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Reduction of Structure Group of Principal Bundles over a Projective Manifold with Picard Number One

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1 Introduction

Let X be a connected projective manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}$. Let G be a connected reductive algebraic group over $\mathbb{C}$. A principal G-bundle will be called *split* if it admits a reduction of structure group to a maximal torus of G. This corresponds to the usual definition of a split vector bundle as a direct sum of line bundles.

Let B be a Borel subgroup of G and $\rho : G \to G'$ an injective homomorphism to a connected reductive algebraic group over $\mathbb{C}$. In Theorem 2.7 we prove the following theorem.

Theorem 1.1. Let E be a principal G-bundle over X such that the principal G'-bundle $E(G') := (E \times G')/G$ obtained by extending the structure group using ρ is split. Then E admits a reduction of structure group to B. □

Furthermore, if X is Fano or it has trivial canonical bundle, then we have the following stronger consequence (Theorem 3.3).

Theorem 1.2. For a G-bundle E on X, if the G'-bundle $E(G')$ is split, then E itself is split. □

If W is a vector bundle on $\mathbb{CP}^n$, $n \geq 2$, such that the restriction $W|_{\mathbb{CP}^2}$ to a plane $\mathbb{CP}^2$ splits as a direct sum of line bundles, then W is already a direct sum of line bundles [12, page 42, Theorem 2.3.2]. This result is a consequence of a splitting criterion

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of G. Horrocks. A corollary of Theorem 1.2 (Corollary 4.1) gives the following analog for G-bundles. If E is a principal G-bundle over $\mathbb{CP}^n$ such that the restriction $E|_{\mathbb{CP}^2}$ is split, then E itself is split. In Lemma 2.2, we show that any split G-bundle over X admits a reduction of structure group to a one-parameter subgroup of G.

A vector bundle W on $\mathbb{CP}^n$ is trivial if there is a point p in $\mathbb{CP}^n$ such that the restriction of W to any line passing through p is trivial [12, page 51, Theorem 3.2.1]. In Corollary 4.2, we prove that a principal G-bundle E over $\mathbb{CP}^n$ is trivial if and only if there is a point p in $\mathbb{CP}^n$ such that the restriction of E to every line $\mathbb{CP}^1 \subset \mathbb{CP}^n$ passing through p is trivial.

2 Criterion for reduction to Borel subgroup

Let G be a connected reductive algebraic group over $\mathbb{C}$. The center of G will be denoted by $Z(G)$. Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus of G.

Let X be a connected smooth projective variety over $\mathbb{C}$ of dimension d. We will assume that the Picard group $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}$.

Fix an ample line bundle ξ on X. The *degree* of a torsionfree coherent sheaf F on X is

$$\deg(F) := \int_X c_1(F)c_1(\xi)^{d-1}. \tag{2.1}$$

Note that if U is a Zariski open subset of X such that $\mathrm{codim}_{\mathbb{C}}(X \setminus U) \geq 2$, then $\deg(i_*i^*F) = \deg(F)$, where $i : U \hookrightarrow X$ is the inclusion map.

Given a group homomorphism $h : H \to G$ and a principal H-bundle E_H, the *extension of structure group* of E_H to G is the G-bundle defined as

$$h_*E_H := (E_H \times G)/H = E_H(G), \tag{2.2}$$

where the quotient is taken using the diagonal action. It may be noted that even in the case h is an embedding, it can happen that h_*E_H and $h_*E'_H$ are isomorphic but E_H and E'_H are not isomorphic.

If H is a closed subgroup, h is the inclusion, and E a principal G-bundle over X, then a *reduction of the structure group* of E to H is a principal H-bundle E_H together with an isomorphism $h_*E_H \cong E$. Equivalently, it is defined by an algebraic section of the fiber bundle

$$E/H \longrightarrow X. \tag{2.3}$$

The quotient E/H corresponds to the restriction to H of the action of G to E. If S is the subvariety of E/H defined by the image of such a section, then we recover E_H as the subset of E given by the inverse image of S for the natural projection

$$E \longrightarrow E/H. \tag{2.4}$$

Note that if $E_H \subset E$ is a reduction of the structure group of E to a subgroup H, then for any $g \in G$ the translation $E_H g$ of E_H by g defines a reduction of the structure group of E to the conjugate $g^{-1}Hg$ of H.

