ON THE DECOMPOSITION OF THE FEYNMAN PROPAGATOR

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ABSTRACT

The Feynman propagator, in momentum representation, is a four-dimensional transform over space and time variables. If the space and time integrations are performed separately, the propagator can be decomposed into two parts, one corresponding to positive and the other to negative energy intermediate state. By the use of this decomposed propagator, the relative contributions of the positive and negative energy intermediate states to the matrix element can be estimated. For example in Compton scattering it leads to the apparently paradoxical result that in the "non-relativistic approximation" it is only the negative energy intermediate state that contributes to the matrix element.

INTRODUCTION

IT is well recognised that one of the great advances in quantum electrodynamics since the Dirac theory of the electron is the Feynman formulation which is characterised essentially by two features: (i) Even when we include pair creation and annihilation of particles the vertices of interaction in perturbation theory can be so ordered in the Feynman sense such that the entire process can be represented through Feynman graphs of single particles thus obviating the use of field theoretic methods and (ii) the perturbation expansions are "inherently four-dimensional" and the covariance of the equations is apparent at every stage of the calculations, the integrations over space and time variables being performed together.

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It has been pointed out by one of the authors (A. R.) that a better understanding of the virtual processes is possible if the space integration is performed first and the time integration subsequently in the "old-fashioned" manner even in the Feynman formulation. It is this method which gives a clearer picture of the contributions due to the different types of virtual processes. It leads to the splitting of the Feynman propagator and while not affecting the elegance of the inherently relativistic approach, reveals the structure of the Feynman propagator in a manner which facilitates the computation of the relative contributions from transitions to positive and negative energy intermediate states. In particular, calculations lead to the apparently paradoxical result that the negative energy states do contribute even if the electron in Compton scattering is non-relativistic. It is conventionally accepted that when the initial particles are non-relativistic, the energy denominators become large for negative energy states and therefore the contribution in such cases can be neglected. This assumption may lead to erroneous results since the numerators may also become large and this can only be studied by the use of the decomposed propagator.[1]

Decomposition of the Feynman Propagator

In the Feynman picture since a negative energy electron travels back in time, the positive and negative energy parts of the intermediate state can be separated by splitting the time integration in the perturbation expansion (*i.e.*, from $t = -\infty$ to $t = +\infty$) into two parts corresponding to the ranges 0 to $+\infty$ with energy $+\mathrm{E}$ and $-\infty$ to 0 with energy $-\mathrm{E}$ respectively. The integration over space variables alone in a perturbation expansion say for a process in which an electron of momentum $\vec{p}$ absorbs a photon of momentum $\vec{q}$, would amount to picking out from the kernel, terms corresponding to momentum $\vec{p} + \vec{q}$ and energy E equal to

$$\pm \mathrm{E}_{p+q} = \pm \sqrt{\left(\vec{p} + \vec{q}\right)^2 + m^2}.$$

[1] The paradoxical result that the negative energy states should contribute even in the non-relativistic case seems to have been well recognized on the ground that the operator γ_μ connects the positive and negative energy states. What we wish to emphasise here is that the relative contributions can be studied by decomposing the Feynman propagator the energy denominators in this case being naturally different from the field theoretic case. This is so because the definition of the intermediate states in field theory refers to systems of electrons, positrons and photons while in the Feynman formalism it relates only to the electron.

If we now perform the time integrations separately corresponding to $+\mathrm{E}_{p+q}$ and $-\mathrm{E}_{p+q}$ we have, for the Feynman propagator in momentum representation,[2]

$$\frac{1}{\not p + \not q - m} = \frac{1}{2}\left[\frac{\overset{+}{\not P} + m}{\left(\mathrm{E}_{\vec{p}+\vec{q}}\right)\left\{\mathrm{E}_{\vec{p}} + \mathrm{E}_{\vec{q}} - \mathrm{E}_{\vec{p}+\vec{q}}\right\}} - \frac{\overline{\not P} + m}{\left(\mathrm{E}_{\vec{p}+\vec{q}}\right)\left\{\mathrm{E}_{\vec{p}} + \mathrm{E}_{\vec{q}} + \mathrm{E}_{\vec{p}+\vec{q}}\right\}}\right] \tag{1}$$

