TORELLI THEOREM FOR THE DELIGNE–HITCHIN MODULI SPACE

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Abstract. Fix integers $g \geq 3$ and $r \geq 2$, with $r \geq 3$ if $g = 3$. Given a compact connected Riemann surface X of genus g, let $\mathcal{M}_{\mathrm{DH}}(X)$ denote the corresponding $\mathrm{SL}(r, \mathbb{C})$ Deligne–Hitchin moduli space. We prove that the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines (up to an isomorphism) the unordered pair $\{X, \overline{X}\}$, where $\overline{X}$ is the Riemann surface defined by the opposite almost complex structure on X.

1. Introduction

Let X be a compact connected Riemann surface of genus g, with $g \geq 2$. We denote by $X_{\mathbb{R}}$ the C^∞ real manifold of dimension two underlying X. Let $\overline{X}$ be the Riemann surface defined by the almost complex structure $-J_X$ on $X_{\mathbb{R}}$; here J_X is the almost complex structure of X.

Fix an integer $r \geq 2$. The main object of this paper is the $\mathrm{SL}(r, \mathbb{C})$ Deligne–Hitchin moduli space

$$\mathcal{M}_{\mathrm{DH}}(X) = \mathcal{M}_{\mathrm{DH}}(X\,, \mathrm{SL}(r, \mathbb{C}))$$

associated to X. This moduli space $\mathcal{M}_{\mathrm{DH}}(X)$ is a complex analytic space of complex dimension $1 + 2(r^2 - 1)(g - 1)$, which comes with a natural surjective holomorphic map

$$\mathcal{M}_{\mathrm{DH}}(X) \longrightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}.$$

We briefly recall from [Si1, page 7] the description of $\mathcal{M}_{\mathrm{DH}}(X)$ (in [Si1], the group $\mathrm{GL}(r, \mathbb{C})$ is considered instead of $\mathrm{SL}(r, \mathbb{C})$).

- The fiber of $\mathcal{M}_{\mathrm{DH}}(X)$ over $\lambda = 0 \in \mathbb{C} \subset \mathbb{CP}^1$ is the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$ of semistable $\mathrm{SL}(r, \mathbb{C})$ Higgs bundles (E, θ) over X (see section 2 for details).
- The fiber of $\mathcal{M}_{\mathrm{DH}}(X)$ over any $\lambda \in \mathbb{C}^* \subset \mathbb{CP}^1$ is canonically biholomorphic to the moduli space $\mathcal{M}_{\mathrm{conn}}(X)$ of holomorphic $\mathrm{SL}(r, \mathbb{C})$ connections (E, ∇) over X. In fact the restriction of $\mathcal{M}_{\mathrm{DH}}(X)$ to $\mathbb{C} \subset \mathbb{CP}^1$ is the moduli space
$$\mathcal{M}_{\mathrm{Hod}}(X) \longrightarrow \mathbb{C}$$
of λ–connections over X for the group $\mathrm{SL}(r, \mathbb{C})$ (see section 3 for details).
- The fiber of $\mathcal{M}_{\mathrm{DH}}(X)$ over $\lambda = \infty \in \mathbb{CP}^1$ is the moduli space $\mathcal{M}_{\mathrm{Higgs}}(\overline{X})$ of semistable $\mathrm{SL}(r, \mathbb{C})$ Higgs bundles over $\overline{X}$. Indeed, the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ is constructed by glueing $\mathcal{M}_{\mathrm{Hod}}(X)$ to the analogous moduli space
$$\mathcal{M}_{\mathrm{Hod}}(\overline{X}) \longrightarrow \mathbb{C}$$

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> of λ–connections over $\overline{X}$. One identifies the fiber of $\mathcal{M}_{\mathrm{Hod}}(X)$ over $\lambda \in \mathbb{C}^*$ with the fiber of $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ over $1/\lambda \in \mathbb{C}^*$; the identification is done using the fact that the holomorphic connections over both X and $\overline{X}$ correspond to representations of $\pi_1(X_{\mathbb{R}})$ in $\mathrm{SL}(r, \mathbb{C})$ (see section 4 for details).

This construction of $\mathcal{M}_{\mathrm{DH}}(X)$ is due to Deligne [De]. In [Hi2], Hitchin constructed the twistor space for the hyper–Kähler structure of the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$; the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ is identified with this twistor space (see [Si1, page 8]).

We note that while both $\mathcal{M}_{\mathrm{Hod}}(X)$ and $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ are complex algebraic varieties, the moduli space $\mathcal{M}_{\mathrm{DH}}(X)$ does not have any natural algebraic structure.

If we replace X by $\overline{X}$, then the isomorphism class of the Deligne–Hitchin moduli space clearly remains unchanged. In fact, there is a canonical holomorphic isomorphism of $\mathcal{M}_{\mathrm{DH}}(X)$ with $\mathcal{M}_{\mathrm{DH}}(\overline{X})$ over the automorphism of $\mathbb{CP}^1$ defined by $\lambda \longmapsto 1/\lambda$.

We prove the following theorem (see Theorem 4.1):

Theorem 1.1. *Assume that $g \geq 3$, and if $g = 3$, then assume that $r \geq 3$. The isomorphism class of the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines uniquely the isomorphism class of the unordered pair of Riemann surfaces $\{X\, , \overline{X}\}$.*

In other words, if $\mathcal{M}_{\mathrm{DH}}(X)$ is biholomorphic to the Deligne–Hitchin moduli space $\mathcal{M}_{\mathrm{DH}}(Y)$ for another compact connected Riemann surface Y, then either $Y \cong X$ or $Y \cong \overline{X}$.

This paper is organized as follows. Higgs bundles are dealt with in Section 2; we also obtain a Torelli theorem for them (see Corollary 2.5). The λ–connections are considered in Section 3, which also contains a Torelli theorem for their moduli space (see Corollary 3.5). Finally, Section 4 deals with the Deligne–Hitchin moduli space; here we prove our main result.

