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Some applications of seesaw duality to branching laws

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The aim of this paper is to give some applications of seesaw duality in the theory of Weil representations to obtain the decomposition of certain representations of a group over a non-Archimedean local field when restricted to a subgroup. We obtain some results on triple product for $GL(2)$ from the theorem of Tunnell [Tu] on the restriction of a representation of $GL(2)$ to a torus corresponding to a quadratic field extension, and an extension of this result proved here. The method of this paper yields sharper results than were obtained in [P1] but are applicable to only certain representations of $GL(2)$, and we are able to say nothing about the corresponding theorems about representations of the invertible elements of the quaternion division algebra. We also use seesaw duality to give branching laws for the decomposition of certain representations of $SO(5)$ over a local field when restricted to $SO(4)$. Here too we are reduced to the smaller pair $(SO(4), SO(3))$. In the final section we reformulate, using seesaw duality, the well known results about theta liftings from two dimensional quadratic spaces to $SL(2)$ to get the tensor product of the Weil representation of (the two fold cover) of $SL(2)$ with any representation of $SL(2)$, and conjecture a similar result for general symplectic groups.

The results obtained in this paper give further evidence to the important role played by the epsilon factors in branching laws involving multiplicity 1 situations, cf. [Tu], [P1], [P2], [P3] for earlier results, and the paper [GP] for some general conjectures.

We now state more precisely the results obtained in this paper. We first recall that the following theorem was proved in [P1]. In this theorem, and in the rest of the paper, we will write the epsilon factor associated to a finite dimensional representation σ of the Weil-Deligne group W_k' of a local field k and a non-trivial additive character ψ of k as $\varepsilon(\sigma, \psi)$. If the determinant of

σ is trivial, then $\varepsilon(\sigma, \psi)$ does not depend on ψ, and we write it simply as $\varepsilon(\sigma)$.

Theorem 1. *Let π_1, π_2, π_3 be three irreducible, admissible, infinite dimensional representations of $GL(2,k)$ for a local field k such that the product of their central characters is trivial. Then the space of $GL(2,k)$-invariant linear forms $\ell : \pi_1 \otimes \pi_2 \otimes \pi_3 \to \mathbf{C}$ is atmost one dimensional, and it is non-zero if and only if $\varepsilon(\sigma_{\pi_1} \otimes \sigma_{\pi_2} \otimes \sigma_{\pi_3}) = 1$ where σ_{π_i} is the two dimensional representation of the Weil-Deligne group W_k' of k associated to the representation π_i of $GL(2,k)$.*

Assume that the representations π_1 and π_2 of $GL(2,k)$ come from characters χ_1 and χ_2 of a quadratic extension K of k. Let $GL(2,k)^+$ be the subgroup of index 2 of $GL(2,k)$ consisting of those elements of $GL(2,k)$ whose determinant is a norm from K^*. The representations π_1 and π_2 decompose into two irreducible components when restricted to $GL(2,k)^+$ which are permuted by any element of $GL(2,k)$ which does not lie in $GL(2,k)^+$. Write

$$\pi_1 = \pi_1^+ \oplus \pi_1^-$$
$$\pi_2 = \pi_2^+ \oplus \pi_2^- .$$

(Once π_1 has been written as $\pi_1 = \pi_1^+ \oplus \pi_1^-$, there is a natural way of writing the decomposition of π_2 as $\pi_2 = \pi_2^+ \oplus \pi_2^-$; this is by requiring π_2^+ to have Whittaker model for the same characters for which π_1^+ has a Whittaker model; the results below do not depend upon the initial indexing of the two components of π_1 as π_1^+ and π_1^-.)

Therefore,

$$\pi_1 \otimes \pi_2 = (\pi_1^+ \otimes \pi_2^+ \oplus \pi_1^- \otimes \pi_2^-) \oplus (\pi_1^+ \otimes \pi_2^- \oplus \pi_1^- \otimes \pi_2^+) .$$

Clearly this is a decomposition of $G[SL(2) \times SL(2)]$-modules where

$$G[SL(2) \times SL(2)] = \{(g_1, g_2) \mid g_1, g_2 \in GL(2), \text{and} \det g_1 = \det g_2\} .$$

Since the space of $GL(2)$-invariant forms in $\pi_1 \otimes \pi_2 \otimes \pi_3$ is atmost one dimensional, only one of $[\pi_1^+ \otimes \pi_2^+ \oplus \pi_1^- \otimes \pi_2^-] \otimes \pi_3$ and $[\pi_1^+ \otimes \pi_2^- \oplus \pi_1^- \otimes \pi_2^+] \otimes \pi_3$ will have a $GL(2)$-invariant form. The next theorem refines Theorem 1 to decide which of these two representations has a $GL(2)$-invariant form. Before coming to this refinement, we observe that the 4-dimensional representation $\sigma_{\pi_1} \otimes \sigma_{\pi_2}$ of the Weil-Deligne group decomposes as a sum of two 2-dimensional representations. Let the two dimensional representations associated to π_1 and π_2 be $\mathrm{Ind}_{K^*}^{W_k} \chi_1$ and $\mathrm{Ind}_{K^*}^{W_k} \chi_2$ respectively. Then

$$\begin{aligned} \sigma_{\pi_1} \otimes \sigma_{\pi_2} &= (\mathrm{Ind}_{K^*}^{W_k} \chi_1) \otimes (\mathrm{Ind}_{K^*}^{W_k} \chi_2) \\ &= \mathrm{Ind}_{K^*}^{W_k} (\chi_1 \chi_2) \oplus \mathrm{Ind}_{K^*}^{W_k} (\chi_1 \bar{\chi}_2) . \end{aligned}$$

We will be using ω_π to denote the central character of an irreducible admissible representation π of $GL(2,k)$ throughout this paper.

Theorem 2. *The representation* $[\pi_1^+ \otimes \pi_2^+ \oplus \pi_1^- \otimes \pi_2^-] \otimes \pi_3$ *has a* $GL(2)$-*invariant form if and only if* $\varepsilon\big[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\chi_2)\big] = \omega_{\pi_3}(-1)$ *and* $\varepsilon\big[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\bar{\chi}_2)\big] = \omega_{\pi_3}(-1)$, *and representation* $[\pi_1^+ \otimes \pi_2^- \oplus \pi_1^- \otimes \pi_2^+] \otimes \pi_3$ *has a* $GL(2)$-*invariant form if and only if* $\varepsilon\big[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\chi_2)\big] = -\omega_{\pi_3}(-1)$ *and* $\varepsilon\big[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\bar{\chi}_2)\big] = -\omega_{\pi_3}(-1)$.

Remark. The two dimensional representation $\mathrm{Ind}_{K^*}^{W_k}\chi$ can also be written as $\mathrm{Ind}_{K^*}^{W_k}\bar{\chi}$ but clearly this does not cause any ambiguity in the above theorem as $\varepsilon\big[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\chi_2)\big]$ and $\varepsilon\big[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\bar{\chi}_2)\big]$ have the same value. However, if one wanted to formulate an analogous statement for the quaternion division algebra, where these two epsilon factors will have opposite value, this ambiguity will have to be fixed directly in terms of the decomposition of $\pi = \pi^+ \oplus \pi^-$, where the ordering of the two components of π without the recourse to Whittaker model, is also ambiguous. One of the possibilities for fixing this is to use the refinement of Tunnell's theorem proved in [P3] where the characters of K^* appearing in π^+ and π^- are described in terms of epsilon factors.

The next refinement of Theorem 1 is when $\pi_1 = \pi_2 = \pi$. In this case

$$\pi \otimes \pi = \mathrm{Sym}^2\pi \oplus \overset{2}{\wedge}\pi\,.$$

Similarly,

$$\sigma_\pi \otimes \sigma_\pi = \mathrm{Sym}^2\sigma_\pi \oplus \overset{2}{\wedge}\sigma_\pi\,.$$

Theorem 3. *For infinite dimensional irreducible admissible representations* π, π' *of* $GL(2)$ *such that* $\omega_\pi^2 \cdot \omega_{\pi'} = 1$, $\mathrm{Sym}^2\pi \otimes \pi'$ *has a* $GL(2)$ *invariant form if and only if* $\varepsilon(\mathrm{Sym}^2\sigma_\pi \otimes \sigma_{\pi'}) = \omega_\pi(-1)$ *and* $\varepsilon(\overset{2}{\wedge}\sigma_\pi \otimes \sigma_{\pi'}) = \omega_\pi(-1)$, *and* $\overset{2}{\wedge}\pi \otimes \pi'$ *has a* $GL(2)$-*invariant linear form if and only if* $\varepsilon(\mathrm{Sym}^2\sigma_\pi \otimes \sigma_{\pi'}) = -\omega_\pi(-1)$ *and* $\varepsilon(\overset{2}{\wedge}\sigma_\pi \otimes \sigma_{\pi'}) = -\omega_\pi(-1)$.

The proof of this theorem will in fact depend on a refinement of Tunnell's theorem to which we now turn.

