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Test vectors for linear forms

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In this note, we refine results of Waldspurger [W], Tunnell [T], and Prasad [P] on the existence of non-zero linear forms ℓ on certain irreducible, admissible complex representations V of p-adic groups. The basic problem we consider is to find an explicit vector v in V, called a test vector, where $\ell(v) \neq 0$. As in [Gr], the test vector will lie on a line fixed by a specific open compact subgroup.

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The first situation we consider is the following. Let F be a local, non-archimedean field with ring of integers A, uniformizing parameter π, and finite residue field $A/\pi A$. Let V_1 be an irreducible, admissible complex representation of the group $\mathrm{GL}_2(F)$, and let ω be the central quasi-character of V_1. We assume that V_1 is infinite dimensional, and when the residual characteristic of F is 2, we assume that V_1 is not super-cuspidal. Let K be a separable quadratic extension of F (we include the case when $K \cong F \times F$), and let V_2 be an irreducible complex representation (= quasi-character χ) of the group $\mathrm{GL}_1(K) \cong K^*$ whose restriction to the subgroup $\mathrm{GL}_1(F) \cong F^*$ is equal to the inverse of the quasi-character ω. We consider the representation $V = V_1 \otimes V_2$ of the group $G = \mathrm{GL}_2(F) \times \mathrm{GL}_1(K)$; this is both admissible and irreducible, and by our hypotheses the subgroup $\Delta\mathrm{GL}_1(F)$ embedded diagonally in G acts trivially on V.

Let H be the subgroup $\Delta\mathrm{GL}_1(K)$ embedded diagonally in G. [To embed $\mathrm{GL}_1(K)$ as a torus in $\mathrm{GL}_2(F)$, we view K as a 2-dimensional vector space over F and let K^* act by left multiplication.] It is not difficult to show, using the theory of Gelfand pairs (cf. [P, Chap. 3]), that the space of H-invariant linear forms $\ell: V \to \mathbf{C}$ has dimension at most one. Waldspurger and Tunnell give a criterion for a non-zero H-invariant linear form to exist.

Specifically, let σ_1 be the two-dimensional representation of the Weil-Deligne group of F associated to V_1 by the unitary Langlands correspondence (normalized as in [D, 3.2.3]) and let σ_2 be the two-dimensional representation of the Weil group of F which is induced from the quasi-character χ of K^*. Then $\det(\sigma_1)$ is equal to ω and $\det(\sigma_2)$ is the product $\alpha_{K/F} \cdot \chi|_{F^*}$, where $\alpha_{K/F}$ is the quadratic character of F^* associated to the extension K/F by local class-field theory, and $\chi|_{F^*}$ is the restriction of χ to F^*. (We identify the abelianized Weil-Deligne group with F^* in the usual manner.) The tensor product $\sigma_1 \otimes \sigma_2$ is then a four-dimensional symplectic representation of the Weil-Deligne group, and has a local root-number $\varepsilon(\sigma_1 \otimes \sigma_2) = \pm 1$. [We have $\varepsilon(\sigma_1 \otimes \sigma_2) = \varepsilon(\sigma_1 \otimes \sigma_2, \psi, dx)$ in the notation of [D2] or [T], where ψ is a non-trivial additive character of F and dx is the unique Haar measure on F which is self-dual for Fourier transform with respect to ψ.]

When $\varepsilon(\sigma_1 \otimes \sigma_2) \neq \alpha_{K/F} \cdot \omega(-1)$, one can show that the representation V_1 is square-integrable and the quadratic extension K is a field [T]. Hence V_1 corresponds to an irreducible complex representation V_1' of the group D^* under the Jaquet-Langlands correspondence [JL, Chap. 14], where D is the quaternion division algebra over F. The representation V_1' has central quasi-character ω and is characterized by the identity of traces: $\operatorname{tr} V_1(x) + \operatorname{tr} V_1'(x) = 0$ on regular elliptic conjugacy classes x. Since the field K embeds as an F-subalgebra of D, the subgroup $H \cong \Delta \mathrm{GL}_1(K)$ acts on the representation $V' = V_1' \otimes V_2$ of the group $G' = D^* \times \mathrm{GL}_1(K)$. Again it is easy to show, using the theory of Gelfand pairs, that the space of H-invariant linear forms on V' has dimension at most one. The criterion of Waldspurger and Tunnell is the following.

Proposition 1.1 [W, T]. *There is a non-zero H-invariant linear form $\ell : V \to \mathbf{C}$ (which is unique up to scalars) if and only if $\varepsilon(\sigma_1 \otimes \sigma_2) = \alpha_{K/F} \cdot \omega(-1)$.*

There is a non-zero H-invariant linear form $\ell' : V' \to \mathbf{C}$ (which is unique up to scalars) if and only if $\varepsilon(\sigma_1 \otimes \sigma_2) = -\alpha_{K/F} \cdot \omega(-1)$.

Remark 1.2. Since V_2 has dimension 1, the existence of an H-invariant linear form $\ell : V \to \mathbf{C}$ is equivalent to the existence of a linear form $\ell_1 : V_1 \to \mathbf{C}$ which satisfies

$$\ell_1(k \cdot v_1) = \chi^{-1}(k) \cdot \ell_1(v_1) \tag{1.3}$$

for all k in $K^* = \mathrm{GL}_1(K)$ and v_1 in V_1. Similarly for the form ℓ' on V'.

Remark 1.4. When K is a *field*, the representation V_1 decomposes as the direct sum of lines $V_1(\eta^{-1})$ on which the torus K^* acts via the characters satisfying $\eta^{-1} = \omega$ on F^* and $\varepsilon(\sigma_1 \otimes \operatorname{Ind} \eta) = \alpha_{K/F} \cdot \omega(-1)$ [T]. If ℓ_1 is a non-zero linear form which satisfies (1.3), then $\ell_1(v_1) \neq 0$ if and only if v_1 has a non-trivial component in the subspace $V_1(\chi^{-1})$. Similarly for the form ℓ_1' on V_1'.

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We will find a test vector for the linear form ℓ (or ℓ') of Proposition 1.1 under the hypothesis that either V_1 or V_2 is unramified. Henceforth we assume:

(2.1) Either V_1 is an unramified principal series representation of $\mathrm{GL}_2(F)$, or $V_2 = \chi$ is an unramified quasi-character of K^*.