A principal G-bundle E over X is called *semistable* if for every maximal parabolic subgroup $P \subset G$ and for every reduction of structure group $\sigma : U \to E/P$ over some Zariski open subset U with $\mathrm{codim}_{\mathbb{C}}(X \setminus U) \geq 2$, the inequality

$$\deg \sigma^*\left(T_{\mathrm{rel}}\right) \geq 0 \tag{2.5}$$

is valid, where T_{rel} is the relative tangent bundle for the natural projection of $E/P|_U$ to U. Since $\mathrm{Pic}(X) \cong \mathbb{Z}$, the semistability condition does not depend on the choice of the polarization ξ.

The condition of semistability has the following equivalent reformulation [14, Lemma 2.1]. A principal G-bundle E over X is semistable if and only if for every parabolic subgroup $P \subset G$ and every holomorphic reduction E_P of the structure group of E to P over some open subset $U \subseteq X$ with $\mathrm{codim}_{\mathbb{C}}(X \setminus U) \geq 2$, and for every nontrivial character $\theta : P \to \mathbb{C}^*$ which is dominant with respect to some Borel subgroup of G contained in P, the inequality $\deg(E_P(\theta)) \leq 0$ is valid, where $E_P(\theta) = E_P(\mathbb{C})$ is the line bundle $(E_P \times \mathbb{C})/P$ over U associated to E_P for the character θ of P.

From the definition, it follows that a $GL(n, \mathbb{C})$-bundle is semistable if and only if the rank n vector bundle associated to it by the standard representation is semistable. We recall that a vector bundle E is called semistable if for any coherent subsheaf F of E, with $0 < \mathrm{rank}(F) < \mathrm{rank}(E)$, the inequality

$$\frac{\deg(F)}{\mathrm{rank}(F)} \leq \frac{\deg(E)}{\mathrm{rank}(E)} \tag{2.6}$$

is valid [10].

A vector bundle is called *split* if it is a direct sum of line bundles, and recall from the introduction that a principal G-bundle over X is defined to be split if it admits a reduction of structure group to the maximal torus T.

Remark 2.1. It is easy to see that a $GL(n, \mathbb{C})$-bundle is split if and only if the vector bundle associated to it by the standard representation of $GL(n, \mathbb{C})$ splits as a direct sum of line bundles. More generally, any extension of the structure group of a split bundle is always split.

A principal G-bundle E over X is said to admit a reduction of structure group to a *one-parameter subgroup of the maximal torus* T if there is a homomorphism

$$\gamma : \mathbb{C}^* \longrightarrow T \tag{2.7}$$

and a principal $\mathbb{C}^*$-bundle $E_{\mathbb{C}^*}$ over X such that the principal G-bundle obtained by extending the structure group of $E_{\mathbb{C}^*}$, using the composition of γ with the inclusion of T in G, is isomorphic to E. Such a G-bundle is obviously split. The following lemma shows that the converse is true.

Lemma 2.2. Any principal T-bundle over X admits a reduction of the structure group to a one-parameter subgroup. □

Proof. Since T is isomorphic to a product of copies of $\mathbb{C}^*$, there is a natural bijection between (isomorphism classes of) T-bundles and vector bundles of the form

$$\zeta_1 \oplus \zeta_2 \oplus \cdots \oplus \zeta_r, \tag{2.8}$$

where $r = \dim T$ and ζ_i are line bundles. Since $\mathrm{Pic}(X) \cong \mathbb{Z}$, it is generated by a fixed line bundle $\mathcal{O}(1)$. So we have $\zeta_i \cong \mathcal{O}(n_i)$ for some $n_i \in \mathbb{Z}$. This means that there is a reduction of the structure group of any principal T-bundle to $\mathbb{C}^*$, with the homomorphism $\gamma : \mathbb{C}^* \to T$ determined by the numbers n_i. More precisely, γ is of the form $\lambda \mapsto (\lambda^{n_1}, \ldots, \lambda^{n_i}, \ldots, \lambda^{n_r})$. This completes the proof of the lemma. ■

Fix an injective homomorphism $\rho : G \to G'$ as in the introduction.

An important fact used in our arguments is that for any connected reductive group G, the center $Z(G)$ is always contained in T. In fact, $Z(G)$ is the intersection of all possible maximal tori of G.

Proposition 2.3. If the principal G'-bundle $E(G')$ is split, then the adjoint vector bundle $\mathrm{ad}(E)$ is split. □

Proof. Take a faithful representation

$$\rho' : G' \longrightarrow \mathrm{GL}(V). \tag{2.9}$$

Let $E(V) := (E(G') \times V)/G'$ be the vector bundle associated to $E(G')$ for ρ'. Since $E(G')$ is the extension of E, the vector bundle $E(V)$ is also an extension of E for $\rho' \circ \rho$.

