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Distinguished Representations for Quadratic Extensions

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Abstract. Let K be a quadratic extension of a field k which is either local field or a finite field. Let G be an algebraic group over k. The aim of the present paper is to understand when a representation of $G(K)$ has a $G(k)$ invariant linear form. We are able to accomplish this in the case when G is the group of invertible elements of a division algebra over k of odd index if k is a local field, and for general connected groups over finite fields.

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1. Introduction

A representation π of a group G is said to be distinguished with respect to a subgroup H if the space of H-invariant linear forms on π is non-zero. Usually, one considers this when H is the subgroup of the fixed points of an involution on G, and when one can prove that the space of H-invariant linear forms on any irreducible representation of G is at most one-dimensional. This concept has been much studied in recent times by H. Jacquet and his collaborators, cf. [7–10], and makes sense for representations of p-adic groups as well as automorphic representations where the requirement on a nonzero H-invariant linear form is replaced by the nonvanishing of a period integral on H. From several of these works as well as the work of Flicker [3], it appears that in many contexts distinguished representations are classified as those representations of G which are obtained via a functorial lift from some other group.

The aim of this article is to discuss a simple situation involving compact p-adic groups where representations of a group having fixed vector under a subgroup are identified exactly to those representations which are obtained via a functorial lifting. The group involved is the group of invertible elements of a division algebra over a quadratic extension K of a local field k, the subgroup being the group of invertible elements of the division algebra over k, and the functorial lifting is from a Unitary group.

The principle we employ to prove our main theorem proves a general result about representations of $G(\mathbf{F}_{q^2})$ which have $G(\mathbf{F}_q)$-invariant vector where G is any connected algebraic group over the finite field $\mathbf{F}_q$. It also proves a result about triple products of representations of $\mathrm{GL}(2)$ over a p-adic field which was earlier proved only in odd residue characteristic by an explicit character calculation, cf. Theorem E of [12].

It should be noted that the results we prove about $G(k)$-invariant forms on a representation space of $G(K)$ reduces to a rather trivial statement if $K = k \oplus k$. In this case our results reduce to say that a representation $\pi_1 \otimes \pi_2$ of $G(k) \times G(k)$ has a $G(k)$-invariant form if and only if $\pi_1^* \cong \pi_2$, and that if $\pi_1^* \cong \pi_2$, then the space of $G(k)$-invariant forms on $\pi_1 \otimes \pi_2$ is exactly 1 dimensional. The present paper attempts to prove the twisted analogue of this.

2. Distinguished Representations for a Division Algebra

We begin by fixing some notation. Let D_k be a division algebra of index n, an odd integer, over a local field k, and with k as its center. Let K be a separable quadratic extension of k. Let V be an irreducible representation of D_K^* for $D_K = D_k \otimes_k K$, and let V^σ denote the irreducible representation of D_K^* obtained from V by using the automorphism of order 2 of D_K induced by the nontrivial automorphism σ of K over k. We abuse notation to denote by σ the automorphism of D_K also.

THEOREM 1. *Let V be an irreducible representation of D_K^* which is trivial on k^*. Then V has a D_k^*-invariant vector if and only if there is an isomorphism $V^* \cong V^\sigma$ of representations of D_K^*.*

Proof. The idea of the proof is to extend the representation $V \otimes V^\sigma$ of D_K^* to a group E containing D_K^* as a subgroup of index 2, and try to see when this representation of E has an E-invariant vector. Since E modulo center is a compact group, this we can do by character theory. The main point of the proof is that the character of this representation of E on the non-trivial coset of D_K^* is related to the character of V at an element of the subgroup D_k^*, in exactly the same way as the character identity of Shintani in the theory of base change, cf. Definition 6.1 of [1].

We now define the group E and its representation on $V \otimes V^\sigma$. Let e be an element of k^* which is not a norm from K^*. Using the element e, one can construct an extension of groups

$$0 \to D_K^* \to E \to \mathbf{Z}/2 \to 0,$$

such that the group E has an element τ with the property that

$$\tau x \tau^{-1} = x^\sigma \quad \text{for all } x \in D_K^*, \quad \text{and} \quad \tau^2 = e.$$

(Since we are considering only those representations of D_K^* which are trivial on k^*, we could have taken E to be a semi-direct product; however, this construction in the style of the Weil group of K over k is more general and natural.)