By an unramified quasi-character of K^* we mean one which is trivial on the open compact subgroup $\mathcal{O}^*$, where $\mathcal{O}$ is the integral closure of A in K. In particular, (2.1) implies that the central quasi-character ω of V_1 is unramified. Let η be an unramified quasi-character of F^* with $\eta^2 = \omega$. Then

$$V \cong (V_1 \otimes \eta^{-1}) \otimes (V_2 \otimes \eta \circ N_{K/F})$$

as a representation of $H = \Delta K^*$. Hence it is no loss of generality in what follows to assume that $\omega = 1$, and hence that χ is a character of K^*/F^*. We define the conductor c of χ as the smallest non-negative integer such that χ is trivial on the subgroup $(A + \pi^c \cdot \mathcal{O})^*$ of K^*.

First assume that V_1 is an unramified principal series representation with $\omega = 1$. Let R be a maximal order in $M_2(F)$ which optimally contains the order $\mathcal{O}_c = A + \pi^c \cdot \mathcal{O}$ of K; for a discussion of optimal embeddings see [Gr, Proposition 3.2], which shows that such maximal orders R exist and are unique up to K^* conjugacy. Let

$$M = R^* \times \mathcal{O}_c^*. \tag{2.2}$$

Then M is an open compact subgroup of $G = \mathrm{GL}_2(F) \times K^*$, and $M \cap \Delta K^* = \Delta \mathcal{O}_c^*$. We will prove the following result in Sect. 3.

Proposition 2.3. *If V_1 is an unramified principal series, the open compact subgroup M of G defined in* (2.2) *fixes a unique line in $V = V_1 \otimes V_2$. If v is any non-zero vector on this line and ℓ is a non-zero H-invariant linear form on V, then $\ell(v) \neq 0$.*

Next assume that V_2 is an unramified representation of $\mathrm{GL}_1(K)$. Let n be the conductor of V_1 in the sense of Jacquet-Langlands [JL, Chap. 2]. When $\varepsilon(\sigma_1 \otimes \sigma_2) = \alpha \cdot \omega(-1) = \alpha(-1)$, let R_n be an order of reduced discriminant (π^n) in $M_2(F)$ which contains $\mathcal{O}$. When $\varepsilon(\sigma_1 \otimes \sigma_2) = -\alpha\omega(-1) = -\alpha(-1)$, we must have $n \geqq 1$. Let R'_n be an order of reduced discriminant (π^n) in D which contains $\mathcal{O}$. For a proof that the respective orders R_n and R'_n exist, and are unique up to conjugacy by K^*, see [Gr, Proposition 3.4]. We note that when K is an unramified field extension of F, we have $\varepsilon(\sigma_1 \otimes \sigma_2) = (-1)^n$. Let

$$M = R_n^* \times \mathcal{O}^*. \tag{2.4}$$

Then M is a compact open subgroup of $G = \mathrm{GL}_2(F) \times K^*$ and $M \cap \Delta K^* = \Delta \mathcal{O}^*$. Similarly we define the compact open subgroup

$$M' = R_n'^* \times \mathcal{O}^* \tag{2.5}$$

of $G' = D^* \times K^*$. The following result will be proved in Sect. 4.

Proposition 2.6. *Assume that V_2 is an unramified representation of* $\mathrm{GL}_1(K)$. *When $n(V_1) \geqq 2$, assume further that the extension K/F is unramified.*

If $\varepsilon(\sigma_1 \otimes \sigma_2) = \alpha_{K/F} \cdot \omega(-1)$ the subgroup M defined in (2.4) *fixes a unique line in V. If v is any non-zero vector on the line fixed by M and ℓ is a non-zero H-invariant linear form on V, then $\ell(v) \neq 0$.*

If $\varepsilon(\sigma_1 \otimes \sigma_2) = -\alpha_{K/F} \cdot \omega(-1)$ the subgroup M' defined in (2.5) *fixes a unique line in V'. If v' is any non-zero vector on the line fixed by M' and ℓ' is a non-zero H-invariant linear form on V', then $\ell'(v') \neq 0$.*

Remark 2.7. When $n(V_1) \geqq 2$ and K/F is ramified, the subgroup M defined in (2.4) fixes a two-dimensional subspace of V when $\varepsilon(\sigma_1 \otimes \sigma_2) = \alpha_{K/F} \cdot \omega(-1)$. There is a unique line in this subspace fixed by the element $\Delta \pi_K$, where π_K is any uniformizing parameter in K. If v is any non-zero vector on this line, we have $\ell(v) \neq 0$. Similarly, the subspace of V' fixed by the subgroup M' defined in (2.5) has dimension two when $\varepsilon(\sigma_1 \otimes \sigma_2) = -\alpha_{K/F} \cdot \omega(-1)$, and $\Delta \pi_K$ fixes a unique line in this subspace, on which $\ell' \neq 0$.

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We now turn to the proof of Proposition 2.3. Since V_1 is unramified, the subspace fixed by R^* has dimension 1, for any maximal order R of $M_2(F)$. [All such R^* are conjugate in $\mathrm{GL}_2(F)$ to the standard maximal compact subgroup $\mathrm{GL}_2(A)$.] Hence the open compact subgroup M defined in (2.2) fixes a unique line in V. We must show that ℓ is non-zero on this fixed line. To do this, we will need to recall several results on unramified principal series representations. Our standard reference is Godement's notes [Go, Chap. 1].

The representation V_1 is induced from an unramified character of the Borel subgroup $B=\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ of $\mathrm{GL}_2(F)$. More precisely, there is an unramified character μ of F^* such that V_1 is isomorphic to the representation of $\mathrm{PGL}_2(F)$ by right translation on the space of functions $f:\mathrm{GL}_2(F)\to\mathbf{C}$ which are invariant by right translation by some open compact subgroup and satisfy

$$f\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g\right)=\mu(a/d)\,|a/d|^{1/2}f(g).$$

The isomorphism class of V_1 is completely determined by the unordered pair of non-zero complex numbers $(\alpha,\alpha^{-1})=(\mu(\pi),\mu^{-1}(\pi))$, and $\alpha^2 \neq q^{\pm 1}$. The eigenvalue of the Hecke operator T_π on the line fixed by R^* is equal to $a_\pi=q^{1/2}(\alpha+\alpha^{-1})$.