Since $\rho' \circ \rho$ is a faithful representation of G, and G is reductive, any irreducible G-module is a direct summand of $E(V)^{\otimes i} \bigotimes (E(V)^*)^{\otimes i'}$ for some $i, i' \geq 0$ [5, page 40, Proposition 3.1(a)]. In other words, any G-module is a direct summand of a finite sum of the type

$$\bigoplus_j \left(E(V)^{\otimes i_j} \otimes \left(E(V)^*\right)^{\otimes i_j'} \right). \tag{2.10}$$

In particular, the adjoint bundle ad(E) is a direct summand of a direct sum of vector bundles of this type.

Since $E(G')$ is split, from Remark 2.1 we know that $E(V)$ is a direct sum of line bundles. Therefore, any $E(V)^{\otimes i} \bigotimes (E(V)^*)^{\otimes i'}$ is also a direct sum of line bundles.

We will show that a direct summand of a split vector bundle over a compact connected projective manifold is split. Let W be a split vector bundle and $W = W^1 \bigoplus W^2$ any decomposition into a direct sum of vector bundles. Let $W^i = \bigoplus_{j \in I_i} V_j^i$, $i = 1, 2$, be the decomposition into indecomposable vector bundles. The existence of such a decomposition is ensured by [2, page 315, Lemma 9]. Consequently, $(\bigoplus_j V_j^1) \bigoplus (\bigoplus_j V_j^2)$ is a decomposition of W into indecomposable vector bundles. A theorem of Atiyah [2, page 315, Theorem 3] says that any given vector bundle over a connected complex projective manifold can be uniquely decomposed as a direct sum of indecomposable vector bundles (unique up to reordering the summation). Now, since W decomposes as a direct sum of line bundles, it follows immediately that each V_j^i is a line bundle.

We have shown that ad(E) is a direct summand of a split vector bundle. Consequently, ad(E) is split. This completes the proof of the proposition. ■

Lemma 2.4. Let E be a semistable G-bundle over X such that the vector bundle ad(E) is split. Then E admits a reduction of the structure group to Z(G). In particular, E is split. □

Proof. Since E is semistable hence ad(E) is semistable.

Now, since $\mathrm{Pic}(X) \cong \mathbb{Z}$, the semistability condition of ad(E) ensures that ad(E) is of the form

$$\mathrm{ad}(E) \cong \zeta^{\oplus N}, \tag{2.11}$$

where ζ is a line bundle over X and $N = \mathrm{rank}(\mathrm{ad}(E))$. Since $\mathrm{Pic}(X) \cong \mathbb{Z}$ and the degree of ad(E) is zero, it follows immediately that the line bundle ζ is trivial and hence ad(E) is a trivial vector bundle. Since ad(E) is isomorphic to the trivial vector bundle, any section of ad(E) over X is a constant section. So if s_1 and s_2 are two sections of ad(E) over X, then their bracket $[s_2, s_2]$ is also constant.

Consequently, $\mathrm{ad}(E)$ is trivial as a Lie algebra bundle. Let $\mathrm{Aut}(\mathfrak{g})$ denote the group of Lie algebra automorphisms of $\mathfrak{g}$. The triviality of $\mathrm{ad}(E)$ as a Lie algebra bundle implies that the principal bundle $E(\mathrm{Aut}(\mathfrak{g}))$ associated to the adjoint representation $G \to \mathrm{Aut}(\mathfrak{g})$ is trivial. The adjoint representation factors is as follows:

$$G \longrightarrow G/Z \xrightarrow{j} \mathrm{Aut}(\mathfrak{g}). \tag{2.12}$$

The group G/Z is the connected component of the identity of $\mathrm{Aut}(\mathfrak{g})$ and j is the inclusion, hence we have a short exact sequence of groups

$$\{e\} \longrightarrow G/Z \xrightarrow{j} \mathrm{Aut}(\mathfrak{g}) \longrightarrow F \longrightarrow \{e\}, \tag{2.13}$$

where F is a discrete group. This gives an exact sequence of pointed sets (see [6, page 153, Section 5], [15, Proposition 11], or [11, page 122, Proposition III.4.5])

$$H^0(X, F) \xrightarrow{\delta} H^1(X, G/Z) \xrightarrow{j_*} H^1(X, \mathrm{Aut}(\mathfrak{g})). \tag{2.14}$$