2. Higgs bundles

Let X be a compact connected Riemann surface of genus g, with $g \geq 3$. Fix an integer $r \geq 2$. If $g = 3$, then we assume that $r \geq 3$. Let

$$\mathcal{M}_{r,\mathcal{O}_X} \tag{2.1}$$

be the moduli space of semistable $\mathrm{SL}(r, \mathbb{C})$–bundles on X. So $\mathcal{M}_{r,\mathcal{O}_X}$ parameterizes all S–equivalence classes of semistable vector bundles E over X of rank r together with an isomorphism $\bigwedge^r E \cong \mathcal{O}_X$. The moduli space $\mathcal{M}_{r,\mathcal{O}_X}$ is known to be an irreducible normal complex projective variety of dimension $(r^2 - 1)(g - 1)$. Let

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \subset \mathcal{M}_{r,\mathcal{O}_X} \tag{2.2}$$

be the open subvariety parameterizing stable $\mathrm{SL}(r, \mathbb{C})$ bundles on X. This open subvariety coincides with the smooth locus of $\mathcal{M}_{r,\mathcal{O}_X}$ according to [NR1, page 20, Theorem 1].

Lemma 2.1. *The holomorphic cotangent bundle*

$$T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$$

does not admit any nonzero holomorphic section.

Proof. Fix a point $x_0 \in X$, and consider the Hecke correspondence

$$\mathcal{M}^{\rm s}_{r,\mathcal{O}_X} \xleftarrow{\ q\ } \mathcal{P} \xrightarrow{\ p\ } \mathcal{U} \subseteq \mathcal{M}_{r,\mathcal{O}_X(x_0)}$$

defined as follows:

- $\mathcal{M}_{r,\mathcal{O}_X(x_0)}$ denotes the moduli space of stable vector bundles F over X of rank r together with an isomorphism $\bigwedge^r F \cong \mathcal{O}_X(x_0)$.
- $\mathcal{U} \subseteq \mathcal{M}_{r,\mathcal{O}_X(x_0)}$ denotes the locus of all F for which every subbundle $F' \subset F$ with $0 < \text{rank}(F') < r$ has negative degree; such vector bundles F are called $(0\,,1)$–*stable* (see [NR2, page 306, Definition 5.1], [BBGN, page 563]).
- $p : \mathcal{P} \longrightarrow \mathcal{U}$ is the $\mathbb{P}^{r-1}$–bundle whose fiber over any vector bundle $F \in \mathcal{U}$ parameterizes all hyperplanes H in the fiber F_{x_0}.
- $q : \mathcal{P} \longrightarrow \mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$ sends any vector bundle $F \in \mathcal{U}$ and hyperplane $H \subseteq F_{x_0}$ to the vector bundle E given by the short exact sequence

  $$0 \longrightarrow E \longrightarrow F \longrightarrow F_{x_0}/H \longrightarrow 0$$

  of coherent sheaves on X; here the quotient sheaf F_{x_0}/H is supported at x_0.

As $\mathcal{M}_{r,\mathcal{O}_X(x_0)}$ is a smooth unirational projective variety (see [Se, page 53]), it does not admit any nonzero holomorphic 1–form. The subset $\mathcal{U} \subseteq \mathcal{M}_{r,\mathcal{O}_X(x_0)}$ is open due to [BBGN, page 563, Lemma 2], and the conditions on r and g ensure that the codimension of the complement $\mathcal{M}_{r,\mathcal{O}_X(x_0)} \setminus \mathcal{U}$ is at least two. Hence also

$$H^0(\mathcal{U}, T^*\mathcal{U}) \,=\, 0$$

due to Hartog's theorem. Since $H^0(\mathbb{P}^{r-1},\, T^*\mathbb{P}^{r-1}) \,=\, 0$, any relative holomorphic 1–form on the $\mathbb{P}^{r-1}$–bundle $p : \mathcal{P} \longrightarrow \mathcal{U}$ vanishes identically. Thus we conclude that

$$H^0(\mathcal{P}, T^*\mathcal{P}) \,=\, 0\,.$$

The same follows for the variety $\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$, because the algebraic map $q : \mathcal{P} \longrightarrow \mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$ is dominant. ☐

We denote by K_X the canonical line bundle on X. Let

$$\mathcal{M}_{\rm Higgs}(X) \,=\, \mathcal{M}_{\rm Higgs}(X\,, \text{SL}(r, \mathbb{C}))$$

denote the moduli space of semistable $\text{SL}(r, \mathbb{C})$ Higgs bundles over X. So $\mathcal{M}_{\rm Higgs}(X)$ parameterizes all S–equivalence classes of semistable pairs $(E\,, \theta)$ consisting of a vector bundle E over X of rank r together with an isomorphism $\bigwedge^r E \cong \mathcal{O}_X$, and a Higgs field $\theta : E \longrightarrow E \otimes K_X$ with $\text{trace}(\theta) \,=\, 0$. The moduli space $\mathcal{M}_{\rm Higgs}(X)$ is an irreducible normal complex algebraic variety of dimension $2(r^2-1)(g-1)$ according to [Si3, page 70, Theorem 11.1].

