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Some Remarks on Representations of a Division Algebra and of the Galois Group of a Local Field

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In this paper we prove a division algebra analogue of a theorem of Jacquet and Rallis about uniqueness of $GL_n(k) \times GL_n(k)$ invariant linear form on an irreducible admissible representation of $GL_{2n}(k)$. We propose a conjecture about when this invariant form exists. We prove some results about self-dual representations of the invertible elements of a division algebra and of Galois groups of local fields. The Shalika model has been studied for principal series representations of $GL_2(D)$ for D a division algebra and a conjecture made regarding its existence in general.

Contents

Let k be a non-Archimedean local field. In a recent paper, Jacquet and Rallis [J-R] have proved that for an irreducible admissible representation π of $GL_{2n}(k)$, $\mathrm{Hom}_H(\pi, 1)$ is atmost one dimensional for $H = GL_n(k) \times GL_n(k)$. It is the purpose of this paper to prove such a theorem in the context of division algebras. We also treat more general characters of the subgroup H. These questions lead one naturally to the study of self-dual representations of the invertible elements of a division algebra and of the Galois group of a local field which is the other motivation for this work.

It is generally believed that the representations π of $GL_{2n}(k)$ for which $\mathrm{Hom}_H(\pi, 1)$ is non-zero for $H = GL_n(k) \times GL_n(k)$ are precisely the ones

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for which the Langlands parameter associated to π is symplectic, cf. [FJ] for some global motivation behind this. In an earlier paper [P-R], D. Ramakrishnan and the author had conjectured that irreducible symplectic representations of the Weil group are precisely those for which the associated representation of the division algebra is orthogonal. Interplay of these two conjectures lies behind some of the considerations in this paper.

1. MULTIPLICITY 1 FOR THE TRIVIAL REPRESENTATION

Let D be a division algebra over a non-Archimedean local field k of index $2n$ (i.e., of dimension $4n^2$). Let K be a quadratic extension of k contained in D, and D_K the centraliser of K in D. (Since the index of D is even, any quadratic extension of k is contained in D.) Then D_K is a division algebra of dimension $2n^2$ over k with center K.

THEOREM 1. *Let π be an irreducible representation of D^*. Then the trivial representation of D_K^* appears in π with multiplicity at most* 1.

Proof. We will prove that the involution $x \to x^{-1}$ preserves all the double cosets of D_K^* in D^*, thereby proving the theorem by the method of Gelfand pairs, cf. [G]. So, we need to prove that for any $x \in D^*$,

$$x^{-1} \in D_K^* x D_K^*.$$

For this, it suffices to prove that for all $x \in D^*$, the vector subspaces xD_Kx and D_K of D intersect non-trivially. It suffices to prove this statement after extending scalars from k to the algebraic closure $\bar{k}$ of k. We therefore assume an identification of $D \otimes \bar{k}$ with $M_{2n}(\bar{k})$ such that $D_K \otimes \bar{k}$ is identified to the sub-algebra $M_n(\bar{k}) \times M_n(\bar{k})$ sitting inside $M_{2n}(\bar{k})$ as the block diagonal matrices. It suffices to prove that for any $g \in GL_{2n}(\bar{k})$,

$$g[M_n(\bar{k}) \times M_n(\bar{k})]\, g \cap [M_n(\bar{k}) \times M_n(\bar{k})] \neq 0.$$

Let $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ where $A, B, C, D \in M_n(\bar{k})$. For a matrix $\left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$ with $X, Y \in M_n(\bar{k})$, we have

$$g \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} g = \begin{pmatrix} AXA + BYC & AXB + BYD \\ CXA + DYC & CXB + DYD \end{pmatrix}.$$

So, to prove that $g[M_n(\bar{k}) \times M_n(\bar{k})]\, g$ and $M_n(\bar{k}) \times M_n(\bar{k})$ intersect non-trivially, we need to prove that there are matrices X and Y in $M_n(\bar{k})$, not both of which are zero, such that

$$AXB + BYD = 0,$$
$$CXA + DYC = 0.$$

This system of equations has to be solved for X and Y. At this point, invoking the *semi-continuity principle* for the existence of solutions of a system of homogeneous equations, we assume that A, B, C, D are all non-singular matrices. Under this assumption, the above system of equations reduces to

$$X = -A^{-1}BYDB^{-1}$$
$$X = -C^{-1}DYCA^{-1}.$$

It is clear that

$$X = -C^{-1}DB^{-1}$$
$$Y = B^{-1}AC^{-1}$$

is a solution to these system of equations, proving the theorem.

The following lemma is true for general Gelfand pairs for which $x \to x^{-1}$ is the involution.

LEMMA 1. *Let H be a subgroup of a finite group G such that $g^{-1} \in HgH$ for every g in G. Then every complex irreducible representation of G which contains a non-zero vector on which H acts trivially is an orthogonal representation.*

Proof. Let V be an irreducible representation of G containing a vector v on which H acts trivially. Put a G-invariant Hermitian structure $\langle\ ,\ \rangle$ on V, and consider the matrix coefficient

$$f(g) = \langle gv, v\rangle.$$

Clearly,

$$f(hgh') = f(g) \qquad \text{for all} \quad g \in G, \quad h \in H, \quad h' \in H.$$

From the condition that $g^{-1} \in HgH$, we get

$$f(g) = f(g^{-1}).$$

But $f(g^{-1})$ is a matrix coefficient of the dual representation. Therefore from the orthogonality relations, $V \cong V^*$.

Now, let B: $V \times V \to \mathbf{C}$ be the unique G-invariant bilinear form. This non-degenerate form must be non-zero on v, forcing B to be symmetric.

COROLLARY 1. *An irreducible representation of D^* containing a fixed vector for D_K^* is an orthogonal representation.*

2. MULTIPLICITY 1 FOR ONE DIMENSIONAL REPRESENTATIONS IF D_K^*

By a theorem of Matsushima and Nakayama, the commutator subgroup of D_K^* is the subgroup of reduced norm 1 elements of D_K^*. Therefore for every character μ of D_K^*, there exists a character χ of K^* such that $\mu(X)=\chi(\mathrm{Nrd}_K X)$ for all X in D_K^* where Nrd_K denotes the reduced norm map.

We would have liked to prove that every 1 dimensional representation of D_K^* appears with multiplicity at most 1 in any irreducible representation of D^*. This we have been unable to do. We are able to treat only those irreducible representations of D^* which have trivial central character. In this case, the characters of D_K^*/k^* are given by characters of K^*/k^{*n}. However, we will be able to prove multiplicity 1 not for all the representations of D_K^*/k^* coming from characters of K^*/k^{*n} but only those representations of D_K^*/k^* coming from K^*/k^*. Finally, we will need a restriction on the division algebra D_K itself. We assume that there is a division algebra D_1 over k such that $D_K=D_1 \otimes K$. In this case the centraliser of D_1 inside D is a quaternion division algebra D_2 containing K, and we have a canonical isomorphism $D \cong D_1 \otimes D_2$. Perhaps we should explain the reason behind this restriction on the division algebra. Our proof below will depend on the existence of an automorphism of order 2 of D^* which preserves K (and therefore its centraliser D_K) and acts by the non-trivial Galois automorphism on K. This can be done if and only if K lies inside a quaternion division algebra which is contained in D.

Presumably these restrictions on the division algebra and on the character are not necessary, but later when we make a conjecture about which of these characters of D_K^* appears in an irreducible representation of D^*, these restrictions do play a role.

THEOREM 2. *Let D_1 be a division algebra over the local field k and D_2 the unique quaternion division algebra over k. Assume that $D=D_1 \otimes D_2$ is a division algebra (which is equivalent to assuming that D_1 has odd degree). Let K be a quadratic subfield of D_2. Denote the centraliser of K inside D by $D_K=D_1 \otimes K$. Let χ be a character of K^*/k^* identified to a character of D_K^*/k^* by composing with the reduced norm map. Then for any irreducible representation π of D^*/k^*, the character χ of D_K^*/k^* appears with multiplicity at most* 1.