This means that $(j_*)^{-1}(e) = \mathrm{image}(\delta)$, where e is the point corresponding to the trivial bundle. Let $E(G/Z)$ be the principal bundle associated to $G \to G/Z$. It gives a point $[E(G/Z)]$ in $H^1(X, G/Z)$, whose image under j_* is equal to $[E(\mathrm{Aut}(\mathfrak{g}))]$. Since this principal bundle is trivial, by exactness we know that $[E(G/Z)]$ is in the image of δ. Recall that $\delta(\sigma)$ is defined as the point corresponding to the principal G/Z-bundle E' given by the Cartesian diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & X \times \mathrm{Aut}(\mathfrak{g}) \arrow[d] \\ X \arrow[r, "\sigma"] & X \times F. \end{tikzcd} \tag{2.15}$$

Since F is discrete, σ has to be constant, and then, for any σ, we have that $\delta(\sigma)$ corresponds to the trivial bundle, so we conclude that $E(G/Z)$ is trivial.

Now, consider the exact sequence of pointed sets

$$H^1(X, Z) \xrightarrow{i_*} H^1(X, G) \longrightarrow H^1(X, G/Z). \tag{2.16}$$

Since the point $[E] \in H^1(X, G)$ maps to the trivial element $[E(G/Z)]$, there is a principal Z-bundle E_Z such that $i_* E_Z \cong E$. In other words, E_Z is a reduction of structure group of E to Z and the proof is complete. ■

Now we are in a position to prove the main result of this section.

Proposition 2.5. Let E be a nonsemistable G-bundle such that the vector bundle $\mathrm{ad}(E)$ is split. Then E admits a reduction of the structure group to a Borel subgroup of G. □

Proof. A nonsemistable G-bundle F admits a *canonical reduction of structure group* [1, Theorem 1.1], [3], which for the case of vector bundles is the usual Harder-Narasimhan filtration [10]. Consider the adjoint vector bundle $\mathrm{ad}(F)$. Since F is not semistable, the vector bundle $\mathrm{ad}(F)$ is not semistable (if $\mathrm{ad}(F)$ is semistable, then F is semistable [13, Theorem 3.18]). Consider the Harder-Narasimhan filtration of $\mathrm{ad}(F)$. Since G is reductive, the Lie algebra $\mathfrak{g}$ admits a nondegenerate symmetric bilinear form invariant under the adjoint action of G. Such a form induces a nondegenerate symmetric bilinear form on $\mathrm{ad}(F)$. Since $\mathrm{ad}(F)^* \cong \mathrm{ad}(F)$, the Harder-Narasimhan filtration is of the form

$$0 = W_{-l} \subset W_{-l+1} \subset \cdots \subset W_{-1} \subset W_0 \subset W_1 \subset \cdots \subset W_{l-1} \subset W_l = \mathrm{ad}(F), \qquad (2.17)$$

where W_{-i} coincides with $W_i^{\perp}$ with respect to the nondegenerate symmetric bilinear form. The canonical reduction $F_P \subset F$ over a Zariski open subset U of X with $\mathrm{codim}_{\mathbb{C}}(X \setminus U) \geq 2$ is determined by the condition that the adjoint bundle $\mathrm{ad}(F_P)$ coincides with W_0. The open subset U is the one over which each W_i is a subbundle of $\mathrm{ad}(F)$. It turns out that W_{-1} coincides with the vector bundle $E_P(\mathfrak{n})$ associated to E_P for the adjoint action of P on the nilpotent radical $\mathfrak{n}$ of the Lie algebra of P.

Let $E_P \subset E$ denote the canonical reduction of E to a proper parabolic subgroup P of G over a Zariski open subset U of X with $\mathrm{codim}_{\mathbb{C}}(X \setminus U) \geq 2$.

Since $\mathrm{ad}(E)$ is a direct sum of line bundles, each term in the Harder-Narasimhan filtration of the adjoint bundle $\mathrm{ad}(E)$ is a subbundle of $\mathrm{ad}(E)$. Therefore, the open subset U coincides with X.

Consider the exact sequence

$$\{e\} \longrightarrow R_u(P) \longrightarrow P \xrightarrow{\psi_P} L \longrightarrow \{e\}, \qquad (2.18)$$

where $R_u(P)$ is the *unipotent radical* of P and L is the *Levi factor*. The Lie algebra of L will be denoted by $\mathfrak{l}$.