There is a natural embedding

$$\iota \,:\, \mathcal{M}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\rm Higgs}(X) \tag{2.3}$$

defined by $E \longmapsto (E\,, 0)$. Let

$$\mathcal{M}^{\rm s}_{\rm Higgs}(X) \,\subset\, \mathcal{M}_{\rm Higgs}(X)$$

be the Zariski open locus of Higgs bundles (E,θ) whose underlying vector bundle E is stable (openness of $\mathcal{M}^{\rm s}_{\rm Higgs}(X)$ follows from [Ma, page 635, Theorem 2.8(B)]). Let

$$\mathrm{pr}_E\,:\,\mathcal{M}^{\rm s}_{\rm Higgs}(X)\,\longrightarrow\,\mathcal{M}^{\rm s}_{r,\mathcal{O}_X} \tag{2.4}$$

be the forgetful map defined by $(E,\theta)\longmapsto E$, where $\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$ is defined in (2.2). One has a canonical isomorphism

$$\mathcal{M}^{\rm s}_{\rm Higgs}(X)\,\xrightarrow{\sim}\,T^*\mathcal{M}^{\rm s}_{r,\mathcal{O}_X} \tag{2.5}$$

of varieties over $\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$, because holomorphic cotangent vectors to a point $E\in\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$ correspond, via deformation theory and Serre duality, to Higgs fields $\theta : E\longrightarrow E\otimes K_X$ with $\mathrm{trace}(\theta)\,=\,0$. In particular, $\mathcal{M}^{\rm s}_{\rm Higgs}(X)$ is contained in the smooth locus

$$\mathcal{M}_{\rm Higgs}(X)^{\rm sm}\,\subset\,\mathcal{M}_{\rm Higgs}(X)\,.$$

We recall that the *Hitchin map*

$$H\,:\,\mathcal{M}_{\rm Higgs}(X)\,\longrightarrow\,\bigoplus_{i=2}^{r}H^0(X,\,K_X^{\otimes i}) \tag{2.6}$$

is defined by sending each Higgs bundle (E,θ) to the characteristic polynomial of θ [Hi1], [Hi2].

The multiplicative group $\mathbb{C}^*$ acts on the moduli space $\mathcal{M}_{\rm Higgs}(X)$ as follows:

$$t\cdot(E\,,\theta)\,=\,(E\,,t\theta)\,. \tag{2.7}$$

On the other hand, $\mathbb{C}^*$ acts on the Hitchin space $\bigoplus_{i=2}^{r}H^0(X,\,K_X^{\otimes i})$ as

$$t\cdot(v_2\,,\cdots,v_i\,,\cdots,v_r)\,=\,(t^2v_2\,,\cdots,t^iv_i\,,\cdots,t^rv_r)\,, \tag{2.8}$$

where $v_i\,\in\,H^0(X,\,K_X^{\otimes i})$ and $i\,\in\,\{2,\ldots,r\}$. The Hitchin map H in (2.6) intertwines these two actions of $\mathbb{C}^*$. Note that there is no nonzero holomorphic function on the Hitchin space which is homogeneous of degree 1 for this action (a function f is homogeneous of degree d if $f(t\cdot(v_2,\cdots,v_r))\,=\,t^df((v_2,\cdots,v_r)))$, because all the exponents of t in (2.8) are at least two.

Lemma 2.2. *The holomorphic tangent bundle*

$$T\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}\,\longrightarrow\,\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$$

does not admit any nonzero holomorphic section.

Proof. The proof of [Hi1, page 110, Theorem 6.2] carries over to this situation as follows. A holomorphic section s of $T\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$ provides (by contraction) a holomorphic function

$$f\,:\,T^*\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}\,\longrightarrow\,\mathbb{C} \tag{2.9}$$

on the total space of the cotangent bundle $T^*\mathcal{M}^{\rm s}_{r,\mathcal{O}_X}$, which is linear on the fibers. Under the isomorphism in (2.5), it corresponds to a function on $\mathcal{M}^{\rm s}_{\rm Higgs}(X)$. The conditions on g and r imply that the complement of $\mathcal{M}^{\rm s}_{\rm Higgs}(X)$ has codimension at least two in $\mathcal{M}_{\rm Higgs}(X)$. Since the latter is normal, the function f in (2.9) extends to a holomorphic function

$$\widetilde{f}\,:\,\mathcal{M}_{\rm Higgs}(X)\,\longrightarrow\,\mathbb{C}\,,$$

for example by [Sc, page 90, Korollar 2]. Since f is linear on the fibers, we know that $\widetilde{f}$ is homogeneous of degree 1 for the action (2.7) of $\mathbb{C}^*$.

On the moduli space $\mathcal{M}_{\text{Higgs}}(X)$, the Hitchin map (2.6) is proper [Ni, Theorem 6.1], and also its fibers are connected. Therefore, the function $\widetilde{f}$ is constant on the fibers of the Hitchin map. Hence $\widetilde{f}$ comes from a holomorphic function on the Hitchin space, which is still homogeneous of degree 1. We noted earlier that there are no nonzero holomorphic functions on the Hitchin space which are homogeneous of degree 1. Therefore, $\widetilde{f} = 0$, and consequently we have $f = 0$ and $s = 0$. ☐

Corollary 2.3. *The restriction of the holomorphic tangent bundle*

$$T\mathcal{M}_{\text{Higgs}}(X)^{\text{sm}} \longrightarrow \mathcal{M}_{\text{Higgs}}(X)^{\text{sm}}$$

to $\iota(\mathcal{M}^{\text{s}}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\text{Higgs}}(X)^{\text{sm}}$ *does not admit any nonzero holomorphic section.*

Proof. Using Lemma 2.2, it suffices to show that the normal bundle of the embedding

$$\iota : \mathcal{M}^{\text{s}}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\text{Higgs}}(X)^{\text{sm}}$$

has no nonzero holomorphic sections. The isomorphism in (2.5) allows us to identify this normal bundle with $T^*\mathcal{M}^{\text{s}}_{r,\mathcal{O}_X}$. Now the assertion follows from Lemma 2.1. ☐

The next step is to show that the above property uniquely characterizes the subvariety $\iota(\mathcal{M}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\text{Higgs}}(X)$. This will follow from the following proposition.