It is easy to see that the representation $V \otimes V^\sigma$ of D_K^* can be extended to a representation of E by $\tau(v_1 \otimes v_2) = ev_2 \otimes v_1 = v_2 \otimes v_1$.

We now calculate the character of the representation $V \otimes V^\sigma$ at an element of the form $d \cdot \tau$ where $d \in D_K^*$. Let $\{v_1, \dots, v_n\}$ be a basis of V. By definition,

$$\begin{aligned}(d \cdot \tau)(v_i \otimes v_j) &= d(v_j \otimes v_i) \\ &= dv_j \otimes d^\sigma v_i \\ &= \left(\sum_l d_{jl} v_l\right) \otimes \left(\sum_m d_{im}^\sigma v_m\right) \\ &= \sum_{l,m} d_{jl} \cdot d_{im}^\sigma v_l \otimes v_m.\end{aligned}$$

Therefore

$$\Theta_{V \otimes V^\sigma}(d \cdot \tau) = \sum_{i,j} d_{ji} \cdot d_{ij}^\sigma = \Theta_V(d^\sigma d).$$

By Schur's lemma, $V \otimes V^\sigma$ can have at most one-dimensional subspace of D_K^*-invariant vectors, and therefore at most one-dimensional subspace of E-invariant vectors. It follows that if we fix a Haar measure on E/K^* such that D_K^*/K^* has volume 1

$$\int_{D_K^*/K^*} \Theta_V \cdot \Theta_{V^\sigma} + \int_{D_K^*/K^*} \Theta_V(d^\sigma d) = 2 \quad \text{or} \ \ 0,$$

depending on whether $V \otimes V^\sigma$ has an E-invariant vector or not. Since

$$\int_{D_K^*/K^*} \Theta_V \cdot \Theta_{V^\sigma} = 1, \quad \text{or} \ \ 0$$

depending on whether $V \otimes V^\sigma$ contains the trivial representation of D_K^*/K^* or not, if $V \otimes V^\sigma$ contains the trivial representation of D_K^*/K^* but not of E, it follows that $\int \Theta_V(d^\sigma d) = -1$ but by Lemma 1 below, $\int \Theta_V(d^\sigma d)$, being the dimension of D_k^*-invariant vectors in V, can never be negative. So if $V \otimes V^\sigma$ has a D_K^*-invariant vector, it has an E-invariant vector and by Lemma 1, $\int_{D_K^*/K^*} \Theta_V(d^\sigma d) = \int_{D_k^*/k^*} \Theta_V(d) = 1$. If, on the other hand, $V \otimes V^\sigma$ does not have D_K^*/K^*-invariant vector, then in particular it will not have E-invariant vector, and therefore $\int_{D_K^*/K^*} \Theta_V \cdot \Theta_{V^\sigma} = \int_{D_k^*/k^*} \Theta_V(d) = 0$. We therefore find that for any irreducible representation V of D_K^* which is trivial on k^*,

$$\int_{D_K^*/K^*} \Theta_V \cdot \Theta_{V^\sigma} = \int_{D_k^*/k^*} \Theta_V(d),$$

as the two sides of the equation are either both 0 or both 1. This completes the proof of the theorem except that we still need to prove the following lemma.

LEMMA 1. *Let f be any class function on D_K^*/k^*. Then for Haar measures on D_k^*/k^* and D_K^*/K^* giving these groups measure* 1, *one has*

$$\int_{D_k^*/k^*} f(x) = \int_{D_K^*/K^*} f(x^\sigma x).$$

Proof. The proof of this lemma will be accomplished by the Weyl integration formula which is available for both conjugacy invariant function, such as f restricted to D_k^*/k^*, and σ-conjugacy invariant function such as $f(x^\sigma x)$ on D_K^*/K^*. (A function F on D_K^* is called σ-conjugacy invariant function if

$$F(x^{-1}yx^\sigma) = F(y), \quad \text{for all } x, y \in D_K^*.)$$

The Weyl integration formula for σ-conjugacy invariant functions says that

$$\int_{D_K^*/K^*} F(g) = \sum \frac{1}{W(T,k)} \frac{1}{\mathrm{vol}(T(k))} \int_{T^{1-\sigma}(K)\backslash T(K)} F(t)\Delta^2(Nt),$$