Proof. It suffices to prove that $D_K^*/k^* \hookrightarrow K^*/k^* \times D^*/k^*$ is a Gelfand pair. Let $j \in D_2^*$ be chosen such that the inner conjugation action of j leaves K invariant and induces the non-trivial Galois automorphism of K over k. We will denote the non-trivial Galois automorphism of K over k by $x \to \bar{x}$. Let $j^2 = \alpha \in k^*$.

We take $(x, y) \to (x, jy^{-1}j^{-1})$ to be the involution on $K^*/k^* \times D^*/k^*$. To prove that this involution is the Gelfand involution for the subgroup D_K^*/k^*, it suffices to prove that for any $(x, y) \in K^*/k^* \times D^*/k^*$, there exists $d_1, d_2 \in D_K^*/k^*$ such that

$$(\mathrm{Nrd}_K\, d_1, d_1)(x, y)(\mathrm{Nrd}_K\, d_2, d_2) = (x, jy^{-1}j^{-1}),$$

i.e., we need to show the existence of $d_1, d_2 \in D_K^*$ such that

$$\mathrm{Nrd}_K(d_1 d_2) = 1, \qquad d_1 y d_2 = jy^{-1}j^{-1}.$$

Both these equations are to be understood up to elements of k^*. It suffices therefore to prove that given any $y \in D^*$, there exists d_1, d_2 in D_K^* such that $d_1 y d_2 = jy^{-1}j^{-1}$ with $\mathrm{Nrd}_K(d_1 d_2)$ belonging to k^*. Our proof below will actually explicitly construct these d_1 and d_2.

Let D_2 be contained in $M_2(K)$ via the inclusion given by $a + bj \to \left(\begin{smallmatrix} a & \alpha\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ for $a, b \in K$, which gives an isomorphism $D_2 \otimes K \cong M_2(K)$. Tensoring this inclusion by D_1, we have

$$D = D_1 \otimes D_2 \hookrightarrow M_2(D_1 \otimes K).$$

Under this inclusion, a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in $M_2(D_1 \otimes K)$ belongs to the division algebra D if and only if

$$\bar{A} = D, \qquad \text{and} \qquad \bar{B} = \alpha C$$

where we have extended the Galois automorphism of K to $D_1 \otimes K$ in the obvious way.

Now, given $y \in D^*$, we have to prove the existence of d_1, d_2 in D_K^* with $d_1 y d_2 = jy^{-1}j^{-1}$, with $\mathrm{Nrd}_K(d_1 d_2)$ in k^*. We write the equation $d_1 y d_2 = jy^{-1}j^{-1}$ as $d_1^{-1} = y d_2 j y j^{-1}$.

Let $y = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $d_2 = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$ be 2×2 matrices with values in $D_1 \otimes K$. Since $j = \left(\begin{smallmatrix} 0 & \alpha I \\ I & 0 \end{smallmatrix}\right)$, we have

$$y d_2 j y j^{-1} = \begin{pmatrix} AXD + \alpha^{-1} BYB & \alpha AXC + BYA \\ CXD + \alpha^{-1} DYB & \alpha CXC + DYA \end{pmatrix}.$$

If this belongs to $D_1 \otimes K$, then

$$\alpha AXC + BYA = 0,$$
$$\alpha CXD + DYB = 0.$$

If A, B, C, D are invertible, then

$$X = -i\alpha^{-1}A^{-1}BD^{-1}$$
$$Y = iD^{-1}CA^{-1}$$

gives a non-trivial solution to the above system of equations where i is an element of K^* whose square belongs to k^*. Since $y = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ belongs to D^*, we have $\bar{A} = D$, and $\bar{B} = \alpha C$. Therefore $X = \bar{Y}$, and $d_2 = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right)$ constructed above belongs to D_K^*. Define d_1 by $d_1^{-1} = yd_2jyj^{-1}$. Then

$$d_2^{-1}d_1^{-1} = \begin{pmatrix} DB^{-1}ABD^{-1}C(C^{-1}D - A^{-1}B) & 0 \\ 0 & AC^{-1}DC(C^{-1}D - A^{-1}B)\,D^{-1}C \end{pmatrix}$$
$$\stackrel{\text{def}}{=} \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix}.$$

Again, since $\bar{A} = D$ and $\bar{B} = \alpha C$, it can be seen that $\bar{W} = Z$, and therefore $d_2^{-1}d_1^{-1} \in D_K^*$, and therefore $d_1 \in D_K^*$. Moreover, $\det W = \det Z$, proving that $\mathrm{Nrd}_K(d_1 d_2)$ belongs to k^*.

Finally, if A or D is not invertible, then both are 0, and then one can easily find appropriate d_1 and d_2. Similarly, if B or C is not invertible, then both are 0, and again one can find appropriate d_1 and d_2.

Proposition 1. *With the notation as in Theorem 2, if a 1-dimensional representation χ of D_K^*/k^* which is obtained via the norm map to K^*/k^* appears in an irreducible representation π of D^*/k^*, then the representation π is an orthogonal representation.*

Proof. The argument of Lemma 1 used for the group $G = K^*/k^* \times D^*/k^*$ and subgroup $H = D_K^*/k^*$ together with the involution in Theorem 2 proves that the representation π is self-dual. Fix a D^*/k^*-invariant non-degenerate bilinear form on the vector space underlying π.

If the character χ is of order 2, then the line on which D_K^*/k^* acts via χ gives a 1-dimensional non-degenerate subspace of π, forcing π to be orthogonal.

If χ is not of order 2, let e_1 be a vector in π on which D_K^*/k^* operates via the character χ. Let j be as in Theorem 2. Clearly, D_K^*/k^* operates via the character χ^{-1} on the vector je_1. In particular, e_1 and je_1 are linearly independent. It is easy to see that the 2-dimensional subspace spanned by

e_1 and je_1 is a non-degenerate subspace of π. Since j^2 belongs to k^*, $j^2e_1 = e_1$. Therefore the matrix of j in the 2-dimensional subspace spanned by e_1 and je_1 is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which has determinant -1! This implies that the D^*-invariant non-degenerate bilinear form on π must be symmetric.

COROLLARY 2. *If D is a quaternion division algebra over a local field k, then any irreducible representation of D^*/k^* is orthogonal.*

Remark 1. This corollary was proved in [P-R] by showing that for every irreducible representation π of D^*/k^*, for D the quaternion division algebra over a local field k, there exists a quadratic extension K of k (depending on π) such that the trivial representation of K^* occurs in π.

PROPOSITION 2. *If D is a quaternion division algebra over a local field k, then any irreducible self-dual representation π of D^* with non-trivial central character is symplectic.*

Proof. Since π is self-dual with non-trivial central character, the central character is of order 2, and therefore determines a quadratic extension K of k by local class field theory. Let j be as in the previous proposition which normalises K^* and acts by the non-trivial element of the Galois group of K over k. If a character χ of K^* appears in π, then so does $\chi(\bar{x}) = \chi^{-1}(x)$. Assume that there is a character χ of K^* of order greater than 2 which appears in π. In this case the subspace of K^* generated by the eigenspaces of K^* with character χ and χ^{-1} is a 2 dimensional non-degenerate subspace of π on which since $\omega_{K/k}(j^2) = -1$, the action of j is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ which has determinant 1. The action of K^* on this 2 dimensional subspace being via character χ and χ^{-1}, is of determinant 1. Therefore if the bilinear form on π is symmetric, the image of K^* and j in this 2 dimensional representation will lie in the special orthogonal group which is abelian. On the other hand, it is clear that the actions of K^* and j don't commute in this 2 dimensional subspace, therefore the invariant bilinear form on π must be symplectic on this 2 dimensional subspace, and therefore on all of π.