Proposition 2.4. *Let Z be an irreducible component of the fixed point locus*

$$\mathcal{M}_{\text{Higgs}}(X)^{\mathbb{C}^*} \subseteq \mathcal{M}_{\text{Higgs}}(X)\,. \tag{2.10}$$

Then $\dim(Z) \leq (r^2-1)(g-1)$, *with equality only for* $Z = \iota(\mathcal{M}_{r,\mathcal{O}_X})$.

Proof. The $\mathbb{C}^*$–equivariance of the Hitchin map H in (2.6) implies

$$\mathcal{M}_{\text{Higgs}}(X)^{\mathbb{C}^*} \subseteq H^{-1}(0),$$

because 0 is the only fixed point in the Hitchin space. We recall that $H^{-1}(0)$ is called the *nilpotent cone.* The irreducible components of $H^{-1}(0)$ are parameterized by the conjugacy classes of the nilpotent elements in the Lie algebra $\text{sl}(r,\mathbb{C})$, and each irreducible component of $H^{-1}(0)$ is of dimension $(r^2-1)(g-1)$ [La].

Thus $\dim(Z) \leq (r^2-1)(g-1)$, and if equality holds, then Z is an irreducible component of the nilpotent cone $H^{-1}(0)$. A result due to Simpson, [Si3, page 76, Lemma 11.9], implies that the only irreducible component of $H^{-1}(0)$ contained in the fixed point locus $\mathcal{M}_{\text{Higgs}}(X)^{\mathbb{C}^*}$ defined in (2.10) is the image $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ of the embedding in (2.3). ☐

Corollary 2.5. *The isomorphism class of the complex analytic space $\mathcal{M}_{\text{Higgs}}(X)$ determines uniquely the isomorphism class of the Riemann surface X, meaning if $\mathcal{M}_{\text{Higgs}}(X)$ is biholomorphic to $\mathcal{M}_{\text{Higgs}}(Y)$ for another compact connected Riemann surface Y of the same genus g, then $Y \cong X$.*

Proof. Let $Z \subset \mathcal{M}_{\text{Higgs}}(X)$ be a closed analytic subset with the following three properties:

- Z is irreducible and has complex dimension $(r^2-1)(g-1)$.

- The smooth locus $Z^{\mathrm{sm}} \subseteq Z$ lies in the smooth locus $\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X)$.
- The restriction of the holomorphic tangent bundle $T\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ to the subspace $Z^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ has no nonzero holomorphic sections.

By Corollary 2.3, the image $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ of the embedding ι in (2.3) has these properties.

The action (2.7) of $\mathbb{C}^*$ on $\mathcal{M}_{\mathrm{Higgs}}(X)$ defines a holomorphic vector field

$$\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}} \longrightarrow T\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}.$$

The third assumption on Z says that any holomorphic vector field on $\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathrm{sm}}$ vanishes on Z^{sm}. Therefore, it follows that the stabilizer of each point in $Z^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Higgs}}(X)$ has nontrivial tangent space at $1 \in \mathbb{C}^*$, and hence the stabilizer must be the full group $\mathbb{C}^*$.

This shows that the fixed point locus $\mathcal{M}_{\mathrm{Higgs}}(X)^{\mathbb{C}^*} \subseteq \mathcal{M}_{\mathrm{Higgs}}(X)$ contains Z^{sm}, and hence also contains its closure Z in $\mathcal{M}_{\mathrm{Higgs}}(X)$. Due to Proposition 2.4, this can only happen for $Z = \iota(\mathcal{M}_{r,\mathcal{O}_X})$. In particular, we have $Z \cong \mathcal{M}_{r,\mathcal{O}_X}$.

We have just shown that the isomorphism class of $\mathcal{M}_{\mathrm{Higgs}}(X)$ determines the isomorphism class of $\mathcal{M}_{r,\mathcal{O}_X}$. The latter determines the isomorphism class of X due to a theorem of Kouvidakis and Pantev [KP, page 229, Theorem E]. $\square$

Remark 2.6. In [BG], an analogous Torelli theorem is proved for Higgs bundles $(E\,,\theta)$ such that the rank and the degree of the underlying vector bundle E are coprime.

3. The λ–connections

In this section, we consider vector bundles with connections, and more generally with λ–connections in the sense of [Si2, page 87] and [Si1, page 4]. We denote by

$$\mathcal{M}_{\mathrm{Hod}}(X) \,=\, \mathcal{M}_{\mathrm{Hod}}(X\,,\mathrm{SL}(r,\mathbb{C}))$$

the moduli space of triples of the form $(\lambda\,,E\,,\nabla)$, where λ is a complex number, and $(E\,,\nabla)$ is a λ–connection on X for the group $\mathrm{SL}(r,\mathbb{C})$. We recall that given any $\lambda \in \mathbb{C}$, a *λ–connection* on X for the group $\mathrm{SL}(r,\mathbb{C})$ is a pair $(E\,,\nabla)$, where

- $E \longrightarrow X$ is a holomorphic vector bundle of rank r together with an isomorphism $\bigwedge^r E \cong \mathcal{O}_X$.
- $\nabla\,:\,E \longrightarrow E\otimes K_X$ is a $\mathbb{C}$–linear homomorphism of sheaves satisfying the following two conditions:
  1. If f is a locally defined holomorphic function on $\mathcal{O}_X$ and s is a locally defined holomorphic section of E, then
  $$\nabla(fs) \,=\, f\cdot\nabla(s) + \lambda\cdot s\otimes df\,.$$
  2. The operator $\bigwedge^r E \longrightarrow (\bigwedge^r E)\otimes K_X$ induced by ∇ coincides with $\lambda\cdot d$.

The moduli space $\mathcal{M}_{\mathrm{Hod}}(X)$ is a complex algebraic variety of dimension $1+2(r^2-1)(g-1)$. It is equipped with a surjective algebraic morphism

$$\mathrm{pr}_\lambda\,:\,\mathcal{M}_{\mathrm{Hod}}(X) \longrightarrow \mathbb{C} \tag{3.1}$$

defined by $(\lambda, E, \nabla) \longmapsto \lambda$.