where the summation is over all the conjugacy classes of maximal tori in D_k^* (i.e. invertible elements of maximal subfields in D_k); $W(T,k)$ denotes the cardinality of the corresponding Weyl group (i.e., the order of the group of automorphisms of the corresponding field); $Nt = t^\sigma t$; Δ is the usual Weyl denominator; $T^{1-\sigma}(K)$ is the subgroup of $T(K)$ consisting of elements of the form $x\sigma(x^{-1})$ for $x \in T(K)$; the measure on $T^{1-\sigma}(K)\backslash T(K)$ is defined using an arbitrary measure on $T(k)$ via the exact sequence

$$1 \to T^{1-\sigma}(K) \to T(K) \to T(k).$$

We note that this formula on page 36 of [1] is only for $\mathrm{GL}(n)$. However the proof for $\mathrm{GL}(n)$ works also for a division algebra once one has noticed that an element of D_K^* whose conjugacy class is defined over k, i.e., an element whose characteristic polynomial is defined over k is conjugate by an element of D_K^* to an element of D_k^*. This follows by Skolem–Noether theorem combined with the fact that a division algebra of index n over a local field k contains a root of all the irreducible polynomials over k of degree dividing n.

The Weyl integration formula applied to $f(x^\sigma x)$ gives,

$$\int_{D_K^*/K^*} f(x^\sigma x) = \sum \frac{1}{W(T,k)} \frac{1}{\mathrm{vol}(T(k))} \int_{T^{1-\sigma}(K)\backslash T(K)} f(Nt)\Delta^2(Nt).$$

Notice that if L is any degree n extension of k contained in D_k giving rise to $T(k) = L^*$, then $T(K) = (L \otimes_k K)^*$, and the norm mapping from $T(K)$ to $T(k)$ is surjective as we are going modulo the center ($= k^*$). Therefore

$$\int_{D_K^*/K^*} f(x^\sigma x) = \sum \frac{1}{W(T,k)} \frac{1}{\mathrm{vol}(T(k))} \int_{T(k)} f(t)\Delta^2(t),$$

which is exactly the Weyl integration formula applied to the class function f restricted to D_k^*

$$\int_{D_k^*/k^*} f(g) = \sum \frac{1}{W(T,k)} \frac{1}{\mathrm{vol}(T(k))} \int_{T(k)} f(t)\Delta^2(t).$$

Remark 1. Representations V of D_K^* which have the property that $V^* \cong V^\sigma$ are precisely those representations of D_K^* whose Langlands parameter comes via base change of the Langlands parameter of a Unitary group over k defined in terms of a Hermitian form in n variables over K, cf. Lemma 15.1.2 [13]. We refer to conjecture on page 143, and Proposition 12 of [3] for related matter.

In the process of the proof of Theorem 1, we have also proved the following proposition for which one can also supply a simple and direct proof by the method of Gelfand pairs. We refer to the article of B. Gross [5] for an exposition on Gelfand pairs.

PROPOSITION 1. *For any irreducible representation V of D_K^*, the dimension of D_k^* invariant vectors is at most one-dimensional.*

3. Finite Groups of Lie Type

Let G be a connected algebraic group over a finite field $\mathbf{F}_q$. The argument that we have employed to study representations of D_K^* which have a D_k^*-invariant vector is a rather formal one depending crucially on Lemma 1. A form of this lemma is actually available for the $\mathbf{F}_{q^2}$ rational points of any connected algebraic group over $\mathbf{F}_q$, cf. Proposition 1.6 of [2]. Before we state this form, we fix some notation. All the notions in the following paragraph are borrowed from [2].

