ON CERTAIN MULTIPLICITY ONE THEOREMS

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Abstract. We prove several multiplicity one theorems in this paper. For k a local field not of characteristic two, and V a symplectic space over k, any irreducible admissible representation of the symplectic similitude group $\mathrm{GSp}(V)$ decomposes with multiplicity one when restricted to the symplectic group $\mathrm{Sp}(V)$. We prove the analogous result for $\mathrm{GO}(V)$ and $\mathrm{O}(V)$, where V is an orthogonal space over k. When k is non-archimedean, we prove the uniqueness of Fourier-Jacobi models for representations of $\mathrm{GSp}(4)$, and the existence of such models for supercuspidal representations of $\mathrm{GSp}(4)$.

Contents

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1. Introduction

In this paper we prove several multiplicity one theorems. Our initial aim when writing this paper was to prove a multiplicity one theorem for the restriction of an irreducible admissible representation of GSp(4) to Sp(4) for the p-adic case. As is well known, such theorems are easy consequences of the uniqueness of Whittaker models, when they exist. But not every representation has a Whittaker model. Our initial attempt was thus to use the analogous concept of Fourier-Jacobi models (recalled below), for which uniqueness was proved by Baruch and Rallis [BR] for the case of Sp(4). This required us to extend their work from Sp(4) to GSp(4), which became a major exercise in itself, useful in its own right.

We now introduce some notation. Let k denote a local field not of characteristic two. Let V denote a finite-dimensional vector space over k with a non-degenerate bilinear form $\langle\ ,\ \rangle$ that is either symmetric or skew-symmetric. Let $\mathrm{U}(V)$ denote the associated automorphism group:

$$\bigl\{ g \in \mathrm{Aut}(V) \bigm| \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle \text{ for all } v_1, v_2 \in V \bigr\}.$$

Let $\mathrm{GU}(V)$ denote the corresponding similitude group:

$$\bigl\{ g \in \mathrm{Aut}(V) \bigm| \exists \lambda_g \in k^\times,\, \forall v_1, v_2 \in V,\, \langle gv_1, gv_2\rangle = \lambda_g \langle v_1, v_2\rangle \bigr\}.$$

We also denote these groups by $\mathrm{Sp}(V)$ and $\mathrm{GSp}(V)$ (resp. $\mathrm{O}(V)$ and $\mathrm{GO}(V)$) if $\langle\ ,\ \rangle$ is skew-symmetric (resp. symmetric). If the dimension of V is $2n$, we write $\mathrm{Sp}(V)$ also as $\mathrm{Sp}(2n)$, and $\mathrm{GSp}(V)$ as $\mathrm{GSp}(2n)$.

We now introduce the Fourier-Jacobi models. Let e_1 be any nonzero vector in V. Let J be the stabilizer of e_1 in $\mathrm{Sp}(2n)$. Then $J \cong \mathrm{Sp}(2n-2) \ltimes H$ where H is the $(2n-1)$-dimensional Heisenberg group. Let $Z \cong k$ be the center of H, and $\psi\colon Z \to \mathbb{C}^\times$ a nontrivial character. Let θ_ψ be the oscillator representation of H with central character ψ. It is well known that θ_ψ can be extended to a representation of $\tilde{J} = \tilde{S}H$ with $\tilde{S}$ the two-fold metaplectic cover of $\mathrm{Sp}(2n-2)$. Let σ be an irreducible admissible genuine representation (i.e., nontrivial on the kernel of the map from $\tilde{S}$ to $\mathrm{Sp}(2n-2)$) of $\tilde{S}$. Then $\sigma \otimes \theta_\psi$ is an irreducible admissible representation of $\tilde{J}$ which, as both σ and θ_ψ are genuine, is in fact a representation of J.

Remark. Any irreducible admissible representation of J on which Z operates via ψ is of this form, cf. theorem 2.6.2 of [BS]; this is part of what is called "Mackey theory", cf. [Ma].

Baruch and Rallis prove the following theorem in [BR].

Theorem 1.1. *Suppose k is non-archimedean. Let π be an irreducible admissible representation of* $\mathrm{Sp}(4)$*. Then for any irreducible admissible representation μ of J on which Z operates via a nontrivial character,* $\dim \mathrm{Hom}_J[\pi, \mu] \leq 1$*.*

Note that in their statement of the theorem, the field k has characteristic zero. However, the proof only requires that the characteristic is not two.

We prove an analogous theorem for $\mathrm{GSp}(4)$. Although our proof is modelled on the proof in [BR], many details are quite different; in particular, the proof for the 'open cell' (see §6) is totally different.

Theorem 1.2. *Suppose k is non-archimedean. Let π be an irreducible admissible representation of* $\mathrm{GSp}(4)$*. Then for any irreducible admissible representation μ of J on which Z operates via a nontrivial character,* $\dim \mathrm{Hom}_J[\pi, \mu] \leq 1$*.*

Remark. By Frobenius reciprocity, a J-invariant map from π to μ is equivalent to an embedding of π into the induced representation $\mathrm{Ind}_J^{\mathrm{GSp}(V)} \mu$, called a Fourier-Jacobi model of π.

One can use the uniqueness theorem for Fourier-Jacobi models to deduce the multiplicity one theorem for restriction from $\mathrm{GSp}(4)$ to $\mathrm{Sp}(4)$. (We omit the details, since we prove a more general theorem via other methods.)

But in a similar vein, i.e., by the method of "models," we give a proof of the multiplicity one theorem for the restriction of an irreducible admissible representation of $\mathrm{GL}(n)$ to $\mathrm{SL}(n)$ due originally to Tadić [T], who proved it by an elaborate analysis using the full classification of irreducible admissible representations of $\mathrm{GL}(n)$ (due to Zelevinsky [Ze]).

Theorem 1.3. *Any irreducible admissible representation of* $\mathrm{GL}(n)$ *decomposes with multiplicity one when restricted to* $\mathrm{SL}(n)$*.*

However, after proving the multiplicity one theorem for GSp(4) by the method of Fourier-Jacobi models, we realized that a more general multiplicity one theorem for restriction from $\mathrm{GU}(V)$ to $\mathrm{U}(V)$ is an easy consequence of a result in linear algebra (of classical groups), combined with the usual formalism of Gelfand pairs adapted to p-adic groups by Gelfand-Kazhdan [GKa] and developed further by Bernstein-Zelevinsky [BZ]. This lemma in linear algebra, valid for any field of characteristic not 2, says (in the symplectic case) that for any g in $\mathrm{GSp}(2n)$, g and ${}^tg$ are conjugate by an element of $\mathrm{GSp}(2n)$ of similitude -1. Forms of this lemma are available for all classical groups in [MVW]. The extension of this result of [MVW] to the symplectic similitude group was observed in [P2]. But for our purposes, its most precise form given in a very recent paper of Vinroot [V1] is what will be essential.

We prove the following theorem in this paper.

Theorem 1.4. *Let V be a finite-dimensional vector space over k with a non-degenerate symmetric or skew-symmetric form $\langle\ ,\ \rangle$. Then any irreducible admissible representation $\widetilde{\pi}$ of $\mathrm{GU}(V)$ decomposes with multiplicity one when restricted to $\mathrm{U}(V)$; i.e., for any irreducible, admissible representation π of $\mathrm{U}(V)$,*

$$\dim \mathrm{Hom}_{\mathrm{U}(V)}[\widetilde{\pi}, \pi] \leq 1.$$

Here is one consequence of the theorem. Suppose k is non-archimedean and has odd residue characteristic. Then Brooks Roberts [R] has constructed a theta correspondence between orthogonal similitude and symplectic similitude groups. He proves that this correspondence is one to one for those representations that decompose with multiplicity one when restricted to the corresponding classical groups. But from the theorem 1.4, this hypothesis is always satisfied.