A 0–connection is a Higgs bundle, so

$$\mathcal{M}_{\mathrm{Higgs}}(X) \,=\, \mathrm{pr}_\lambda^{-1}(0) \subset \mathcal{M}_{\mathrm{Hod}}(X)$$

is the moduli space of Higgs bundles considered in the previous section. In particular, the embedding (2.3) of $\mathcal{M}_{r,\mathcal{O}_X}$ into $\mathcal{M}_{\mathrm{Higgs}}(X)$ also gives an embedding of $\mathcal{M}_{r,\mathcal{O}_X}$ into $\mathcal{M}_{\mathrm{Hod}}(X)$. Slightly abusing notation, we denote this embedding again by

$$\iota \,:\, \mathcal{M}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\mathrm{Hod}}(X)\,. \tag{3.2}$$

It maps the stable locus

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \,\subset\, \mathcal{M}_{r,\mathcal{O}_X}$$

into the smooth locus

$$\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \,\subset\, \mathcal{M}_{\mathrm{Hod}}(X)\,. \tag{3.3}$$

We let $\mathbb{C}^*$ act on $\mathcal{M}_{\mathrm{Hod}}(X)$ as

$$t\cdot(\lambda, E, \nabla) \,=\, (t\cdot\lambda, E, t\cdot\nabla)\,. \tag{3.4}$$

This extends the $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Higgs}}(X)$ introduced above in formula (2.7).

Proposition 3.1. *Let Z be an irreducible component of the fixed point locus*

$$\mathcal{M}_{\mathrm{Hod}}(X)^{\mathbb{C}^*} \,\subseteq\, \mathcal{M}_{\mathrm{Hod}}(X).$$

Then $\dim(Z) \,\leq\, (r^2-1)(g-1)$, *with equality only for* $Z \,=\, \iota(\mathcal{M}_{r,\mathcal{O}_X})$.

Proof. A point $(\lambda, E, \nabla) \,\in\, \mathcal{M}_{\mathrm{Hod}}(X)$ can only be fixed by $\mathbb{C}^*$ if $\lambda \,=\, 0$. Hence Z is automatically contained in $\mathcal{M}_{\mathrm{Higgs}}(X)$. Now the claim follows from Proposition 2.4. ☐

A 1–connection is a holomorphic connection in the usual sense, so

$$\mathcal{M}_{\mathrm{conn}}(X) \,:=\, \mathrm{pr}_\lambda^{-1}(1) \,\subset\, \mathcal{M}_{\mathrm{Hod}}(X) \tag{3.5}$$

is the moduli space of $\mathrm{SL}(r,\mathbb{C})$ holomorphic connections (E,∇) over X. We denote by

$$\mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X) \subset \mathcal{M}_{\mathrm{conn}}(X) \qquad \text{and} \qquad \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \subset \mathcal{M}_{\mathrm{Hod}}(X)$$

the Zariski open subvarieties where the underlying vector bundle E is stable (openness follows from [Ma, page 635, Theorem 2.8(B)]).

Proposition 3.2. *The forgetful map*

$$\mathrm{pr}_E \,:\, \mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X) \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \tag{3.6}$$

defined by $(E\,,\nabla) \longmapsto E$ *admits no holomorphic section.*

Proof. This map pr_E is surjective, because a criterion due to Atiyah and Weil implies that every stable vector bundle E on X of degree zero admits a holomorphic connection. In fact, E admits a unique unitary holomorphic connection according to a theorem of Narasimhan and Seshadri [NS]; this defines a canonical C^∞ section

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X) \tag{3.7}$$

of the map pr_E. Since any two holomorphic $\mathrm{SL}(r,\mathbb{C})$–connections on E differ by a Higgs field $\theta : E \longrightarrow E \otimes K_X$ with $\mathrm{trace}(\theta) = 0$, the map pr_E in (3.6) is a holomorphic torsor under the holomorphic cotangent bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$.

Given a complex manifold $\mathcal{M}$, we denote by $T_{\mathbb{R}}\mathcal{M}$ the tangent bundle of the underlying real manifold $\mathcal{M}_{\mathbb{R}}$, and by

$$J_{\mathcal{M}} : T_{\mathbb{R}}\mathcal{M} \longrightarrow T_{\mathbb{R}}\mathcal{M}$$

the almost complex structure of $\mathcal{M}$. Let

$$\varpi : \mathcal{X} \longrightarrow \mathcal{M} \tag{3.8}$$

be a holomorphic torsor under a holomorphic vector bundle $\mathcal{V} \longrightarrow \mathcal{M}$. To each C^∞ section $s : \mathcal{M} \longrightarrow \mathcal{X}$ of ϖ, we can associate a $(0,1)$–form

$$\overline{\partial} s \in C^\infty(\mathcal{M},\, \Omega^{0,1}\mathcal{M} \otimes \mathcal{V})$$

in the following way. The vector bundle homomorphism

$$\widetilde{ds} := ds + J_{\mathcal{X}} \circ ds \circ J_{\mathcal{M}} : T_{\mathbb{R}}\mathcal{M} \longrightarrow s^*T_{\mathbb{R}}\mathcal{X}$$

satisfies the identity

$$J_{\mathcal{X}} \circ \widetilde{ds} + \widetilde{ds} \circ J_{\mathcal{M}} = J_{\mathcal{X}} \circ ds - ds \circ J_{\mathcal{M}} - J_{\mathcal{X}} \circ ds + ds \circ J_{\mathcal{M}} = 0\,, \tag{3.9}$$

and, since ϖ is holomorphic, we also have

$$d\varpi \circ \widetilde{ds} = d\varpi \circ ds + J_{\mathcal{M}} \circ d\varpi \circ ds \circ J_{\mathcal{M}} = \mathrm{id} - \mathrm{id} = 0\,. \tag{3.10}$$