Let π be an irreducible admissible representation of D^* where D is a quaternion algebra over k, and let K be a separable quadratic algebra over k (so K could be $k \oplus k$) which is contained in D. Let NK^* denote the normaliser of K^* in D^*. Let ρ be a character of k^*. Assume that the central character, ω_π, of π satisfies $\omega_\pi(x) = \rho(x^2)$ for $x \in k^*$. Tunnell's theorem [Tu] gives conditions under which the representation $\rho \circ \det$ of K^* appears in π. We extend this theorem of Tunnell to give conditions under which the representation $\rho \circ \det$ of NK^* appears in π.

Theorem 4. *Let* π *be an irreducible admissible representation of* D^* *where* D *is a quaternion algebra over a local field* k, *and assume that* π *is infinite dimensional if* D *is a matrix algebra. Then the representation* $\rho \circ \det$ *of* NK^* *appears in* π *if and only if* $\varepsilon(\sigma_\pi \otimes \rho^{-1}) = \varepsilon_D$ *and* $\varepsilon(\sigma_\pi \otimes \rho^{-1} \otimes \omega_{K/k}) = \omega_{K/k}(-1)$ *where* $\varepsilon_D = 1$ *if* D *is the matrix algebra, and* -1 *if* D *is a division algebra,* *and $\omega_{K/k}$ is the quadratic character of k^* associated to the quadratic algebra K (so $\omega_{K/k} \equiv 1$ if $K = k \oplus k$).*

We now turn to the decomposition of a representation of $SO(5)$ when restricted to $SO(4)$. It will, however, be simpler to consider instead the decomposition of a representation of $GSp(4)$ when restricted to $G(SL(2) \times SL(2)) = \{(x, y) \in GL(2) \times GL(2) : \det(x) = \det(y)\}$, and this is what we shall do.

For representations V_1, V_2 of groups G_1, G_2, we let $V_1 \otimes V_2$ denote the tensor product representation of $G_1 \times G_2$. In the following theorem, the notion of L-packets is taken from Vigneras [Vi]; see section 5 for more details.

Theorem 5. *Let $\{\pi\}$ be a generic L-packet on $GSp(4)$ over a local field k with Langlands parameter $\sigma(\pi) : W_k' \to GSp(4, \mathbb{C})$. Let $\tau = \tau_1 \otimes \tau_2$ be an irreducible representation of $GL(2) \times GL(2)$ with both τ_1 and τ_2 infinite dimensional, and with Langlands parameters $\sigma(\tau_1), \sigma(\tau_2) : W_k' \to GL(2, \mathbb{C})$. Assume that the similitude factor associated to $\sigma(\pi) : W_k' \to GSp(4, \mathbb{C})$ is the product $\det \sigma(\tau_1) \cdot \det \sigma(\tau_2)$. Then there exists atmost one representation, say π, in the L-packet $\{\pi\}$ and atmost one irreducible representation τ' of $G(SL(2) \times SL(2)) = \{(x, y) \in GL(2) \times GL(2) : \det(x) = \det(y)\}$ appearing in $\tau_1 \otimes \tau_2$ such that*

$$\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi, \tau'] \neq 0 .$$

Moreover, such a pair (π, τ') exists if and only if $\varepsilon(\sigma(\pi) \otimes \sigma(\tau_1)^* \otimes \sigma(\tau_2)^*) = 1$.

There is an analogous branching law to the subgroup $GL(2, K)^\#$ of $GSp(4, k)$ where K is a separable quadratic extension of k with Galois automorphism $x \to \bar{x}$, and $GL(2, K)^\# = \{g \in GL(2, K) | \det g \in k^*\}$. Recall that for a representation τ of $GL(2, K)$ with Langlands parameter $\sigma(\tau) : W_K' \to GL(2, \mathbb{C})$, we have constructed in [P2] a 4-dimensional representation of W_k' by a process which we called *multiplicative induction* there, and denoted by $M_K^k \sigma(\tau)$; the restriction of $M_K^k \sigma(\tau)$ to W_K' is $\sigma(\tau) \otimes \sigma(\bar{\tau})$ where $\bar{\tau}$ is the representation of $GL(2, K)$ obtained from τ by applying the Galois automorphism of $GL(2, K)$.

Theorem 6. *Let $\{\pi\}$ be a generic L-packet on $GSp(4)$ over a local field k with Langlands parameter $\sigma(\pi) : W_k' \to GSp(4, \mathbb{C})$. Let τ be an infinite dimensional irreducible representation of $GL(2, K)^\#$ with Langlands parameter $\sigma(\tau) : W_K' \to GL(2, \mathbb{C})$. Assume that the similitude factor associated to $\sigma(\pi) : W_k' \to GSp(4, \mathbb{C})$ is the product $\det \sigma(\tau) \cdot \det \sigma(\bar{\tau})$. Then there exists atmost one representation, say π, in the L-packet $\{\pi\}$ and atmost one irreducible representation τ' of $GL(2, K)^\#$ appearing in τ such that*

$$\mathrm{Hom}_{GL(2,K)^\#}[\pi, \tau'] \neq 0 .$$

Moreover, such a pair (π, τ') exists if and only if $\varepsilon(\sigma(\pi) \otimes M_K^k \sigma(\tau)^*) = 1$.

2

We summarize the Howe duality correspondence and the seesaw duality for similitude groups in this section.

Let (V, q) be a finite dimensional vector space V over k together with a quadratic form q on it. Define

$$GO(V) = \{g \in GL(V) | q(gv) = \lambda_g q(v) \text{ for some } \lambda_g \in k^*\} ;$$

λ_g is called the similitude factor associated to g. Similarly, for a finite dimensional symplectic space W, define $GSp(W)$ and the notion of similitude factor for elements of this group.

Let G be a subgroup of $GO(V)$ containing $O(V)$, and let H be a subgroup of $GSp(W)$ containing $Sp(W)$. Assume that $\lambda_G = \lambda_H$ where λ_G is the subgroup of k^* defined by

$$\lambda_G = \{x \in k^* | x = \lambda_g \text{ for some } g \in G\} ,$$

and λ_H is defined similarly.

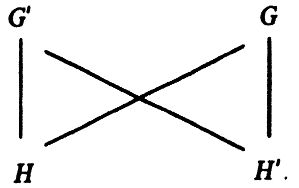
Define $\mathscr{R}(G \times H)$, a subgroup of $G \times H$, by

$$\mathscr{R}(G \times H) = \{(g, h) \in G \times H | \lambda_g \cdot \lambda_h = 1\} .$$

There is a natural map of $\mathscr{R}(G \times H)$ into $Sp(V \otimes W)$ over which the metaplectic covering of the symplectic group splits. The Weil representation ω of $Sp(V \otimes W)$ will be considered to be a representation of $\mathscr{R}(G \times H)$ via this splitting. There is an explicit description of this representation of $\mathscr{R}(G \times H)$, cf. [H-K, 5.1.5], in terms of the action of $Sp(W)$, and the natural action of $GO(V)$ on functions on $V \otimes W_1$ where W_1 is a maximal isotropic subspace of W; this description also serves to prove the splitting of the metaplectic cover restricted to $\mathscr{R}(G \times H)$.

Let π be an irreducible admissible representation of H. Via the natural surjection $\mathscr{R}(G \times H) \to H$, π can be thought of as a representation of $\mathscr{R}(G \times H)$. It follows that $\mathrm{Hom}(\omega, \pi)$ is a representation space for $\mathscr{R}(G \times H)$, and the space of $Sp(W)$-invariant vectors in it, i.e., $\mathrm{Hom}_{Sp(W)}(\omega, \pi)$ is a representation space for $\mathscr{R}(G \times H)/Sp(W) = G$. This representation space is not a smooth G-module, and is in fact the algebraic dual $\theta_0(\pi)^*$ of a smooth (admissible) G-module $\theta_0(\pi)$. When k is of odd residue characteristic and $\lambda_G = \lambda_H = \{e\}$, $\theta_0(\pi)$ has a unique irreducible quotient by a theorem of Waldspurger, and the case of general similitude groups can be reduced to this case (cf. [R]) in many cases to prove that $\theta_0(\pi)$ has a unique irreducible quotient. We will denote this irreducible quotient of $\theta_0(\pi)$ by $\theta(\pi)$; the representation $\theta(\pi)$ is called the Howe lift of π. If $\pi = \sum_\alpha \pi_\alpha$ is a sum of irreducible representations of H, then define $\theta(\pi) = \sum_\alpha \theta(\pi_\alpha)$.

A pair (G, H) and (G', H') of dual reductive pairs in a symplectic similitude group is called a seesaw pair if $H \subset G'$ and $H' \subset G$. Under such a condition $\lambda_G = \lambda_H = \lambda_{G'} = \lambda_{H'}$. Such a pair (of dual pairs) is usually pictorially depicted by the following diagram where vertical arrows denote inclusion, and slanted arrows connect members of dual reductive pairs



With the notation as above, here is the basic lemma about seesaw pairs.