Let $\mathbf{E}$ be an extension of degree n of a finite field $\mathbf{F}$. Let G be a connected algebraic group over $\mathbf{F}$. Let σ denote the Frobenius automorphism of $\overline{\mathbf{F}}$ over $\mathbf{F}$. Two elements x and y of $G(\mathbf{E})$ are said to be σ-conjugate if there exists z in $G(\mathbf{E})$ such that $x = zy\sigma(z)^{-1}$. An element $z \in G(\mathbf{E})$ is said to be a σ-centraliser of $y \in G(\mathbf{E})$ if $zy\sigma(z)^{-1} = y$. Write any element of $G(\mathbf{E})$ as $y^{-1}\sigma(y)$ for some y in $G(\overline{\mathbf{F}})$. This is possible by Lang's theorem. Define a mapping N, called the norm mapping, from the set of σ-conjugacy classes in $G(\mathbf{E})$ to the set of conjugacy classes in $G(\mathbf{F})$ by $N(y^{-1}\sigma(y)) = y\sigma^n(y)^{-1}$. It can be seen that the norm mapping N is a bijection from the set of σ-conjugacy classes in $G(\mathbf{E})$ to the set of conjugacy classes in $G(\mathbf{F})$. It is easy to see that $x \in G(\mathbf{E})$ is a σ-centraliser of $y^{-1}\sigma(y) \in G(\mathbf{E})$ if and only if $yxy^{-1} \in G(\mathbf{F})$. If $x \in G(\mathbf{E})$ and $yxy^{-1} \in G(\mathbf{F})$, it can be seen that yxy^{-1} commutes with $y\sigma^n(y)^{-1}$. Therefore there exists a bijection between the σ-centraliser of $x \in G(\mathbf{E})$ and the centraliser of $N(x)$ in $G(\mathbf{F})$. This proves the following lemma.

LEMMA 2. *Let G be a connected algebraic group over a finite field* $\mathbf{F}$. *Suppose that* $\mathbf{E}$ *is a degree n extension of* $\mathbf{F}$, *and N the mapping introduced above from the* *set of σ-conjugacy classes in $G(\mathbf{E})$ to the set of conjugacy classes in $G(\mathbf{F})$. Let f be a class function on $G(\mathbf{E})$. Then*

$$\frac{1}{|G(\mathbf{F})|}\sum_{G(\mathbf{F})} f(x) = \frac{1}{|G(\mathbf{E})|}\sum_{G(\mathbf{E})} f(N(x)).$$

For our application, we need a form of this lemma in which instead of the norm mapping, one takes the closely related mapping $x \to x\sigma(x)$ for $\mathbf{E}$ quadratic over $\mathbf{F}$. If $x = y^{-1}\sigma(y)$, then $x\sigma(x) = y^{-1}\sigma^2(y)$. It follows that $x\sigma(x)$ is conjugate to $N(x)^{-1}$ by an element from $G(\overline{\mathbf{F}})$. This brings us to the following definition.

DEFINITION. Let G be a connected algebraic group over a finite field $\mathbf{F}$. A representation of $G(\mathbf{F})$ is called *stable* if its character takes the same value on any two conjugacy classes of $G(\mathbf{F})$ which become the same in $G(\overline{\mathbf{F}})$. A class function is called *stable* if it takes the same value on any two conjugacy classes of $G(\mathbf{F})$ which become the same in $G(\overline{\mathbf{F}})$. A conjugacy class in $G(\mathbf{F})$ containing an element x of $G(\mathbf{F})$ is called *stable* if the intersection of the conjugacy class in $G(\overline{\mathbf{F}})$ containing x with $G(\mathbf{F})$ is the conjugacy class in $G(\mathbf{F})$ containing x.

QUESTION 1. What is the relationship between the number of unstable characters and unstable conjugacy classes? We note that for $\mathrm{SL}_n(\mathbf{F})$, stable characters are exactly those which are left invariant under the conjugation action of $\mathrm{GL}_n(\mathbf{F})$, and similarly stable conjugacy classes are those which are left invariant under the inner conjugation action of $\mathrm{GL}_n(\mathbf{F})$. Therefore for $\mathrm{SL}_n(\mathbf{F})$, the number of unstable characters and the number of unstable conjugacy classes is the same.

Remark 2. Since the centraliser of any element of $\mathrm{GL}_n(\overline{\mathbf{F}})$ is connected, it follows from Lang's theorem that any class function on $\mathrm{GL}_n(\mathbf{F}_q)$ or $U_n(\mathbf{F}_q)$ is stable.

Since $x\sigma(x)$ and hence $\sigma(x)x$ is conjugate to $N(x)^{-1}$ by an element from $G(\overline{\mathbf{F}})$, our previous lemma gives the following for a stable class function.