where $\mathrm{P} = p + q$ and $\overset{+}{\not p}$ is the Feynman four vector with energy $+\mathrm{E}_{\vec{p}+\vec{q}}$ and $\overline{\not p}$ has the fourth component equal to $-\mathrm{E}_{\vec{p}+\vec{q}}$.

We shall now make use of the above expression for the propagator and calculate the cross-sections for Compton scattering taking into account intermediate states of positive energy only, forbidding intermediate states of negative energy and *vice-versa.*

The matrix element representing the scattering of an incident photon of momentum q_1 by an electron (at rest) to momentum q_2, the electron having a momentum p_2, is obtained by considering the two possibilities (1) the photon q_1 being absorbed and q_2 being emitted subsequently in the Feynman sense and (2) q_2 being emitted and q_1 absorbed subsequently.

The propagators for the two cases are given by

$$\frac{1}{\not p_1 + \not q_1 - m} = \frac{1}{2}\left[\frac{\overset{+}{\not P}_1 + m}{\left(\mathrm{E}_{\vec{p}_1+\vec{q}_1}\right)\left\{\mathrm{E}_{\vec{p}_1} + \mathrm{E}_{\vec{q}_1} - \mathrm{E}_{\vec{p}_1+\vec{q}_1}\right\}} - \frac{\overline{\not P}_1 + m}{\mathrm{E}_{\vec{p}_1+\vec{q}_1}\left\{\mathrm{E}_{\vec{p}_1} + \mathrm{E}_{\vec{q}_1} + \mathrm{E}_{\vec{p}_1+\vec{q}_1}\right\}}\right] \tag{2}$$

where $\overset{+}{\not p}_1$ has energy component $\mathrm{E}_{\vec{p}_1+\vec{q}_1}$ and

[2] Throughout this paper, we work in natural units, *i.e.*, $c = 1$, $\hbar = 1$ and use the conventional Feynman notation: If p denotes the four vector with components $p_x, p_y, p_z, \mathrm{E}_{\vec{p}}\left(\vec{p}, \mathrm{E}_{\vec{p}}\right)$ then the Feynman four vector $\not p = p_\mu \gamma_\mu = \mathrm{E}_{\vec{p}}\gamma_t - \vec{p}\cdot\vec{\gamma}$, where γ's are the usual Dirac matrices and $\not p \not q = \mathrm{E}_p \omega - \vec{\omega}\cdot\vec{p}$, where q is another four vector with components $\left(\omega, \vec{\omega}\right)$.

$$\frac{1}{\not{p}_1 - \not{q}_2 - m} = \frac{1}{2}\left[\frac{\overset{+}{\not{p}}_2 + m}{\left(E_{\vec{p}_1-\vec{q}_2}\right)\left\{E_{\vec{p}_1} - E_{\vec{q}_2} - E_{\vec{p}_1-\vec{q}_2}\right\}}\right.$$

$$\left. - \frac{\overset{-}{\not{p}}_2 + m}{\left(E_{\vec{p}_1-\vec{q}_2}\right)\left\{E_{\vec{p}_1} - E_{\vec{q}_2} + E_{\vec{p}_1-\vec{q}_2}\right\}}\right] \tag{3}$$

where $\not{p}_2$ has energy component $E_{\vec{p}_1-\vec{q}_2}$.