The equation in (3.10) means that $\widetilde{ds}$ maps into the subbundle of vertical tangent vectors in $s^*T_{\mathbb{R}}\mathcal{X}$, which is canonically isomorphic to $\mathcal{V}_{\mathbb{R}}$ (the real vector bundle underlying the complex vector bundle $\mathcal{V}$). Thus we can consider $\widetilde{ds}$ as a real 1–form

$$\widetilde{ds} \in C^\infty(\mathcal{M},\, T^*_{\mathbb{R}}\mathcal{M} \otimes \mathcal{V}_{\mathbb{R}})\,.$$

Identify $T_{\mathbb{R}}\mathcal{M}$ with $T^{0,1}\mathcal{M}$ using the $\mathbb{R}$–linear isomorphism defined by

$$v \longmapsto v - \sqrt{-1}\cdot J_{\mathcal{M}}(v)\,,$$

and also identify $\mathcal{V}_{\mathbb{R}}$ with $\mathcal{V}$ using the identity map. From (3.9) it follows that $\widetilde{ds}$ is actually a $\mathbb{C}$–linear homomorphism from $T^{0,1}\mathcal{M}$ to $\mathcal{V}$ in terms of these identifications. Let

$$\overline{\partial} s \in C^\infty(\mathcal{M},\, \Omega^{0,1}_{\mathcal{M}} \otimes \mathcal{V})\,.$$

be the $(0\,,1)$–form with values in $\mathcal{V}$ defined by $\widetilde{ds}$. From the construction of $\overline{\partial} s$ it is clear that

- $\overline{\partial} s$ vanishes if and only if s is holomorphic, and
- $\overline{\partial} s$ is $\overline{\partial}$–closed.

Therefore, $\overline{\partial} s$ defines a Dolbeault cohomology class

$$[\varpi] := [\overline{\partial} s] \in H^{0,1}_{\overline{\partial}}(\mathcal{M},\, \mathcal{V}) \cong H^1(\mathcal{M},\, \mathcal{V})\,. \tag{3.11}$$

Since $\mathcal{V}$ acts on $\varpi : \mathcal{X} \longrightarrow \mathcal{M}$, each section $v \in C^\infty(\mathcal{M}, \mathcal{V})$ acts on the sections of ϖ; we denote this action by $s \longmapsto v + s$. The above construction implies that

$$\overline{\partial}(v+s) = \overline{\partial}v + \overline{\partial}s\,. \tag{3.12}$$

Consequently, the Dolbeault cohomology class $[\varpi]$ in (3.11) does not depend on the choice of the C^∞ section s. From (3.12) it also follows that $[\varpi]$ vanishes if and only if the torsor ϖ in (3.8) admits a holomorphic section.

We now take ϖ to be the torsor pr_E in (3.6) under the cotangent bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, and we take s to be the C^∞ section in (3.7). For this case, the class

$$[\overline{\partial}s] \in H^1(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X},\, T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \tag{3.13}$$

has been computed in [BR, page 308, Theorem 2.11]; the result is that it is a nonzero multiple of $c_1(\Theta)$, where Θ is the ample generator of $\mathrm{Pic}(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X})$. In particular, the cohomology class (3.13) of the torsor pr_E in question is nonzero. Therefore, pr_E does not admit any holomorphic section. □

We note that the forgetful map pr_E defined in Proposition 3.2 extends canonically from $\mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X)$ to $\mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X)$. Slightly abusing notation, we denote this extended map again by

$$\mathrm{pr}_E : \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \longrightarrow \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}\,.$$

This map is defined by $(\lambda, E, \nabla) \longmapsto E$, and it also extends the map pr_E in (2.4).

Corollary 3.3. *The only holomorphic map*

$$s : \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \longrightarrow \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X)$$

with $\mathrm{pr}_E \circ s = \mathrm{id}$ *is the restriction*

$$\iota : \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X)$$

of the embedding ι *defined in* (3.2).

Proof. The composition

$$\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \xrightarrow{s} \mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \xrightarrow{\mathrm{pr}_\lambda} \mathbb{C}\,,$$

where pr_λ is the projection in (3.1), is a holomorphic function on $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, and hence it is a constant function. Up to the $\mathbb{C}^*$ action in (3.4), we may assume that this constant is either 0 or it is 1.

If this constant were 1, then s would factor through $\mathrm{pr}_\lambda^{-1}(1) = \mathcal{M}^{\mathrm{s}}_{\mathrm{conn}}(X)$, which would contradict Proposition 3.2.

Hence this constant is 0, and s factors through $\mathrm{pr}_\lambda^{-1}(0) = \mathcal{M}^{\mathrm{s}}_{\mathrm{Higgs}}(X)$. Thus s corresponds, under the isomorphism (2.5), to a holomorphic global section of the vector bundle $T^*\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$. But any such section vanishes due to Lemma 2.1; this means that s is indeed the restriction of the canonical embedding ι in (3.2). □

Corollary 3.4. *As in* (3.3)*, let* $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$ *be the smooth locus of* $\mathcal{M}_{\mathrm{Hod}}(X)$*. The restriction of the holomorphic tangent bundle*

$$T\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$$

to $\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$ *does not admit any nonzero holomorphic section.*

Proof. We denote the holomorphic normal bundle of the restricted embedding

$$\iota \,:\, \mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X} \hookrightarrow \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$$

by $\mathcal{N}$. Due to Lemma 2.2, it suffices to show that this vector bundle $\mathcal{N}$ over $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ has no nonzero holomorphic sections.