It remains to treat the case when all the characters of K^* appearing in π are of order 2 or 1. However, we will see that this case never occurs. In fact more is true: no character of K^* appearing in π is of order 2 or 1. To see this, let there be a character χ of K^* appearing in π with $\chi^2 = 1$. Then χ-eigenspace will be a one dimensional non-degenerate subspace of π. This will force the bilinear form on π to be symmetric. Under the condition that $\chi^2 = 1$, it is clear that the 1 dimensional χ eigenspace of K^* is invariant under j. Since the orthogonal group in 1 variable consists only of ± 1, j will operate on the χ eigenspace by ± 1. Therefore j^2 will operate by 1. This leads to a contradiction as j^2 operates by $\omega_{K/k}(-1) = -1$, and proves that this case never occurs.

Remark 2. The reason for restricting ourselves to quaternion division algebra in Proposition 2 unlike Proposition 1 is that we have proved multiplicity 1 theorem (Theorem 2) for only trivial central character. If we had multiplicity 1 theorem available for one dimensional representations of D_K^* obtained from characters of $K^*/\mathrm{Nrd}\, K^*$ via the norm mapping then we could again conclude that an irreducible representation of D^* of non-trivial central character containing such a character is symplectic (again using crucially that the index of D is $2n$ for an odd integer n).

3. SELF-DUAL REPRESENTATIONS OF GALOIS GROUPS

The local Langlands correspondence establishes a bijection between irreducible n dimensional representations of the Weil–Deligne group of a local field k and irreducible representations of D^* where D is any division algebra with center k of index n. This correspondence takes self-dual representations of the Weil–Deligne group to self-dual representations of D^*. The question arises as to how this correspondence behaves for orthogonal and symplectic representations. Based on considerations of Poincaré duality on the middle dimensional cohomology of a certain rigid analytic space, the author and D. Ramakrishnan provided a conjectural answer in [P-R]. Here is the conjecture suitably enlarged to cover some cases not considered in [P-R]. (The part of this conjecture which needs the Weil–Deligne group instead of just the Weil group stands on rather thin ice.)

Conjecture 1. The Langlands correspondence between irreducible n-dimensional representations of the Weil–Deligne group of a local field k and irreducible representations of D^* where D is a division algebra of index n over its center k takes orthogonal representations of the Weil–Deligne group to symplectic representations of D^* and symplectic representations of the Weil–Deligne group to orthogonal representations of D^* if n is even. If n is odd, any irreducible self-dual representation of D^* is orthogonal.

We will prove some results on representations of Galois groups of local fields in this section and use those to reinterpret this conjecture in some cases so as not to directly talk about the Langlands correspondence!

PROPOSITION 3. *Let k be a local field of residue characteristic $p \neq 2$. Then any irreducible representation σ of W_k of dimension $2n$, n odd, can be obtained by induction from an irreducible representation of dimension n of W_K for a quadratic field extension K of k.*

Proof. The proof of the proposition will be by induction on n, the case of $n = 1$ being well known. By a theorem of Koch, [Ko, Theorem 2.1], the restriction of a primitive representation of W_k restricted to the wild inertia subgroup is irreducible. (We recall that by a primitive representation one means a representation which is not induced from any proper subgroup.) Since the wild inertia subgroup is a pro-p group, the dimension of any of its irreducible representations is a power of p. Since σ is even dimensional and the residue characteristic of k is odd, we conclude that σ is not irreducible when restricted to the wild inertia subgroup, and therefore σ is not a primitive representation. It follows that there exists a field extension k' of k such that σ is induced from a representation, say σ', of $W_{k'}$. There are 2 cases to consider now.

Case 1. Dimension of σ' is even.

Case 2. Dimension of σ' is odd.

In Case 1, by induction hypothesis, the representation σ' is induced from a representation σ'' of the Weil group $W_{k''}$ for a field extension k'' of k' of degree 2. It follows that σ is induced from the representation σ'' of $W_{k''}$ where the degree of k'' over k is even. In the second case, the degree of k' is already even. The following simple lemma therefore completes the proof.

LEMMA 2. *Let k be a local field of residue characteristic $p \neq 2$, and K a field extension of k of degree $2m$, m odd. Then K contains a quadratic field extension of k.*

Proof. Let K_u (resp. K_t) denote the maximal unramified (resp. tame) extension of k contained in K. The degree of K_t over k is even. If the degree of K_u over k is even, the lemma is clear. So assume that the degree of K_u is odd over k. The extension K_t is a totally ramified tame extension of K_u, and therefore K_t is obtained from K_u by attaching an mth root of an element of K_u where m is the degree of K_t over K_u which is even. So, by taking the 2nd root of this element, we get a degree 2 extension of K_u contained in K_t. Now it is easy to see that for any odd degree extension L of a local field k of odd residue characteristic, the natural map

$$k^*/k^{*2} \rightarrow L^*/L^{*2},$$

is an isomorphism, proving that the degree 2 extension of K_u comes from a degree 2 extension of k contained in K_t.

The following Lemma will be used in the main theorem of this section (Theorem 4 below). The invariant $c(W)$ in this lemma was introduced by Rogawski in [Ro, Lemma 15.1.1]. We will omit the simple proof of this lemma.

LEMMA 3. *Let H be a subgroup of index 2 of a group G. Let s be an element of G which does not lie in H. Let W be a finite dimensional irreducible representation of H such that W^s is isomorphic to W^* where W^s denotes the* *representation of H on W in which $h \in H$ operates via $h^s = shs^{-1}$. Fix a non-degenerate bilinear form (which is unique up to scaling) B: $W \times W \to \mathbf{C}$ such that $B(hw_1, h^s w_2) = B(w_1, w_2)$. Then*

(a) *There exists a constant $c(W) \in \{\pm 1\}$ independent of s such that*

$$B(v, w) = c(W)\, B(w, s^2 v) \qquad \textit{for all} \quad v, w \in W.$$

(b) *The representation $V = \operatorname{Ind}_H^G W$ of G is self-dual, and is orthogonal if and only if $c(W) = 1$.*

(c) *If the dimension of W is odd, then $c(W) = \det(s^2)$.*

We are now ready to prove the main theorem of this section.

THEOREM 4. *Let σ be an irreducible self-dual representation of W_k of dimension $2n$ for an odd integer n where k is a local field of odd residue characteristic. Then σ is an orthogonal representation if and only if* $\det \sigma$ *is non-trivial, and σ is a symplectic representation if and only if* $\det \sigma = 1$.

Proof. Since the special orthogonal group in 2 variables is abelian, the theorem is trivial if $n = 1$. We will therefore assume in the rest of the proof that $n > 1$. Using Proposition 3, write $\sigma = \operatorname{Ind}_{W_K}^{W_k} \tau$ where τ is an n dimensional representation of W_K for a quadratic field extension K of k. Let τ^θ be the representation of W_K obtained from τ by the conjugation action on W_K of an element of W_k not lying in W_K. Since σ is self-dual, either τ is self-dual or $\tau^\theta \cong \tau^*$. However, as we will see in Proposition 4 below, W_K does not have an irreducible self-dual representation of dimension $n > 1$. We therefore assume that $\tau^\theta \cong \tau^*$. This relation implies that $\det \tau$ restricted to k^* is trivial on the norms from K^*. The previous Lemma implies that σ is orthogonal if and only if the restriction of the determinant of τ to k^* is trivial. We relate the determinants of σ and τ using the following general lemma about the determinant of an induced representation due to Gallagher, cf. paragraph following Prop. 1.2 in [D].

LEMMA 4. *Let W be a finite dimensional representation of a subgroup H of finite index of a group G. Let* Ind *W denote the representation of G obtained by inducing the representation W of H. Then*

$$\det \operatorname{Ind} W(g) = \varepsilon^{\dim W} \det W(t(g)),$$

where t is the transfer map from G^{ab} to H^{ab}, and ε is the determinant of the permutation representation of G on G/H.

By local class field theory, the transfer map from W_k to W_K can be identified to the inclusion of k^* into K^*. Therefore, $\det \sigma = \det \tau|_{k^*} \cdot \omega_{K/k}$ where $\omega_{K/k}$ is the quadratic character of k^* associated by local classfield theory to the extension K. Since the determinant of τ restricted to k^* is already known to be either trivial or $\omega_{K/k}$ (from the isomorphism $\tau^\theta \cong \tau^*$), it follows that $\det \sigma = 1$, or $\omega_{K/k}$ depending on whether σ is symplectic or orthogonal, completing the proof of Theorem 4.