In §2, we give the rather simple proofs of theorems 1.3 and 1.4. We also state a conjecture concerning multiplicity one restriction for more general groups. From §3 on, we will assume that k is non-archimedean, and will work exclusively with the group GSp(4). We prove theorem 1.2 about the uniqueness of the Fourier-Jacobi model for its representations. For completeness, we also prove the existence of Fourier-Jacobi models for supercuspidal representations.

Acknowledgement: Theorem 1.4 has long been expected, at least in the rank-two symplectic case. We thank Paul Sally for suggesting that we study it, and for enthusiastic encouragement on several occasions. We also thank E. Moshe Baruch and David Soudry for helpful correspondence. We thank the Institute of Mathematical Sciences, Singapore for the invitation to participate in their special semester activity on representation theory in the summer of 2002, where the two authors shared an office, and the collaboration on this theorem was conceived. The first-named author also thanks the Tata Institute for an invitation in the summer of 2003 to continue our work there.

2. Multiplicity one upon restriction

In this section, we (re)prove theorem 1.3 and prove theorem 1.4. We deal with the archimedean case first, since it is easy, and from then on assume that k is non-archimedean.

2.1. **Archimedean case.** We recall some general Clifford theory.

Lemma 2.1. *If G is a group with center Z, and H is a normal subgroup with G/ZH a finite cyclic group, then any irreducible representation of G decomposes with multiplicity one when restricted to H. (If G is a real Lie group, then by a representation of G, we mean either a continuous representation in a Frechet space, or a Harish-Chandra module.)*

Proof. Most of this follows from Lemma 2.1.1 of [GKn]. The multiplicity one result is just as for finite groups: the "Mackey obstruction" vanishes for cyclic quotients. □

Suppose k is archimedean. Then the lemma implies both theorem 1.3 and theorem 1.4, since the quotient in the lemma has order 1 or 2.

Therefore, assume for the rest of this section that k is non-archimedean.

2.2. **Restriction from $\mathrm{GL}(n)$ to $\mathrm{SL}(n)$.** In this section only, there is no restriction on the characteristic of k.

Our proof of theorem 1.3 depends on the following theorem of Zelevinsky, corollary 8.3 of [Ze].

Theorem 2.2. *Let $\widetilde{\pi}$ be an irreducible admissible representation of $\mathrm{GL}(n)$. Let U_n be the group of upper-triangular unipotent matrices in $\mathrm{GL}(n)$. Then there exists a character $\theta\colon U_n \to \mathbb{C}^\times$ such that $\mathrm{Hom}_{U_n}[\widetilde{\pi}, \theta] \cong \mathbb{C}$.*

Proof of Theorem 1.3. If $\mathrm{Hom}_{U_n}[\widetilde{\pi}, \theta] \cong \mathbb{C}$, then $\mathrm{Hom}_{U_n}[\pi, \theta]$ is also isomorphic to $\mathbb{C}$ for some irreducible admissible representation π of $\mathrm{SL}(n)$ which appears in the restriction of $\widetilde{\pi}$ with multiplicity exactly one. Since the set of irreducible admissible representations π of $\mathrm{SL}(n)$ such that $\mathrm{Hom}_{\mathrm{SL}(n)}[\widetilde{\pi}, \pi] \neq 0$ lies in a single $\mathrm{GL}(n)$-orbit (for the inner conjugation action of $\mathrm{GL}(n)$ on $\mathrm{SL}(n)$, and hence on representations of $\mathrm{SL}(n)$), this completes the proof of the theorem. □

2.3. **Restriction from $\mathrm{GU}(V)$ to $\mathrm{U}(V)$.** We will prove theorem 1.4 by applying the method of Gelfand pairs:

Theorem 2.3. *Suppose G is the group of k-points of an algebraic k-group, H is the group of k-points of a closed k-subgroup, and G/H carries a G-invariant distribution. Suppose that τ is an algebraic anti-involution of G that preserves H and takes each H-conjugacy class in G into itself. Then*

(a) Every H-invariant distribution on G is τ-invariant.
(b) For any irreducible, smooth representations $\widetilde{\pi}$ of G and π of H, with smooth duals $\widetilde{\pi}^\vee$ and $\pi^\vee$ respectively, let $m(\widetilde{\pi}, \pi)$ denote the dimension of the space of H-invariant linear maps from $\widetilde{\pi}$ to π. Then $m(\widetilde{\pi}, \pi) m(\widetilde{\pi}^\vee, \pi^\vee) \leq 1$.

Proof. Both parts of the theorem are due to Gelfand-Kazhdan as refined by Bernstein-Zelevinsky. For part (a) we refer to theorem 6.13 of [BZ], and for part (b) we refer to lemma 4.2 of [P1]. □

Suppose that $G = \mathrm{GU}(V)$, $H = \mathrm{U}(V)$, and $\widetilde{\pi}$ is an irreducible, admissible representation of G. From generalities, we know that as a representation of H, $\widetilde{\pi}$ decomposes into a finite direct sum of irreducible representations:

$$\widetilde{\pi} \cong \pi_1 \oplus \cdots \oplus \pi_\ell.$$

Then

$$\widetilde{\pi}^\vee \cong \pi_1^\vee \oplus \cdots \oplus \pi_\ell^\vee.$$

Thus, $m(\widetilde{\pi}, \pi) = m(\widetilde{\pi}^\vee, \pi^\vee)$ for any summand π of $\widetilde{\pi}$. Since $m(\widetilde{\pi}, \pi) \geq 1$, theorem 2.3 will imply $m(\widetilde{\pi}, \pi) = 1$ as long as there exists an anti-involution τ as in the theorem.

Thus, we will have proved theorem 1.4 if we can find a suitable anti-involution τ. This is provided by the following lemmas, which follow from the work of Vinroot (see corollary 1 of [V1] in the symplectic case, and [V2] in the orthogonal case.)

Lemma 2.4. *Suppose V is a symplectic space. Fix $d \in \mathrm{GSp}(V)$ of similitude -1. Let τ be the anti-involution on $\mathrm{GSp}(V)$ defined by $\tau(g) = d^tgd^{-1}$. Then for any $g \in \mathrm{GSp}(V)$, g and $\tau(g)$ are conjugate by an element of $\mathrm{Sp}(V)$.*

Lemma 2.5. *Suppose V is an orthogonal space. Let τ be the anti-involution on $\mathrm{GO}(V)$ defined by $\tau(g) = {}^tg$. Then for any $g \in \mathrm{GO}(V)$, g and $\tau(g)$ are conjugate by an element of $\mathrm{O}(V)$.*

2.4. **A conjecture on multiplicity one restriction.** In this paper we have proved a multiplicity one theorem for restriction from $\mathrm{GU}(V)$ to $\mathrm{U}(V)$ (where $\mathrm{U}(V)$ is symplectic or orthogonal), as well as reproved a theorem (originally due to Tadić) about restriction from $\mathrm{GL}(n)$ to $\mathrm{SL}(n)$. We note that the theorem about $\mathrm{GU}(V)$ has been proved by a generality valid for all fields not of characteristic two, whereas the theorem on $\mathrm{GL}(n)$ is proved by both Tadić and ourselves using non-archimedean local fields (as the general lemma from linear algebra that one may wish to be true, i.e., for h a fixed element of $\mathrm{GL}(n)$, A and h^tAh^{-1} are conjugate via $\mathrm{SL}(n)$, does not hold, as one can easily see). There is, however, the possibility that such a lemma holds for distributions on $\mathrm{GL}(n)$, and therefore the multiplicity one theorem can indeed by proved by the method of Gelfand pairs, as developed by Gelfand and Kazhdan [GKa]. This suggests the possibility that, just like the uniqueness of Whittaker models, proved for all quasi-split groups, the following too is true in this generality, and could be proved by analyzing invariant distributions.