LEMMA 3. *Let G be a connected algebraic group over a finite field $\mathbf{F}_q$. Let f be a stable class function on $G(\mathbf{F}_{q^2})$. Then for the automorphism σ of $G(\mathbf{F}_{q^2})$ induced by the nontrivial Galois automorphism of $\mathbf{F}_{q^2}$ over $\mathbf{F}_q$, one has*

$$\frac{1}{|G(\mathbf{F}_q)|}\sum_{G(\mathbf{F}_q)} f(x) = \frac{1}{|G(\mathbf{F}_{q^2})|}\sum_{G(\mathbf{F}_{q^2})} f(x^\sigma x).$$

As in the proof of Theorem 1, this lemma implies that one has the following theorem in which for a representation V of $G(\mathbf{F}_{q^2})$, V^σ denotes the representation of $G(\mathbf{F}_{q^2})$ obtained by applying the nontrivial Galois automorphism of $\mathbf{F}_{q^2}$ over $\mathbf{F}_q$.

THEOREM 2. *Let G be a connected algebraic group over a finite field $\mathbf{F}_q$. Let V be an irreducible stable representation of $G(\mathbf{F}_{q^2})$. Then V has a $G(\mathbf{F}_q)$-invariant vector if and only if there is an isomorphism $V^* \cong V^\sigma$ of representations of $G(\mathbf{F}_{q^2})$.*

Remark 3. Theorem 2 is not true for all representations of $G(\mathbf{F}_{q^2})$. For instance, it is false for some (unstable) representations of $\mathrm{SL}_2(\mathbf{F}_{q^2})$. We recall that to a nontrivial characters θ of the group of norm 1 elements of $\mathbf{F}_{q^4}$ over $\mathbf{F}_{q^2}$ which is not of order 2 there is associated a discrete series representation of $\mathrm{SL}_2(\mathbf{F}_{q^2})$; a character and its inverse giving the same representation of $\mathrm{SL}_2(\mathbf{F}_{q^2})$. If θ is nontrivial but of order 2, there are two representations of $\mathrm{SL}_2(\mathbf{F}_{q^2})$ associated to θ which are permuted by any element of $\mathrm{GL}_2(\mathbf{F}_{q^2})$ whose determinant is not a square in $\mathbf{F}_{q^2}$. The sum of these two representations, say Π_1 and Π_2, is invariant under $V \to V^{*\sigma}$, as θ is. This means that to check $\Pi_1 \cong \Pi_1^{*\sigma}$, all we need to check is that the characters of Π_1 and $\Pi_1^{*\sigma}$ are the same on the unipotent element $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ where a is any element of $\mathbf{F}_{q^2}$ which is not a square. Since -1 is always a square in $\mathbf{F}_{q^2}$, and the Galois automorphism of $\mathbf{F}_{q^2}$ takes squares to squares, it follows that Π_1 and $\Pi_1^{*\sigma}$ have the same character values on the element $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, and therefore $\Pi_1 \cong \Pi_1^{*\sigma}$. On the other hand, since the group of norm one elements of $\mathbf{F}_{q^4}$ has $q^2 + 1$ elements, the nontrivial character of order 2 on it is nontrivial on ± 1, and therefore on the center of $\mathrm{SL}_2(\mathbf{F}_{q^2})$. In particular, these representations of $\mathrm{SL}_2(\mathbf{F}_{q^2})$ will not have a $\mathrm{SL}_2(\mathbf{F}_q)$-invariant vector.

Our proof of Theorem 2 gives also the following multiplicity 1 theorem.

THEOREM 3. *Let G be a connected algebraic group over a finite field $\mathbf{F}_q$. An irreducible stable representation of $G(\mathbf{F}_{q^2})$ has at most* 1 *dimensional space of $G(\mathbf{F}_q)$ invariant vectors.*

Remark 4. Theorem 3 is not true for all representations of $G(\mathbf{F}_{q^2})$. For instance it is false for a (unstable) representation of $\mathrm{SL}_2(\mathbf{F}_{q^2})$. We recall that the principal series representation of $\mathrm{SL}_2(\mathbf{F}_{q^2})$ associated to the unique character of order 2 of its maximal torus which is isomorphic to $\mathbf{F}_{q^2}^*$ is sum of two irreducible representations. One of these does not have $\mathrm{SL}_2(\mathbf{F}_q)$-invariant vector whereas the other has a 2-dimensional subspace of $\mathrm{SL}_2(\mathbf{F}_q)$-invariant vectors.