The cross-sections for the entire process taking the effect of positive energy alone ignoring the negative energy part in the intermediate state can be calculated by using the sum of the first terms of (2) and (3) as the propagator. Thus the matrix element after substituting for $\overset{+}{\not{p}}_1$, etc., is

$$M^{(+)} = \frac{(-i)}{2}(4\pi e^2)\,\bar{u}_2\left[\not{e}_2\frac{\gamma_t\sqrt{\omega_1^2+m^2} - \gamma_x\omega_1 + m}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}}\not{e}_1\right.$$

$$\left. + \not{e}_1\frac{\gamma_t\sqrt{\omega_2^2+m^2}+\gamma_x\omega_2\cos\theta+\gamma_y\omega_2\sin\theta+m}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}}\not{e}_2\right]u_1. \tag{4}$$

$$M^{(-)} = \frac{+i}{2}(4\pi e^2)\,\bar{u}_2\left[\not{e}_2\frac{-\gamma_t\sqrt{\omega_1^2+m^2} - \gamma_x\omega_1 + m}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\not{e}_1\right.$$

$$\left. + \not{e}_1\frac{-\gamma_t\sqrt{\omega_2^2+m^2}+\gamma_x\omega_2\cos\theta+\gamma_y\omega_2\sin\theta+m}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\not{e}_2\right]u_1 \tag{5}$$

Taking the incoming photon to be along the x direction we have for polarisation

(A) $\not{e}_1 = \gamma_z$ or (B) $\not{e}_1 = \gamma_y$

and for the outgoing photon

(A′) $\not{e}_2 = \gamma_z$ or (B′) $\not{e}_2 = \gamma_y\cos\theta - \gamma_x\sin\theta$.

Thus the matrix elements $M^{(+)}$ and $M^{(-)}$ for various polarisation combinations are given by

$M^{(+)}$ (AB′ or BA′)

$$= \frac{1}{\sqrt{F_1F_2}}\,\frac{(-i4\pi e^2)}{2}\left[\frac{F_1F_2\{i\sqrt{\omega_1^2+m^2}-im\}e^{i\theta}+i\omega_1F_1p_{2+}e^{i\theta}}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}}\right.$$

$$\left. + \frac{F_1F_2\{-i\sqrt{\omega_2^2+m^2}+im\}e^{i\theta}+i\omega_2F_1p_{2+}}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}}\right] \tag{6}$$

where $p_{2+} = p_{2x} + ip_{2y}$

and

$M^{(-)}$ (AB′ or BA′)

$$= \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2\,(i\sqrt{\omega_1^2+m^2}+im)\,e^{i\theta}-i\omega_1F_1p_{2+}e^{i\theta}}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left.+\frac{F_1F_2\,\{-i\sqrt{\omega_2^2+m^2}-im\}\,e^{i\theta}-i\omega_2F_1p_{2+}}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\right]. \tag{7}$$

$M^{(+)}$ (AA′)

$$= \frac{1}{\sqrt{F_1F_2}}\frac{(-i4\pi e^2)}{2}\left[\frac{F_1F_2\,\{\sqrt{\omega_1^2+m^2}-m\}-\omega_1F_1p_{2-}}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left.+\frac{F_1F_2\,\{\sqrt{\omega_2^2+m^2}-m\}+\omega_2F_1p_{2-}e^{i\theta}}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}}\right]. \tag{8}$$

$M^{(-)}$ (AA′)

$$= \frac{1}{\sqrt{F_1F_2}}\left(\frac{-i4\pi e^2}{2}\right)\left[\frac{F_1F_2\,\{\sqrt{\omega_1^2+m^2}+m\}+\omega_1F_1p_{2-}}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left.+\frac{F_1F_2\,\{\sqrt{\omega_2^2+m^2}+m\}-\omega_2F_1p_{2-}e^{i\theta}}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\right]. \tag{9}$$