Conjecture 2.6. *Let $\mathbf{G}$ be a quasi-split reductive algebraic group over a local field k. Let $\widetilde{\mathbf{G}}$ be a reductive algebraic group containing $\mathbf{G}$ such that the derived groups of $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ are the same, and such that $\widetilde{\mathbf{G}}/\mathbf{G}$ is connected. Then multiplicity one holds for restriction of irreducible admissible representations of $\widetilde{\mathbf{G}}(k)$ to $\mathbf{G}(k)$.*

Remark. It is well known that multiplicity one is not true for restriction from $D^\times$ to $SL_1(D)$, D a division algebra over a local field, cf. section 4 of [LL]. So the quasi-splitness assumption seems necessary.

Remark. One example for which the conjecture would be especially useful is where $\widetilde{\mathbf{G}}$ is a unitary group $\mathrm{U}(n)$, and $\mathbf{G} = \mathrm{SU}(n)$. Just like its close cousin $(\mathrm{GL}(n), \mathrm{SL}(n))$, multiplicity one cannot be proved purely by methods of linear algebra, but will require careful analysis of invariant distributions.

3. Fourier-Jacobi models: Basic setup and notation

Assume from now on that k is non-archimedean. Let

$$j = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix},$$

so that $\mathrm{Sp}(4)$ is the subgroup of $\mathrm{GL}(4)$ defined by

$${}^t g j g = j,$$

and $\mathrm{GSp}(4)$ is the subgroup of $\mathrm{GL}(4)$ defined by

$${}^t g j g = \lambda(g) j \quad \text{for some } \lambda(g) \in k^\times.$$

Let C denote the center of $\mathrm{GSp}(4)$. For $\lambda \in k^\times$, $v \in k^2$, $A \in \mathrm{GL}_2(k)$, $z \in k$, and $B \in M_2(k)$ with $B_{11} = B_{22}$, let

$$\begin{aligned}
\mathsf{m}(\lambda, A) &= \begin{pmatrix} \lambda & & & \\ & A_{11} & A_{12} & \\ & A_{21} & A_{22} & \\ & & & \lambda^{-1}\det A \end{pmatrix} \\
\mathsf{h}(v, z) &= \begin{pmatrix} 1 & v_1 & v_2 & z \\ & 1 & & v_2 \\ & & 1 & -v_1 \\ & & & 1 \end{pmatrix} \\
\mathsf{q}(A, B, \lambda) &= \begin{pmatrix} A & \\ & \lambda A^* \end{pmatrix} \begin{pmatrix} I & B \\ & I \end{pmatrix}, \\
&\qquad \text{where } A^* = \omega\, {}^t A^{-1} \omega^{-1},\ \omega = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \\
\mathsf{n}(B) &= \mathsf{q}(I, B, 1)
\end{aligned}$$

$$w' = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} = \mathsf{q}(\omega, I, 1)$$

$$w'' = \begin{pmatrix} & & & -1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix}.$$

Let M, H, and N denote the images of m, h, and n, respectively. Then $P = MH$ is the Klingen parabolic subgroup of GSp(4), and its unipotent radical H is the Heisenberg group. The image of q is the Siegel parabolic subgroup Q, whose unipotent radical is N. Let

$$\begin{aligned} M' &= \left\{ \mathsf{m}(1, m) \,\middle|\, m \in \mathrm{SL}(2) \right\} \\ J &= M'H \quad \text{(the } \textbf{Fourier-Jacobi} \text{ group)} \\ L &= \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \lambda & \\ & & & \lambda \end{pmatrix} \,\middle|\, \lambda \in k^\times \right\} \\ Z &= \left\{ \mathsf{h}(0, z) \,\middle|\, z \in k \right\}. \end{aligned}$$

Note that Z is the center of both H and J. Note also that $M = CLM'$ (in any order), and thus $P = JCL$.

Let τ be the involution on $\mathrm{GSp}(4) \times J$ defined by

$$\tau(g, h) = (d^{-1} j^{-1} {}^t g j d, d^{-1} h^{-1} d),$$

where

$$d = \begin{pmatrix} -1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix}.$$

Clearly, d normalizes the subgroup J, and the involution τ when restricted to the center Z of the Heisenberg group H is trivial. We will abuse notation to denote the restriction of τ to any τ-invariant subgroup of GSp(4) also by τ. We note that

$$\tau(\mathsf{m}(\lambda, A)) = \mathsf{m}(\lambda^{-1} \det A, d_1^{-1} {}^t A d_1),$$

where $d_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

Let ΔJ denote the image of J under the diagonal embedding $J \to \mathrm{GSp}(4) \times J$. By the method of Gelfand pairs, as developed by Gelfand and Kazhdan [GKa] (and applied, for example, in [BR] and in §2.3), to prove theorem 1.2, it suffices to show that any distribution on $\mathrm{GSp}(4) \times J$ which is bi-invariant under $\Delta J \subset \mathrm{GSp}(4) \times J$, and ψ-quasi-invariant under translations of the second variable by Z, is fixed by the involution τ.

Just as in theorem 2.6 of [BR], this is equivalent to proving the following theorem.

Theorem 3.1. *Let T be a distribution on* $\mathrm{GSp}(4)$ *which is invariant under inner-conjugation by J, and $L_z \cdot T = \psi(z)T$ for all $z \in Z$ (where L_z is left translation by z). Then T is fixed by τ.*

This is clearly equivalent to the following:

Theorem 3.2. *Let T be as in theorem 3.1, and suppose in addition that T is τ-skew-invariant. Then T is identically zero.*

4. General strategy for proving uniqueness

We outline the general strategy of our proof of the theorem 3.2. Implicitly, it involves decomposing $\mathrm{GSp}(4) = P \cup Pw'P \cup Pw''P$ into a disjoint union of J-invariant, τ-invariant subsets $X_0, X_1, \ldots, X_m = P$, such that $Y_i = \cup_{j \geq i} X_j$ is a closed subset of $\mathrm{GSp}(4)$, and X_i is open in Y_i. We begin by showing that T vanishes on the open subset X_0, and thus restricts to its complement. Continuing in this way, we will show in turn that T vanishes on the complement of Y_i for all i (the final case, i.e., vanishing of T on P is the subject of §8), completing the proof that $T = 0$. We emphasize that the method used in the ith step will vary with i. In many cases, we will show that every J-orbit in X_i is τ-stable, and use the following lemma of Bernstein, cf. lemma 2.7 of [BR].

Lemma 4.1. *Let X be the set of k-points of a k-variety on which a group J acts, as well as an automorphism τ of order two normalizing the action of J, i.e., in the automorphism group of X, $\tau J \tau^{-1} = J$. If every J-orbit in X is stable under τ, then every J-invariant distribution on X is τ-invariant.*

In some cases we will show that every J-orbit in X_i is stable under left multiplication by Z, and appeal to the following lemma, cf. lemma 2.8 of [BR].

Lemma 4.2. *Let X be a J-stable subvariety of* $\mathrm{GSp}(4)$ *which is stable under Z (where J acts by conjugation and Z by right translation). If every J orbit in X is stable under Z, then a distribution on X which is J-invariant, and on which Z operates by ψ, is trivial.*

These properties of the orbits imply that all distributions on X_i (not just those on the closure of X_i) with our invariance properties must vanish on X_i, a stronger result than we need. In a few cases, we will have to use more delicate means to show that T vanishes on X_i, but these cases can be reduced to [BR], which is what we do in this paper.