Let X denote the homogeneous tree associated to $\mathrm{PGL}_2(F)$, whose vertices correspond bijectively to maximal orders in $M_2(F)$, and hence to maximal compact subgroups of $\mathrm{GL}_2(F)$ [S, Chap. 2, Sect. 1]. The maximal orders in $M_2(F)$ are the endomorphism rings of the homothety classes of rank 2 lattices in F^2.

Lemma 3.1. *Let $x\in X$ be the point on the tree fixed by R^*. For an integer $c\geqq 1$, let $V_1(c)$ be the subspace of V_1 generated by those vectors which are fixed by some maximal compact subgroup of $\mathrm{GL}_2(F)$ corresponding to a point on the tree at distance $\leqq c$ from x. Then $V_1(c)$ has dimension $(q+1)q^{c-1}$.*

Proof. The intersection of the maximal compact subgroups corresponding to vertices of the tree at distance $\leqq c$ from x is the group $R(c)^*$ of elements in R^* which are congruent to 1 modulo π^c. It follows that $V_1(c)$ is contained in the space of vectors fixed by $R(c)^*$. Since V_1 is an unramified representation, the space of vectors fixed by $R(c)^*$ is, as an R^*-module, isomorphic to the space of functions on $\mathbf{P}^1(R/\pi^c)$ which is $(q+1)q^{(c-1)}$-dimensional. It therefore suffices to show that $V_1(c)$ is exactly the space of vectors fixed under $R(c)^*$. Observe that functions on $\mathbf{P}^1(R/\pi^{c-1})$ sit naturally in the space of functions on $\mathbf{P}^1(R/\pi^c)$, and that the quotient is irreducible. Therefore we only need to prove that $V_1(c)$ is not contained in the space of functions on $\mathbf{P}^1(R/\pi^{(c-1)})$. But if $V_1(c)$ were contained in this space of functions then in particular any vector in V stabilised by a maximal compact at distance c from x will be invariant under $R(c-1)^*$. Therefore $R(c-1)^*$ will be contained in the intersection of maximal compact subgroups corresponding to points of distance c from x, i.e. $R(c-1)^*\subset R(c)^*$. This contradiction completes the proof of the lemma.

We begin the proof of Proposition 2.3 in the case when K is a field. Then by Remark 1.4 we have a decomposition of the space V_1 into one-dimensional eigenspaces $V_1(\eta^{-1})$ for the torus K^*, and we must show that the fixed vector for R^* has a non-trivial component on the line $V_1(\chi^{-1})$.

When K/F is an unramified field extension, there is a unique maximal order R_0 in $M_2(F)$ which contains $\mathcal{O}$. We first treat the case when $c=0$, so $M=R_0^* \times \mathcal{O}^*$ and a vector v fixed by M is the tensor product $v_1 \otimes 1$, with v_1 fixed by R_0^* in V_1. Since the subspace of V_1 fixed by $\mathcal{O}^*$ is equal to $V_1(\chi^{-1})$ for χ unramified, it has dimension 1 and is spanned by v_1. Hence $\ell_1(v_1) \neq 0$.

Now assume that $c \geqq 1$. We have seen that the maximal orders of $M_2(F)$ correspond to the vertices of the homogeneous tree associated to $\mathrm{PGL}_2(F)$; the maximal orders R which optimally contain $\mathcal{O}_c$ correspond to the $(q+1)\cdot q^{c-1}$ vertices of distance c from R_0. These points are permuted simply-transitively by the action of the group $K^*/F^*(1+\pi^c\cdot\mathcal{O})=\mathcal{O}^*/\mathcal{O}_c^*$. If v_1 is fixed by some R^* and ℓ_1 satisfies (1.3) we must show that $\ell_1(v_1)\neq 0$. But if $\ell_1(v_1)=0$, then ℓ_1 is equal to zero on the span on all lines fixed by subgroups S^*, where the distance of the maximal order S from R_0 in the tree is $\leqq c$. This span has dimension $(q+1)\cdot q^{c-1}$ by Lemma 3.1 and is therefore precisely the subspace fixed by $\mathcal{O}_c^*$ in V_1. This is a contradiction, as ℓ_1 is non-zero on the line $V_1(\chi^{-1})$, which is contained in the subspace fixed by $\mathcal{O}_c^*$.

When K/F is a ramified field extension, there are two maximal orders R_0 and R_0' which contain $\mathcal{O}$. [There is a unique Eichler order $R_0 \cap R_0'$ of discriminant (π) which contains $\mathcal{O}$.] The maximal orders R which optimally contain $\mathcal{O}_c$ correspond to the $2\cdot q^c$ vertices of distance c from either R_0 or R_0' in the tree, and these points are permuted simply transitively by the group $K^*/F^*(1+\pi^c\cdot\mathcal{O})$. Hence, arguing as above, we see that a non-zero vector v_1 fixed by R^* has non-trivial component in the eigenspace $V_1(\chi^{-1})$, for all quasi-characters χ of conductor c of K^*/F^*.

Finally, we consider the case when $K\cong F\times F$. Here we will use the Kirillov model for V_1 where the action of the split torus K^* is evident. Again we use Godement's notes [Go] as our basic reference. The Kirillov model occurs in the space of complex functions $\mathscr{F}$ on F^* which are locally constant and have compact support on F; its precise definition depends on the choice of a non-trivial additive character ψ of F, which we normalize to have kernel equal to A. The Borel subgroup B of $\mathrm{GL}_2(F)$ acts on the space $\mathscr{F}$ by the formula

$$\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} f\right](x)=\psi(bx/d)f(ax/d) \tag{3.2}$$

and the subspace $\mathscr{F}_0$ of locally constant functions with compact support in F^* is B-stable, and has codimension 2 in V_1. Moreover, the subspace $V_1^{\pm}$ of functions f in $\mathscr{F}$ with $f(x)\sim c\cdot|x|^{1/2}\cdot\mu(x)^{\pm}$ as $x\to 0$ has codimension 1 in V_1 and is B-stable, and the Borel subgroup acts on the quotients $V_1/V_1^{\pm}$ by the characters $|a/d|^{1/2}\mu(a/d)^{\mp}$ of the split torus K^*. Finally, a function f_1 in V_1 which is fixed by the maximal compact subgroup $\mathrm{GL}_2(A)=R_0^*$ is supported on A and is given by the formula:

$$f_1(x)=q^{-(n/2)}(\alpha^n+\alpha^{n-2}+\ldots-\alpha^{-n}), \tag{3.3}$$

where $\alpha=\mu(\pi)$ and $n=\mathrm{ord}(x)\geqq 0$.