Remark 5. The referee has pointed out that our Theorems 2 and 3 for GL_n and U_n are known, and are due to R. Gow, cf. [4].

4. General p-adic Case

Since the statement of Theorem 2 is rather general, and makes sense for all groups G over a p-adic field k with K as a quadratic extension, it is natural to investigate if something similar might be true in the p-adic case too. It seems very likely that this will be the case, except that we need to take extra care about certain things which are evident in the finite field case, and indeed as we illustrate for GL_2, the situation is not quite as in the finite field case. If our Lemma 1 is to have any analogue at all for general p-adic groups, then the image of the norm mapping from $G(K)$ to $G(k)$ should be a subgroup of $G(k)$. From Lemma 1.4 of [1] it follows that the regular elliptic elements in the image of the norm mapping from $\mathrm{GL}_n(K)$ to $\mathrm{GL}_n(k)$ is exactly those whose determinant is a norm from K^*, and an element of the diagonal torus is a norm if and only if each of its entries is a norm. One can make a general statement for any semi-simple element in any quasi-split reductive p-adic group with simply-connected derived group along similar direction. It follows that even if we did not worry about convergence problems, the integral $\int_{G(K)} f(x^\sigma x)$ will not be an integral over a subgroup of $G(k)$. However, there might be cases where the character might be zero at elements of a particular subgroup of $G(k)$ which do not come from norms of $G(K)$, and in those cases the integral $\int_{G(K)} f(x^\sigma x)$ can be written as an integral over a subgroup of $G(k)$, and the existence of an invariant form on a representation Π of $G(K)$ for that subgroup of $G(k)$ might be equivalent to $\Pi^* \cong \Pi^\sigma$. One can in any case ask the following question in which we use the notion of a stable character which is defined for p-adic groups too, but we do not define it here.

QUESTION 2. Let G be a connected reductive quasi-split group over a local field k whose derived group is simply-connected. Let K be a quadratic extension of k and Π an irreducible admissible *stable* representation of $G(K)$. Prove that Π has at most one-dimensional space of $G(k)$-invariant forms, and if Π has a $G(k)$-invariant linear form, $\Pi^* \cong \Pi^\sigma$.

Here is a theorem due to Jacquet, Hakim and Flicker, cf. [3], in this direction for $\mathrm{GL}(2)$. It is the local version of a global theorem proved in [6].

THEOREM 4. *An irreducible discrete series representation* Π *of* $\mathrm{GL}_2(K)$ *with trivial central character has a* $\mathrm{GL}_2(k)$ *invariant linear form if and only if* Π *is a base change of a representation of* $\mathrm{GL}_2(k)$ *with nontrivial central character. A principal series representation* Π *of* $\mathrm{GL}_2(K)$ *with trivial central character has a* $\mathrm{GL}_2(k)$ *invariant linear form if and only if* Π *is either a base change of a principal series representation of* $\mathrm{GL}_2(k)$, *or is the base change of a discrete series representation of* $\mathrm{GL}_2(k)$ *with nontrivial central character*.

We remark that for a representation Π of $\mathrm{GL}_2(K)$ with trivial central character $\Pi^* \cong \Pi$, and therefore the analogue of Theorem 2 will ask for an isomorphism between Π and Π^σ as the necessary and sufficient condition for Π to have $\mathrm{GL}_2(k)$ invariant form, which is exactly the condition for Π to be a base change from a representation of $\mathrm{GL}_2(k)$. However, Theorem 3 requires this condition on base change together with an extra condition on the central character of the representation it is a base change of, if Π is a discrete series representation.

Remark 6. The referee has pointed out a rather precise conjecture in the case of GL_n which is that a representation Π of $GL_n(K)/k^*$ for n even has a linear form on which $GL_n(k)$ operates trivially or via the quadratic character of k^* associated to K if and only if $\Pi^* \cong \Pi^\sigma$. If n is odd, $\Pi^* \cong \Pi^\sigma$ if and only if Π has a $GL_n(k)$-invariant form.

We end this section with the following general criterion for the existence of invariant forms which is a simple consequence of Mackey theory.