5. Using the result of Baruch and Rallis on Sp(4)

Let $\mathcal{G} = k^\times \mathrm{Sp}(4)$. Clearly $\mathcal{G}$ is an open subgroup of $\mathrm{GSp}(4)$, and therefore any distribution on $\mathrm{GSp}(4)$ can be restricted to it. Let T be a distribution on $\mathrm{GSp}(4)$ with invariance properties under J and Z as in the statement of theorem 3.2, and which transforms under $k^\times$ by a given character (the central character), and is τ skew-invariant. The restriction to $\mathcal{G}$ of such a distribution is equivalent (by a form of Frobenius reciprocity) to a distribution on $\mathrm{Sp}(4)$ with invariance under J and Z and which further is τ skew-invariant. The $\mathrm{Sp}(4)$ theorem of Baruch and Rallis implies that this distribution on $\mathrm{Sp}(4)$ is zero. Hence our distribution T is zero on this subgroup $\mathcal{G}$. In the next sections, we analyze the possible support for the distribution T.

6. Open cell

This section is devoted to proving the following result:

Lemma 6.1. *Every distribution on $Pw''P$ satisfying the invariance properties of theorem 3.1 is τ-invariant.*

6.1. **Transferring the problem from $Pw''P$ to smaller spaces.** Let $X = Pw''P$. Clearly, $X = HMw''H$, with H acting on X by conjugation. Thus H-invariant distributions on

$$X = HMw''H \cong HMH$$

can be identified with distributions on $P = MH$ under the map

$$h_1 m w'' h_2 \in HMw''H \mapsto m h_2 h_1 \in MH.$$

Since w'' commutes with $M' \cong \mathrm{SL}(2)$, for elements $m_1 \in \mathrm{SL}(2)$,

$$m_1(h_1mw''h_2)m_1^{-1} = (m_1h_1m_1^{-1})(m_1mm_1^{-1})w''(m_1h_2m_1^{-1}).$$

Therefore under the identification of H-invariant distributions on $X = HMw''H$ with distributions on MH, the J-invariant distributions correspond to distributions on $M \times H$ on which $\mathrm{SL}(2)$ operates in the natural way by the inner-conjugation action.

It can be checked that $\tau(w'') = w''$. Therefore, for $g = h_1mw''h_2$, $\tau(g) = \tau(h_2)w''\tau(m)\tau(h_1) = \tau(h_2)w''\tau(m)w''^{-1}w''\tau(h_1)$. Therefore under the identification of distributions on $X = HMw''H$ with distributions on MH through the map $(h_1mw''h_2) \mapsto (m, h_2h_1)$, the involution τ on X corresponds to the involution

$$(m, h) \mapsto (w''\tau(m)w''^{-1}, \tau(h)).$$

Thus we are reduced to proving that $\mathrm{SL}(2)$-invariant distributions on $M \times H$ are invariant under this latter involution.

Actually, we are looking at distributions on $X = HMw''H$ on which Z acts on the left via ψ. Clearly, distributions on X which are H-invariant and (Z, ψ)-invariant correspond to distributions on MH which are (Z, ψ)-invariant. These correspond to distributions on $MH/Z \cong M \times k^2$.

Since $w''\tau(m)w''^{-1} = \mathsf{m}(\lambda, d_1^{-1\,t}Ad_1)$, for $m = \mathsf{m}(\lambda, A)$, we are finally reduced to proving the following result:

Lemma 6.2. *An* $\mathrm{SL}(2)$*-invariant distribution on* $\mathrm{GL}(2) \times k^2$ *is invariant under* $\tau' : (g, v) \mapsto (d_1^{-1\,t}gd_1, d_2v)$*, where* $d_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ *and* $d_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Before we proceed further, we note the following lemma.

Lemma 6.3. *For any* $g, w \in \mathrm{GL}(2)$ *with* $\det w = -1$*, the matrices* g *and* w^tgw^{-1} *are conjugate by an element of* $\mathrm{SL}(2)$.

This is a special case of lemma 2.4 (and is not difficult to prove directly).

Remark. It follows from this lemma that representations of $\mathrm{GL}(2)$ restrict to $\mathrm{SL}(2)$ without multiplicity, something that is already clear from Whittaker model considerations. Unfortunately, there is no analogue of lemma 6.3 for higher n, and therefore there is no Gelfand pairs proof of Tadić's theorem.

6.2. **On a certain quadratic form.** Let k^2 be the 2-dimensional vector space over k with the standard symplectic structure $\langle\ ,\ \rangle$. Associated to any $g \in \mathrm{GL}_2(k)$, we have a quadratic form Q_g on k^2 defined by

$$Q_g(v) = \langle gv, v\rangle.$$

It can be seen that Q_g is a non-degenerate quadratic form if and only if the eigenvalues of g (in the algebraic closure $\bar{k}$ of k) are distinct. However, we will not have any occasion to use this fact.

Lemma 6.4. *For an element $g \in \mathrm{GL}(2)$, let $Z(g)$ denote its centralizer in $\mathrm{GL}(2)$. Then we have $\mathrm{SO}(Q_g) = Z(g) \cap \mathrm{SL}(2)$.*

Proof. Clearly,

$$\begin{aligned}
t \in \mathrm{SO}(Q_g) &\iff Q_g(tv) = Q_g(v) && \text{for all } v \in k^2, \text{ and } \det t = 1,\\
&\iff \langle gtv, tv\rangle = \langle gv, v\rangle && \text{for all } v \in k^2, \text{ and } \det t = 1,\\
&\iff \langle t^{-1}gtv, v\rangle = \langle gv, v\rangle && \text{for all } v \in k^2, \text{ and } \det t = 1,\\
&\iff \langle [g - t^{-1}gt]v, v\rangle = 0 && \text{for all } v \in k^2, \text{ and } \det t = 1.
\end{aligned}$$

Observe that for $v \neq 0$, $\langle w, v\rangle = 0$ if and only if $w = \lambda v$ for some $\lambda \in k$. Therefore, an element $t \in \mathrm{SL}(2)$ belongs to $\mathrm{SO}(Q_g)$ if and only if for any $v \in k^2$, $[g - t^{-1}gt]v = \lambda_v v$ for some $\lambda_v \in k$.

It is well known that if every vector of a vector space is an eigenvector for a given linear operator, then the linear operator must be a multiple of the identity. Therefore,

$$g - t^{-1}gt = \lambda I, \quad \text{for some } \lambda \in k.$$

Taking the trace, we find that λ must be zero, i.e., $t \in Z(g)$. The lemma follows. □

Lemma 6.5. *For any quadratic form q on k^2 and vectors v_1, v_2 with $q(v_1) = q(v_2) \neq 0$, there exists an element $g \in \mathrm{SO}(q)$ with $gv_1 = v_2$.*

Proof. This is the usual Witt's theorem, except for the conclusion that g can be chosen to have determinant 1. Given v_1, v_2 with $q(v_1) = q(v_2) \neq 0$, there exist $w_1 \perp v_1$ and $w_2 \perp v_2$. Since the discriminant of q is equal to $q(v_1)q(w_1)$, as well to $q(v_2)q(w_2)$, we may assume that that $q(w_1) = q(w_2)$. Clearly, the transformations that take v_1 to v_2 and w_1 to $\pm w_2$ are in $\mathrm{O}(q)$, and one of them has determinant 1. □

6.3. **Proof of Lemma 6.2.** Let

$$\begin{array}{rccc} \pi\colon & \mathrm{GL}(2)\times k^2 & \to & k\times k\times k \\ & (g,v) & \mapsto & (\mathrm{tr}(g),\det(g),\langle gv,v\rangle), \end{array}$$

where $\langle\ ,\ \rangle$ is the standard symplectic form on k^2 with $\langle e_1,e_2\rangle = 1 = -\langle e_2,e_1\rangle$, and $\langle e_1,e_1\rangle = \langle e_2,e_2\rangle = 0$.

It is easy to see that this mapping is SL(2)-invariant and is also τ'-invariant. (For τ' invariance, we note that $\langle gv,w\rangle = \langle v, d''{}^t g d''^{-1} w\rangle$, for $d'' = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and that $d'' = d_1 d_2$.) We will prove the proposition by showing that any SL(2)-invariant distribution supported on a fiber of π is τ'-invariant. This is sufficient by the Bernstein Localization theorem (lemma 4.1). We will achieve this by dividing the possible fibers into three cases. But first we introduce the following notation.