Now let $\chi=\xi(a/d)$ be a character of K^* whose restriction to F^* is trivial, and let ℓ_1 be a non-zero linear form on V_1 which satisfies (1.3). First we assume that ξ is unramified. If $\xi^{-1}=|\,|^{1/2}\cdot\mu^{\pm}$, the form ℓ_1 has kernel equal to the subspace $V_1^{\mp}$ since $f_1=v_1$ does not lie in either of these hyperplanes, we have $\ell_1(v_1)\neq 0$ as claimed. Hence we may assume that $\xi^{-1}\neq|\,|^{1/2}\cdot\mu^{\pm}$, and consequently that ℓ_1 is non-zero when restricted to the subspace $\mathscr{F}_0$. It must then be given by the formula

$$\ell_1(f)=\int_{F^*}\xi(x)\cdot f(x)\cdot dx/|x| \quad \text{for all } f \text{ in } \mathscr{F}_0. \tag{3.4}$$

In particular, ℓ_1 is non-zero on the characteristic function of A^*. But this characteristic function is a linear combination

$$q^{-1}f_1(\pi^{-2}x)-q^{-1/2}(\alpha+\alpha^{-1})f_1(\pi^{-1}x)+f_1(x)$$

of translates of $f_1=v_1$ by elements of K^*, by (3.2)–(3.3). Hence $\ell_1(v_1)\neq 0$ also.

Finally, assume that ξ has conductor $n\geqq 1$. Then ℓ_1 is again non-zero on $\mathscr{F}_0$ and is given by the integral (3.4). Let $R_0=M_2(A)$ be a maximal order optimally containing

$$\mathcal{O}=\left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}: a,d\in A\right\},$$

and for $j\in A/\pi^n$, let

$$R_j=\begin{pmatrix}\pi^n & j\\ 0 & \pi^n\end{pmatrix} R_0 \begin{pmatrix}\pi^n & j\\ 0 & \pi^n\end{pmatrix}^{-1}.$$

If j is a unit in A/π^n, R_j is a maximal order which optimally contains

$$\mathcal{O}_n=\left\{\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}: a\equiv d(\operatorname{mod}\pi^n)\right\}.$$

All such maximal orders are obtained in this way and are conjugate under $K^*/\mathcal{O}_n^*\cdot F^*=F^*/(1+\pi^n A)$. If f_0 is a vector in V fixed by R_0^*, then

$$f_j=\begin{pmatrix}\pi^n & j\\ 0 & \pi^n\end{pmatrix} f_0=\psi(jx/\pi^n)f_0(x)$$

is fixed by R_j^*. Let $\ell: V\to\mathbf{C}$ be in the χ-eigenspace for K^*. If j is divisible by π, then R_j contains an order strictly larger than $\mathcal{O}_n$ and $\ell(f_j)=0$. It follows that if $\ell(f_j)=0$ for one j which is a unit, $\ell(f_j)=0$ for all $j\in A$. In particular ℓ vanishes on the linear combination

$$\sum_{j\in A/\pi_n A}\psi(-j/\pi^n)f_j(x)=\left[\sum_{j\in A/\pi^n A}\psi(j(x-1)/\pi^n)\right]f_0(x).$$

Since $f_0(x)$ vanishes outside A, and equals 1 on A^*, this linear combination is q^n times the characteristic function of $(1+\pi^n A)$. But ℓ is non-zero on the characteristic function of $(1+\pi^n A)$ by (3.4), so $\ell(f_j)\neq 0$ when $j\not\equiv 0(\operatorname{mod}\pi)$.

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We now turn to the proof of Proposition 2.6. First assume that K is a field. When K/F is unramified, there is a unique order R_n (or R_n') of reduced descriminant (π^n) which contains $\mathcal{O}$. The subspace of V_1 fixed by R_n^* is one-dimensional, and equal to the line fixed by $\mathcal{O}^*$ [Gr, Proposition 6.4]. Since ℓ_1 is non-zero on this line, the result easily follows. A similar argument covers the case when $\varepsilon(\sigma_1\otimes\sigma_2)=-\alpha_{K/F}\omega(-1)$, using the line fixed by $R_n'^*$ in V_1'.

When K/F is ramified, our hypothesis states that the conductor $n(V_1)$ is less than or equal to 1. Since we have treated the case of an unramified representation V_1 in the last section, we may assume that $n(V_1)=1$. But V_1 has trivial central character, so must be an unramified twist of the special representation. In this case there is a unique Eichler order $R_n=R_1$ in $M_2(F)$ which contains $\mathcal{O}$ and the vector v_1 fixed by

R_1^* spans the eigenspace $V_1(\chi^{-1})$ for K^*. Hence $\ell_1(v_1) \neq 0$. The argument is trivial in the division algebra case, as V_1' is one-dimensional.

When $K \cong F \times F$, we again use the Kirillov model for V_1. Since we have treated the case of an unramified representation in the last section, we may assume that $n(V_1) \geqq 1$.

Lemma 4.1. *Assume that V_1 has trivial central character and conductor $n(V_1) \geqq 1$.*

a) *If $n(V_1)=1$, then V_1 is the twist of the special representation by an unramified quadratic character η. The new vector in the Kirillov model is supported on A and is given by the function $f(x)=\eta(x)|x|$ there.*

b) *If $n(V_1) \geqq 2$, then the new vector in the Kirillov model is given by the characteristic function of A^*.*

Proof. We recall that we always take the Kirillov model for V_1 with respect to an additive character ψ which is trivial on A. If $n(V_1)=1$, then V_1 is either an unramified twist of the special representation or is a principal series representation with ramified central character. Since V_1 has trivial central character by assumption, this eliminates the second case. If V_1 is the twist of the special representation by an unramified character η, then the space $\mathscr{F}_0$ of locally constant functions with compact support on k^* has codimension 1 in the Kirillov model and the Borel subgroup acts on the quotient space by the character $\eta(ad) \cdot |a/d|$. The new vector v_1 fixed by $R_1^* = \Gamma_0(\pi)$ is supported on A, and is given by the function $f_1(x)=\eta(x)|x|$ there. This follows from the explicit construction of the Kirillov model for the special representation as given in Godement [Go].