Lemma. *For a seesaw pair of dual reductive pairs* (G,H) *and* (G',H') *as above, let* π *be an irreducible representation of* H, *and* π' *of* H', *then we have the following isomorphism:*

$$\mathrm{Hom}_H[\theta_0(\pi'),\pi] \cong \mathrm{Hom}_{H'}[\theta_0(\pi),\pi'] \,.$$

Proof. For any smooth representation V of an ℓ-group, let V^* denote the algebraic dual of V, and let $V^\vee$ denote the smooth dual of V. The context will always make clear the underlying ℓ-group with respect to which the smooth dual is being taken.

Let ω be the Weil representation of the big symplectic group in which both the pairs (G,H) and (G',H') are contained. Let H_1' denote the subgroup of H' where the similitude factor is 1. We have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{R}(H\times H')}(\pi^\vee \otimes \pi'^\vee,\omega^*) &\cong \mathrm{Hom}_H(\pi^\vee, \mathrm{Hom}_{H_1'}(\pi'^\vee,\omega^*))\\ &\cong \mathrm{Hom}_H(\pi^\vee,\theta_0(\pi')^*)\\ &\cong \mathrm{Hom}_H(\theta_0(\pi'),\pi)\,.\end{aligned}$$

Similarly,

$$\mathrm{Hom}_{\mathscr{R}(H\times H')}(\pi^\vee \otimes \pi'^\vee,\omega^*) \cong \mathrm{Hom}_{H'}(\theta_0(\pi),\pi')\,,$$

proving the lemma.

Remarks. 1. Let (G_1,H_1) and (G_2,H_2) be two dual reductive pairs in a symplectic similitude group. Assume that $G_1 \subseteq G_2$ and the isometry groups associated to G_1 and G_2 are the same, and similarly $H_1 \subseteq H_2$ and the isometry groups associated to H_1 and H_2 are the same. Then it is clear from the definitions that for a representation π of G_2, the restriction to H_1 of the Howe lift of π to H_2 is the same as the Howe lift to H_1 of the restriction of π to G_1.

2. If V is a 2 dimensional orthogonal space with the quadratic form the norm form of a quadratic field K, and $G = GO(V)$ then $\lambda_G = \{x \in k^* | \omega_{K/k}(x) = 1\}$. If W is a 2 dimensional symplectic space, then we take H to be $GL(2)^+ = \{g \in GL(2) | \omega_{K/k}(\det g) = 1\}$. In this case Howe duality correspondence gives a 1-1 correspondence between irreducible admissible representations of $GO(V)$ and certain representations of $GL(2)^+$. These representations of $GL(2)^+$ remain irreducible when induced to $GL(2)$, and it is

customary to call the induced representation of $GL(2)$ also to be the Howe lift of the representation of $GO(V)$.

3. In the basic seesaw duality lemma above, it is the representation $\theta_0(\pi)$ and not the irreducible representation $\theta(\pi)$ which appears, and makes the method of seesaw pairs restrictive. By Kudla [K2], $\theta_0(\pi) = \theta(\pi)$ if π is supercuspidal, and in many other cases $\theta_0(\pi) = \theta(\pi)$ as for example when lifting an infinite dimensional representation of $GL(2)$ which is not special to $GO(4)$. *Our applications of seesaw will deal only in the situations that we already know $\theta_0(\pi) = \theta(\pi)$ except in one situation where we are able to deduce this equality from branching laws arising out of seesaw duality.*

3

We will prove Theorem 2 in this section. Let K be a separable quadratic field extension of a local field k with the Galois automorphism $x \longrightarrow \bar{x}$. Let $D = K \oplus Kj$ with $\mathrm{Nm}(j) = -j^2 = a$, and $jxj^{-1} = \bar{x}$. Then D is a division algebra if and only if $\omega_{K/k}(-a) = -1$. Define, $\varepsilon_D = 1$ if D is a matrix algebra, and $\varepsilon_D = -1$ if D is a division algebra. Therefore $\varepsilon_D = \omega_{K/k}(-a)$.

For a general quadratic space V with quadratic form q on it, recall that $GO(V)$ denotes the group $GO(V) = \{g \in \mathrm{Aut}\, V \,|\, q(gv) = \lambda_g q(v)$ for all $v \in V\}$. The mapping $g \mapsto \lambda_g$ on $GO(V)$ is a group homomorphism whose kernel is $O(V)$. Clearly $(\det g)^2 = \lambda_g^n$ where $n = \dim V$. If n is even, define $GSO(V)$ to be the subgroup of $GO(V)$ with $\det g = \lambda_g^{\frac{n}{2}}$. For quadratic spaces (q_1, V_1) and (q_2, V_2), let $G[O(V_1) \times O(V_2)]$ denote the subgroup of $GO(V_1) \times GO(V_2)$ consisting of (g_1, g_2) with $\lambda g_1 = \lambda g_2$. Clearly $G[O(V_1) \times O(V_2)] \hookrightarrow GO(V_1 \oplus V_2)$. Define $G[SO(V_1) \times SO(V_2)]$ to be $G[O(V_1) \times O(V_2)] \cap [GSO(V_1) \times GSO(V_2)]$.

If D^* is the group of invertible elements of D, there is a homomorphism from $[D^* \times D^*]/\Delta k^*$ to $\mathrm{Aut}(D)$ given by $(g_1, g_2)X = g_1 X g_2^{-1}$ for $g_1, g_2 \in D^*$, and $X \in D$. This gives an isomorphism of $[D^* \times D^*]/\Delta k^*$ with $GSO(D)$.

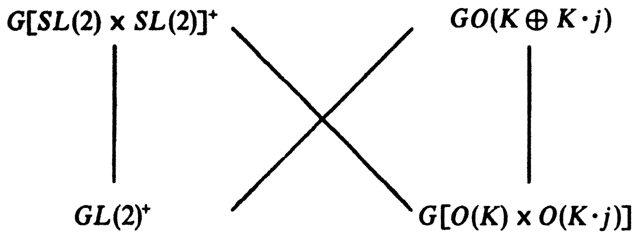
Both $GSO(K)$ and $GSO(K \cdot j)$ are isomorphic to K^*, and since any element of K^1 is of the form $y/\bar{y}$ for $y \in K^*$, an arbitrary element of $G[SO(K) \times SO(K \cdot j)]$ is of the form $(x \cdot y, x \cdot \bar{y})$.

We have the inclusion of $G[SO(K) \times SO(K \cdot j)]$ into $D^* \times D^*/\Delta k^*$ given by

$$G[SO(K) \times SO(K \cdot j)] \hookrightarrow GSO(K \oplus Kj) = GSO(D) \cong D^* \times D^*/\Delta k^* .$$

It can be seen that under this inclusion, the element $(xy, x\bar{y})$ of $G[SO(K) \times SO(K \cdot j)]$ goes to the element (x, y^{-1}) of $[D^* \times D^*]/\Delta k^*$. If a character of the group $G[SO(K) \times SO(K \cdot j)]$ (thought of as a subgroup of $K^* \times K^*$) is given by a pair of characters (χ_1, χ_2) of K^*, then it corresponds to the character $(\chi_1\chi_2, (\chi_1\bar{\chi}_2)^{-1})$ of $K^* \times K^*$ contained in $[D^* \times D^*]/\Delta k^*$.

We will now use the following seesaw diagram



where $GL(2)^+ = \{g \in GL(2) | \omega_{K/k}(\det g) = 1\}$, and $G[SL(2) \times SL(2)]^+ = \{(g_1, g_2) \in GL(2)^+ \times GL(2)^+ | \det g_1 = \det g_2\}$.

It is known from [J-L] that for a representation π_3 of $GL(2)$, its Howe lift to $GO(D)$ remains irreducible when restricted to $GSO(D)$, and under the isomorphism $GSO(D) \cong D^* \times D^*/\Delta k^*$, it is $\pi_3 \otimes \pi_3^*$ if $D = GL(2)$, and is $\pi_3' \otimes \pi_3'^*$ if D is the quaternion division algebra and π_3' is the representation of D^* associated to the representation π_3 of $GL(2)$ by the Jacquet-Langlands correspondence (so $\pi_3' = \{0\}$ if π_3 is not a discrete series representation). For a character χ of K^* such that $\chi(x) \neq \chi(\bar{x})$ for some $x \in K^*$, the character χ induces to give a two dimensional irreducible representation of $GO(K)$. The Howe lift of this two dimensional representation of $GO(K)$ to $GL(2,k)$ is the representation of $GL(2,k)$ associated to the character χ of K^*.

Let π_{χ_1,χ_2} be the representation of $G[SL(2) \times SL(2)]^+$ obtained by the Howe lift of the representation of $G[O(2) \times O(2)]$ which restricts to the character $\chi_1 \times \chi_2$ on $G[SO(2) \times SO(2)]$.