Remark 3. It is a curious consequence of Theorem 4 that there are no irreducible representations of the Galois group of a local field of odd residue characteristic with values in $SO(2n, \mathbf{C})$ with n odd.

Since an irreducible self-dual representation of the Weil–Deligne group $W'_k = W_k \times SL_2(\mathbf{C})$ is the tensor product of a self-dual representation of W_k and one of $SL_2(\mathbf{C})$, it is easy to see that the conclusion of Theorem 4 continues to hold good for the representations of the Weil–Deligne group too. Therefore, we can reinterpret Conjecture 1 for $n = 2m$, m odd as follows.

Conjecture 2. If $n = 2m$ for an odd integer m and the residue characteristic of k is odd, an irreducible self-dual representation of D^* is orthogonal if and only if its central character is trivial.

Remark 4. We compare the above conjecture to what is known about representations of compact Lie groups. Suppose that G is a compact connected Lie group. Then there is an element h in the centre of G of order 1 or 2 with the property that an irreducible self-dual representation of G is orthogonal if and only if the action of h is trivial on the representation, cf. [St, Lemma 79]. In particular, an irreducible self-dual representation of a compact connected Lie group whose center is trivial is always orthogonal.

We remark that Theorem 4 could also be proved in the tame case using the following result of Moy.

THEOREM 5 (Moy). *Let k be a local field of residue characteristic p which is an odd prime. Let σ be an n-dimensional irreducible self-dual representation of the Weil group of k such that $(n, p) = 1$. Then σ is induced from a character θ of E^* where E is a degree n field extension of k. Moreover, E contains a subfield F with $[E : F] = 2$ with Galois automorphism $x \to \bar{x}$ such that $\theta(\bar{x}) = \theta^{-1}(x)$ for all $x \in E^*$. The representation σ is symplectic if and only if θ restricted to F^* is non-trivial.*

It is a consequence of the theorem of Moy recalled above that there is no irreducible self-dual odd dimensional representation of the Galois group of a local field of odd residue characteristic whose dimension is greater than 1 and is coprime to the residue characteristic of the local field. We give an independent proof of this fact, extending it slightly in the following proposition.

PROPOSITION 4. *There is no irreducible self-dual odd dimensional representation of the Galois group of a local field of odd residue characteristic of dimension greater than* 1.

Proof. Assume that V is an odd dimensional irreducible representation of the Galois group G of a finite Galois extension L of the local field k. We note that a group of odd order has no irreducible non-trivial self-dual representation, cf. [Se, Exercise 13.9]. Therefore an odd dimensional self-dual representation of a group of odd order must contain the trivial representation of the group. Applying this to the wild inertia subgroup of G which is a normal subgroup of G, we find that the wild inertia subgroup of G operates trivially on V. Next, the inertia subgroup of G modulo wild inertia subgroup is a cyclic group. Since V is self-dual, whenever a character χ of the inertia subgroup appears in V, so does χ^{-1}. Since dim V is odd, this means that $\chi^2 = 1$ for some character of the inertia subgroup, and therefore as the inertia subgroup is a normal subgroup, $\chi^2 = 1$ for all characters of the inertia group. Since G modulo inertia subgroup is cyclic, this forces the dimension of V to be less than or equal to 2, completing the proof.

4. SELF-DUAL REPRESENTATIONS FOR DIVISION ALGEBRAS OF ODD-INDEX

PROPOSITION 5. *Let D be a division algebra of odd index n over a local field k of odd residue characteristic. Then D^* has no self-dual irreducible representations of dimension greater than* 1.

Proof. It is easy to see that one can reduce to the case where the central character is trivial. The proof of this proposition then follows once we have checked that any finite quotient of D^*/k^* is a group of odd order. (As recalled earlier, by [Se, Exercise 13.9], a group of odd order has no non-trivial irreducible self-dual representation.) Let R denote the maximal compact subring of D and $P = \pi R$ the unique maximal ideal in R. If $\mathbf{F}_q$ is the residue field of k, then R/P is isomorphic to $\mathbf{F}_{q^n}$. Define the standard decreasing filtration $D^*(i)$ on D^* by $D^*(i) = \{x \in R^* \mid (x-1) \in P^i\}$ for $i > 0$. Then $D^*(1)$ is a pro-p group where p is the characteristic of the residue field of k. The cardinality of $D^*/k^*D^*(1)$ is

$$n \cdot \frac{q^n - 1}{q - 1} = n \cdot [1 + q + \cdots q^{n-1}]$$

which is an odd number.

Remark 5. It is well known that the number of irreducible self-dual representations of a finite group is the same as the number of conjugacy classes in a finite group which are invariant under $x \to x^{-1}$, cf. [Se, Exercise 13.9]. From this it is easy to see that any division algebra over a non-Archimedean field of residue characteristic 2 has non-trivial irreducible self-dual representations, as for instance because all the elements of $D^*(n)/D^*(n+1)$ are of order 2.

5. NUMBER OF SELF-DUAL REPRESENTATIONS

We saw in the previous section that there are no self-dual representations of dimension greater than 1 of the invertible elements of a division algebra of odd index over local fields of odd residue characteristic. In this section we count the number of irreducible self-dual representations of a division algebra in the simplest case when the division algebra is of index 3 over a local field of characteristic 0 and of residue characteristic 2. This gives the division algebra analogue of a theorem of Weil, cf. [Tu2, theorem 5.2] according to which the number of irreducible 2 dimensional primitive representations of the Galois group of the algebraic closure of a local field k of residue characteristic 2 is, up to twisting by one dimensional characters,

$$\frac{4}{3}[q^{2\,\mathrm{val}2} - 1],$$

where q is the cardinality of the residue field of k, and val 2 denotes the valuation of 2 in k.

The relationship of 2 dimensional Galois representations to self-dual representations is via the exact sequence:

$$1 \to \mathbf{C}^* \to GL(2, \mathbf{C}) \to PGL(2, \mathbf{C}) = SO(3, \mathbf{C}) \to 1.$$

By a well-known theorem of Tate, it follows that the set of 2 dimensional irreducible representations up to twisting is in bijective correspondence with the conjugacy classes of homomorphisms of the Galois group into $PGL(2, \mathbf{C}) = SO(3, \mathbf{C})$. It is easy to see that 2 dimensional primitive representations are precisely the ones for which the associated 3 dimensional representation of the Galois group is irreducible. Therefore Weil's theorem recalled above together with the Langlands correspondence for division algebras of index 3 over a local field of residue characteristic 2 (which is known as we are in the tame case) implies the following result for which we give an independent proof.

PROPOSITION 6. *Let D be a cubic division algebra over a local field k of characteristic* 0, *and of residue characteristic* 2, *and let q be the cardinality of the residue field. Then the number of irreducible self-dual representations of D^*/k^* of dimension greater than* 1 *is*

$$\frac{4}{3}[q^{2\,\mathrm{val}2}-1].$$

Proof. We count the number of irreducible self-dual representations of D^*/k^* using the well-known fact that the number of irreducible self-dual representations of a finite group is equal to the number of conjugacy classes in the group which map into themselves under the involution $x \to x^{-1}$. From the structure of tame extensions of a local field, it is easy to see that k has exactly 3 ramified cubic extensions which are all Galois conjugate if k does not have 3rd roots of unity, and these are all cyclic if k has 3rd roots of unity. Therefore if k does not contain 3rd roots of unity then there is a single conjugacy class of ramified cubic extensions in D, and if k has 3rd roots of unity, then D has 3 conjugacy classes of ramified cubic subfields in D.