If two elements (g_1,v_1) and (g_2,v_2) are in the same SL(2)-orbit, i.e., there exists $s\in \mathrm{SL}(2)$ such that $(g_2,v_2) = (sg_1s^{-1}, sv_1)$, we write $(g_1,v_1)\sim_{\mathrm{SL}(2)} (g_2,v_2)$.

Case 1: Consider a fiber of π lying over (x,y,z), where $z\neq 0$. We will show any SL(2)-orbit in such a fiber is τ'-invariant. That is, we will prove that

$$(g,v)\sim_{\mathrm{SL}(2)} (d_1^{-1}{}^t g d_1, d_2 v),$$

for any (g,v) such that $\langle gv,v\rangle\neq 0$ where $d_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $d_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

By lemma 6.3, there exists $s\in\mathrm{SL}(2)$ such that $d_1^{-1}{}^t g d_1 = sgs^{-1}$. Further by combining lemmas 6.4 and 6.5, there exists $t\in Z(g)\cap \mathrm{SL}(2)$ such that $s^{-1}d_2 v = tv$. Therefore,

$$\begin{array}{rcl} (d_1^{-1}{}^t g d_1, d_2 v) & = & (sgs^{-1}, d_2 v) \\ & \sim_{\mathrm{SL}(2)} & (g, s^{-1}d_2 v) \\ & = & (g, tv) \\ & \sim_{\mathrm{SL}(2)} & (g,v). \end{array}$$

Case 2: We next look at the fiber over an element (x,y,z) with $x^2\neq 4y$ and $z=0$. Since $z=0$, for an element (g,v) in the fiber, $\langle gv,v\rangle = 0$, and therefore v is an eigenvector of g. Since $x^2\neq 4y$, eigenvalues of g (in $\bar{k}$) are distinct, hence g is diagonalizable over k

with distinct eigenvalues. Call such a fiber $F_{(x,y,z)}$. We have a map

$$\begin{array}{rccc} \mu\colon & F_{(x,y,z)} & \rightarrow & \mathrm{GL}(2) \\ & (g,v) & \mapsto & g. \end{array}$$

Since g is diagonalizable over k, any conjugate of g by GL(2) is in fact conjugate by SL(2). Therefore the image of $F_{(x,y,z)}$ in GL(2) is a homogeneous space for the SL(2) action, and can be taken to be SL(2)$/T$ where T is the subgroup of SL(2) consisting of those elements that commute with g. We assume that $g = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right)$. Therefore $\mu^{-1}(g) \cong \left\{ (v_1, v_2) \in k^2 \,\middle|\, v_1 v_2 = 0 \right\}$. Since $\tau' v = d_2 v$, τ' maps (v_1, v_2) to (v_2, v_1).

From a form of Frobenius reciprocity, as given for instance in [Be], cf. lemma on page 60, the SL(2)-invariant distributions on $F_{(x,y,z)}$ are in natural correspondence with the T-invariant distributions on $\mu^{-1}(g) \cong \left\{ (v_2, v_1) \in k^2 \,\middle|\, v_1 \cdot v_2 = 0 \right\}$, where T is the diagonal subgroup of SL(2) which acts on $\mu^{-1}(g)$ by $t \cdot (v_1, v_2) = (t v_1, t^{-1} v_2)$. The following lemma therefore suffices to prove that any SL(2)-invariant distribution on $F_{(x,y,z)}$ is τ'-invariant. This simple and basic lemma has appeared in many people's works on invariant distributions; we refer to lemma 4.6 of [P1].

Lemma 6.6. *Let* $X = \left\{ (v_1, v_2) \,\middle|\, v_1 v_2 = 0 \right\} \subset k^2$. *Let* $k^\times$ *operate on* X *by* $t \cdot (v_1, v_2) = (t v_1, t^{-1} v_2)$. *Then any distribution on* X *which is invariant under* $k^\times$ *is invariant under the involution* $(v_1, v_2) \mapsto (v_2, v_1)$.

Case 3: We finally look at the fiber over an element (x, y, z) with $x^2 = 4y$ and $z = 0$. We assume without loss of generality that $(x, y) = (2, 1)$, so that we are dealing with unipotent matrices. The fiber is thus

$$F_{(2,1,0)} = \left\{ (g, v) \,\middle|\, g \text{ is unipotent and } gv = v \right\}.$$

In this case, we will again prove that

$$(g, v) \sim_{\mathrm{SL}(2)} (d_1^{-1t} g d_1, d_2 v),$$

for any (g, v) in such a fiber, and therefore that any SL(2)-invariant distribution supported on such a fiber is τ'-invariant.

We will find it more convenient to check invariance under the involution $\tau'' : (g, v) \mapsto (d_2^{-1t} g d_2, d_1 v)$, which differs from τ' by an element of SL(2).

By (the proof of) lemma 6.3, we can assume that $d_2^{-1t} g d_2 = sgs^{-1}$ with $s = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$. Therefore,

$$(d_2^{-1t} g d_2, d_1 v) = (sgs^{-1}, d_1 v) \sim_{\mathrm{SL}(2)} (g, s^{-1} d_1 v).$$

We will be done if $s^{-1} d_1 v = v$, or $sv = d_1 v$. Since we have $z = 0$, $s^{-1} d_1 v$ is in any case an eigenvector of g. We will assume that g is a unipotent matrix which is not identity, as the other case is trivial. Therefore, g has a unique eigenvector up to scaling. Therefore,

$$\lambda s^{-1} d_1 v = v$$

for some $\lambda \in k^\times$. It suffices to prove that λ can be taken to be 1. We write out the equation, $\lambda s^{-1} d_1 v = v$, or $sv = \lambda d_1 v$, assuming that v is the column vector (v_1, v_2), explicitly:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Equivalently,

$$\begin{pmatrix} v_1 + n v_2 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}.$$

Therefore if $v_2 \neq 0$, $\lambda = \pm 1$. By changing s to $-s$, we then can assume that $\lambda = 1$, and we are done. If $v_2 = 0$, then again $\lambda = 1$.

7. Middle cell

We will prove the following:

Lemma 7.1. *Every distribution on $P \cup Pw'P$ satisfying the invariance properties of theorem 3.2 vanishes on $Pw'P$.*

As for the open cell, we need to examine the various J-orbits in $Pw'P$. Recall that we are using $\langle\ ,\ \rangle$ to denote the symplectic form given by the skew-symmetric matrix

$$j = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}.$$

Let $\{e_1, e_2, e_3, e_4\}$ denote the standard basis of k^4. With this notation, since an element p of P has the property that pe_1 is a multiple of e_1, it follows that for $g = p_1 w' p_2 \in Pw'P$,

$$\begin{aligned} g_{41} = -\langle ge_1, e_1\rangle = -\langle p_1 w' p_2 e_1, e_1\rangle &= -\langle w'\lambda_2 e_1, \lambda_1 e_1\rangle \\ &= -\langle \lambda_2 e_2, \lambda_1 e_1\rangle = 0, \end{aligned}$$

where λ_1 and λ_2 are scalars. Clearly, g_{41} is zero for elements of P too. On the other hand, it can be easily checked that $\langle ge_1, e_1\rangle \neq 0$ for $g = p_1 w'' p_2 \in Pw''P$. Thus, $P \cup Pw'P$ consists exactly of those elements g of $\mathrm{GSp}(4)$ with $g_{41} = 0$.