$M^{(+)}$ (BB′)

$$= \frac{1}{\sqrt{F_1F_2}}\frac{(-i4\pi e^2)}{2}\left[\frac{F_1F_2\,\{\sqrt{\omega_1^2+m^2}e^{-i\theta}-me^{-i\theta}\}-\omega_1F_1p_{2-}e^{i\theta}}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1-\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left.+\frac{F_1F_2\,\{\sqrt{\omega_2^2+m^2}e^{i\theta}-me^{i\theta}\}+\omega_2F_1p_{2-}}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2-\sqrt{\omega_2^2+m^2}\}}\right] \tag{10}$$

and

$M^{(-)}$ (BB′)

$$= \frac{1}{\sqrt{F_1F_2}}\frac{(-i4\pi e^2)}{2}\left[\frac{F_1F_2\,\{\sqrt{\omega_1^2+m^2}e^{-i\theta}+me^{-i\theta}\}+\omega_1F_1p_{2-}e^{i\theta}}{\sqrt{\omega_1^2+m^2}\,\{m+\omega_1+\sqrt{\omega_1^2+m^2}\}}\right.$$
$$\left.+\frac{F_1F_2\,\{\sqrt{\omega_2^2+m^2}e^{i\theta}+me^{i\theta}\}-\omega_2F_1p_{2-}}{\sqrt{\omega_2^2+m^2}\,\{m-\omega_2+\sqrt{\omega_2^2+m^2}\}}\right] \tag{11}$$

where

$$F_2 = E_2 + m, \quad F_1 = E_1 + m.$$

For any one of the polarization cases considered above, we sum over the initial spin states of the electron and average over the final spin states to obtain the cross-section. We shall now list $|M|^2$ for the various combinations of polarization.

$|M^{(+)}(AB' \text{ or } BA')|^2$

$$= \frac{1}{4F_1F_2}(4\pi e^2)^2 \left[\frac{F_1^2F_2^2(\omega_1^2 + 2m^2 - 2m\sqrt{\omega_1^2+m^2}) + F_1^2\omega_1^2\vec{p}_2^{\,2} + (\sqrt{\omega_1^2+m^2} - m) \times 2F_1^2F_2\omega_1 p_{2x}}{(\omega_1^2+m^2)\{2m^2 + 2\omega_1^2 + 2m\omega_1 - 2(m+\omega_1)\sqrt{\omega_1^2+m^2}\}} \right.$$

$$+ \frac{F_1^2F_2^2(\omega_2^2 + 2m^2 - 2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} - 2F_1^2F_2\omega_2 \times (p_{2x}\cos\theta + p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2} - m)}{(\omega_2^2+m^2)\{2m^2 + 2\omega_2^2 - 2m\omega_2 - 2(m-\omega_2)\sqrt{\omega_2^2+m^2}\}}$$

$$\left. + \frac{\begin{array}{l} -2F_1^2F_2^2(\sqrt{\omega_1^2+m^2} - m)(\sqrt{\omega_2^2+m^2} - m) + 2F_1^2\omega_1\omega_2\vec{p}_2^{\,2}\cos\theta \\ \quad - 2F_1^2F_2p_{2x}\omega_1(\sqrt{\omega_2^2+m^2} - m) + 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta) \\ \quad \times (\sqrt{\omega_1^2+m^2} - m) \end{array}}{\sqrt{\omega_1^2+m^2}\sqrt{\omega_2^2+m^2}\{m+\omega_1 - \sqrt{\omega_1^2+m^2}\}\{m - \omega_2 - \sqrt{\omega_2^2+m^2}\}} \right]. \tag{12}$$

Denoting the denominators of the three terms as $D_1^{(+)}$, $D_2^{(+)}$ and $D_3^{(+)}$ respectively, we have