We fix some notation for the division algebra. Let R denote the maximal compact subring of D and $P=\pi R$ the unique maximal ideal in R. If $\mathbf{F}_q$ is the residue field of k, then R/P is isomorphic to $\mathbf{F}_{q^3}$. Define the standard decreasing filtration $D^*(i)$ on D^* by $D^*(i)=\{x \in R^* \mid (x-1) \in P^i\}$ for $i>0$. Let $D^*(0)=R^*$. It can be easily see that for $i>0$, $D^*(i)/D^*(i+1)$ is isomorphic to $\mathbf{F}_{q^3}$. Clearly, D^* leaves $D^*(i)$ invariant under the inner conjugation action, and therefore acts on $D^*(i)/D^*(i+1) \cong \mathbf{F}_{q^3}$ $(i>0)$. This action of D^* on $\mathbf{F}_{q^3}$ factors through $D^*/D^*(1)$. For any subfield M of D, let $M^*(i)=M \cap D^*(i)$. We note that this filtration on M^* need not be the same as the one usually associated to a local field.

Let L be a cubic unramifield extension of k contained in D. We fix p to be a uniformising parameter in k. By the known structure theory of division algebras over local fields, we can assume that π normalises L and acts by a non-trivial automorphism, to be denoted by σ, of L over k. We will use σ to denote the automorphism of the residue field of L which can also be identified to the residue field of D. We will let $K=k(\pi)$. If k contains 3rd roots of unity, then let $K=K_1$, $K_2=k(\omega\pi)$, and $K_3=k(\omega^2\pi)$ where ω is a root of unity in k^* whose image in $\mathbf{F}_q^*/\mathbf{F}_q^{*3}$ is non-trivial. With this notation we will prove the following lemma.

LEMMA 5. *If k contains 3rd roots of unity, any non-trivial element of $D^*/k^*D^*(n)$ is conjugate to an element of $K_i^*/k^*K_i^*(n)$ for $i=1,2,3$ or to an element of $L^*/k^*L^*(n)$. The element of $K_i^*/k^*K_i^*(n)$ or of $L^*/k^*L^*(n)$ is unique up to Galois action. If k does not contain 3rd roots of unity, any* *non-trivial element of* $D^*/k^*D^*(n)$ *is conjugate to a unique element of* $K^*/k^*K^*(n)$ *or to an element of* $L^*/k^*L^*(n)$ *which is unique up to Galois action.*

COROLLARY 4. *A conjugacy class in* $D^*/k^*D^*(n)$ *which is invariant under the involution* $x \to x^{-1}$ *is represented by an element of order* 2.

Proof. This is clear if the field to which the element belonged had no non-trivial automorphism. The other possibility will give $\sigma(x) = x^{-1}$ for a non-trivial automorphism σ of a degree 3 extension of k. This implies $\sigma^2(x) = x$. Since $\sigma^3 = 1$, we get $x = \sigma(x) = x^{-1}$, completing the proof.

Assuming the Lemma, we complete the proof of the proposition. Suppose first we are in the case when k does not contain a non-trivial 3rd root of unity. It can be seen that if $n \geqslant 6\mathrm{val}(2)$, an element x of $K^*/k^*K^*(n)$ is of order 2 if and only if after multiplying x by an element of k^*, one can assume that $x \equiv 1 \bmod \pi^{n-r}$ where $r = 3\mathrm{val}(2)$. We will assume that $n \geqslant 6\,\mathrm{val}(2)$ in the rest of this paragraph. Under this assumption, it follows that the number of elements x of $K^*/k^*K^*(n)$ of order $\leqslant 2$ is $q^r/q^{r/3} = q^{2\,\mathrm{val}(2)}$. Similarly, the elements of order $\leqslant 2$ in $L^*/k^*L^*(n)$ are precisely those which are represented by $L^*(n-r)$. The order of $k^*L^*(n-r)/k^*L^*(n)$ is again $q^{2\,\mathrm{val}(2)}$. Therefore the set of non-trivial elements of order 2 of $L^*/k^*L^*(n)$ modulo Galois action has cardinality $[q^{2\,\mathrm{val}(2)} - 1]/3$. Adding the contribution coming from the non-trivial elements of order 2 in $K^*/k^*K^*(n)$, we get the desired result. The case when the 3rd roots of unity are in k is similar except that there are three fields $K_i^*/k^*K_i^*(n)$ each contributing $[q^{2\,\mathrm{val}(2)} - 1]/3$ non-trivial elements of order 2.

We now return to the proof of the Lemma.

Proof of Lemma 5. Since any element of D^* which is not contained in k^* is contained in a cubic extension of k, any element of D^* is conjugate to one of the subgroups defined in the lemma. The main point of the lemma therefore is the way the conjugacy classes intersect the subgroups defined in the lemma. For this purpose, we will continue to use the notation introduced in the proposition. We will prove the lemma assuming that k does not contain 3rd roots of unity so that K is, up to the conjugation action, the unique ramified cubic extension of k contained in D. We first check that a non-trivial element x in $K^*/k^*K^*(n)$ is not conjugate to a non-trivial element y in $L^*/k^*L^*(n)$. Assume, if possible, that x is conjugate to y as elements of $D^*/k^*D^*(n)$; so, let $x = gyg^{-1}\lambda \bmod D^*(n)$ for some $g \in D^*$, $\lambda \in k^*$. Multiplying x and y by elements of k^*, we can assume that we actually have $x = gyg^{-1} \bmod D^*(n)$ with $x \in K^* \cap D^*(r)$ and $y \in L^* \cap D^*(r)$ for some $r > 0$. Furthermore we can assume that r is the maximal integer s with the property that $x \cdot u \in K^* \cap D^*(s)$ for some unit u in k^*. Clearly,

$\mathrm{val}_D(x-1)$ is congruent to 1 or 2 mod 3 whereas $\mathrm{val}_D(y-1)$ is congruent to 0 modulo 3. (Here val_D denotes the valuation in D^*.) From the equation $x = gyg^{-1} \bmod D^*(n)$, we have $(x-1) - g(y-1)\,g^{-1} \in D(n)$, therefore $\mathrm{val}_D(x-1) = \mathrm{val}_D(y-1)$, leading to a contradiction.

The proof of the lemma will be completed if we can prove that if two elements of $K^*/k^*K^*(n)$ (resp. of $L^*/k^*L^*(n)$) are conjugate as elements of $D^*/k^*D^*(n)$, then they are equal (resp., are Galois conjugate). This is what we do below.

We will write any element in R^* uniquely as $g = g_0 + g_1\pi + g_2\pi^2 + \cdots$ where g_i is either a root of unity in L^* of odd order, or is zero. The element π operates on such elements as $\pi g_i \pi^{-1} = \sigma(g_i)$. Let

$$x = 1 + x_1\pi^m + x_2\pi^{m+1} + \cdots$$

$$y = 1 + y_1\pi^m + y_2\pi^{m+1} + \cdots,$$

where x_i, y_i all belong to k be two elements of K^*. Since all the statements in the lemma are modulo k^*, we will assume that the coefficient of π^j is zero whenever j is divisible by 3, and that x_1 and y_1 are not zero, in particular 3 does not divide m. Let $n = m + r$, $r > 0$. Suppose that x and y are conjugate as elements of $D^*/k^*D^*(m+r)$ by an element g in D^*. We will prove that $x = y$. Notice that we can assume that $g \in R^*$. We will inductively prove that for $s \leqslant r$ if $x_i = y_i$ for $1 \leqslant i \leqslant s$, and $g = g_0 + g_1\pi + \cdots$ with $g_i \in k^*$ for $i \leqslant s-1$ then $x_i = y_i$ for $1 \leqslant i \leqslant s+1$, and $g = g_0 + g_1\pi + \cdots$ with $g_i \in k^*$ for $i \leqslant s$. We start the induction by proving that $x_1 = y_1$, and $g_0 \in k^*$. For this, look at the equation $gxg^{-1} = y$ modulo $k^*D^*(m+1)$. We find that $g_0 x_1 \pi^m = y_1 \pi^m g_0$, or $g_0 \cdot \sigma^m(g_0^{-1}) \in k^*$. However, since we are assuming that the 3rd roots of unity are not in k, $g_0\sigma^m(g_0)^{-1}$ which is a norm 1 element and belongs to k^* must be trivial. Since m is not divisible by 3, this implies that g_0 belongs to k^*. Now we assume that $x_i = y_i$ for $1 \leqslant i \leqslant s$, and $g = g_0 + g_1\pi + \cdots$ with $g_i \in k^*$ for $i \leqslant s-1$. Since g_i belongs to k^* for $i \leqslant s-1$, $(g_0 + g_1\pi + \cdots + g_{s-1}\pi^{s-1})$ belongs to K^*, and therefore in particular commutes with x. This implies that g times the inverse of $(g_0 + g_1\pi + \cdots + g_{s-1}\pi^{s-1})$ continues to satisfy the equation $gxg^{-1} = y$. By this modification we can assume that $g_0 = 1$, and $g_i = 0$ for $1 \leqslant i \leqslant s-1$. So, the equation $gxg^{-1} = y$ in $D^*/k^*D^*(m+s+1)$ becomes,