From seesaw duality and Frobenius reciprocity

$$\mathrm{Hom}_{GL(2)^+}[\pi_{\chi_1,\chi_2}, \pi_3^*]$$

$$\cong \mathrm{Hom}_{G[O(K)\times O(K\cdot j)]}\left[\pi_3^* \otimes \pi_3, \mathrm{Ind}_{G[SO(K)\times SO(K\cdot j)]}^{G[O(K)\times O(K\cdot j)]}(\chi_1 \times \chi_2)\right]$$

$$\cong \mathrm{Hom}_{G[SO(K)\times SO(K\cdot j)]}[\pi_3^* \otimes \pi_3,\ \chi_1 \times \chi_2]$$

$$\cong \mathrm{Hom}_{K^*}[\pi_3^*, \chi_1\chi_2] \otimes \mathrm{Hom}_{K^*}[\pi_3, \chi_1^{-1}\bar{\chi}_2^{-1}]\,.$$

Using Tunnell's theorem, it follows that $\mathrm{Hom}_{GL(2)^+}[\pi_{\chi_1,\chi_2}, \pi_3^*] \neq 0$ if and only if $\varepsilon\left[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\chi_2)\right] = \omega_{\pi_3}(-1) \cdot \omega_{K/k}(-1)\varepsilon_D$ and $\varepsilon\left[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\bar{\chi}_2)\right] = \omega_{\pi_3}(-1) \cdot \omega_{K/k}(-1)\varepsilon_D$. In the notation introduced before Theorem 2, it is clear from the construction of π_{χ_1,χ_2}, that $\pi_{\chi_1,\chi_2} = \pi_1^+ \otimes \pi_2^+$ if $a = 1$, and $\pi_{\chi_1,\chi_2} = \pi_1^+ \otimes \pi_2^-$ if $\omega_{K/k}(a) = -1$. From the relation $\varepsilon_D = \omega_{K/k}(-a)$, it follows that $\mathrm{Hom}_{GL(2)^+}[\pi_1^+ \otimes \pi_2^+, \pi_3^*] \neq 0$ if and only if $\varepsilon\left[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\chi_2)\right] = \omega_{\pi_3}(-1)$ and $\varepsilon\left[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\bar{\chi}_2)\right] = \omega_{\pi_3}(-1)$. Similarly, $\mathrm{Hom}_{GL(2)^+}[\pi_1^+ \otimes \pi_2^-, \pi_3^*] \neq 0$ if and only if $\varepsilon\left[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\chi_2)\right] = -\omega_{\pi_3}(-1)$ and $\varepsilon\left[\sigma_{\pi_3} \otimes \mathrm{Ind}_{K^*}^{W_k}(\chi_1\bar{\chi}_2)\right] = -\omega_{\pi_3}(-1)$. Since $\pi_1^+ \otimes \pi_2^+ \oplus \pi_1^- \otimes \pi_2^- = \mathrm{Ind}_{GL(2)^+}^{GL(2)}(\pi_1^+ \otimes \pi_2^+)$, and $\pi_1^+ \otimes \pi_2^- \oplus \pi_1^- \otimes \pi_2^+ = \mathrm{Ind}_{GL(2)^+}^{GL(2)}(\pi_1^+ \otimes \pi_2^-)$, another application of Frobenius reciprocity completes the proof of Theorem 2. Observe that even though we are able to obtain results here only for triple products of $GL(2)$, we need to use Tunnell's theorem both for $GL(2)$ and the quaternion division algebra.

4

We prove Theorem 4 in this section. It is easily seen that the theorem can be reduced to the case when the central character of π is trivial. We will assume this to be the case and take $\rho = 1$.

It is a theorem of Frohlich and Queyrut that for a character χ of L^* which is trivial on k^* where L is a quadratic extension of a local field k, and for a character ψ_L of L of the form $\psi_L(x) = \psi(\operatorname{tr} x)$ for ψ any additive character of k,

$$\varepsilon(\chi, \psi_L) = \chi(\Delta),$$

where Δ is any element of L^* whose trace to k is zero. We consider Theorem 4 as a generalisation of this theorem to the situation of quaternion algebras. In fact the proof of Deligne [D] generalises easily to this situation. We recall that Deligne proved the theorem of Frohlich and Queyrut using the local functional equation of Tate:

$$\frac{\int_{L^*} \hat{f}(x)\|x\|^{1-s}\chi^{-1}(x)\,d^*x}{L(\chi^{-1}, 1-s)} = \varepsilon(\chi, \psi_L, s)\,\frac{\int_{L^*} f(x)\|x\|^{s}\chi(x)\,d^*x}{L(\chi, s)}$$

where f is any compactly supported function on L with $\hat{f}$ its Fourier transform taken with respect to $\psi_L(x) = \psi(\operatorname{tr} x)$, $d^*x = \frac{dx}{\|x\|}$ where dx is the Haar measure on L self dual for the character ψ_L.

In our present situation D plays the role of L and K plays the role of k, and the local functional equation of Tate is replaced by the following functional equation of Jacquet-Langlands

$$\frac{\int_{D^*} \hat{f}(x)\|\det x\|^{\frac{3}{2}-s}\phi(x^{-1})\,d^*x}{L(\pi^*, 1-s)} = \varepsilon(\pi, \psi_D, s)\varepsilon_D\,\frac{\int_{D^*} f(x)\|\det x\|^{\frac{1}{2}+s}\phi(x)\,d^*x}{L(\pi, s)}\,.$$

Here $\varepsilon_D = +1$ if $D = M(2,k)$, and -1 if D is the quaternion division algebra, f is a compactly supported function on D, $\hat{f}$ its Fourier transform using the character $\psi_D(x) = \psi(\operatorname{tr}[x])$ and ϕ is a matrix coefficient of π which we take to be $\phi(x) = \langle xv_0, v_0\rangle$ where v_0 is the vector in π on which K^* operates trivially, $d^*x = \frac{dx}{\|\det x\|^2}$ where dx is the Haar measure on D self dual for the character ψ_D.

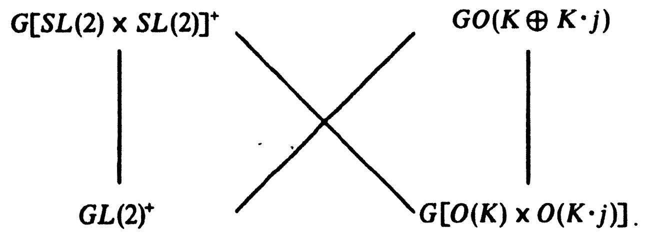
In the present proof the role of Δ in the theorem of Frohlich and Queyrut is played by the element $j \in D^*$ introduced in the last section such that $D = K \oplus K \cdot j$ is a direct sum of quadratic spaces. Clearly $\phi(xgy) = \phi(g)$ for all $x, y \in K^*$, and $\phi(j) = 1$ if and only if NK^* operates trivially on v_0. We omit to give complete translation of Deligne's proof to the present situation except pointing out that it works for any supercuspidal representation of $GL(2)$ and any representation of D^* where D is the quaternion division algebra, and yields that for a representation of D^* containing a vector v_0 on which K^* operates trivially, NK^* also operates trivially if and only if $\varepsilon(\pi) = \varepsilon_D$. Combining this with Tunnell's theorem according to which there exists a vector v_0 on which K^* operates trivially if and only if $\varepsilon(\pi) \cdot \varepsilon(\pi \otimes \omega_{K/k}) = \varepsilon_D \omega_{K/k}(-1)$, we get Theorem 4 for such representations.

If π is the principal series of $PGL(2)$ induced from the characters (χ, χ^{-1}), then $\varepsilon(\pi) = \varepsilon(\chi \oplus \chi^{-1}) = \chi(-1)$. From the decomposition $BK^* = GL(2)$ where B is any Borel subgroup of $GL(2)$, one can easily calculate the action of NK^* on the vector fixed by K^*, and we find that NK^* operates trivially on this vector if and only if $\chi(-1) = \varepsilon(\pi) = 1$. The case of special representation can also be treated in the same way.

Finally, if $K = k \oplus k$, then K^* is the diagonal subgroup of $GL(2)$. The linear form which is invariant under the diagonal subgroup is most conveniently described in the Kirillov model. If π is supercuspidal, then the Kirillov model consists of compactly supported functions on k which vanish at the origin. In this case $f \mapsto \int_{k^*} f(x) d^*x$ is the linear form invariant under the diagonal subgroup. The action of Weyl group is also given by a simple formula involving the epsilon factor, and this proves the theorem for supercuspidal representations. For principal series the theorem could be proved by looking directly at the action of K^* on G/B; we omit the details.

5

We prove Theorem 3 in this section. When the representation π is either a principal series or is a special representation, then the theorem can be easily proved using the orbit method to write the symmetric square of π in terms of explicit induced representation and applying Theorem 4; we omit the details of this. For supercuspidal representations, we will be able to prove the theorem only for representations π coming from a character χ of a quadratic field extension K of k. The proof in this case will again depend on the seesaw pair used in section 3, the notation of which we will continue to follow here:



We recall from section 3 that $D = K \oplus Kj$ with $\mathrm{Nm}(j) = -j^2 = a$, and as seen there, the isomorphism of $[D^* \times D^*]/\Delta k^*$ with $GSO(D)$ induces the isomorphism of $[K^* \times K^*]/\Delta k^*$ with $G[SO(K) \times SO(K \cdot j)]$, taking the character $\chi \times \chi$ of $G[SO(K) \times SO(K \cdot j)]$ to the character $[\chi^2, (\chi\bar{\chi})^{-1}]$ of $K^* \times K^* \subseteq$

$D^* \times D^*$. The automorphism $J_a : K \oplus Kj \to K \oplus Kj$ given by $x + yj \to ay + xj$ belongs to $GSO(D)$, and under the mapping $D^* \times D^* \to GSO(D)$, this element of $GSO(D)$ can be represented by $(x_0, j^{-1}x_0)$ where $x_0 \in K^*$ with $\mathrm{tr}(x_0) = 0$. Let $(K^* \times K^*)^e$ denote the group generated by $K^* \times K^*$ and $(x_0, j^{-1}x_0)$ inside $D^* \times D^*$, and let $[\chi^2 \times (\chi\bar{\chi})^{-1}]^e$ denote the extension of the character $[\chi^2 \times (\chi\bar{\chi})^{-1}]$ to $(K^* \times K^*)^e$ by declaring $[\chi^2 \times (\chi\bar{\chi})^{-1}]^e(x_0, j^{-1}x_0) = \chi(a)$.