$$\begin{aligned}
|M^{(+)}(AA')|^2 \\
= \frac{1}{4F_1F_2}(4\pi e^2)^2 \Bigg[\frac{1}{D_1^{(+)}} \{ & F_1^2F_2^2(\omega_1^2 + 2m^2 - 2m\sqrt{\omega_1^2+m^2}) \\
& + F_1^2\omega_1^2\vec{p}_2^{\,2} - 2F_1^2F_2\omega_1p_{2x}(\sqrt{\omega_1^2+m^2} - m)\} \\
+ \frac{1}{D_2^{(+)}} \{ & F_1^2F_2^2(\omega_2^2 + 2m^2 - 2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} \\
& + 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta) \\
& \times (\sqrt{\omega_2^2+m^2} - m)\} \\
+ \frac{1}{D_3^{(+)}} \{ & 2F_1^2F_2^2(\sqrt{\omega_1^2+m^2} - m)(\sqrt{\omega_2^2+m^2} - m) \\
& + 2F_1^2F_2\omega_2(\sqrt{\omega_1^2+m^2} - m) \\
& \times (p_{2x}\cos\theta + p_{2y}\sin\theta) - 2\omega_1\omega_2F_1^2\vec{p}_2^{\,2}\cos\theta \\
& - 2F_1^2F_2\omega_1p_{2x}(\sqrt{\omega_2^2+m^2} - m)\} \Bigg].
\end{aligned} \tag{13}$$

$$
\begin{aligned}
&|M^{+}(BB')|^2 \\
&= \frac{1}{4F_1F_2}(4\pi e^2)^2 \Big[\frac{1}{D_1^{(+)}}\{F_1^2F_2^2(\omega_1^2 + 2m^2 - 2m\sqrt{\omega_1^2+m^2}) \\
&\qquad + F_1^2\vec{p}_2^{\,2}\omega_1^2 - 2F_1^2F_2\omega_1(\sqrt{\omega_1^2+m^2} - m) \\
&\qquad \times (p_{2x}\cos 2\theta + p_{2y}\sin 2\theta)\} \\
&\quad + \frac{1}{D_2^{(+)}}\{F_1^2F_2^2(\omega_2^2 + 2m^2 - 2m\sqrt{\omega_2^2+m^2}) + F_1^2\vec{p}_2^{\,2}\omega_2^2 \\
&\qquad + 2F_1^2F_2\omega_2(\sqrt{\omega_2^2+m^2} - m) \\
&\qquad \times (p_{2x}\cos\theta - p_{2y}\sin\theta)\} \\
&\quad + \frac{1}{D_3^{(+)}}\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2} - m)(\sqrt{\omega_2^2+m^2} - m) \\
&\qquad \times \cos 2\theta - 2F_1^2F_2\omega_1(\sqrt{\omega_2^2+m^2} - m)p_{2x} \\
&\qquad - 2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta + 2F_1^2F_2\omega_2 \\
&\qquad \times (\sqrt{\omega_1^2+m^2} - m)(p_{2x}\cos\theta + p_{2y}\sin\theta)\}\Big].
\end{aligned} \tag{14}
$$

Similarly in the case of negative energy, we have

$$
\begin{aligned}
&|M^{-}(AB' \text{ or } BA')|^2 \\
&= \frac{(4\pi e^2)^2}{4F_1F_2}\Bigg[\frac{F_1^2F_2^2(\omega_1^2 + 2m^2 + 2m\sqrt{\omega_1^2+m^2}) + F_1^2\omega_1^2\vec{p}_2^{\,2} - 2F_1^2F_2\omega_1 p_{2x} \times (\sqrt{\omega_1^2+m^2} + m)}{(\omega_1^2+m^2)(2m^2 + 2\omega_1^2 + 2m\omega_1 + 2\sqrt{\omega_1^2+m^2}(m+\omega_1))} \\
&\quad + \frac{F_1^2F_2^2(\omega_2^2 + 2m^2 + 2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} + 2F_1^2F_2\omega_2 \times (p_{2x}\cos\theta + p_{2y}\sin\theta)(\sqrt{\omega_2^2+m^2} + m)}{(\omega_2^2+m^2)\{2m^2 + 2\omega_2^2 - 2m\omega_2 + 2\sqrt{\omega_2^2+m^2}(m-\omega_2)\}} \\
&\quad + \frac{\begin{array}{l}- 2F_1^2F_2^2(\sqrt{\omega_1^2+m^2} + m)(\sqrt{\omega_2^2+m^2} + m) + 2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta \\ + 2F_1^2F_2\omega_1 p_{2x}(\sqrt{\omega_2^2+m^2} + m) - 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta) \\ \times (\sqrt{\omega_1^2+m^2} + m)\end{array}}{\sqrt{\omega_1^2+m^2}\sqrt{\omega_2^2+m^2}\{m + \omega_1 + \sqrt{\omega_1^2+m^2}\}\{m - \omega_2 + \sqrt{\omega_2^2+m^2}\}}\Bigg].
\end{aligned} \tag{15}
$$