Now assume that $n(V_1) \geqq 2$. The proof of the formula for the new vector will follow the argument in Casselman [Ca], which implicitly treats the case of supercuspidal representations. We begin by showing that the new vector always lies in the subspace $\mathscr{F}_0$. Since this is all of V_1 for a supercuspidal representation, we may assume that V_1 is either a principal series or a special representation. In this case, by our hypothesis on the conductor and central character, the characters of the split torus acting on the (semi-simplification) of the quotient $V_1/\mathscr{F}_0$ are ramified. But the new vector is fixed by the subgroup of diagonal matrices with a and d in A^*. Hence it goes to zero in the quotient, and lies in $\mathscr{F}_0$.

A vector v in V_1 is invariant under the subgroup $\Gamma_0(\pi^m)$ of $\mathrm{GL}_2(A)$ if and only if both v and $H_m v$ are invariant under the subgroup

$$B(A)=\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in A^*,\ b \in A\right\},$$

where $H_m=\begin{pmatrix} 0 & 1 \\ -\pi^m & 0 \end{pmatrix}$ [Ca]. We note that by formula (3.2), a function $f(x)$ in $\mathscr{F}_0$ is invariant under $B(A)$ if and only if f is supported on A and $f(x)$ depends only on $\mathrm{ord}(x)$.

Let du be the Haar measure of volume one on the compact group A^*. For a character v of A^* and an integer n, define the Fourier transform of a function f in $\mathscr{F}_0$ by

$$\hat{f}_n(v)=\int_{A^*} f(u\pi^n)v(u)\,du.$$

These transforms are collected in the Laurent polynomial

$$\hat{f}(v,t)=\sum_{n\in\mathbb{Z}} t^n \hat{f}_n(v).$$

A restatement of the fact that f is invariant under $B(A)$ is that $\hat{f}(v,t)$ is equal to zero for $v \neq 1$, and that $\hat{f}(1,t)$ is a polynomial in t.

The action of the Weyl group element $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on $\mathscr{F}_0$ in V_1 is given by the local functional equation of Jacquet and Langlands [J-L, 2.10, 2.18] (we recall that our central character $\omega = 1$):

$$\widehat{wf}(v,t) = C(v,t)\hat{f}(v^{-1},t^{-1}), \quad \text{with}$$

$$C(1, q^{s-1/2}) = L(1-s, V_1) \cdot \varepsilon(s, V_1, \psi)/L(s, V_1).$$

Since $n = n(V_1) \geqq 2$ and $\omega = 1$, we have $L(s, V_1) = 1$ for all s and $\varepsilon(s, V_1, \psi) = A \cdot q^{-n(s-1/2)}$, with A a non-zero complex number. Hence $C(1,t) = A \cdot t^{-n}$. Since $H_m = w \cdot \begin{pmatrix} \pi^m & 0 \\ 0 & 1 \end{pmatrix}$, we find that

$$\widehat{H_m f}(1,t) = A \cdot t^{m-n} \cdot \hat{f}(1, t^{-1}).$$

Now assume that f is a new vector, so both f and $H_n f$ are fixed by $B(A)$. The both $f(1,t)$ and $f(1,t^{-1})$ are polynomials in t. Hence $f(1,t)$ is a constant. Since $f(v,t) = 0$ for $v \neq 1$, we see that f is a multiple of the characteristic function of A^*.

Remark 4.2. The same result holds if we only assume that the central character of V_1 is unramified.

We now complete the proof of Proposition 2.6. Assume that $n(V_1) = 1$. The new vector in the Kirillov model is given by the function f_1 of Lemma 4.1. Hence the linear combination of K^* translates of this vector $-\eta(\pi) \cdot q^{-1} \cdot f_1(\pi^{-1}x) + f_1(x)$ lies in the subspace $\mathscr{F}_0$ and is the characteristic function of A^*. The argument now separates into two cases, as in the previous section. Write $\chi = \xi(a/d)$. If $\xi^{-1} = \eta \cdot |\,|$, then ℓ_1 has kernel the codimension one subspace $\mathscr{F}_0$ and is non-zero on the new vector f_1, which does not have compact support. If $\xi^{-1} \neq \eta \cdot |\,|$, then ℓ_1 is non-zero on $\mathscr{F}_0$ and hence given by the integral (3.4). Since ξ is unramified, this integral is non-zero on the characteristic function of A^*, and hence $\ell_1(v_1) \neq 0$.

If $n(V_1) \geqq 2$, the new vector fixed by R_n^* is represented by the characteristic function of A^* in the Kirillov model, by Lemma 4.1. The linear form ℓ_1 must be non-zero when restricted to $\mathscr{F}_0$ as the torus acts on the quotient via ramified characters. Hence ℓ_1 is given by the integral formula (3.4) and $\ell_1(v_1) \neq 0$. This completes the proof of Proposition 2.6 in the split case.

5

The second general situation involving test vectors for linear forms which we will consider is the following. Let V_1, V_2, and V_3 be three irreducible, infinite dimensional, admissible complex representations of the group $\mathrm{GL}_2(F)$ whose central characters satisfy $\omega_1 \cdot \omega_2 \cdot \omega_3 = 1$. When the residual characteristic of F is 2, we assume that at least one V_i is not super-cuspidal. We consider the representation $V = V_1 \otimes V_2 \otimes V_3$ of the group $G = \mathrm{GL}_2(F) \times \mathrm{GL}_2(F) \times \mathrm{GL}_2(F)$. By our hypotheses this is admissible and irreducible, and the subgroup $\Delta\mathrm{GL}_1(F)$ embedded diagonally in G acts trivially on V. The theory of Gelfand pairs shows that the space of $H = \Delta\mathrm{GL}_2(F)$ invariant linear forms $\ell: V \to \mathbf{C}$ has dimension at most one, and Prasad [P] gives a criterion for a non-zero H-invariant linear form to exist.

Let σ_i be the two-dimensional representations of the Weil-Deligne group of F associated to the irreducible representations V_i. The triple tensor product $\sigma_1\otimes\sigma_2\otimes\sigma_3$ is an eight-dimensional symplectic representation of the Weil-Deligne group, and has a local root-number $\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=\pm 1$. When $\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=-1$, one can show that the representations V_i are all square-integrable. Hence we obtain the corresponding representations V_i' of the group D^*, and an irreducible representation $V'=V_1'\otimes V_2'\otimes V_3'$ of $D^*\times D^*\times D^*$. Again, by the theory of Gelfand pairs, the space of $H'=\Delta D^*$ invariant linear forms on V' has dimension at most one.