Let $\pi = \pi^+ \oplus \pi^-$ as in the introduction, so π^+ and π^- are representations of $GL(2)^+$. Let $g_a = \begin{pmatrix} 1 & 0 \\ 0 & a^{-1} \end{pmatrix}$. Define $\pi^a = \pi^+$ if $\omega_{K/k}(a) = 1$, and $\pi^a = \pi^-$ if $\omega_{K/k}(a) = -1$. The action of g_a on π takes π^+ to π^a. The Howe lift of $(\chi \times \chi)$ (by which we mean the Howe lifts of all irreducible representations of $G(O(K) \times O(K \cdot j))$ which contain the restriction of the character $\chi \times \chi$ to $G(SO(K) \times SO(K \cdot j)))$ gives the representation $\pi^+ \otimes \pi^a$ of $G[SL(2) \times SL(2)]^+$.

By Frobenius reciprocity,

$$\mathrm{Hom}_{GL(2)}[\pi \otimes \pi, \pi'^*] = \mathrm{Hom}_{GL(2)^+}[\pi^+ \otimes \pi^+, \pi'^*] \oplus \mathrm{Hom}_{GL(2)^+}[\pi^+ \otimes \pi^-, \pi'^*]\,.$$

From this it can be checked that,

$$\mathrm{Hom}_{GL(2)}[\mathrm{Sym}^2(\pi), \pi'^*]$$
$$= \mathrm{Hom}_{GL(2)^+}[\mathrm{Sym}^2(\pi^+), \pi'^*] \oplus \mathrm{Hom}_{GL(2)^+ \times \sigma}[\pi^+ \otimes \pi^-, \pi'^*],$$

where σ is the involution of order 2 on $\pi^+ \otimes \pi^-$ commuting with $GL(2)^+$ action, given by $v_1 \otimes v_2 \to g_a^{-1} v_2 \otimes g_a v_1$ where $a \in k^*$ with $\omega_{K/k}(a) = -1$; if we use such a g_a with $\omega_{K/k}(a) = -1$ to identify π^+ to π^-, then this automorphism of $\pi^+ \otimes \pi^-$ becomes the automorphism $x \otimes y \to y \otimes x$ of $\pi^+ \otimes \pi^+$.

The Weil representation ω of the big symplectic group in which the seesaw pair under consideration is contained in, operates on $\mathcal{S}(K) \otimes \mathcal{S}(K)$. The element (g_a, J_a) belongs to this symplectic group (without similitude) and from the formula 5.1.5 in [H-K], it follows that (g_a, J_a) acts on $\mathcal{S}(K) \otimes \mathcal{S}(K)$ by $f_1 \otimes f_2 \to \chi(a) f_2 \otimes f_1$ (after the identification of π^+ to π^- by g_a where a will be a fixed element of k^* with $\omega_{K/k}(a) = -1$). Therefore the $\chi(a)$ eigenspace of this automorphism on $\mathcal{S}(K) \otimes \mathcal{S}(K)$ consists exactly of symmetric functions.

The natural action of (g_a, J_a) on $GL(2)^+ \times (K^* \times K^*)$ defines a semi-direct product which will, for typographical reason, be written as $[GL(2)^+ \times (K^* \times K^*)] \times (g_a, J_a)$. As π' is a representation of $GL(2)$, the representation $\pi' \otimes [\chi^2 \times (\chi\bar{\chi})^{-1}]$ of $GL(2)^+ \times (K^* \times K^*)$ has a natural extension to a representation of $[GL(2)^+ \times (K^* \times K^*)] \times (g_a, J_a)$ (using the action of J_a defined earlier) which will be denoted by $\pi' \otimes [\chi^2 \times (\chi\bar{\chi})^{-1}]^e$. Calculation of the $\pi' \otimes [\chi^2 \times (\chi\bar{\chi})^{-1}]^e$ isotypical representation of $\mathcal{R}(GL(2)^+ \times (K^* \times K^*)) \times (g_a, J_a)$ in ω, as in the seesaw duality lemma yields the following isomorphism:

$$\mathrm{Hom}_{(K^* \times K^*)^e}[\pi'^* \otimes \pi', [\chi^2 \times (\chi\bar{\chi})^{-1}]^e] = \mathrm{Hom}_{GL(2)^+ \times \{\sigma\}}[\pi^+ \otimes \pi^-, \pi'^*]\,.$$

Therefore the question of when π'^* appears in $\mathrm{Sym}^2(\pi)$ reduces to the question: when does the representation $[\chi^2 \times (\chi\bar{\chi})^{-1}]^e$ of $(K^* \times K^*)^e$ appear in $\pi'^* \otimes \pi'$.

By theorem 4, $j^{-1}x_0$ operates on the $(\chi\bar{\chi})^{-1}$ eigenspace in π' by $\chi(-ax_0^{-2})$ if and only if (recall that $\varepsilon_D = \omega_{K/k}(-a)$)

$$\varepsilon(\sigma_{\pi'} \otimes \chi|_{k^*}) = \omega_{K/k}(-a)$$
$$\varepsilon(\sigma_{\pi'} \otimes \chi|_{k^*} \otimes \omega_{K/k}) = \omega_{K/k}(-1)\,.$$

It follows that $(x_0, j^{-1}x_0)$ operates by $\chi(-a)$ on the $[\chi^2, (\chi\bar{\chi})^{-1}]$ eigenspace in $\pi'^* \otimes \pi'$ if and only if

$$\varepsilon(\sigma_{\pi'} \otimes \mathrm{Ind}_{K^*}^{W_k}\chi^2) = \omega_{K/k}(a)$$
$$\varepsilon(\sigma_{\pi'} \otimes \chi|_{k^*}) = \omega_{K/k}(-a)$$
$$\varepsilon(\sigma_{\pi'} \otimes \chi|_{k^*} \otimes \omega_{K/k}) = \omega_{K/k}(-1)\,,$$

and therefore, $(x_0, j^{-1}x_0)$ operates by $\chi(a)$ on the $[\chi^2, (\chi\bar{\chi})^{-1}]$ eigenspace in $\pi'^* \otimes \pi'$ if and only if

$$\varepsilon(\sigma_{\pi'} \otimes \mathrm{Ind}_{K^*}^{W_k}\chi^2) = \omega_{K/k}(a)$$
$$\varepsilon(\sigma_{\pi'} \otimes \chi|_{k^*}) = \chi(-1)\omega_{K/k}(-a)$$
$$\varepsilon(\sigma_{\pi'} \otimes \chi|_{k^*} \otimes \omega_{K/k}) = \chi(-1)\omega_{K/k}(-1)\,.$$

Since $\sigma_\pi = \mathrm{Ind}_{K^*}^{W_k}\chi$,

$$\mathrm{Sym}^2\sigma_\pi = \mathrm{Ind}_{K^*}^{W_k}(\chi^2) \oplus \chi|_{k^*}$$
$$\overset{2}{\wedge}\sigma_\pi = \omega_\pi = \chi|_{k*} \cdot \omega_{K/k}\,,$$

Theorem 4 follows.

6

We will prove theorem 5 in this section. We begin by summarising the work of Vigneras [Vi] on the Langlands parameter of discrete series representations of $GSp(4)$, structure of their L-packets, and the Howe lifting between $GO(4)$ and $GSp(4)$.

We recall that the L-group of $GSp(4)$ is $GSp(4, \mathbb{C})$. We let V^s denote a four dimensional quadratic space which is split, and let V^a denote a four dimensional quadratic space which is anisotropic. Since k is non-archimedean, V^a is unique up to isomorphism. We note that $GSO(V^s) \cong \{GL(2) \times GL(2)\}/\Delta(k^*)$, and similarly $GSO(V^a) \cong \{D^* \times D^*\}/\Delta(k^*)$ where D is the unique quaternion division algebra over k.

For a quadratic field extension K of k with the Galois automorphism $x \to \bar{x}$, let V be the 4-dimensional space $V = \{X \in M(2,K) | X = {}^t\bar{X}\}$ together with the determinant as the quadratic form (which takes values in k). For an element $g \in GL(2,K)$, we have the automorphism $X \to gX{}^t\bar{g}$ which lies in $GSO(V)$, and for the action of k^* on V by scaling, we have the isomorphism

$[GL(2,K) \times k^*]/K^* \cong GSO(V)$ where K^* is included as scalar matrices in $GL(2,K)$, and through the inverse of the norm in k^*.