$$\begin{aligned}(1 + g_s\pi^s + \cdots)(1 + x_1\pi^m + \cdots + x_s\pi^{m+s-1} + x_{s+1}\pi^{m+s})(1 + g_s\pi^s + \cdots)^{-1} \\ = 1 + x_1\pi^m + \cdots + x_s\pi^{m+s-1} + y_{s+1}\pi^{m+s}.\end{aligned}$$

After some simplification, this leads to

$$x_{s+1} - y_{s+1} = x_1[\sigma^m(g_s) - g_s].$$

The right hand side of this equation is of trace 0, whereas the left hand side of this equation belongs to k. This implies that both sides are zero, proving the inductive step that $x_{s+1} = y_{s+1}$, and $g_s \in k^*$. We omit the other cases of the lemma as they follow the same pattern.

Remark 6. Lemma 5 says that the conjugacy classes in $D^*/D^*(n)$ are described in terms of field extensions in the obvious way. We believe that it is true for all division algebras whose index is coprime to the residue characteristic but do not have a proof of it.

6. ANOTHER PROOF OF PROPOSITION 6

The referee has kindly pointed out that one can count the number of irreducible, self-dual representations of D^* using Howe's work. We give the details here.

Let D be any division algebra over a local field k whose center is k and index n which is coprime to the residue characteristic of k. According to Howe, cf. [C-H], there exists a bijective correspondence

$$[K/k, \theta] \to \pi(K/k, \theta)$$

between Galois orbits of pairs $(K/k, \theta)$ where K is a field extension of k with $[K:k] \mid n$, θ an admissible character of K^*, and irreducible representations $\pi(K/k, \theta)$ of D^*. We recall that a character θ of K^* is called admissible if it does not factor through the norm mapping to an intermediate field, and if θ restricted to $K^*(1)$ factorises via norms to a field extension E of k, then K is unramified over E.

One has $\pi(K/k, \theta)^* = \pi(K/k, \theta^{-1})$, cf. 1.5.8 of [K-Z]. Therefore $\pi(K/k, \theta)$ is self-dual if and only if $(K/k, \theta^{-1})$ is in the Galois orbit of $(K/k, \theta)$ which means that $\theta = \theta^{-1}$ if $[K:k]$ is odd. Assume now that n is odd and that the residue characteristic of k is even. If $\theta^2 = 1$ and θ restricted to $K^*(1)$ factorises through the norms to an intermediate field, then since the number of invertible elements in the residue field of K is odd, it follows that θ itself factorises through the norms. Therefore the number of irreducible self-dual representations of D^* are in bijective correspondence with the pairs $(K/k, \theta)$ where $\theta^2 = 1$ and θ does not factor through the norms to an intermediate field.

Now we note that the image of K^* under the norm mapping is a subgroup of k^* of odd index. Therefore there exists a bijective correspondence between characters of order 2 on K^* which factorise through the norm mapping to k^*, and characters of order 2 on k^*.

It is clear that for a character ω of k^*/k^{*2}, and irreducible representation π of D^*, $\pi \otimes \omega$ has central character which is ω times the central character of π. Therefore twisting by quadratic characters of k^* has no fixed points on representations of D^*, and there is exactly 1 in the orbit of any irreducible self-dual representation π of D^* with trivial central character. It is easy to see that for any degree d extension of $\mathbf{Q}_2$,

$$[k^* : k^{*2}] = 2^{2+d}.$$

Assume now $n=3$. The number of self-dual representations contributed by a cubic field K to representations of D^*/k^* is therefore $2^{-(2+d)}(2^{2+3d} - 2^{2+d})/3 = (2^{2d}-1)/3$ or $2^{-(2+d)}(2^{2+3d} - 2^{2+d}) = (2^{2d}-1)$ depending on whether K is Galois over k or not. Recalling that there are exactly 3 ramified cubic extensions which are either all Galois or all are Galois conjugate, and there is a unique unramified extension of degree 3, we again get the number of irreducible self-dual representations of D^*/k^* to be

$$\tfrac{4}{3}(q^{2\,\mathrm{val}\,2} - 1).$$

7. MULTIPLICITY ONE FOR $GL_2(D_1)$ AND SHALIKA MODELS

In this section we briefly point out certain multiplicity one results for the group $GL_2(D_1)$ where D_1 is an odd degree division algebra over k. Let K be a quadratic extension of k. Then $(D_1 \otimes K)^*$ is a subgroup of $GL_2(D_1)$, and every character of $(D_1 \otimes K)^*$ obtained from a character of K^*/k^* appears with multiplicity atmost 1 in any irreducible representation of $GL_2(D_1)$. This follows exactly as the proof of Theorem 2.

One can also consider the subgroup $D_1^* \times D_1^*$ of $GL_2(D_1)$. From this point on in this section, we do not assume that the index of D_1 is odd. There is a natural map from $D_1^* \times D_1^*$ to $(k^* \times k^*)/k^*$, and any representation of $D_1^* \times D_1^*$ obtained from a character of $(k^* \times k^*)/k^*$ appears with multiplicity atmost 1 in any irreducible representation of $GL_2(D_1)$. However, this time, the proof of Theorem 2 does not work as there are double cosets which are not preserved under the involution. (In the notation of Theorem 2, these double cosets are those for which B or C may be zero.) To take care of this case, we need to show that any distribution on the group which is bi-invariant under the subgroup is invariant under the involution, cf. Lemma 4.2 in [P]. We use the technique of Bernstein to do this which is to cut the space up in parts each of which is bi-invariant under the subgroup, and the involution, and prove the result on distributions on each piece. On the part of the group where BC is not zero, any double coset is invariant under the involution, and there is no problem. On the part where $BC=0$, one gets reduced to checking that a distribution supported on $D_1 \times 0 \cup 0 \times D_1 \subset D_1 \times D_1$ which is invariant under $(t, t_2) \to (xt_1 y^{-1}, yt_2 x^{-1})$ for all $x, y \in D_1^*$ is invariant under the involution $(t_1, t_2) \to (t_2, t_1)$. This is the analogue of lemma 4.6 of [P], and is easy to see.

As in Jacquet, Rallis [JR], one can deduce the uniqueness of the *Shalika model* which we define below.

DEFINITION (Shalika Model). A representation π of $GL_2(D_1)$ realised on a vector space V is said to Shalika model if there exists a non-zero linear form ℓ on V such that $\ell([\begin{smallmatrix} A & AX \\ 0 & A \end{smallmatrix}]v) = \psi(\operatorname{tr} X)\,\ell(v)$, for a non-trivial additive character ψ of k, and for all $v \in V$, $A \in D_1^*$, and $X \in D_1$.

The following proposition characterises principal series representations with Shalika model.