Proposition 5.1 [P]. *There is a non-zero H-invariant linear form $\ell: V\to\mathbf{C}$ (which is unique up to scalars) if and only if $\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=+1$.*

There is a non-zero H'-invariant linear form $\ell': V'\to\mathbf{C}$ (which is unique up to scalars) if and only if $\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=-1$.

6

We will find a test vector for the linear form ℓ (or ℓ') of Proposition 5.1 under the hypothesis that

(6.1) Either the V_i are all unramified principal series representations of $\mathrm{GL}_2(F)$, or the V_i are all unramified twists of the special representation of $\mathrm{GL}_2(F)$.

In particular, the central quasi-characters ω_i are all unramified; let η_i be unramified quasi-characters of F^* with $\eta_i^2=\omega_i$ and $\eta_1\eta_2\eta_3=1$. Then

$$V\cong(V_1\otimes\eta_1^{-1})\otimes(V_2\otimes\eta_2^{-1})\otimes(V_3\otimes\eta_3^{-1})$$

as a representation of $H=\Delta\mathrm{GL}_2(F)$. Hence it is no loss of generality in what follows to assume that all $\omega_i=1$.

Proposition 6.1. *If the representations V_i are all unramified principal series, then the open compact subgroup $M=\mathrm{GL}_2(A)\times\mathrm{GL}_2(A)\times\mathrm{GL}_2(A)$ of $G=\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$ fixes a unique line in $V=V_1\times V_2\times V_3$. If v is any non-zero vector on this line and ℓ is a non-zero H-invariant linear form on V, then $\ell(v)\neq 0$.*

Next assume that the V_i are all unramified twists of the special representation; since the central quasi-characters ω_i are all trivial, each representation has the form $V_i\cong\mathrm{Sp}\otimes\eta_i$ where Sp is the special representation – which occurs in the locally constant functions on $\mathbf{P}^1(F)$ modulo the constant functions – and η_i is an unramified quadratic character of F^*. Hence V_i is completely determined by the value $\eta_i(\pi)=\pm 1$. We have the following formula for the local root-number [P, Proposition 8.6]:

(6.2)
$$\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=-\eta_1(\pi)\cdot\eta_2(\pi)\cdot\eta_3(\pi).$$

Proposition 6.3. *Assume that the V_i are unramified twists of the special representation, for $i=1, 2, 3$.*

If $\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=+1$, let R be an Eichler order of reduced discriminant (π) in $M_2(F)$. Then the open compact subgroup $M=R^*\times R^*\times R^*$ of $G=\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)\times\mathrm{GL}_2(F)$ fixes a unique line in $V=V_1\times V_2\times V_3$. If v is any non-zero vector on this line and ℓ is a non-zero H-invariant linear form on V, then $\ell(v)\neq 0$.

If $\varepsilon(\sigma_1\otimes\sigma_2\otimes\sigma_3)=-1$ let R' be the unique maximal order in D. Then the open compact subgroup $M'=R'^*\times R'^*\times R'^*$ of $G'=D^*\times D^*\times D^*$ fixes a unique line in $V'=V_1'\times V_2'\times V_3'$. If v' is any non-zero vector on this line and ℓ' is a non-zero H'-invariant linear form on V', then $\ell'(v')\neq 0$.

We remark that an example of an Eichler order of reduced discriminant (π) is the subring of integral matrices with $c\equiv 0 \pmod{\pi}$. Hence R^* is conjugate to the Iwahori subgroup $\Gamma_0(\pi)$ of $\mathrm{GL}_2(A)$ in the first case. We will see that the line fixed by M is also fixed by the triple product (g,g,g), where g is a non-trivial element in the normalizer of R^*. Similarly, the line fixed by M' is also fixed by the triple product (g',g',g'), where $g'=\pi_D$ is a non-trivial element in the normalizer of the units R'^* of the maximal order of D.

7

We now turn to the proofs of Propositions 6.1 and 6.3. For 6.1 we will only sketch the main ideas; the details appear in [P, Theorem 5.10]. Let $G=\mathrm{GL}_2(F)$. We may identify our linear form on $V_1\otimes V_2\otimes V_3$ with a non-zero linear map ℓ in the space $\mathrm{Hom}_G(V_1\otimes V_2, \tilde V_3)$, where $\tilde V_3$ is the contragredient of V_3. If v_i are new vectors fixed by $\mathrm{GL}_2(A)$ in V_i, it suffices to show that $\ell(v_1\otimes v_2)\neq 0$ in $\tilde V_3$. (It will then be a new vector in $\tilde V_3$ which pairs non-trivially with v_3.)

Write $V_i=\mathrm{Ind}_B^G\chi_i$ with $\chi_i\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}=\mu_i(a/d)$ as in Sect. 3, so V_i consists of functions satisfying $f\left(\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}g\right)=\chi_i\delta^{1/2}\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}f(g)$ with $\delta\begin{pmatrix} a & b\\ 0 & d\end{pmatrix}=|a/d|$. We then have an exact sequence of G-modules, where ind_T^G denotes compact induction from the split torus $T\subsetneq B$:

$$0\to \mathrm{ind}_T^G\chi_1\chi_2^{-1}\to V_1\otimes_G V_2\to \mathrm{Ind}_B^G\chi_1\chi_2\delta^{1/2}\to 0\,. \tag{7.1}$$

Indeed, $V_1\otimes_G V_2=\mathrm{Res}_G\mathrm{Ind}_{B\times B}^{G\times G}(\chi_1\times\chi_2)$ and the action of G on $B\times B\backslash G\times G=\mathbf{P}^1(F)\times\mathbf{P}^1(F)$ has precisely two orbits. The open orbit $(x\neq y)$ can be identified with $T\backslash G$ and the closed orbit $(x=y)$ with $B\backslash G$.

There are two cases to consider. If $\chi_1\chi_2\delta^{1/2}=\chi_3^{\pm}$ the map ℓ is a scalar multiple of the surjection in (7.1). Since $v_i(bk)=\chi_i(b)\delta(b)^{1/2}$ for $b\in B$, $k\in\mathrm{GL}_2(A)$, we see that $v_1\otimes v_2$ goes to a non-zero new vector in $\mathrm{Ind}_B^G(\chi_1,\chi_2\delta^{1/2})$. Hence $\ell(v_1\otimes v_2)\neq 0$.