From these descriptions of the orthogonal groups we find in particular that the L-packets of $GSO(V^s)$, $GSO(V^a)$, and $GSO(V)$ are singleton sets. The following theorem is a paraphrase of the results of Vigneras in [Vi].

Theorem 7. *The following is a complete list of the Langlands parameters of discrete series representations of $GSp(4)$, over a non-archimedean local field k of odd residue characteristic, together with the structure of their L-packets and the information about which of these representations could be lifted from $GO(4)$ and the associated representation of $GO(4)$ (corresponding to the quadratic spaces V^s, V^a, and V).*

1. *$\sigma = \sigma_1 \oplus \sigma_2$ where $\sigma_1 \neq \sigma_2$ are 2-dimensional irreducible representations of the Weil-Deligne group W_k' with $\det \sigma_1 = \det \sigma_2$. This parameter is contained in the cuspidal subgroup $G(SL(2) \times SL(2))$ of $GSp(4,\mathbb{C})$. The L-packet of representations of $GSp(4,k)$ corresponding to such a parameter σ has two elements, exactly one of which is generic. The generic representation is obtained as the Howe lift from a representation of $GO(V^s)$ which is induced from the representation $\pi_1 \otimes \pi_2^*$ of $[GL(2) \times GL(2)]/\Delta k^* \cong GSO(V^s)$ where π_1 and π_2 are discrete series representations of $GL(2)$ with parameters σ_1 and σ_2. The representation which is not generic is obtained as the Howe lift from a representation of $GO(V^a)$ which is induced from the representation $\pi_1' \otimes \pi_2'^*$ of $[D^* \times D^*]/\Delta k^* \cong GSO(V^a)$ where π_1' and π_2' are the representations of D^* with parameters σ_1 and σ_2.*

2. *$\sigma = \mathrm{Ind}_{W_K'}^{W_k'} \theta$ where θ is a two dimensional representation of W_K' for a quadratic field extension K of k, which does not extend to W_k' but such that $\det \theta$ does extend to W_k'. The representation σ with values in $GL(4,\mathbb{C})$ can be embedded in $GSp(4,\mathbb{C})$ in exactly two in-equivalent way. The L-packet corresponding to either of these two representations $W_k' \to GSp(4,\mathbb{C})$ consists exactly of one element, and both the representations can be obtained as Howe liftings from $GO(V)$. (Actually the duality correspondence is between $GSp(4)^+$ and $GO(V)$, but we follow remark 2 of section 2 which applies here also, to construct correspondence between $GSp(4)$ and $GO(V)$.) The corresponding two representations of $GO(V)$ are obtained by induction from $GSO(V)$ of representations which remain irreducible when restricted to $GL(2,K)$ (recall that $GSO(V) \cong [GL(2,K) \times k^*]/K^*$), and has the parameter θ (so the two representations of $GSO(V)$ are the two ways of extending this representation of $GL(2,K)$ to $GSO(V)$).*

3. *$\sigma = \tau \otimes sp(2)$ where τ is an irreducible two dimensional representation of W_k with values in $GO(2)$ (this condition is automatic in our situation of odd residue characteristic), and $sp(2)$ is the two dimensional irreducible representation of W_k' which is trivial on W_k. The corresponding L-packet has exactly one element, and it does not come from Howe lifting from $GO(4)$ for any 4-dimensional quadratic space.*

4. *$\chi \otimes sp(4)$ which is the 4-dimensional irreducible representation of W_k' on which W_k operates by the character χ. This corresponds to the Steinberg*

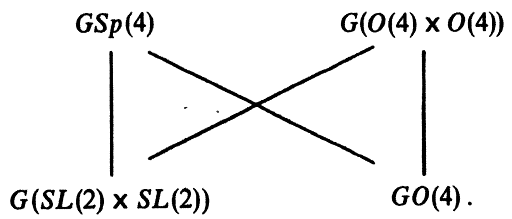
representation of $GSp(4)$ and does not come from the Howe lifting from $GO(4)$ for any 4-dimensional quadratic space.

Remark. In cases 1 and 2 of the above theorem, supercuspidal representations of $GSO(4)$ lift to supercuspidal representations of $GSp(4)$. This follows from the work of Kudla [Ku] together with the known correspondence between $GO(4)$ and $GL(2)$.

Remark. To be strictly correct, Vigneras constructed certain representations of $GSp(4)$ by lifting them from a quadratic form in four variables, and constructed certain other representations as subquotients of principal series and checked that the L-functions and epsilon factors (constructed by Piatetski-Shapiro and Soudry) of these after twisting by an arbitrary representation of $GL(1)$ and $GL(2)$ is what is predicted by the parameters given in the above theorem. So she did not prove that the representations she constructs have these parameters, nor did she prove that this list exhausts all discrete series representations but she did prove that if Langlands' conjecture on the parametrisation of representations of $GSp(4)$ is true then the list is complete in odd residue characteristic. Our methods say nothing about representations of $GSp(4)$ which are not lifted from a quadratic space of dimension 4.

We now come to the proof of Theorem 5. We will be proving this theorem for only those discrete series representation of $GSp(4)$ which are obtained as Howe lift of a representation of an orthogonal group in four variables; in particular we will not be able to handle cases 3 and 4 of theorem 7. For principal series representations induced from supercuspidal representation of a parabolic, the standard Mackey theory can be used to reduce theorem 5 to the situation of triple products for $GL(2)$ but we will not do that here. Similarly, Steinberg representation being part of a principal series whose other factors are easy to describe, can be taken care of. However, the representations in case 3, though also part of a principal series (induced from a supercuspidal representation of the Levi of a parabolic which stabilises a line), we do not know how to handle as the other component of the principal series is as mysterious.

Our proof of theorem 5 will use the seesaw pair originally used by Harris and Kudla [HK] involving the dual pairs $(GSp(4), GO(4))$ and $(G(SL(2) \times SL(2)), G(O(4) \times O(4)))$:



Assume that the L-packet $\{\pi\}$ of representations of $GSp(4)$ is as in case 1 of Theorem 6 above with Langlands parameter $\sigma(\pi) = \sigma_1 \oplus \sigma_2$. Write $\{\pi\} = \{\pi^1, \pi^2\}$ with π^1 the generic element which is obtained from $GO(V^s)$ as the

Howe lift of the representation $\pi_1 \otimes \pi_2^*$ of $[GL(2) \times GL(2)]/\Delta k^* \cong GSO(V^s)$ as there. Let τ' be a representation of $G(SL(2) \times SL(2))$ appearing in $\tau = \tau_1 \otimes \tau_2$, and let $\theta_0(\tau')$ be its Howe lift to $G(O(V^s) \times O(V^s))$. By the seesaw duality theorem, and Frobenius reciprocity, we have

$$\begin{aligned}\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi^1, \tau'] &\cong \mathrm{Hom}_{GO(V^s)}[\theta_0(\tau'), \mathrm{Ind}_{GSO(V^s)}^{GO(V^s)} \pi_1 \otimes \pi_2^*] \\ &\cong \mathrm{Hom}_{GSO(V^s)}[\theta_0(\tau'), \pi_1 \otimes \pi_2^*] .\end{aligned}$$

Summing over all the representations τ' of $G(SL(2) \times SL(2))$ in the representation $\tau_1 \otimes \tau_2$ of $GL(2) \times GL(2)$, and noting that the Howe lift to $GO(V^s)$ of a representation τ_1 of $GL(2)$ remains irreducible when restricted to $GSO(V^s)$, and is $\tau_1 \otimes \tau_1^*$ under the isomorphism $GSO(V) \cong \{GL(2) \times GL(2)\}/\Delta(k^*)$, we have

$$\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi^1, \tau_1 \otimes \tau_2] \cong \mathrm{Hom}_{GL(2)\times GL(2)}[(\tau_1 \otimes \tau_1^*) \otimes (\tau_2 \otimes \tau_2^*), \pi_1 \otimes \pi_2^*] .$$

Therefore $\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi^1, \tau_1 \otimes \tau_2] \neq 0$ if and only if $\mathrm{Hom}_{GL(2)}[\tau_1 \otimes \tau_2, \pi_1] \neq 0$ and $\mathrm{Hom}_{GL(2)}[\tau_1^* \otimes \tau_2^*, \pi_2^*] \neq 0$. Therefore by the multiplicity one theorem for triple product, cf. [P1], $\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi^1, \tau'] \neq 0$ for atmost one representation τ' of $G(SL(2) \times SL(2))$ appearing in $\tau_1 \otimes \tau_2$, and by the theorem on epsilon factors proved in [P1], we get that $\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi^1, \tau_1 \otimes \tau_2] \neq 0$ if and only if $\varepsilon(\sigma_1 \otimes \sigma(\tau_1)^* \otimes \sigma(\tau_2)^*) = 1$ and $\varepsilon(\sigma_2 \otimes \sigma(\tau_1)^* \otimes \sigma(\tau_2)^*) = 1$.