One has a canonical isomorphism

$$\mathcal{M}^{\mathrm{s}}_{\mathrm{Hod}}(X) \xrightarrow{\sim} \mathcal{N} \tag{3.14}$$

of varieties over $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$, defined by sending any (λ, E, ∇) to the derivative at $t = 0$ of the map

$$\mathbb{C} \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X)\,, \qquad t \longmapsto (t \cdot \lambda\,, E\,, t \cdot \nabla)\,.$$

Using this isomorphism, from Corollary 3.3 we conclude that vector bundle $\mathcal{N}$ over $\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}$ does not have any nonzero holomorphic sections. This completes the proof. ☐

Corollary 3.5. *The isomorphism class of the complex analytic space* $\mathcal{M}_{\mathrm{Hod}}(X)$ *determines uniquely the isomorphism class of the Riemann surface* X*.*

Proof. The proof is similar to that of Corollary 2.5. Let $Z \subset \mathcal{M}_{\mathrm{Hod}}(X)$ be a closed analytic subset satisfying the following three conditions:

- Z is irreducible and has complex dimension $(r^2-1)(g-1)$.
- The smooth locus $Z^{\mathrm{sm}} \subseteq Z$ lies in the smooth locus $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{Hod}}(X)$.
- The restriction of the holomorphic tangent bundle $T\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$ to the subspace Z^{sm} has no nonzero holomorphic sections.

From Corollary 3.4 we know that $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ satisfies all these conditions.

Consider the vector field on $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}}$ given by the action of $\mathbb{C}^*$ on $\mathcal{M}_{\mathrm{Hod}}(X)$ in (3.4). From the third condition on Z we know that this vector field vanishes on Z^{sm}. This implies that the fixed point locus $\mathcal{M}_{\mathrm{Hod}}(X)^{\mathbb{C}^*}$ contains Z^{sm}, and hence also contains its closure Z. Therefore, using Proposition 3.1 it follows that $Z = \iota(\mathcal{M}_{r,\mathcal{O}_X})$; in particular, Z is isomorphic to $\mathcal{M}_{r,\mathcal{O}_X}$. Finally the isomorphism class of X is recovered from the isomorphism class of $\mathcal{M}_{r,\mathcal{O}_X}$ using [KP, page 229, Theorem E]. ☐

4. THE DELIGNE–HITCHIN MODULI SPACE

We recall Deligne's construction [De] of the Deligne–Hitchin moduli space $\mathcal{M}_{\mathrm{DH}}(X)$, as described in [Si1, page 7].

Let $X_{\mathbb{R}}$ be the C^∞ real manifold of dimension two underlying X. Fix a point $x_0 \in X_{\mathbb{R}}$. Let

$$\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \,:=\, \mathrm{Hom}(\pi_1(X_{\mathbb{R}}, x_0), \mathrm{SL}(r,\mathbb{C}))/\!\!/\mathrm{SL}(r,\mathbb{C})$$

denote the moduli space of representations $\rho \,:\, \pi_1(X_{\mathbb{R}}, x_0) \longrightarrow \mathrm{SL}(r,\mathbb{C})$; the group $\mathrm{SL}(r,\mathbb{C})$ acts on $\mathrm{Hom}(\pi_1(X_{\mathbb{R}}, x_0), \mathrm{SL}(r,\mathbb{C}))$ through the adjoint action of $\mathrm{SL}(r,\mathbb{C})$ on itself. Since the fundamental groups for different base points are identified up to an inner automorphism, the space $\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$ is independent of the choice of x_0. Hence we will omit any reference to x_0.

The Riemann–Hilbert correspondence defines a biholomorphic isomorphism

$$\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \xrightarrow{\sim} \mathcal{M}_{\mathrm{conn}}(X)\,. \tag{4.1}$$

It sends a representation $\rho : \pi_1(X_{\mathbb{R}}) \longrightarrow \mathrm{SL}(r,\mathbb{C})$ to the associated holomorphic $\mathrm{SL}(r,\mathbb{C})$–bundle E^X_ρ over X, endowed with the induced connection ∇^X_ρ. The inverse of (4.1) sends a connection to its monodromy representation, which makes sense because any holomorphic connection on a Riemann surface is automatically flat.

Given $\lambda \in \mathbb{C}^*$, we can similarly associate to a representation

$$\rho\,:\,\pi_1(X_{\mathbb{R}}) \longrightarrow \mathrm{SL}(r,\mathbb{C})$$

the λ–connection $(E^X_\rho\,,\lambda\cdot\nabla^X_\rho)$. This defines a holomorphic open embedding

$$\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X) \tag{4.2}$$

onto the open locus $\mathrm{pr}_\lambda^{-1}(\mathbb{C}^*) \subset \mathcal{M}_{\mathrm{Hod}}(X)$ of all triples $(\lambda\,,E\,,\nabla)$ with $\lambda \neq 0$.

Let J_X denote the almost complex structure of the Riemann surface X. Then $-J_X$ is also an almost complex structure on $X_{\mathbb{R}}$; the Riemann surface defined by $-J_X$ will be denoted by $\overline{X}$.

We can also consider the moduli space $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ of λ–connections on $\overline{X}$, etcetera.

Now one defines the Deligne–Hitchin moduli space

$$\mathcal{M}_{\mathrm{DH}}(X) := \mathcal{M}_{\mathrm{Hod}}(X) \cup \mathcal{M}_{\mathrm{Hod}}(\overline{X})$$

by glueing $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ to $\mathcal{M}_{\mathrm{Hod}}(X)$, along the image of $\mathbb{C}^*\times\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$ for the map in (4.2). More precisely, one identifies, for each $\lambda \in \mathbb{C}^*$ and each representation $\rho \in \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$, the two points

$$(\lambda\,,E^X_\rho\,,\lambda\cdot\nabla^X_\rho) \in \mathcal{M}_{\mathrm{Hod}}(X) \qquad \text{and} \qquad (\lambda^{-1}\,,E^{\overline{X}}_\rho\,,\lambda^{-1}\cdot\nabla^{\overline{X}}_\rho) \in \mathcal{M}_{\mathrm{Hod}}(\overline{X})\,.$$

This identification yields a complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ of dimension $2(r^2-1)(g-1)+1$. This analytic space does not possess a natural algebraic structure since the Riemann–Hilbert correspondence (4.1) is holomorphic and not algebraic.

