. Similarly, when K is the quadratic unramified extension of k, the representation π restricted to L^*/k^* lifts to the spin group if and only if $q \equiv 3 \bmod (4)$ and f even. Finally, if L is the quadratic unramified extension of k with $q \equiv 3 \bmod (4)$, then the representation π restricted to L^*/k^* lifts to the spin group if and only if $q \equiv 7 \bmod (8)$ as follows from a similar analysis. All these conclusions combine to prove the proposition.

We next consider the lifting of a representation π of D_k^*/k^* associated to a quadratic field K when restricted to K^*/k^*. In this case the obstruction to lifting is related to the epsilon factor of π. We will assume that the reader is familiar with the basic properties of the epsilon factor for which we refer to [T]. We, however, do want to state two theorems about epsilon factors which will be crucial to our calculations; the first due to Deligne

[D2, Lemma 4.1.6] describes how epsilon factor changes under twisting by a character of small conductor, and the second is a theorem of Frohlich and Queyrut [F-Q, Theorem 3].

Lemma 6. *Let α and β be two multiplicative characters of a local field K such that* $\mathrm{cond}(\alpha) \geqslant 2\,\mathrm{cond}(\beta)$. *For an additive character ψ of K, let y be an element of K such that $\alpha(1+x) = \psi(xy)$ for all $x \in K$ with $\mathrm{val}(x) \geqslant \frac{1}{2}\,\mathrm{cond}(\alpha)$ if conductor of α is positive; if conductor of α is 0, let $y = \pi_k^{-\mathrm{cond}(\psi)}$ where π_k is a uniformising parameter of k. Then*

$$\varepsilon(\alpha\beta, \psi) = \beta^{-1}(y)\varepsilon(\alpha, \psi).$$

Lemma 7. *Let K be a separable quadratic extension of a local field k, and ψ an additive character of k. Let ψ_K be the additive character of K defined by $\psi_K(x) = \psi(\mathrm{tr}\,x)$. Then for any character χ of K^* which is trivial on k^*, and any $x_0 \in K^*$ with $\mathrm{tr}(x_0) = 0$*

$$\varepsilon(\chi, \psi_K) = \chi(x_0).$$

In the next proposition we analyse the lifting of a representation π of D_k^*/k^* associated to a quadratic field K when restricted to K^*/k^*.

PROPOSITION 4

Let π be an irreducible representation of D_k^*/k^* with values in $O(V)$ associated to a character χ of K^* for a quadratic extension K of k. Then the associated representation with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group, $Spin(V \oplus \mathbf{C})$, when restricted to K^*/k^* if and only if $\varepsilon(\pi) = -\omega(2)$ if K is ramified, and $\omega(-1)^f \varepsilon(\pi) = 1$ if K is unramified and the conductor of π is 2f.

Proof. The proof of this proposition is very similar to that of Proposition 3. Since the proof is essentially the same in the case when K is unramified or ramified, and in fact since the unramified case is much simpler, we will assume in the rest of the proof that K is ramified.

Since k has odd residue characteristic, K^*/k^* has exactly one character of order 2 which is an unramified character of K^* taking the value -1 on a uniformising parameter π_K of K; denote this character by ν. We fix π_K such that $\pi_k = \pi_K^2$ belongs to k so that $K = k(\sqrt{\pi_k})$. Clearly π_K is the unique element of K^*/k^* of order 2.

Let us now write the decomposition of π as K^*-module as in Proposition 1:

$$\pi = \sum_{\mu \in X} \mu \oplus \sum_{\mu \in X} \mu^{-1} \oplus a \cdot 1 \oplus b \cdot \nu \tag{i}$$

where a and b are integers $0 \leqslant a, b \leqslant 1$, and X is a finite set of characters of K^*/k^* of order $\geqslant 3$. Since the dimension of π is $(q+1)q^{f-1}$, it is in particular even. Therefore $a = b$.

By Lemma 3, the representation π of K^*/k^* with values in $SO(V \oplus \mathbf{C})$ lifts to the spin group $\mathrm{Spin}(V \oplus \mathbf{C})$ if and only if

$$\left(\nu^{a} \cdot \prod_{\mu \in X} \mu\right)(\pi_K) = 1.$$

As π_K has order 2 in K^*/k^*, all the characters of K^*/k^* take the value ± 1 on π_K. Let r be the number of characters μ from X such that $\mu(\pi_K) = 1$, and let s be the number of characters μ from X such that $\mu(\pi_K) = -1$. Therefore from the decomposition of π as in (i) we get,

$$\dim(\pi) = 2(r+s) + 2a \tag{ii}$$

$$\Theta_\pi(\pi_K) = 2(r-s), \tag{iii}$$

$$\left(\nu^a \cdot \prod_{\mu \in X} \mu\right)(\pi_K) = (-1)^{s+a}. \tag{iv}$$

From (ii) and (iii),

$$\dim(\pi) - \Theta_\pi(\pi_K) = 4s + 2a. \tag{v}$$

Using Lemma 5 for the character of π at π_K we get

$$\Theta_\pi(\pi_K) = -2G_\chi \cdot \omega(2)\chi(\pi_K),$$

and as $\dim(\pi) = (q+1)q^{f-1}$, we get from (v) that

$$(q+1)q^{f-1} + 2G_\chi \cdot \omega(2)\chi(\pi_K) = 4s + 2a. \tag{vi}$$

We next calculate the epsilon factor $\varepsilon(\pi)$. As the associated representation of the Weil group is induced from the character χ of K^*.

$$\begin{aligned}\varepsilon(\pi) &= \varepsilon(\mathrm{Ind}_{K^*}^{W_k}\chi, \psi_k)\\ &= \varepsilon(\mathrm{Ind}_{K^*}^{W_k}(\chi - 1), \psi_k)\cdot\varepsilon(\mathrm{Ind}_{K^*}^{W_k}1, \psi_k)\\ &= \varepsilon(\chi, \psi_K)\varepsilon(\omega_{K/k}, \psi_k)\end{aligned}$$

Here ψ_k is any additive character of k, and ψ_K is the additive character of K obtained from ψ_k using the trace map from K to k.