Next note that the function $\gamma\colon \mathrm{GSp}(4) \to k$ defined by $\gamma(g) = -\langle g^2 e_1, e_1\rangle$ is invariant under J, i.e., $\gamma(tgt^{-1}) = \gamma(g)$ for all $t \in J$. For $g \in Pw'P$, since $g_{41} = 0$,

$$\gamma(g) = (g^2)_{41} = g_{42}g_{21} + g_{43}g_{31}.$$

It can be easily checked that γ is invariant under the action of τ on $\mathrm{GSp}(4)$.

Since w' normalizes L, we have

$$Pw'P = JCLw'LCJ = JCLw'J,$$

every element of which is J-conjugate to an element of $JCLw'$. So, modulo C, every element of $Pw'P$ is J-conjugate to an element of the form $hm'w'$, where $h \in H$ and $m' \in M'L$. Write

$$g = c\,\mathsf{h}((a', b'), z')\,\mathsf{m}(1, m)\,w',$$

where $c \in k^\times$ and $m \in \mathrm{GL}(2)$. Let $\lambda = \det m$. Then $\gamma(g) = c^2 m_{21}\lambda$.

Lemma 7.2. *Suppose $\gamma(g) \neq 0$. Then the J-orbit of g is τ-invariant.*

Proof. Without loss of generality, assume $c = 1$. Let $\gamma_0 = \gamma(g) = m_{21}$. Let $r = -m_{11}/\gamma_0$ and $q = m_{22}/\gamma_0$. Then q and r are the unique values so that

$$g' := \mathsf{n}\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\, g\, \mathsf{n}\begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}^{-1} = \mathsf{h}((a, b''), z'')\,\mathsf{m}(1, \begin{pmatrix} 0 & -\lambda\gamma_0^{-1} \\ \gamma_0 & 0 \end{pmatrix})\,w'$$

for some a, b'', and z''. Let $s = -\lambda a/\gamma_0$. Then s is the unique value so that

$$g'' := \mathsf{n}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\, g'\, \mathsf{n}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}^{-1} = \mathsf{h}((0, b), z)\,\mathsf{m}(1, \begin{pmatrix} 0 & -\lambda\gamma_0^{-1} \\ \gamma_0 & 0 \end{pmatrix})\,w'$$

for some b and z. In other words, g'' is the unique element in the J-conjugacy class of g having this form.

Since $\tau(g'') \in Pw'P$ and $\gamma(\tau(g'')) = \gamma_0$, we see that, from the calculations above, $\tau(g'')$ (like g) has a unique J-conjugate of the form

$$\mathsf{h}((0,b'),z')\,\mathsf{m}(1,\begin{pmatrix}0 & -\lambda\gamma_0^{-1}\\ \gamma_0 & 0\end{pmatrix})\,w'$$

for some b' and z'. The characteristic polynomial of this element (and thus of $\tau(g'')$) is

$$X^4 - b'\gamma_0 X^3 + z'\gamma_0 X^2 - b'\gamma_0 X + \lambda.$$

One can similarly compute the characteristic polynomial of g'' (and thus of g). But since g and $\tau(g'')$ must have the same characteristic polynomial, we must have that $b = b'$ and $z = z'$. $\square$

From now on, assume that $\gamma(g) = 0$. Then $m_{21} = 0$, so we may write

$$g = \mathsf{q}(A, \left(\begin{smallmatrix} r & s\\ t & r\end{smallmatrix}\right), \lambda) \in Q \smallsetminus P$$

for some $r, s, t \in k$ and $\lambda \in k^\times$. Since all considerations in the rest of the section depend only on g up to scalars, we assume that $A = \left(\begin{smallmatrix} au & 1\\ u & 0\end{smallmatrix}\right) \in \mathrm{GL}_2(k)$. Let β_1 and β_2 denote the (generalized) eigenvalues of A.

Lemma 7.3. *Suppose that for all $i, j \in \{1, 2\}$, we have $\beta_i\beta_j \neq \lambda$. Then the J-orbit of g contains Zg.*

Proof. Write $g = \mathsf{q}(A, B, \lambda)$. We would like to solve, for $T \in N$ and $S \in Z$, the equation

$$\mathsf{n}(T)\mathsf{q}(A,B,\lambda)\mathsf{n}(-T) = S\mathsf{q}(A,B,\lambda),$$

or,

$$\begin{aligned}\begin{pmatrix} I & T\\ 0 & I\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & \lambda A^*\end{pmatrix}\begin{pmatrix} I & B\\ 0 & I\end{pmatrix}\begin{pmatrix} I & -T\\ 0 & I\end{pmatrix}\\ = \begin{pmatrix} I & S\\ 0 & I\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & \lambda A^*\end{pmatrix}\begin{pmatrix} I & B\\ 0 & I\end{pmatrix},\end{aligned}$$

or,

$$\begin{pmatrix} I & T\\ 0 & I\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & \lambda A^*\end{pmatrix}\begin{pmatrix} I & -T\\ 0 & I\end{pmatrix} = \begin{pmatrix} I & S\\ 0 & I\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & \lambda A^*\end{pmatrix},$$

or,

$$\lambda TA^* - AT = \lambda SA^*.$$

Observe that for matrices L_1 and L_2, the transformation

$$X \mapsto L_1X - XL_2$$

is singular if and only if an eigenvalue of L_1 is the same as an eigenvalue of L_2. This implies that the equation

$$\lambda TA^* - AT = \lambda SA^*$$

can be solved for T if A and λA^* do not share an eigenvalue, i.e., if the eigenvalues of A are $\{\beta_1, \beta_2\}$, then

$$\{\beta_1, \beta_2\} \cap \lambda\{\beta_1^{-1}, \beta_2^{-1}\} = \phi,$$

i.e., $\lambda \notin \{\beta_1^2, \beta_1\beta_2, \beta_2^2\}$.

We actually need to solve for T with

$$\omega^t T\omega = T.$$

For this we write the earlier equation as

$$\lambda T - ATA^{*-1} = \lambda S.$$

If $T \mapsto \omega^t T\omega$ is denoted by σ, then the above equation becomes:

$$\lambda T - AT\sigma(A) = \lambda S.$$

Applying σ to this equation, we obtain

$$\lambda\sigma(T) - A\sigma(T)\sigma(A) = \lambda S.$$

Adding the two previous equations,

$$\lambda[T + \sigma(T)] - A[T + \sigma(T)]\sigma(A) = 2\lambda S.$$

Now $\frac{1}{2}(T + \sigma(T))$ is of the desired form, completing the proof of the lemma. □

Suppose from now on that the hypothesis of lemma 7.3 does not hold. Then we can divide the rest of the proof into three cases.

Case 1: $\beta_i^2 = \lambda$ for precisely one value of $i \in \{1, 2\}$. Assume without loss of generality that $\beta_1^2 = \lambda \neq \beta_2^2$. Since β_1 and β_2 thus have different minimal polynomials over k, they must both lie in $k^\times$. Therefore, for $A' = A/\beta_1$, $A'^* = \beta_1 A^* = (\lambda A^*)/\beta_1$. Thus for $g' = g/\beta_1$, the 2×2 block diagonal matrices are (A', A'^*), i.e., up to scaling g belongs to $\mathrm{Sp}(4)$. Furthermore, one of the eigenvalues of A'

is 1. By appealing to [BR], we will see in §10 that we don't have to worry about these elements.

Case 2: $\lambda = \beta_1^2 = \beta_2^2$. Then $\beta_2 = \pm\beta_1$. If $\beta_2 = \beta_1$, then $\lambda = \beta_1\beta_2 = \det(A)$ and $\operatorname{tr}(A) \neq 0$. If $\beta_2 = -\beta_1$, then $\lambda = -\beta_1\beta_2 = -\det(A)$ and $\operatorname{tr}(A) = 0$.