Repeating all the analysis done above with the seesaw pair after replacing V^s by V^a, we find that for π^2, the representation of $GSp(4)$ obtained by lifting from the corresponding representation of $GO(V^a)$, $\mathrm{Hom}_{G(SL(2)\times SL(2))}[\pi^2, \tau_1 \otimes \tau_2] \neq 0$ if and only if $\varepsilon(\sigma_1 \otimes \sigma(\tau_1)^* \otimes \sigma(\tau_2)^*) = -1$ and $\varepsilon(\sigma_2 \otimes \sigma(\tau_1)^* \otimes \sigma(\tau_2)^*) = -1$. This proves Theorem 5 for L-packets of $GSp(4)$ as in case 1 of Theorem 6 except that it assumes, in the notation of section 2, $\theta_0(\pi_1 \otimes \pi_2^*) = \theta(\pi_1 \otimes \pi_2^*)$, which as mentioned in section 2, one knows apriori only for supercuspidal representations $\pi_1 \otimes \pi_2^*$. We will check below that this is the case when π_1 is supercuspidal and π_2 is a special representation, but have not been able to treat the case when both π_1 and π_2 are special representations.

The idea of the proof for $\theta_0(\pi_1 \otimes St) = \theta(\pi_1 \otimes St)$, is, amusingly enough, to use the same seesaw duality, and to conclude that if they are not equal, then $\theta_0(\pi_1 \otimes St)$ will have more irreducible components of $G(SL(2) \times SL(2))$ than is allowed by the branching law using the seesaw.

For simplicity of notation, let $M = \pi_1 \otimes St$ where π_1 is a supercuspidal representation of $GL(2)$ with trivial central character, and therefore isomorphic to its own dual. The Jacquet module of $M \otimes \theta_0(M)$ with respect to the unipotent radical of the parabolic $[GL(2) \times B]/\Delta k^*$ of $GSO(4) = [GL(2) \times GL(2)]/\Delta k^*$ where B is the upper triangular subgroup of $GL(2)$, is $\pi_1 \otimes |x/y|^{1/2} \otimes \theta_0(M)$ as a $GL(2) \times \{k^* \times k^*\} \times GSp(4)$ module where $|x/y|$ is the obvious character of $k^* \times k^*$. From Kudla [Ku, Theorem 2.8] it follows that the π_1-isotypical part of the corresponding Jacquet module of the Weil representation is

$$\pi_1 \otimes |x/y|^{1/2} \otimes \mathrm{Ind}_P^{GSp(4)}(\pi_1 |\cdot|^{-1/2}) ,$$

where P is the parabolic in $GSp(4)$ stabilising an isotropic plane with Levi subgroup $GL(2) \times k^*$ and $\pi_1| \cdot |^{-1/2}$ is a representation of $GL(2) \times k^*$ which is trivial on k^*. It follows that $\theta_0(M)$ is a quotient of $\mathrm{Ind}_P^{GSp(4)}(\pi_1| \cdot |^{-1/2})$.

By Mackey theory, and results on triple product for $GL(2)$ in terms of epsilon factors, it is easy to see that for supercuspidal representations τ_1, τ_2 of $GL(2)$, an irreducible component τ' of $\tau_1 \otimes \tau_2$ restricted to $G(SL(2) \times SL(2))$ appears in $\mathrm{Ind}_P^{GSp(4)}(\pi_1| \cdot |^{-1/2})$ if and only if $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(\pi_1)) = 1$. On the other hand, by seesaw duality argument given before, we know that an irreducible component τ' of $\tau_1 \otimes \tau_2$ restricted to $G(SL(2) \times SL(2))$ appears in $\theta_0(M)$ if and only if $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(\pi_1)) = 1$ and $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(St)) = 1$. If we can now prove that there exists supercuspidal representations τ_1 and τ_2 of $GL(2)$ with $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(\pi_1)) = 1$ but $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(St)) \neq 1$, then it would follow that $\theta_0(M)$ is not equal to $\mathrm{Ind}_P^{GSp(4)}(\pi_1| \cdot |^{-1/2})$, and as $\mathrm{Ind}_P^{GSp(4)}(\pi_1| \cdot |^{-1/2})$ is known to have only two irreducible components, we will have proved that $\theta_0(M)$ is irreducible, and is the unique irreducible quotient of $\mathrm{Ind}_P^{GSp(4)}(\pi_1| \cdot |^{-1/2})$. From [P1], looked at from the point of view of division algebra, it is clear that $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(St)) \neq 1$ if and only if $\tau_1 = \tau_2^*$, and $\varepsilon(\sigma(\tau_1) \otimes \sigma(\tau_2) \otimes \sigma(\pi_1)) = 1$ if and only if the representation π_1' of the division algebra does not appear in $\tau_1^{*'} \otimes \tau_2^{*'}$. If π_1' has level at least 2 (i.e. π_1' and all its twists by one dimensional characters are not trivial on the first congruence subgroup of the division algebra), then taking $\tau_1 = \tau_2^*$ of level one does the trick, and even in the case when π_1' has level one a little more careful choice of $\tau_1 = \tau_2^*$ of level one works.

We remark that by a similar analysis, one can prove that the representation $\pi_1 \otimes 1$ of $GSO(4)$ appears in the duality correspondence with $GSp(4)$, and one has $\theta_0(\pi_1 \otimes 1) = \theta(\pi_1 \otimes 1)$ which is equal to the other irreducible component of $\mathrm{Ind}_P^{GSp(4)}(\pi_1| \cdot |^{-1/2})$. Moreover, the seesaw duality can be used to give the decomposition of this representation too restricted to $G(SL(2) \times SL(2))$.

We now turn our attention to representations of $GSp(4)$ obtained using Howe lift from representations of $GO(V)$ where V is a four dimensional quadratic space whose discriminant algebra is a quadratic field extension K of k. In this case $GSO(V) \cong [GL(2,K) \times k^*]/K^*$, and the lifting of a representation η from $GL(2,k)$ to $GO(V)$ remains irreducible when restricted to $GSO(V)$, and is $\eta|_K \otimes \omega_\eta$ where $\eta|_K$ is the base change of η to $GL(2,K)$, and ω_η is the central character of η, cf. [Co]. (Actually, we should be using the group $GL(2,k)^+$ instead of $GL(2,k)$ for the duality correspondence.) If $\sigma(\theta)$ is the two dimensional representation of W_K' associated to a representation θ of $GL(2,K)$ (which is not invariant under the Galois automorphism of $GL(2,K)$ but whose central character is), then the Langlands parameter (with values in $GL(4,\mathbb{C})$) of the two representations of $GSp(4)$ obtained by Howe lift is $\mathrm{Ind}_{W_K}^{W_k}\sigma(\theta)$, and a similar seesaw argument together with the following identity of epsilon factors proves theorem 5 in this case. (We will need to use Frobenius reciprocity too to take care of the difference between $GL(2,k)^+$ and $GL(2,k)$.)

$$\varepsilon([\mathrm{Ind}_{W_K}^{W_k}\sigma(\theta)] \otimes \sigma(\tau_1)^* \otimes \sigma(\tau_2)^*) = \varepsilon(\sigma(\theta) \otimes \sigma(\tau_1)^*|_{W_K} \otimes \sigma(\tau_2)^*|_{W_K}).$$

Remark. The proof given above of Theorem 5 infact gives more information as predicted by [G-P1] but not incorporated in Theorem 5. It shows that when the L-packet $\{\pi\}$ of $GSp(4)$ has more than one element, then one can decide which representation in the L-packet of $GSp(4)$ contains a particular representation (or rather the sum of representations in an L-packet) of $G(SL(2) \times SL(2))$ in terms of epsilon factors.

7

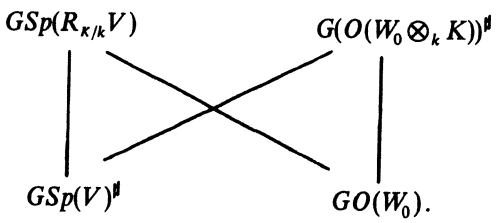
We will prove Theorem 6 in this section. As the arguments are very similar to the proof of Theorem 5, we will be very brief.

For a vector space V over K, let $R_{K/k}V$ be the same space V but now thought of as a vector space over k. If V is given with an alternating form $\langle\,,\rangle$, then $R_{K/k}V$ acquires the alternating form $\mathrm{tr}\circ\langle\,,\rangle$. If W_0 is a quadratic space over k, then clearly $W_0 \otimes_k K$ is a quadratic space over K. Define $GSp(V)^\sharp$ to be the subgroup of $GSp(V)(K)$ where the similitude factor takes values in k^*; similarly define $GO(W_0 \otimes_k K)$.

We have the following isomorphism of symplectic spaces:

$$R_{K/k}[V \otimes_K (W_0 \otimes_k K)] \cong (R_{K/k}V) \otimes_k W_0 \,.$$

This gives rise to the following seesaw diagram.