We now use the theorem of Frohlich and Queyrut to calculate $\varepsilon(\chi, \psi_K)$. As the restriction of χ to k^* is $\omega_{K/k}$ and not the trivial character, we cannot directly apply this theorem. However, a slight modification works. For this observe that as k has odd residue characteristic, the quadratic character $\omega_{K/k}$ of k^* is trivial on $1 + \pi_k\mathcal{O}_k$ where $\mathcal{O}_k$ (respectively $\mathcal{O}_K$) is the maximal compact subring of k (respectively K). Also, since K is a ramified extension,

$$\mathcal{O}_K^*/(1 + \pi_K\mathcal{O}_K) \cong \mathcal{O}_k^*/(1 + \pi_k\mathcal{O}_k).$$

Use this isomorphism to extend $\omega_{K/k}$ from $\mathcal{O}_k^*$ to $\mathcal{O}_K^*$ and then extend this characterof $\mathcal{O}_K^* \cdot k^*$ to K^* in one of the two possible ways. Denote this extension of $\omega_{K/k}$ to K^* by $\tilde{\omega}$. As the conductor of $\tilde{\omega}$ is 1, by Lemma 6,

$$\begin{aligned}\varepsilon(\pi) &= \varepsilon(\chi\cdot\tilde{\omega}\cdot\tilde{\omega}^{-1}, \psi_K)\cdot\varepsilon(\omega_{K/k}, \psi_k)\\ &= \varepsilon(\chi\cdot\tilde{\omega}, \psi_K)\cdot\tilde{\omega}(y)\cdot\varepsilon(\omega_{K/k}, \psi_k)\\ &= (\chi\cdot\tilde{\omega})(\pi_K)\cdot\tilde{\omega}(y)\cdot\varepsilon(\omega_{K/k}, \psi_k)\end{aligned} \tag{vii}$$

where y is the element of K^* with the property that

$$\chi\cdot\tilde{\omega}(1+x) = \psi(xy) \quad \text{for all } x \text{ with } \mathrm{val}(x) \geqslant \tfrac{1}{2}\mathrm{cond}\,\chi,$$

therefore $y = \pi_K^{-(2f+1)}a_0(\chi) +$ higher order terms. It follows that

$$\chi(1 + \pi_K^{2f-1}x) = \psi(\pi_k^{-1}a_0(\chi)\cdot x).$$

From the definition of epsilon factors,

$$\sum_{x\in(\mathcal{O}_k/\pi_k)^*} \omega(x)\psi(\pi_k^{-1}x) = \sqrt{q}\omega_{K/k}(\pi_k)\varepsilon(\omega_{K/k},\psi_k),$$

and therefore,

$$\sum_{x\in(\mathcal{O}_k/\pi_k)^*} \omega(x)\chi(1+\pi_K^{2f-1}x) = \sqrt{q}\omega_{K/k}(a_0(\chi)\cdot\pi_k)\varepsilon(\omega_{K/k},\psi_k).$$

Comparing with the definition of G_χ, we get

$$G_\chi = \omega_{K/k}(a_0(\chi)\cdot\pi_k)\cdot\varepsilon(\omega_{K/k},\psi_k).$$

Using (vii),

$$\begin{aligned}\varepsilon(\pi) &= (\chi\cdot\tilde{\omega})(\pi_K)\tilde{\omega}(\pi_K^{-(2f+1)}a_0(\chi))\cdot\varepsilon(\omega_{K/k},\psi_k)\\ &= \chi(\pi_K)\cdot\omega(-1)^{f+1}G_\chi.\end{aligned}$$

Finally, we can use (vi) to give the value of s as follows:

$$4s+4a=(q+1)q^{f-1}+2a+2\varepsilon(\pi).$$

We note that by Tunnell's theorem, the trivial character of K^* appears in π if and only if

$$\varepsilon(\pi)\cdot\varepsilon(\pi\otimes\omega_{K/k}) = -\omega_{K/k}(-1).$$

But since $\pi\cong\pi\otimes\omega_{K/k}$, and $\varepsilon(\pi)=\pm1$, the trivial character of K^* appears in π, i.e. $a=1$, if and only if $\omega_{K/k}(-1)=-1$. Now the proposition can be deduced by a case-by-case analysis depending on the values of $\omega(2)$ and $\omega(-1)$.

Propositions 3 and 4 can now be combined using Lemma 4 to give the following theorem.

Theorem 1. *Let π be an irreducible representation of D_k^*/k^* with values in $O(V)$ associated to a character χ of K^* for a quadratic extension K of k. Then the associated representation with values in $SO(V\oplus\mathbf{C})$ lifts to the spin group, $Spin(V\oplus\mathbf{C})$, if and only if $\omega(-2)=-1$ and $\varepsilon(\pi)=\omega(-1)$ if K is ramified, and $\omega(-1)^{f-1}=-1$ and $\varepsilon(\pi)=-1$ if K is unramified and the conductor of π is $2f$.*

Remark 2. We do not know when an orthogonal representation of a connected compact Lie group can be lifted to the spin group, say in terms of the highest weight of the representation. The question is interesting for finite groups too, for instance the symmetric group all whose representations are known to be orthogonal, or for finite groups of Lie type.

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