Denoting the denominations in (15) by $D_1^{(-)}$, $D_2^{(-)}$ and $D_3^{(-)}$ respectively, we have

$$
\begin{aligned}
&|M^-(AA')|^2 \\
&= \frac{(4\pi e^2)^2}{4F_1F_2}\Bigg[\frac{1}{D_1^{(-)}}\{F_1^2F_2^2(\omega_1^2+2m^2+2m\sqrt{\omega_1^2+m^2}) \\
&\qquad + F_1^2\omega_1^2\vec{p}_2^{\,2} + 2F_1^2F_2p_{2x}\omega_1(\sqrt{\omega_1^2+m^2}+m)\} \\
&\quad + \frac{1}{D_2^{(-)}}\{F_1^2F_2^2(\omega_2^2+2m^2+2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} \\
&\qquad - 2F_1^2F_2\omega_2(p_{2x}\cos\theta + p_{2y}\sin\theta) \\
&\qquad \times(\sqrt{\omega_2^2+m^2}+m)\} \\
&\quad + \frac{1}{D_3^{(-)}}\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}+m)(\sqrt{\omega_2^2+m^2}-m) \\
&\qquad - 2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta + 2F_1^2F_2p_{2x}\omega_1 \\
&\qquad \times(\sqrt{\omega_2^2+m^2}+m) - 2F_1^2F_2\omega_2 \\
&\qquad \times(p_{2x}\cos\theta + p_{2y}\sin\theta)(\sqrt{\omega_1^2+m^2}+m)\}\Bigg].
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&|M^-(BB')|^2 \\
&= \frac{(4\pi e^2)^2}{4F_1F_2}\Bigg[\frac{1}{D_1^{(-)}}\{F_1^2F_2^2(\omega_1^2+2m^2+2m\sqrt{\omega_1^2+m^2}) \\
&\qquad + F_1^2\omega_1^2\vec{p}_2^{\,2} + 2F_1^2F_2\omega_1' \\
&\qquad \times(p_{2x}\cos 2\theta + p_{2y}\sin 2\theta)(\sqrt{\omega_1^2+m^2}+m)\} \\
&\quad + \frac{1}{D_2^{(-)}}\{F_1^2F_2^2(\omega_2^2+2m^2+2m\sqrt{\omega_2^2+m^2}) + F_1^2\omega_2^2\vec{p}_2^{\,2} \\
&\qquad - 2F_1^2F_2\omega_2(p_{2x}\cos\theta - p_{2y}\sin\theta) \\
&\qquad \times(\sqrt{\omega_2^2+m^2}+m)\} \\
&\quad + \frac{1}{D_3^{(-)}}\{2F_1^2F_2^2(\sqrt{\omega_1^2+m^2}+m)(\sqrt{\omega_2^2+m^2}+m) \\
&\qquad \times\cos 2\theta - 2F_1^2\vec{p}_2^{\,2}\omega_1\omega_2\cos\theta + 2F_1^2F_2\omega_1 \\
&\qquad \times(\sqrt{\omega_2^2+m^2}+m)p_{2x} - 2F_1^2F_2\omega_2 \\
&\qquad \times(\sqrt{\omega_1^2+m^2}+m)(p_{2x}\cos\theta + p_{2y}\sin\theta)\}\Bigg].
\end{aligned} \tag{17}
$$