If $\chi_1\chi_2\delta^{1/2}\neq\chi_3^{\pm}$, $F=T_\pi(v_1\otimes v_2)-[\chi_1\chi_2(\pi)-q\chi_1^{-1}\chi_2^{-1}(\pi)](v_1\otimes v_2)$ lies in the subspace $\mathrm{ind}_T^G\chi_1\chi_2^{-1}$, see [P]. If $\ell(v_1\otimes v_2)=0$ then, since ℓ is G-invariant, $\ell(F)=0$ also. But F, viewed as a function on G/T, takes a non-zero constant value on a *single* $\mathrm{GL}_2(A)$ orbit [the set of points whose reduction $(\mathrm{mod}\,\pi)$ is not on the diagonal of $\mathbf{P}^1(\mathbf{F}_q)\times\mathbf{P}^1(\mathbf{F}_q)$]. From this we will derive a contradiction.

Indeed, by Frobenius reciprocity, we have an isomorphism between $\mathrm{Hom}_G(\mathrm{ind}_T^G\chi_1\chi_2^{-1},\tilde V_3)$ and the one-dimensional space of linear forms m on V_3 satisfying $m(t^{-1}v)=\chi_1\chi_2^{-1}(t)m(v)$. Associated to m we have the G-linear map:

$$\ell_m(f)(v)=\int_{T\backslash G} f(g)m(gv)dg \tag{7.2}$$

for $f\in\mathrm{ind}_T^G\chi_1\chi_2^{-1}$ and $v\in V_3$.

By Proposition 2.3 we have $m(v_3)\neq 0$. Since v_3 is fixed by $\mathrm{GL}_2(A)$ and F takes a constant non-zero value on a single $\mathrm{GL}_2(A)$ orbit in $T\backslash G$, we may conclude from (7.2) that $\ell_m(F)(v_3)\neq 0$. Hence $\ell(F)\neq 0$ in $\tilde V_3$.

We now turn to the proof of 6.3. The second statement is clear, as V' has dimension 1. To prove the first, we will show that if V_3 is *any* representation of $G=\mathrm{GL}_2(F)$ with trivial central character and conductor $\leqq 1$ which is *not* isomorphic to the special representation Sp, then the unique G-invariant linear form on $\mathrm{Sp}\otimes\mathrm{Sp}\otimes V_3$ takes a non-zero value on the product of new vectors $v_1\otimes v_2\otimes v_3$. The vectors v_1 and v_2 are assumed to be fixed by the Iwahori subgroup $\Gamma_0(\pi)$; v_3 is fixed by $\Gamma_0(\pi)$ when $n(V_3)=1$, and by a maximal compact subgroup containing $\Gamma_0(\pi)$ when $n(V_3)=0$.

The action of $\Gamma_0(\pi)$ on $\mathbf{P}^1(F)$ has two orbits, which contain 0 and ∞ respectively. Let f_0 and f_∞ be the characteristic functions of these orbits. In the representation Sp, on the locally constant functions on $\mathbf{P}^1(F)$ modulo the constant functions, we have the linear relation $f_0+f_\infty=0$, and $f_0=-f_\infty$ is a new vector fixed by $\Gamma_0(\pi)$. The function $f_0\times f_\infty$ on $\mathbf{P}^1\times\mathbf{P}^1$ vanishes on the diagonal, so lies in the Schwartz space $S(T\backslash G)$ of the open orbit. It is non-zero on a single $\Gamma_0(\pi)$-orbit in $T\backslash G$, where it takes the value 1.

But $S(T\backslash G)$ is easily seen [P, Lemma 5.4] to lie in an exact sequence of G-modules

$$0\to\mathrm{Sp}\to S(T\backslash G)\to\mathrm{Sp}\otimes\mathrm{Sp}\to 0. \tag{7.3}$$

Hence the space of G-invariant linear forms on $\mathrm{Sp}\otimes\mathrm{Sp}\otimes V_3$ is isomorphic to $\mathrm{Hom}_G[S(T\backslash G),\tilde{V}_3]$, under the hypothesis that V_3 (hence $\tilde{V}_3$) is not isomorphic to Sp. By Frobenius reciprocity, there is an isomorphism between the (one-dimensional) space of T-invariant linear forms m on V_3 and $\mathrm{Hom}_G[S(T\backslash G),\tilde{V}_3]$, sending a T-invariant linear form m on V_3 to $\ell_m\in\mathrm{Hom}_G[S(T\backslash G),\tilde{V}_3]$ given by

$$\ell_m(f)(v)=\int_{T\backslash G} f(g)m(gv)dg \tag{7.4}$$

for $f\in S(T\backslash G)$ and $v\in V_3$. By Proposition 2.6 we have $m(v_3)\neq 0$. Since v_3 is also $\Gamma_0(\pi)$-invariant and $f=f_0\times f_\infty$ takes a non-zero constant value on a single $\Gamma_0(\pi)$ orbit, we may conclude from (7.4) that $\ell_m(f_0\times f_\infty)(v_3)\neq 0$. Hence $\ell(v_1\otimes v_2\otimes v_3)\neq 0$.

Remark 7.5. If V_1 and V_2 are unramified, and V_3 has conductor $n\geqq 1$, the non-zero linear form ℓ on $V=V_1\otimes V_2\otimes V_3$ which is G-invariant must vanish on $v_1\otimes v_2\otimes v_3$, where v_1 and v_2 are fixed by $\mathrm{GL}_2(A)$ and v_3 by $\Gamma_0(\pi^n)$. Indeed, the restriction of ℓ to $v_1\otimes v_2\otimes V_3$ would be a $\mathrm{GL}_2(A)$-invariant form on V_3, which is necessarily zero. Perhaps one should study the value of ℓ on $v_1\otimes v_2^*\otimes v_3$, where v_2^* is a new vector fixed by the units R^* in a maximal order with $R^*\cap\mathrm{GL}_2(A)=\Gamma_0(\pi^n)$.