We now specialise to the case when V is a 2-dimensional vector space over K and W_0 is 4-dimensional over k. If the discriminant algebra associated to W_0 is K, then for K^1 the norm one subgroup of K^*, we have the inclusion $GL(2,K)/K^1 \hookrightarrow GO(W_0)$, and the embedding $GO(W_0) \hookrightarrow GO(W_0 \otimes_k K)$ gives rise to $GL(2,K)/K^1 \hookrightarrow [GL(2,K) \times GL(2,K)]/\Delta K^*$ given by $g \to (g, \bar{g})$.

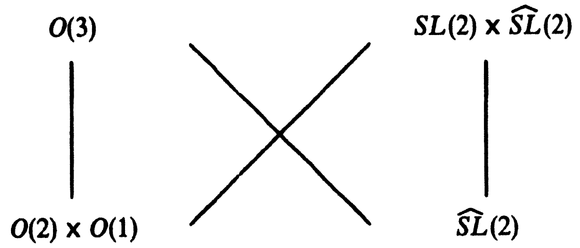
If the representation π of $GSp(4,k)$ comes as the Howe lift of a representation θ of $GL(2,K)$, with parameter $\sigma(\theta) : W_K' \to GL(2,\mathbb{C})$, and if $\sigma(\tau)$ is the parameter of a representation τ of $GL(2,K)$, then it is clear from the above seesaw diagram that some irreducible component τ' of the restriction of τ to $GL(2,K)^\sharp$ appears in π if and only if $\mathrm{Hom}_{GL(2,K)}[\tau \otimes \bar{\tau}, \theta] \neq 0$, or from [P1], if and only if $\varepsilon(\sigma(\tau) \otimes \sigma(\bar{\tau}) \otimes \sigma(\theta)^*) = 1$. The following identity now proves Theorem 6 for the representation π.

$$\varepsilon([\mathrm{Ind}_{W_K}^{W_k}\sigma(\theta)] \otimes M_K^k \sigma(\tau)^*) = \varepsilon(\sigma(\tau) \otimes \sigma(\bar{\tau}) \otimes \sigma(\theta)^*) \,.$$

When the L-packet $\{\pi\}$ on $GSp(4,k)$ comes from split orthogonal group in 4-variables, then again theorem 6 can be reduced to the situation studied in [P2] on the decomposition of a $GL(2, K)$ representation when restricted to $GL(2, k)$; however, as the theorem there on epsilon factors was not proved in all cases, the same gap remains here.

8

We will use the following seesaw pair in this section to obtain the tensor product of the Weil representation (of the two fold cover) of $SL(2)$ with any representation of $SL(2)$.



We begin by summarising the well known Howe duality correspondence between $O(2)$ and $SL(2)$ which has been studied for a long time starting with the works of Shalika, Tannaka, Casselman, etc. The following theorem follows from their work though it does not seem to have been explicitly stated anywhere.

Theorem 8. *Let $O(Q)$ denote the orthogonal group in 2 variable corresponding to an orthogonal space Q. Let $\{\alpha \cdot Q\}$ be the set of isomorphism classes of quadratic forms which are multiples of Q (so by local class field theory, the set $\{\alpha \cdot Q\}$ has cardinality 1 or 2 depending on whether Q represents a zero or not). All the groups $O(\alpha Q)$ are isomorphic, and given an irreducible representation χ of $SO(Q)$ (a character in fact as $SO(Q)$ is abelian), let S be the set of irreducible representations of $O(\alpha Q)$ which contains this character. Then the Howe lift to $SL(2, k)$ of this set S of representations is an L-packet on $SL(2, k)$. Here the Howe lifting is with respect to the symplectic space $\alpha Q \otimes W$ where W is any fixed 2-dimensional symplectic space, and the additive character ψ used to define the Weil representation of $Sp(\alpha Q \otimes W)$ is arbitrary but fixed.*

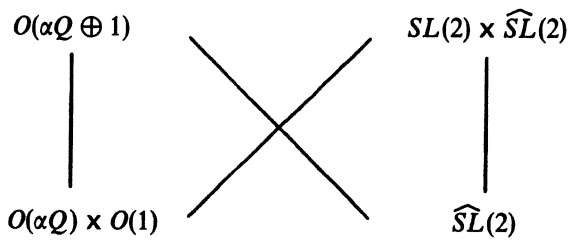
We will use the notation introduced in Theorem 8 above in the rest of this section. We note that for Q an-isotropic, as $\{\alpha Q\}$ runs over the set of isomorphism classes of quadratic forms which are multiples of Q, the quadratic spaces $\alpha Q \oplus 1$ run over the distinct quadratic spaces of dimension 3 with a given discriminant. So, $O(\alpha Q \oplus 1)$ represents both the groups $D_k^*/k^* \times \{\pm 1\}$ and $PGL(2, k) \times \{\pm 1\}$ exactly once. It is a theorem of Waldspurger [W] that for Q an-isotropic, the Howe lifts from $O(\alpha Q \oplus 1)$ for different α give disjoint sets of representations of $\widehat{SL}(2, k)$, and exhaust all the genuine representations of $\widehat{SL}(2,k)$ (i.e. those representations which do not factor through $SL(2,k)$). Moreover, if *det* denotes the quadratic character of $O(\alpha Q \oplus 1)$ obtained by taking the determinant, then for any representation V of $O(\alpha Q \oplus 1)$ exactly one of V or $V \otimes det$ appears in the duality correspondence with $\widehat{SL}(2)$.

Let Q_1 and Q_2 denote the two distinct isomorphism classes of quadratic spaces of dimension 3 of a given discriminant. Define a genuine L-packet on $\widehat{SL}(2)$ to be the set of representations consisting of Howe lifts of representations of $O(Q_1)$ and $O(Q_2)$ whose restrictions to $SO(Q_1)$ and $SO(Q_2)$ (which are isomorphic to D_k^*/k^* and $PGL(2,k)$ in some order) are related by the Jacquet-Langlands correspondence. The L-packets thus defined have at most two elements.

If the quadratic form Q is the norm form of a quadratic algebra K, then $SO(Q) \cong K^*/k^*$ and it can be seen that the embedding of $SO(\alpha Q)$ in $SO(\alpha Q \oplus 1)$ is the natural embedding of K^*/k^* in $PGL(2)$ or D_k^*/k^*. If NK^*/k^* denotes the normaliser of K^*/k^* in $PGL(2)$ or D_k^*/k^*, then the image of the natural embedding of $O(\alpha Q) \times O(1)$ in $O(\alpha Q \oplus 1)$ which is $PGL(2) \times \{\pm 1\}$, or $D_k^*/k^* \times \{\pm 1\}$, is $NK^*/k^* \times \{\pm\}$.

It is well known that if π is a discrete series representation of $PGL(2)$ then any character of K^*/k^* appears in exactly one of the representations π of $PGL(2)$ or π' of D_k^*/k^* where π and π' are related by the Jacquet-Langlands correspondence, and any character of K^*/k^* appears in a principal series representation of $PGL(2)$.

Using the seesaw pair,



the above theorem, and the various observations made in this section we get the following theorem. (We will also need to use the trivial case of seesaw duality that when a representation of $\widehat{SL}(2)$ does not occur in the duality correspondence with $O(\alpha Q \oplus 1)$, then it also does not appear in the tensor product of a representation of $SL(2)$ which is lifted from $O(\alpha Q)$ with the Weil representation of $\widehat{SL}(2)$.)

Theorem 9. *Let ω_ψ be the Weil representation of $\widehat{SL}(2)$ associated to a character ψ of k, and $\{\pi_1\}$ an L-packet of infinite dimensional representations of $SL(2,k)$ coming from a quadratic algebra and $\{\pi_2\}$ a genuine L-packet on $\widehat{SL}(2,k)$ which does not factor through $SL(2,k)$. Assume that the Howe lift of π_2 to $O(V)$ is not one dimensional for some $\pi_2 \in \{\pi_2\}$, and V 3-dimensional* *isotropic quadratic space. Then there is exactly one representation π_2 in $\{\pi_2\}$ which appears as a quotient of $\widehat{SL}(2,k)$-modules*

$$\omega_\psi \otimes \left(\sum_{\pi \in \{\pi_1\}} \pi \right) \to \pi_2 .$$

Motivated by the above theorem and the multiplicity 1 theorem for the restriction of an irreducible admissible representation of $O(n)$ to $O(n-1)$ of [GPR], we end the paper with the following conjecture which the above theorem answers for all non-exceptional representations of $SL(2,k)$. It is the analogue of conjecture 8.6 of [G-P] for Weil representations.

Conjecture. *Let $\{\pi\}$ be an L-packet of representations of $Sp(2n,k)$ and let ω_ψ denote the Weil representation of the metaplectic group $\widehat{Sp}(2n,k)$ associated to a non-trivial additive character of k. Then any representation of $\widehat{Sp}(2n,k)$ appears (as a quotient) in $\left(\bigoplus_{\pi \in \{\pi\}} \pi \right) \otimes \omega_\psi$ with multiplicity at most one. In particular, for any representation π of $Sp(2n,k)$, $\omega_\psi \otimes \pi$ decomposes with multiplicity at most one.*

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