PROPOSITION 7. *A principal series representation π of $GL_2(D_1)$ induced from the representation $\pi_1 \otimes \pi_2$ of the Levi subgroup $D_1^* \times D_1^*$ of the minimal parabolic subgroup P of $GL_2(D_1)$ consisting of upper triangular matrices has a Shalika model if and only if $\pi_1 \cong \pi_2^*$.*

Proof. It is easy to see that P has 2 orbits on $GL_2(D_1)/P$, one closed orbit consisting of the single point represented by P, and the other an open orbit, consisting of the complement of this point. The open orbit can be taken to be the orbit of the point ωP in $GL_2(D_1)$ for $\omega = (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix})$. The stabiliser of ωP in P is $H = \{(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}) \mid a, d \in D_1^*\}$.

By the Mackey theory of restriction of an induced representation, we have an exact sequence of P-modules:

$$0 \to \operatorname{ind}_H^P(\pi_1 \otimes \pi_2 \otimes \delta) \to \pi \to \pi_1 \otimes \pi_2 \otimes \delta \to 0,$$

where the character δ of H is defined by $\delta(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}) = |ad^{-1}|$, and appears here because of its presence in the definition of (normalised) induction.

Let $N = \{(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) \mid b \in D_1\}$. Since $P = HN$, it is easy to see from the definition of induced representation that

$$\operatorname{ind}_H^P(\pi_1 \otimes \pi_2 \otimes \delta) = \left\{ \begin{array}{l} f\colon N \to \pi_1 \otimes \pi_2 \otimes \delta \text{ where } f \text{ is a compactly} \\ \text{supported locally constant function} \end{array} \right\}$$

$$\cong \mathscr{S}(N) \otimes \pi_1 \otimes \pi_2 \otimes \delta.$$

Here $\mathscr{S}(N)$ denotes the Schwartz space of locally constant, compactly supported functions on N which is a natural representation of N under translation, and of H via its action on N as a normal subgroup of P. The isomorphism above is one of P modules in which P acts on $\pi_1 \otimes \pi_2 \otimes \delta$ via $P/N \cong H$.

For any representation V of N, let $V_{N,\psi}$ define the twisted Jacquet functor

$$V_{N,\psi} = V/\{n \cdot n - \psi(\mathrm{tr}(n))\, v \mid v \in V, n \in N\}.$$

From the description of the induced representation $\mathrm{ind}_H^P(\pi_1 \otimes \pi_2 \otimes \delta)$, we find that

$$\begin{aligned}[\mathrm{ind}_H^P(\pi_1 \otimes \pi_2 \otimes \delta)]_{N,\psi} &\cong \mathscr{S}(N)_{N,\psi} \otimes \pi_1 \otimes \pi_2 \otimes \delta \\ &\cong \pi_1 \otimes \pi_2 \otimes \delta.\end{aligned}$$

Here we have used the isomorphism $\mathscr{S}(N)_{N,\psi} \cong \mathbf{C}$ which can be deduced from the Frobenius reciprocity

$$\mathrm{Hom}_N[\mathrm{ind}_{\{e\}}^N \mathbf{C}, \mathbf{C}_\psi] \cong \mathrm{Hom}_{\{e\}}[\mathbf{C}, \mathbf{C}] \cong \mathbf{C},$$

where $\mathbf{C}_\psi$ denotes the 1 dimensional representation of N on which $N \cong D_1$ operate via $\psi \circ \mathrm{tr}$.

Since $V \to V_{N,\psi}$ is an exact functor, we have the short exact sequence

$$0 \to [\mathrm{ind}_H^P(\pi_1 \otimes \pi_2 \otimes \delta)]_{N,\psi} \to \pi_{N,\psi} \to [\pi_1 \otimes \pi_2 \otimes \delta]_{N,\psi} \to 0.$$

Since N operates trivially on $\pi_1 \otimes \pi_2 \otimes \delta$, $[\pi_1 \otimes \pi_2 \otimes \delta]_{N,\psi} = 0$. Therefore from the exact sequence above together with the earlier calculation of the twisted Jacquet functor of $\mathrm{ind}_H^P(\pi_1 \otimes \pi_2 \otimes \delta)$, we find the isomorphism $\pi_{N,\psi} \cong \pi_1 \otimes \pi_2 \otimes \delta$. It is easy to see that this isomorphism is one of H_0-modules for $H_0 = \{(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) \mid a \in D_1^*\} \cong D_1^*$. Since $\delta(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}) = 1$, we have the isomorphism of H_0-modules

$$\pi_{N,\psi} \cong \pi_1 \otimes \pi_2.$$

We note that π has a Shalika model if and only if $\pi_{N,\psi}$ has H_0-invariant linear form, and therefore π has a Shalika model if and only if $\pi_1 \cong \pi_2^*$.

The following conjecture is motivated by its global counter-part in [J-S].

Conjecture 3. Let D be a division algebra over a local field k and π an irreducible admissible representation of $GL_2(D)$ with trivial central character which is either an irreducible principal series, or is square integrable. Then π has a Shalika model if and only if π is a self-dual representation whose Langlands parameter is symplectic.

Remark 7. According to Tadic, [Ta], the principal series representation of $GL_2(D)$ induced from the representation $(\pi \otimes \delta^{1/2}) \otimes (\pi \otimes \delta^{-1/2})$ is of length 2, with an irreducible sub-representation, to be denoted by $\mathrm{St}(\pi)$, which is square-integrable and an irreducible quotient, to be denoted by $\mathrm{Sp}(\pi)$, which is non-tempered:

$$0 \to \mathrm{St}(\pi) \to V \to \mathrm{Sp}(\pi) \to 0.$$

Applying the twisted Jacquet functor and using the earlier calculation for the principal series, we have

$$0 \to \mathrm{St}(\pi)_{N,\psi} \to \pi \otimes \pi \to \mathrm{Sp}(\pi)_{N,\psi} \to 0.$$

We do not know how to calculate the twisted Jacquet functors $\mathrm{St}(\pi)_{N,\psi}$ and $\mathrm{Sp}(\pi)_{N,\psi}$ even in the simplest case when D is a quaternionic division algebra. It will be very interesting to be able to calculate these twisted Jacquet functors. However, even without the calculation of the twisted Jacquet functors, it follows from the above exact sequence that atmost one of the representations $\mathrm{St}(\pi)$, or $\mathrm{Sp}(\pi)$ has a Shalika model, and exactly one has Shalika model if π is self-dual. The conjecture above predicts which one has Shalika model. To make this explicit, suppose that the Langlands parameter of π is the irreducible representation $\sigma \otimes Sp_r$ of the Weil–Deligne group $W'_k = W_k \times SL(2)$ where Sp_r is the unique irreducible representation of $SL(2)$ of dimension r. Then the Langlands parameter of $\mathrm{St}(\pi)$ is $\sigma \otimes Sp_{2r}$. This representation is symplectic if and only if σ is orthogonal. Some of the conclusions of the conjecture are therefore as follows. First when the index of D is odd. In this case D^* has self-dual representation of dimension greater than 1 only in residue characteristic 2, and for any such π, our conjecture predicts $\mathrm{St}(\pi)$ to have Shalika model. If D is quaternionic, and $\dim \pi$ is greater than 1, our conjecture predicts $\mathrm{St}(\pi)$ to have Shalika model exactly when π is a self-dual representation of D^* whose central character is non-trivial. More generally, for division algebras over local fields of odd residue characteristic and of index $2n$ for an odd integer n, our conjecture predicts that $\mathrm{St}(\pi)$ has a Shalika model if and only if π is self-dual with non-trivial central character.

Remark 8. We believe that conjecture 3 is true for representations of $GL(2n, F)$, F a finite field, which have a Whittaker model (where Shalika model is defined in an analogous manner), but this also we have not been able to prove. The general question one would like to answer is: what is the twisted Jacquet functor for such representations?

8. A CONJECTURE ABOUT MULTIPLICITIES

We assume in this section that $D = D_1 \otimes D_2$ is a division algebra with D_2 a quaternion division algebra.

From the results proved in the previous sections, we know that for any character χ of K^*/k^*, the corresponding 1 dimensional representation of D_K^*/k^* appears with multiplicity atmost 1 in any irreducible representation of D^*, and also in any irreducible representation of $GL_2(D_1)$. In this section we make a conjecture regarding this multiplicity, and prove it in some cases.