LEMMA 4. *If a cuspidal representation Π of a p-adic group G is obtained from a finite dimensional representation W of a compact open subgroup $\mathcal{K}$ by induction, then Π has an invariant form for a subgroup H of G if and only if $\int_H f^g \neq 0$, for some $g \in G$, and f a matrix coefficient of the representation W of $\mathcal{K}$ thought of as a matrix coefficient of G; here $f^g(h) = f(ghg^{-1})$.*

5. Triple Product for $\mathbf{GL(2)}$

By a method similar to the one we employed in the proof of our Theorem 1, we can also prove the following theorem. The analogue of this theorem for $GL(2)$ was stated without proof in [12] as Theorem E and it was noted there that it sufficed to prove the theorem for division algebras. At that time the author had proved this result only in odd residue characteristic by an explicit calculation with the character formula but did not publish that proof in the hope for a more conceptual proof.

THEOREM 5. *Let K be a cubic cyclic extension of a local field k. Let D be a quaternion division algebra over k. Let π be an irreducible representation of D_K^*/k^* where $D_K = D \otimes_k K$. Let σ be a generator of the Galois group of K over k. Extend the automorphism σ of K over k to an automorphism of D_K and denote the resulting automorphism of D_K again by σ. Let π^σ and π^{σ^2} denote the representations of D_K^* obtained from π by applying the automorphism σ and σ^2 of D_K^*. Then π has a D^*-invariant vector if and only if the representation $\pi\otimes\pi^\sigma\otimes\pi^{\sigma^2}$ of D_K^* has a D_K^*-invariant form.*

Proof. Let E be the semi-direct product of D_K^* with $\mathbf{Z}/3$ such that the inner conjugation action of a generator, say τ, of $\mathbf{Z}/3$ acts on D_K^* via σ. The representation $\pi \otimes \pi^\sigma \otimes \pi^{\sigma^2}$ of D_K^* can be extended to a representation of the group E such that

$$\tau(v_1 \otimes v_2 \otimes v_3) = (v_3 \otimes v_1 \otimes v_2).$$

Let $\{v_1, \ldots, v_n\}$ be a basis of π. By definition

$$\begin{aligned}(d \cdot \tau)(v_i \otimes v_j \otimes v_k) &= d(v_k \otimes v_i \otimes v_j)\\ &= dv_k \otimes d^\sigma v_i \otimes d^{\sigma^2} v_j\\ &= \left(\sum_l d_{kl} v_l\right) \otimes \left(\sum_m d^\sigma_{im} v_m\right) \otimes \left(\sum_n d^{\sigma^2}_{jn} v_n\right)\end{aligned}$$

$$= \sum_{l,m,n} d_{kl} \cdot d^{\sigma}_{im} \cdot d^{\sigma^2}_{jn} v_l \otimes v_m \otimes v_n.$$

Therefore

$$\Theta_{V\otimes V^{\sigma}\otimes V^{\sigma^2}}(d\cdot\tau) = \sum_{i,j,k} d_{ki}\cdot d^{\sigma}_{ij}\cdot d^{\sigma^2}_{jk} = \Theta_V(d^{\sigma}d^{\sigma^2}d).$$

Similarly

$$\Theta_{V\otimes V^{\sigma}\otimes V^{\sigma^2}}(d\cdot\tau^2) = \Theta_V(d^{\sigma^2}d^{\sigma}d).$$

Since $V\otimes V^{\sigma}\otimes V^{\sigma^2}$ can have at most one-dimensional subspace of D^*_K-invariant vectors by Theorem 1.1 of [11], the dimension of E-invariant vectors in $V\otimes V^{\sigma}\otimes V^{\sigma^2}$ is also at most one-dimensional. If we therefore fix a Haar measure on E/K^* such that D^*_K/K^* has volume 1

$$\int_{D^*_K/K^*}\Theta_V\Theta_{V^{\sigma}}\Theta_{V^{\sigma^2}} + \int_{D^*_K/K^*}\Theta_V(d^{\sigma}d^{\sigma^2}d) + \int_{D^*_K/K^*}\Theta_V(d^{\sigma^2}d^{\sigma}d)$$

is equal to 3 or 0 depending on whether $V\otimes V^{\sigma}\otimes V^{\sigma^2}$ has an E-invariant vector or not.