Lemma 7.4. *If* $\operatorname{tr}(A) \neq 0$*, then the* J*-orbit of* $g = \mathsf{q}(A, B, \det A)$ *is* τ*-invariant.*

Proof. In this case, $u = -\lambda$. Let $n = \mathsf{n}\left(\begin{smallmatrix} x & y \\ z & x \end{smallmatrix}\right)$, where $x, y, z \in k$ are to be determined. Then

$$ngn^{-1} = \mathsf{q}(\left(\begin{smallmatrix} au & 1 \\ u & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} R & S \\ T & R \end{smallmatrix}\right), \lambda),$$

where (R, S, T) is an affine function of (x, y, z) that takes the value (r, s, t) at the origin and has gradient

$$\begin{pmatrix} \lambda u^{-1} - 1 & 0 & -a\lambda u^{-1} \\ 0 & -1 & \lambda u^{-2} \\ -2a\lambda & \lambda & a^2\lambda - 1 \end{pmatrix}.$$

Since $a \neq 0$, the first and third rows are independent (look at the second and third columns). Thus, we may choose x, y, and z to give R and T any desired value, so g is N-conjugate (and thus J-conjugate) to

$$g' = \mathsf{q}(\left(\begin{smallmatrix} -a\lambda & 1 \\ -\lambda & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & s' \\ 0 & 0 \end{smallmatrix}\right), \lambda)$$

for some $s' \in k$. Let

$$p = \mathsf{h}(0, -s')\, \mathsf{h}((a, 0), 0)\, \mathsf{m}(1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)) \in J.$$

Then $pg'p^{-1} = \tau(g')$. Thus, the J-orbit of g is τ-invariant. $\square$

Lemma 7.5. *If* $\operatorname{tr}(A) = 0$*, then the* J*-orbit of* $g = \mathsf{q}(A, B, \pm\det A)$ *is* τ*-invariant.*

Proof. Let $h = \mathsf{h}(0, -t\lambda^{-1})$. Let

$$m = \begin{cases} \mathsf{m}(1, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)) & \text{if } \lambda = -\det(A), \\ \mathsf{m}(1, \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)) & \text{if } \lambda = \det(A). \end{cases}$$

Let $g' = hgh^{-1}$. Then

$$g' = \mathsf{q}(\left(\begin{smallmatrix} 0 & 1 \\ u & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} r & s' \\ 0 & r \end{smallmatrix}\right), \lambda)$$

for some $s' \in k$. Let $h' = \mathsf{h}(0, -s')$. Then $h'm\tau(g')m^{-1}h'^{-1} = g'$. Thus, the J-orbit of g is τ-invariant. $\square$

Case 3: $\lambda \neq \beta_i^2$ for $i = 1, 2$. Then $\lambda = \beta_1\beta_2 = \det(A)$ (since we are assuming that the hypothesis of lemma 7.3 is not satisfied). Therefore either lemma 7.4 or lemma 7.5 applies.

8. Closed cell

In this section we prove the following:

Proposition 8.1. *Any distribution on P that is J-invariant and ψ invariant for a non-trivial character ψ of Z must be invariant under the involution τ.*

Proof. Let $p = \mathsf{m}(\lambda, A)h$ be an element of $P = MH$. It is easy to see that for $z \in Z$,

$$pzp^{-1} = (\lambda^2/\det A)z.$$

Therefore if $\lambda^2 \neq \det A$, then for any $z_0 \in Z$, there is $z \in Z$ such that

$$pzp^{-1} = zz_0,$$

or, $z^{-1}pz = z_0 p$, implying that the J orbit of such a p is Z stable.

On the other hand, if $\lambda^2 = \det A$, then the J-orbit of p is τ-invariant. This can be either checked as an easy exercise, or else observe that in this case p/λ in fact belongs to $\mathrm{Sp}(4)$, and therefore one can use the calculation of lemma 5.4 of [BR]. $\square$

9. Constructible sets

Before we put all of the pieces together to prove our main theorem, we need a bit of general topology that does not seem to have been carefully written down anywhere that we could find. The reason for our need is that a distribution can be restricted to an open set, and we can try to decide if the distribution is zero or not on it. If zero, then the question becomes one on the complementary closed set. And we can proceed inductively trying to prove that a distribution is zero on the whole set, the kind of goal we have set ourselves to in this paper.

However, situations might arise where a space is decomposed not into an open and a complementary closed set, but into a slightly more complicated subset, a constructible set, and its complement. In our case this arises when we are considering all J-conjugates of elements for which the hypothesis of lemma 7.3 holds, where we would like to apply lemma 4.2 to say that the distribution restricted to such elements is zero, except that it does not make sense to restrict distributions to such general subsets (a certain union of orbits).

First let us recall that a subset Y of a topological space X is said to be **constructible** if Y is a finite union of locally closed subsets. (A subset is **locally closed** if it is the intersection of a closed set with an open set.)

The reason for the importance of constructible sets in p-adic groups arises from the following theorem, which is a variation of a well-known theorem due to Chevalley in algebraic geometry. We refer to [BZ] for a proof.

Theorem 9.1. *Let $\underline{X}$ and $\underline{Y}$ be algebraic varieties over a non-archimedean local field k, and $\underline{f}$ be a morphism of algebraic varieties between $\underline{X}$ and $\underline{Y}$. Denote the corresponding k-valued points, and the morphism between the k-valued points by removing the underline. Then $f(X)$ is a constructible subset of the topological space Y.*

For our purposes, the following lemma is of utmost importance.

Lemma 9.2. *Let Y be a constructible subset of a topological space X. Then there are finitely many closed subsets $X_1 \subset \cdots \subset X_n = X$ such that $X_{i+1} \smallsetminus X_i$ is an open subset of X_{i+1} which is either contained in Y or in $X \smallsetminus Y$. Further, this decomposition is* canonical *in the sense that if a group operates on X preserving Y, then it also preserves each of the sets X_i.*

Proof. We recall the following well-known decomposition of a constructible set, cf. [BZ], into a disjoint union of locally closed subsets.

For any subset A of X, define $C^1(A) = C(A) = \bar{A} \smallsetminus \overline{\bar{A} \smallsetminus A}$. Clearly, $C(A)$ is a locally closed subset of X, and is contained in A. Further, $C(A) = A$ if and only if A is locally closed. For $i > 1$, inductively define $C^i(A)$ to be $C(A \smallsetminus [C(A) \cup C^2(A) \cup \cdots \cup C^{i-1}(A)])$. It is easy to see that if A is constructible, then $C^i(A)$ is empty for large i, and therefore such an A is a finite *disjoint* union of the locally closed sets $C^i(A)$.

Renaming the indices, let $Y = \cup_{i=1}^{n-2} Y_i$, a disjoint union of locally closed subsets Y_i with $Y_i = Z_i \smallsetminus W_i$ where Z_i and W_i are closed subsets of X. Now define $X_1 = \cup_{i=1}^{n-2} W_i$, $X_i = X_{i-1} \cup Z_{i-1}$ for $1 < i < n$, and $X_n = X$.

Clearly the X_i's have the property desired. These sets are canonically constructed, and therefore are preserved under any group action preserving Y. $\square$

10. Proof of uniqueness of Fourier-Jacobi models completed

We now have all the pieces necessary to complete the proof of theorem 3.2. We start with a distribution T with invariance properties as in the statement of this theorem. Our aim is to prove that such a distribution is identically zero. By appealing to [BR] as in §5, we already know that T is zero on $\mathcal{G} = k^\times \operatorname{Sp}(4)$. By §6, T is zero on the open cell, thus T is supported on the union $P \cup Pw'P$ of the closed and the middle cell. Let $Y = Pw'P$, an open subset of this union. Write $Y = Y_{\mathrm{o}} \cup Y_{\mathrm{c}}$, with Y_{o} the (open) subset of Y on which $\gamma(g) \neq 0$.