The differential cross-sections $d\sigma$ are given by $2\pi\,|M|^2 \times$ (density of states). After substituting for p_2^2, etc., in terms of ω_1 and ω_2 from the following relations we may compare the cross-sections for positive and negative intermediate energies.

$$F_2 = 2m + \omega_1 - \omega_2 \quad \text{(from energy conservation)}$$

$$\vec{p}_2^{\,2} = E_2^2 - m^2 = (2m + \omega_1 - \omega_2)(\omega_1 - \omega_2)$$

$$F_1^2 = 4m^2$$

$$p_{2x} = \omega_1 - \omega_2 \cos\theta \quad \text{(from momentum conservation)}$$

$$p_{2y} = -\,\omega_2 \sin\theta.$$

Discussion

Since the expressions are rather complicated, it is difficult to compare the cross-sections for the positive and negative energy intermediate states directly. However we shall take up some special cases to get an idea of the relative contributions:

(1) Non-relativistic case, *i.e.*, $\omega_1 \ll m$ and $\omega_1 \sim \omega_2$. In the case of positive energy intermediate states

$$d\sigma^+(AA') = \frac{e^4}{16m^4}\Big[(-2\omega_1)(\omega_1 - \omega_2\cos\theta)\left(1 + \frac{\omega_1}{m}\right) + (2\omega_2)(\omega_2 - \omega_1\cos\theta)\left(1 - \frac{\omega_2}{m}\right) - 2\omega_2(\omega_2\cos\theta - \omega_1) - 2\omega_1^2\cos\theta\Big]$$

(neglecting terms of order ω_1^2/m^2)

$$\simeq \frac{e^4}{16m^2}\frac{\omega_1^2}{m^2} \sim 0. \tag{18}$$

Similarly

$$d\sigma^+(BB' \text{ or } AB' \text{ or } BA') \sim 0.$$

Now we shall calculate the cross-section for the intermediate negative energies

$$d\sigma^-(\mathrm{AA}')$$

$$= \frac{e^4}{16m^2}\left[\frac{1}{4m^4\left(1+\frac{\omega_1}{m}\right)}\left\{2m\left[2m(4m^2+2\omega_1^2)\right.\right.\right.$$

$$\left.\left.+\,2\omega_1\left(2m+\frac{\omega_1^2}{2m}\right)(\omega_1-\omega_2\cos\theta)\right]\right\}$$

$$+\frac{1}{4m^4\left(1-\frac{\omega_2}{m}\right)}\left\{2m\left[2m(4m^2+2\omega_2^2)\right.\right.$$

$$\left.\left.+\,2\omega_2\left(2m+\frac{\omega_2^2}{2m}\right)(\omega_2-\omega_1\cos\theta)\right]\right\}$$

$$+\frac{1}{4m^4}\left\{2m\left[4m\left(2m+\frac{\omega_1^2}{2m}\right)\left(2m+\frac{\omega_1^2}{2m}\right)\right.\right.$$

$$+\,2(\omega_1-\omega_2\cos\theta)\,\omega_1\left(2m+\frac{\omega_1^2}{2m}\right)$$

$$\left.\left.\left.+\,2(\omega_2-\omega_1\cos\theta)\,\omega_2\left(2m+\frac{\omega_2}{2m}\right)\right]\right\}\right].$$

Neglecting ω_1^2/m^2, we have

$$d\sigma^-(\mathrm{AA}') \simeq \frac{e^4}{16m^2}\left[\frac{16m^4+16m^4+32m^4}{4m^4}\right]$$

$$\simeq \frac{e^4}{m^2}. \tag{19}$$

Similarly

$$d\sigma^-(\mathrm{BB}') = \frac{e^4}{m^2}(\cos^2\theta) \tag{20}$$

$$d\sigma^-(\mathrm{AB}' \text{ or } \mathrm{BA}') = 0. \tag{21}$$

(ii) Extreme relativistic case (*a*) small angles, *i.e.*, $\omega_1 \sim \omega_2 \gg m$. The results for different polarization are tabulated below, (*b*) large angles $\omega_1 \gg m \sim \omega_2$. In this case the results are similar to the previous ones excepting that here

$$d\sigma(\mathrm{AB}' \text{ or } \mathrm{BA}') = d\sigma(\mathrm{BB}') \neq d\sigma(\mathrm{AA}').$$

This is to be expected since $e_1 \cdot e_2$ for BB$' \sim 0$ since $\cos\theta \sim 0$.