Remark 7.6. We give an example to show that even for the equal conductor case, the invariant linear form may vanish on the tensor product of new forms in $V_1\otimes V_2\otimes V_3$. Let V_1, V_2, V_3 be supercuspidal representations of trivial central character and of conductor 2. The representations V_i when restricted to $\mathrm{GL}_2(A)$ have pieces W_i which are irreducible discrete series representations for $\mathrm{GL}_2(\mathbf{F}_q)$. So the linear form on $V_1\otimes V_2\otimes V_3$ gives rise to a $\mathrm{GL}_2(\mathbf{F}_q)$-invariant linear form on $W_1\otimes W_2\otimes W_3$. The new forms in V_i can be thought of as the unique vectors in W_i invariant under the split torus of $\mathrm{GL}_2(\mathbf{F}_q)$. Now assume that $V_2=V_3$ (this assumption is not essential but makes the argument more transparent), and therefore $W_2=W_3$, and assume that W_1 does not lie in the symmetric product of W_2 but rather lies in the exterior product of W_2 (it is easy to see that there are many discrete series representations W_1 with this property, and that any W_1 appears in some V_1). This clearly implies that the unique linear form takes the value zero on the product of new forms in this case.

8

We end by giving some global applications of the local results in Propositions 6.1 and 6.3. There are similar applications of Propositions 2.3 and 2.6, whose formulation we leave to the reader.

Let E be a global field, with no complex places. Let $\mathbf{A}$ be the ring of adeles of E, and let D be a quaternion division algebra over E which is ramified at every real place. For each place $\wp$ of E, we define $\varepsilon_\wp(D) = +1$ if $\wp$ is split in D, and $\varepsilon_\wp(D) = -1$ if $\wp$ is ramified in D. Then $\prod_\wp \varepsilon_\wp(D) = 1$ by global reciprocity.

Let V_1, V_2, and V_3 be admissible irreducible representations of the adelic group $G = D^*_{\mathbf{A}}/E^*_{\mathbf{A}}$, which is the direct product of a compact subgroup with a locally compact, totally disconnected subgroup. Let $V = V_1 \otimes V_2 \otimes V_3$ be the corresponding irreducible representation of G^3.

Proposition 8.1. *The space of G-invariant linear forms $\ell: V \to \mathbf{C}$ has dimension $\leqq 1$, with equality holding if and only if $\varepsilon_\wp(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \varepsilon_\wp(D)$ for all places $\wp$ of E.*

Proof. The representation V is the restricted tensor product of local representations $V_\wp = V_{1,\wp} \otimes V_{2,\wp} \otimes V_{3,\wp}$ of $(D \otimes E_\wp)^{*3}/E_\wp^{*3} = G_\wp^3$. Since the space of $G_\wp$-invariant forms on $V_\wp$ has dimension $\leqq 1$, this shows that the G-invariant forms have dim $\leqq 1$ on V. A necessary condition for equality is that dim $\mathrm{Hom}_{G_\wp}(V_\wp, \mathbf{C}) = 1$ for all $\wp$; by Proposition 5.1 (and the analogous result at real completions [P]) this holds if and only if $\varepsilon_\wp(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \varepsilon_\wp(D)$.

In fact, the necessary condition on the existence of local forms $\ell_\wp \neq 0$ which are $G_\wp$-invariant is sufficient, by Proposition 6.1. Indeed, the tensor product $\ell = \otimes \ell_\wp$ is non-zero and G-invariant on V.

We henceforth assume that the irreducible representations V_i of G satisfy the further conditions:

(8.2) At every $\wp$ which is ramified in D we have dim $V_{i,\wp} = 1$.

(8.3) At every $\wp$ which is split in D we have equal conductors $n_\wp = n(V_{1,\wp}) = n(V_{2,\wp}) = n(V_{3,\wp}) \leqq 1$.

Let v_i^K denote the global new vectors in V_i, which are fixed by the open compact subgroup:

$$K = \prod_{\wp \text{ ram.}} R_\wp^* \times \prod_{\wp \text{ split}} \Gamma_0(\pi^{n_\wp}) \tag{8.4}$$

of G. Then the v_i^K are unique up to scalars, and $v^M = v_1^K \otimes v_2^K \otimes v_3^K$ is a basis for the line V^M fixed by the open compact subgroup $M = K^3$ of G^3.

Proposition 8.5. *Let ℓ be a G-invariant linear form on V. Then the new vector v^M is a test vector for ℓ: we have $\ell \neq 0$ in* $\mathrm{Hom}_G(V, \mathbf{C})$ *if and only if $\ell(v^M) \neq 0$ in* $\mathbf{C}$.

Proof. We have $\ell = \otimes \ell_\wp$ so $\ell(v^M) = \prod \ell_\wp(v_\wp^M)$. But we have seen in Propositions 6.1 and 6.3 that $\ell_\wp \neq 0$ implies $\ell_\wp(v_\wp^M) \neq 0$.

Proposition 8.5 is often useful in the study of *automorphic* representations V of G^3. We say that V_i is an automorphic representation of $G = D^*_{\mathbf{A}}/E^*_{\mathbf{A}}$ if there exists a D^*-invariant linear form $m_i: V \to \mathbf{C}$. The theorem of "multiplicity-one" shows that when m_i exists, it is unique up to scalars. Since the quotient $D^* E^*_{\mathbf{A}} \backslash D^*_{\mathbf{A}}$ is compact, the integral

$$(8.6) \qquad \ell(v_1 \otimes v_2 \otimes v_3) = \int_{D^* E^*_{\mathbb{A}} \backslash D^*_{\mathbb{A}}} m_1(gv_1) m_2(gv_2) m_3(gv_3) dg$$

gives a G-invariant linear form on an automorphic representation V.

Of course, a necessary condition for the form ℓ defined by (8.6) to be non-zero is that $\varepsilon_\wp(\sigma_1 \otimes \sigma_2 \otimes \sigma_3) = \varepsilon_\wp(D)$ for all $\wp$, by Proposition 8.1. But this collection of local conditions is not sufficient. Jacquet conjectured, and Kudla-Harris [K-H] proved, that ℓ is non-zero provided the local conditions are met *and* the Garrett L-function $L(V, s)$ is non-zero at its central critical point $s = \frac{1}{2}$. For the definition and analytic properties of $L(V, s)$, see [Ga] and [PS-R].

In the case when V is automorphic and satisfies (8.2)–(8.3) we deduce that $L(V, \frac{1}{2})$ and $\ell(v^M)$ are either both zero or both non-zero. In fact, one can give a precise formula for $L(V, \frac{1}{2})$ in which the only term which can vanish is the value $\ell(v^M)$ on our global test vector [Gr-K].

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