We note that by the work of Deligne–Kazhdan–Vigneras, there is a one-to-one correspondence between square-integrable representations π of $GL_2(D_1)$ and finite dimensional representations π' of D^*. Under this correspondence, the characters of the representations π and π' are negative of each other at the regular conjugacy classes shared by the two groups D^* and $GL_2(D_1)$. (One may add that at the moment the Deligne–Kazhdan–Vigneras correspondence is known only in characteristic 0.)

Conjecture 4. Let π be an irreducible admissible representation of $GL_2(D_1)$ with trivial central character which is either an irreducible principal series, or is square integrable. If π is a square integrable representation of $GL_2(D_1)$, let π' be the representation of D^* associated by Deligne–Kazhdan–Vigneras, and let $\pi'=0$ otherwise. For the character μ of D_K^*/k^* obtained from a character χ of K^*/k^* via the norm map, let $m(\pi,\mu)$, $m(\pi',\mu)$ denote the multiplicity of μ in π, π' respectively. Then

(a) The sum $m(\pi,\mu)+m(\pi',\mu)$ is independent of the character χ of K^*/k^*. Therefore as $m(\pi',\mu)$ can be non-zero for atmost finitely many characters χ, the sum $m(\pi,\mu)+m(\pi',\mu)$ is either 0 or 1.

(b) The sum $m(\pi,\mu)+m(\pi',\mu)=1$ if and only if the Langlands parameter σ associated to π which is a $2n$-dimensional representation of the Weil–Deligne group W_k of k is symplectic, equivalently, if and only if the representation π' of D^* is orthogonal.

(c) The representation $\sigma\otimes \mathrm{Ind}_{K^*}^{W_k}\chi$ of W_k is a symplectic representation of the Weil–Deligne group, and therefore its epsilon factor is independent of the choice of the additive character of the field k. The representation μ of D_K^*/k^* appears in π if and only if $\varepsilon(\sigma\otimes \mathrm{Ind}_{K^*}^{W_k}\chi)=\omega_{K/k}(-1)$, where $\omega_{K/k}$ is the quadratic character of k^* associated to K.

Needless to say, the conjecture is based on the theorem of Tunnell and Saito in the context of $GL(2)$, cf. [Tu1], [S].

PROPOSITION 8. *The conjecture above is true if either π is an irreducible principal series, or is a twist of the Steinberg representation.*

Proof. We first take up the case of irreducible principal series. Suppose that the irreducible principal series representation of $GL_2(D_1)$ is obtained by inducing the representation $V_1\otimes V_2$ of the Levi component $D_1^*\times D_1^*$ of the standard parabolic P inside $GL_2(D_1)$. Since $(D_1 \otimes K)^*$ acts transitively on $GL_2(D_1)/P$ with D_1^* as the stabiliser, the restriction of an irreducible principal series to $(D_1 \otimes K)^*$ is obtained by inducing the representation $V_1 \otimes V_2$ of D_1^* to $(D_1 \otimes K)^*$. By Frobenius reciprocity, if the principal series is to contain a character μ of $(D_1 \otimes K)^*$ as in the conjecture above (meaning those which come from characters of K^*/k^* via the norm mapping), then the representation $V_1 \otimes V_2$ of D_1^* must contain the trivial representation, i.e. we must have $V_1 \cong V_2^*$. Conversely, if $V_1 \cong V_2^*$, then all the character μ of $(D_1 \otimes K)^*$ as in the conjecture do appear in the principal series. Clearly, the parameter of the principal series representation is symplectic if and only if V_1 is isomorphic to the dual of V_2.

We now check the condition on the epsilon factor. Let the Langlands parameter associated to the representation V_1 be σ_1. So, the parameter of the principal series representation is $\sigma_1 \oplus \sigma_1^*$. For simplicity of notation, let $\rho = \operatorname{ind}_{K^*}^{W_k} \chi$. From the standard properties of epsilon factors, cf. [T], we find

$$\begin{aligned} \varepsilon[(\sigma_1 \oplus \sigma_1^*) \otimes \rho] &= \det(\sigma_1 \otimes \rho)(-1) \\ &= \det(\sigma_1)^{\dim \rho}(-1)(\det \rho)^{\dim \sigma_1}(-1) \\ &= \omega_{K/k}(-1). \end{aligned}$$

Here we have used that the dimension of σ_1 is n which is odd, and that the determinant of ρ is $\omega_{K/k}$.

Next, we look at the Steinberg representation which is obtained on the space of locally constant functions on $GL_2(D_1)/P$ modulo the constant functions. The representation π' is the trivial representation of D^*. We find that any representation of D_K^*/k^* obtained from a character of K^*/k^* appears exactly in one of the representations π or π', and it appears in π' if and only if the representation of D_K^*/k^* is trivial. We check that this conclusion matches with the epsilon factors.

We note that since $\rho = \operatorname{Ind}_{K^*}^{W_k} \chi$ is self-dual with determinant $\omega_{K/k}$, the general relation, $\varepsilon(\rho) \cdot \varepsilon(\rho^*) = (\det \rho)(-1)$ implies that $\varepsilon(\rho)^2 = \omega_{K/k}(-1)$.

Let Sp_n denote the representation of the Weil–Deligne group corresponding to the Steinberg representation of $GL_n(k)$. With this notation, and a property of epsilon factors, we find

$$\varepsilon[Sp_{2n} \otimes \rho] = \varepsilon(\rho)^{2n} \det(-F, \rho^I)^{2n-1},$$

where I is the inertia subgroup of W_k, and F denotes a Frobenius element in W_k. If ρ is an irreducible 2 dimensional representation of W_k, it is clear that $\rho^I = 0$, proving that

$$\varepsilon[Sp_{2n} \otimes \rho] = \omega_{K/k}(-1).$$

It remains to treat the case when ρ is reducible, which is the case if and only if the character χ of K^* comes from a character, say ν of k^* via the norm mapping. In this case, $\rho = \nu + \nu \cdot \omega_{K/k}$. It follows that $\det(-F, \rho^I) = -1$ if and only if χ is the trivial character of K^*, proving the proposition for the Steinberg representation.

We next prove the following finiteness theorem necessary for our Conjecture 4.

PROPOSITION 9. *There are only finitely many characters χ of K^* trivial on k^* for which*

$$\varepsilon(\sigma \otimes \mathrm{Ind}\chi) = -\omega_{K/k}(-1).$$

Proof. Let $\mathbf{1}$ denote the trivial representation of W_k. Since the epsilon factor is preserved under induction for representations of dimension 0,

$$\varepsilon([\sigma - 2n \cdot \mathbf{1}] \otimes \mathrm{Ind}(\chi)) = \varepsilon([\sigma_K - 2n \cdot \mathbf{1}] \otimes \chi),$$

where σ_K denotes the restriction of σ to the Weil–Deligne group W_K of K.
Therefore

$$\varepsilon(\sigma \otimes \mathrm{Ind}\chi) = \varepsilon(\sigma_K \otimes \chi) \frac{\varepsilon(\mathrm{Ind}\chi)^{2n}}{\varepsilon(\chi)^{2n}}.$$

Since for any representation V of the Weil group, $\varepsilon(V)\,\varepsilon(V^*) = \det V(-1)$, we have

$$\varepsilon(\chi)^2 = 1, \qquad \varepsilon(\mathrm{Ind}\chi)^2 = \omega_{K/k}(-1).$$

Therefore

$$\varepsilon(\sigma \otimes \mathrm{Ind}\chi) = \varepsilon(\sigma_K \otimes \chi)\, \omega_{K/k}(-1).$$

Since σ is a symplectic representation, in particular its determinant is trivial, it follows by a theorem of Deligne that when the conductor of χ is very large,

$$\varepsilon(\sigma_K \otimes \chi) = \varepsilon(\chi)^{2n} = 1.$$

Therefore for characters χ of large conductor, $\varepsilon(\sigma \otimes \mathrm{Ind}\chi) = \omega_{K/k}(-1)$.

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