The forgetful map pr_λ in (3.1) extends to a natural holomorphic morphism

$$\mathrm{pr}\,:\,\mathcal{M}_{\mathrm{DH}}(X) \longrightarrow \mathbb{CP}^1 = \mathbb{C}\cup\{\infty\} \tag{4.3}$$

whose fiber over $\lambda \in \mathbb{CP}^1$ is canonically biholomorphic to

- the moduli space $\mathcal{M}_{\mathrm{Higgs}}(X)$ of $\mathrm{SL}(r,\mathbb{C})$ Higgs bundles on X if $\lambda = 0$,
- the moduli space $\mathcal{M}_{\mathrm{Higgs}}(\overline{X})$ of $\mathrm{SL}(r,\mathbb{C})$ Higgs bundles on $\overline{X}$ if $\lambda = \infty$,
- the moduli space $\mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}})$ of equivalence classes of representations

  $$\mathrm{Hom}(\pi_1(X_{\mathbb{R}},x_0),\mathrm{SL}(r,\mathbb{C}))/\!\!/\mathrm{SL}(r,\mathbb{C})$$

  if $\lambda \neq 0\,,\infty$.

Now we are in a position to prove the main result.

Theorem 4.1. *The isomorphism class of the complex analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines uniquely the isomorphism class of the unordered pair of Riemann surfaces $\{X\,,\overline{X}\}$.*

Proof. We denote by $\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)$ the smooth locus, and by

$$T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \longrightarrow \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$$

its holomorphic tangent bundle. Since $\mathcal{M}_{\mathrm{Hod}}(X)$ is open in $\mathcal{M}_{\mathrm{DH}}(X)$, Corollary 3.4 implies that the restriction of $T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$ to

$$\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_X}) \subset \mathcal{M}_{\mathrm{Hod}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \tag{4.4}$$

does not admit any nonzero holomorphic section. The same argument applies if we replace X by $\overline{X}$. Since $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ is also open in $\mathcal{M}_{\mathrm{DH}}(X)$, the restriction of $T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$ to

$$\iota(\mathcal{M}^{\mathrm{s}}_{r,\mathcal{O}_{\overline{X}}}) \subset \mathcal{M}_{\mathrm{Hod}}(\overline{X})^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \tag{4.5}$$

does not admit any nonzero holomorphic section either. Here $\mathcal{M}_{r,\mathcal{O}_{\overline{X}}}$ is the moduli space of holomorphic $\mathrm{SL}(r,\mathbb{C})$–bundles E on $\overline{X}$, and ι denotes, as in (2.3) and in (3.2), the canonical embedding of $\mathcal{M}_{r,\mathcal{O}_{\overline{X}}}$ into $\mathcal{M}_{\mathrm{Higgs}}(\overline{X}) \subset \mathcal{M}_{\mathrm{Hod}}(\overline{X})$ defined by $E \longmapsto (E,0)$.

The rest of the proof is similar to that of Corollary 2.5. We will extend the $\mathbb{C}^*$ action on $\mathcal{M}_{\mathrm{Hod}}(X)$ in (3.4) to $\mathcal{M}_{\mathrm{DH}}(X)$. First consider the action of $\mathbb{C}^*$ on $\mathcal{M}_{\mathrm{Hod}}(\overline{X})$ defined as in (3.4) by substituting $\overline{X}$ in place of X. Note that the action of any $t \in \mathbb{C}^*$ on the open subset $\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \longrightarrow \mathcal{M}_{\mathrm{Hod}}(X)$ in (4.2) coincides with the action of $1/t$ on $\mathbb{C}^* \times \mathcal{M}_{\mathrm{rep}}(X_{\mathbb{R}}) \longrightarrow \mathcal{M}_{\mathrm{Hod}}(\overline{X})$. Therefore, we get an action of $\mathbb{C}^*$ on $\mathcal{M}_{\mathrm{DH}}(X)$. Let

$$\eta : \mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \longrightarrow T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \tag{4.6}$$

be the holomorphic vector field defined by this action of $\mathbb{C}^*$.

Let $Z \subset \mathcal{M}_{\mathrm{DH}}(X)$ be a closed analytic subset with the following three properties:

- Z is irreducible and has complex dimension $(r^2-1)(g-1)$.
- The smooth locus $Z^{\mathrm{sm}} \subseteq Z$ lies in the smooth locus $\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}} \subset \mathcal{M}_{\mathrm{DH}}(X)$.
- The restriction of the holomorphic tangent bundle $T\mathcal{M}_{\mathrm{DH}}(X)^{\mathrm{sm}}$ to the subspace Z^{sm} has no nonzero holomorphic sections.

We noted above that both $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ and $\iota(\mathcal{M}_{r,\mathcal{O}_{\overline{X}}})$ (see (4.4) and (4.5)) satisfy these conditions.

The third condition on Z implies that the vector field η in (4.6) vanishes on Z^{sm}. It follows that the fixed point locus $\mathcal{M}_{\mathrm{DH}}(X)^{\mathbb{C}^*}$ contains Z^{sm}, and hence also contains its closure Z. Therefore, using Proposition 3.1 we conclude that Z is one of $\iota(\mathcal{M}_{r,\mathcal{O}_X})$ and $\iota(\mathcal{M}_{r,\mathcal{O}_{\overline{X}}})$. Using [KP, page 229, Theorem E] we now know that the isomorphism class of the analytic space $\mathcal{M}_{\mathrm{DH}}(X)$ determines the isomorphism class of the unordered pair of Riemann surfaces $\{X\,,\overline{X}\}$. This completes the proof of the theorem. ☐

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