By the analogue of Lemma 1 in the present situation (with Haar measures on D^*_K/K^* and D^*_k/k^* to have volume 1),

$$\int_{D^*_K/K^*}\Theta_V(d^{\sigma}d^{\sigma^2}d) = \int_{D^*_K/K^*}\Theta_V(d^{\sigma^2}d^{\sigma}d) = \int_{D^*_k/k^*}\Theta_V(d).$$

We would like to observe here that if f is a class function on D^*_K, then $F_1(x) = f(x^{\sigma}x^{\sigma^2}x)$ is σ conjugacy invariant, and $F_2(x) = f(x^{\sigma^2}x^{\sigma}x)$ is σ^2 conjugacy invariant functions. There seems no simple relationship between F_1 and F_2, and therefore the fact that the first two integrals are equal follows only after the analogue of Lemma 1 is proved, and not apriori. (The σ-conjugacy form of Weyl integration formula is used for the σ-conjugacy invariant function $F_1(x) = f(x^{\sigma}x^{\sigma^2}x)$ just as in Lemma 1 to compare the integral of $F_1(x) = f(x^{\sigma}x^{\sigma^2}x)$ on D^*_K/K^* with the usual Weyl integration formula for the integral of $f(x)$ on D^*_k/k^*.)

It follows that

$$\int_{D^*_K/K^*}\Theta_V\Theta_{V^{\sigma}}\Theta_{V^{\sigma^2}} + 2\int_{D^*_k/k^*}\Theta_V(d) = 3 \quad \text{or} \quad 0, \tag{$*$}$$

depending on whether $V\otimes V^{\sigma}\otimes V^{\sigma^2}$ has an E-invariant vector or not. Because $\int_{D^*_k/k^*}\Theta_V(x)$ is a nonnegative integer, it follows from the equation $(*)$ that

$$\int_{D^*_k/k^*}\Theta_V(x) = 0 \quad \text{or} \quad 1$$

and, moreover,

$$\int_{D_K^*/K^*} \Theta_V \Theta_{V^\sigma} \Theta_{V^{\sigma^2}} = \int_{D_k^*/k^*} \Theta_V(d),$$

completing the proof of the theorem.

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References

1. Arthur, J. and Clozel, L.: *Simple Algebras, Base Change, and the Advanced Theory of the Trace Formula*, Ann. of Math. Stud., Princeton University Press, Princeton.
2. Digne, F.: Shintani descent and L-functions on Deligne Lusztig varieties, *Proc. Sympos. Pure Math*. **47**, part 1, (1987), 61–68.
3. Flicker, Y.: On distinguished representations, *Crelle J*. **418** (1992), 139–172.
4. Gow, R.: Two multiplicity-free permutation of the general linear group $\mathrm{GL}(n, q^2)$, *Math. Z.* **188** (1984), 45–54.
5. Gross, B.: Some applications of Gelfand pairs to number theory, *Bull. Amer. Math. Soc*. **124** (1991), 277–303.
6. Harder, G., Langlands, R. and Rapoport, M.: Algebraische zyklen auf Hilbert-Blumenthal flaachen, *J. Reine Angew. Math*. **366** (1986), 53–120.
7. Jacquet, H. and Lai, K.: A relative trace formula, *Compositio Math*. **54** (1985), 243–310.
8. Jacquet, H., Lai, K.F. and Rallis, S.: A trace formula for symmetric spaces, *Duke Math. J*. **70** (1993), 305–372.
9. Jacquet, H. and Rallis, S.: Symplectic periods, *Crelle J*. **423** (1992), 175–197.
10. Jacquet, H. and Ye, Y.: Distinguished representations and quadratic base change, *Trans. Amer. Math. Soc*. **348** (1996), 913–939.
11. Prasad, D.: Trilinear forms for representations of $\mathrm{GL}(2)$ and local ε-factors, *Compositio Math*. **75** (1990), 1–46.
12. Prasad, D.: Invariant forms for representations of GL_2 over a local field, *Amer. J. Math*. **114** (1992), 1317–1363.
13. Rogawski, J.: *Automorphic Representations of Unitary Groups in Three Variables*, Ann. of Math. Stud., Princeton Univ. Press, Princeton.