By lemmas 7.2 and 4.1, T is zero on Y_{o}, thus T is supported on $P \cup Y_{\mathrm{c}}$. Since anyway we know that the support of T is outside $\mathcal{G}$, we get that the support of T is contained in the closed subset $Y_{\mathrm{cc}} = Y_{\mathrm{c}} \smallsetminus (Y_{\mathrm{c}} \cap \mathcal{G})$ of Y_{c}. Let S denote the set of elements of the form appearing in lemma 7.3. This is the set of rational points of a k-variety. One can write $Y_{\mathrm{cc}} = Y_1 \cup Y_2$, where Y_1 is the subset of Y_{cc} for which the hypothesis of lemma 7.3 holds, i.e., it consists of J-conjugates of elements of S. Applying theorem 9.1 to the map from $S \times J$ to $\operatorname{GSp}(4)$ defined by $(s, j) \mapsto jsj^{-1}$, we see that Y_1 is a constructible subset of Y_{cc}. Applying lemma 9.2 to the topological space $X = Y_{\mathrm{cc}}$, and $Y = Y_1$, we are able to write Y_{cc} as an increasing union of closed sets such that the successive differences are either in Y_1 or Y_2, to which we can apply now lemmas 4.2 and 4.1 respectively (for the first, J orbits are Z-invariant, and for the second, J-orbits are τ-invariant by lemmas 7.4 and 7.5) to conclude that the distribution is zero on Y_{cc}, thus is supported in P.

We remind the reader that by removing $Y_{\mathrm{c}} \cap \mathcal{G}$ from Y_{c}, we have removed from consideration elements with eigenvalues $\{c, c\alpha, c\alpha^{-1}, c\}$, where $\alpha, c \in k^\times$ and $\alpha \neq \pm 1$, for which J-orbits are in fact not τ-invariant, and were a source of difficulty for [BR]; for us, luckily, we can just use [BR] instead of having to redo this part of their argument.

Finally, by proposition 8.1, T is zero on P, completing the proof of theorem 3.2. $\square$

11. Existence of Fourier-Jacobi models

Having shown that an irreducible admissible representation π of GSp(4) has at most one Fourier-Jacobi model, it would be desirable to know that such models actually exist. Unlike the Whittaker model, which may not exist for some representations, Fourier-Jacobi models (should) always exist as long as π is not one dimensional. Unfortunately we are able to prove this only for supercuspidal representations of GSp(4).

We begin with the following general lemma.

Lemma 11.1. *Let π be a smooth representation of $N \cong k$, on which N acts nontrivially. Then there exists a nontrivial additive character $\psi\colon N \to \mathbb{C}^\times$, and a nonzero linear form $\ell\colon \pi \to \mathbb{C}$ such that*

$$\ell(nv) = \psi(n)\ell(v),$$

for all $v \in \pi, n \in N$.

Proof. It clearly suffices to prove that the twisted Jacquet module

$$\pi_\psi := \frac{\pi}{\left\{ nv - \psi(n)v \;\middle|\; n \in N, v \in \pi \right\}}$$

is nonzero for some nontrivial additive character $\psi\colon N \to \mathbb{C}^\times$. This will be a simple consequence of the exactness of the Jacquet functor, denoted $\pi \mapsto \pi_N$ defined as above but for $\psi = 1$, and of the twisted Jacquet functor $\pi \mapsto \pi_\psi$. Let $\pi[N]$ be the kernel of the map from π to π_N. Clearly $\pi[N]$ is an N-module, which by the exactness of the Jacquet functor has trivial Jacquet module. Since any finitely-generated representation has an irreducible quotient, cf. [BZ] lemma 2.6(a), any smooth representation has an irreducible *subquotient*. By Schur's lemma, which is valid for any smooth representation, cf. [BZ] lemma 2.11, any irreducible representation of N is one dimensional. Therefore $\pi[N]$ has an irreducible subquotient which is one dimensional, and therefore given by a non-trivial character $\psi\colon N \to \mathbb{C}^\times$. The exactness of the twisted Jacquet functor then implies that the twisted Jacquet module of $\pi[N]$, and hence of π, is non-trivial. $\square$

Lemma 11.2. *Let π be an irreducible, smooth representation of* $\mathrm{GSp}(4)$ *that is not one dimensional. Let ψ be a nontrivial character of k, thought of as a character of Z, the center of the unipotent radical of the Klingen parabolic subgroup of* $\mathrm{GSp}(4)$*. Then*

$$\mathrm{Hom}_Z(\pi, \psi) \neq 0.$$

Proof. It is easy to see that a Levi subgroup of the Klingen parabolic operates transitively on the set of all non-trivial characters of Z. Hence, if the conclusion of the lemma is true for one ψ, it is true for all ψ. The previous lemma gives that if $\mathrm{Hom}_Z(\pi, \psi) = 0$, then Z acts trivially on π, and hence the normal subgroup generated by Z also acts trivially on π. But it is a standard fact that $\mathrm{Sp}(4)$ has no normal subgroup besides the center, and hence $\mathrm{Sp}(4)$ must act trivially on π, and therefore π must be one dimensional, concluding the proof of the lemma. □

We still have the task of proving that the representation

$$\pi_\psi := \frac{\pi}{\left\{ zv - \psi(z)v \;\middle|\; z \in Z, v \in \pi \right\}},$$

of J, which we know now is nonzero, has nonzero irreducible quotients. We would have liked to believe that this is obvious, but we did not succeed in finding a general proof. Here is a proof for the case where π is supercuspidal.

In the next two lemmas, G is a general ℓ-group, countable at infinity, in the sense of [BZ]. This hypothesis is satisfied by algebraic groups over non-archimedean local fields. We let $\mathcal{S}(G)$ denote the Schwartz space of locally constant, compactly supported functions on G, thought of as a left G-module. We let dg denote a Haar measure on G.

We recall proposition 2.12 of [BZ]:

Lemma 11.3. *Let G be an ℓ-group, and f a compactly supported function on G. Then there is an irreducible smooth representation π of G such that the action of f on π is non-trivial.*

We combine this lemma with the following trivial lemma:

Lemma 11.4. *Let π be a smooth irreducible representation of G. Then for every vector $v \in \pi$, there is a homomorphism of G-modules*

$\mathcal{S}(G) \to \pi$ given by

$$f \mapsto \int_G f(g)\pi(g)v\,dg.$$

For a function $f \in \mathcal{S}(G)$, the image of f under this homomorphism is non-zero for some choice of $v \in \pi$ if and only if the action of f on π is nontrivial.

Proposition 11.5. *Let π be a supercuspidal representation of* $\mathrm{Sp}(4)$*. Then the representation π_ψ of J has an irreducible quotient.*

Proof. Observe that the supercuspidal representation π can be realized on a space of functions in $\mathcal{S}(\mathrm{Sp}(4))$. Fix one such realization, and think of elements of π now as functions on $\mathrm{Sp}(4)$. Restricting these functions to the Fourier-Jacobi group J, we get a space of locally constant, compactly supported functions on J. For a function g of this kind, and for any element $z \in Z$, $f = zg - g$ is another such function. We can (and do) choose $z \in Z$ so that f is nonzero. By the previous two lemmas, there is an irreducible representation ρ of J on which f acts nontrivially. By generalities (cf. [BZ], proposition 2.11), ρ has a central character (i.e., Schur's lemma holds). Hence, Z operates by a character on ρ. This character cannot be trivial, as f was chosen to be of the form $zg - g$. $\square$

Question. It would be interesting to understand π_ψ as a representation of J. Of the irreducible representations of J with central character ψ (which are parametrized by irreducible representations of $\tilde{S}$, the two-fold cover of $\mathrm{SL}(2)$), which ones occur as a quotient in π_ψ? We expect that if π is a generic representation of $\mathrm{GSp}(4)$, then every irreducible representation of $\tilde{S}$ appears. Further, if π is a degenerate representation, then we expect π_ψ to be a representation of finite length as a J-module, with a unique irreducible quotient.

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