Differential cross-sections for Compton effect

Energy / Polarization	Non-relativistic ($\omega_1 \sim \omega_2 \ll m$)		Extreme relativistic ($\omega_1 \gg m$; $\omega_1 \sim \omega_2$) small angles	
	Positive Intermediate Energy	Negative Intermediate Energy	Positive Intermediate Energy	Negative Intermediate Energy
AA′	$\frac{e^4}{m^2}\left(\frac{\omega_1^2}{m^2}\right) \sim 0$	$\frac{e^4}{m^2}$	$\frac{e^4}{4m^2}\left(1 - \frac{m}{\omega_1} - \frac{m}{\omega_2}\right)$	$\frac{e^4}{4m^2}\left(1 + \frac{m}{\omega_1} + \frac{m}{\omega_2}\right)$
BB′	Do.	$\frac{e^4}{m^2}\cos^2\theta$	$\frac{e^4}{4m^2}\left(1 - \frac{m}{\omega_2} - \frac{m}{\omega_1}\cos 2\theta\right)$	$\frac{e^4}{4m^2}\left(1 + \frac{m}{\omega_2} + \frac{m}{\omega_1}\cos 2\theta\right)$
AB′	Do.	$\frac{e^4}{m^2}\left(\frac{\omega_1^2}{m^2}\right) \sim 0$	$\frac{e^4}{4m^2}$	$\frac{e^4}{4m^2}$
BA′	Do.	Do.	Do.	Do.

[*cf. Quantum Theory of Radiation*, by W. Heitler, (1954)].

It is interesting to note that the contributions arising from the negative energy intermediate states dominates the cross-sections in the non-relativistic limit for the electron. In fact the contribution from positive energy intermediate states is almost zero so that

$$\tfrac{1}{2}\, d\sigma^{-}(\mathrm{AA}' + \mathrm{BB}') = \tfrac{1}{2}\frac{e^4}{m^2}(1 + \cos^2\theta) = \tfrac{1}{2}\, d\sigma\,(\mathrm{AA}' + \mathrm{BB}')$$

$$\text{since } d\sigma^{+} = 0$$

which is the actual cross-section (*i.e.*, including both positive and negative energy intermediate states).

In the extreme relativistic case, we find that contributions arise from both positive and negative energy intermediate states. We also find that in case (*a*) $d\sigma^{+}(\mathrm{AB}') = d\sigma^{-}(\mathrm{AB}')$ though $d\sigma\,(\mathrm{AB}'$ or $\mathrm{BA}')$ is zero. This can be easily seen from the fact $\mathrm{M}^{+}(\mathrm{AB}') = -\,\mathrm{M}^{-}(\mathrm{AB}')$ and hence $|\mathrm{M}^{+} + \mathrm{M}^{-}|^2 = 0$. Similarly for $d\sigma^{+}(\mathrm{AA}') + d\sigma^{-}(\mathrm{AA}') = e^4/2m^2$ though $d\sigma\,(\mathrm{AA}') = e^4/m^2$. This is again obvious since in this case $\mathrm{M}^{+}(\mathrm{AA}') \simeq +\,\mathrm{M}^{-}(\mathrm{AA}')$ so that $d\sigma = |\mathrm{M}^{+} - \mathrm{M}^{-}|^2 = |2\mathrm{M}^{+}(\mathrm{AA}')|^2 = e^4/m^2$. Calculations on other electrodynamic processes will be given in a later contribution.

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