Let $E_L = (\psi_P)_* E_P$ be the extension of the structure group by ψ_P. Its adjoint bundle $E_L(\mathfrak{l})$ will be denoted by $\mathrm{ad}(E_L)$. From the construction of canonical reduction it follows that $\mathrm{ad}(E_L)$ is semistable. In fact, $\mathrm{ad}(E_L)$ coincides with W_0/W_{-1}, if $\{W_i\}$ is the Harder-Narasimhan filtration of $\mathrm{ad}(E)$. Therefore, the L-bundle E_L is semistable [13, Theorem 3.18].

Since $\mathrm{ad}(E)$ is a direct sum of line bundles, each quotient W_i/W_{i-1} is naturally a direct summand of $\mathrm{ad}(E)$, where $\{W_i\}$ is the Harder-Narasimhan filtration of $\mathrm{ad}(E)$.

(This follows from the construction of Harder-Narasimhan filtration of a direct summand of line bundles.) Consequently, $\mathrm{ad}(E_L)$ is a direct summand of $\mathrm{ad}(E)$. We saw in the proof of Proposition 2.3 that a direct summand of a split vector bundle is again split. Therefore, $\mathrm{ad}(E_L)$ is a split vector bundle.

Since E_L is semistable and $\mathrm{ad}(E_L)$ is split, from Proposition 2.3 and Lemma 2.4 it follows that the L-bundle E_L admits a reduction of the structure group to its center hence to the maximal torus $T(L)$ of L.

Let $E_{T(L)} \subset E_L$ be a $T(L)$-bundle giving a reduction of the structure group of E_L to $T(L)$.

Let $\overline{\psi} : E_P \to E_L$ denote the projection induced by the natural projection ψ_P of P to L. The inverse image

$$\overline{\psi}^{-1}(E_{T(L)}) \subset E_P \tag{2.19}$$

defines a reduction of the structure group of the P-bundle E_P to the subgroup $\psi_P^{-1}(T(L)) \subset P$. But $\psi_P^{-1}(T(L))$ lies in a Borel subgroup of G contained in P. This completes the proof. ∎

Remark 2.6. In the proof of Proposition 2.5, it is possible to directly prove that the bundle $\mathrm{ad}(E_L)$ is trivial from the description of the canonical filtration of the G-bundle E in terms of the canonical filtration of $\mathrm{ad}(E)$.

Combining all these results, we obtain Theorem 1.1 stated in the introduction.

Theorem 2.7. Let E be a principal G-bundle over X such that the principal G'-bundle $E(G') := (E \times G')/G$ obtained by extending the structure group using ρ is split. Then E admits a reduction of structure group to B. □

Proof. By Proposition 2.3, $\mathrm{ad}(E)$ is split. If E is semistable, then by Lemma 2.4 it is split (in particular, it has a reduction to a Borel subgroup). If E is not semistable, then we apply Proposition 2.5. ∎

Remark 2.8. It is not true in general that if $\mathrm{ad}(E)$ is split then E is split. Take two elements $g_1, g_2 \in SU(2)$ such that $g_1 g_2 g_1^{-1} g_2^{-1} = -1$. Let Y be a Riemann surface of genus two. Let Γ denote the free group generated by a_1, b_1, a_2, b_2. Take a standard presentation of the fundamental group $\pi_1(Y)$ as the quotient of Γ by the normal subgroup generated by $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$. Consider the homomorphism

$$\beta : \pi_1(Y) \longrightarrow SU(2) \tag{2.20}$$

defined by $\beta(a_i) = g_i$ and $\beta(b_i) = 1$, $i = 1, 2$. Since g_1 and g_2 do not commute, β defines an irreducible unitary representation. Therefore, the rank two vector bundle V on Y associated to this representation is stable. In particular, V is indecomposable. Since the vector bundle V is indecomposable, the principal $SL(2, \mathbb{C})$-bundle defined by V is not split. But the representation of $\pi_1(Y)$ on the Lie algebra $\mathfrak{su}(2)$ defined by β is reducible. Indeed, as the adjoint action of $-1 \in SU(2)$ on the Lie algebra $\mathfrak{su}(2)$ is trivial, the adjoint actions of g_1 and g_2 commute. Therefore, the adjoint vector bundle $ad(V)$ defined by the sheaf of trace zero endomorphisms of V decomposes as a direct sum of line bundles.

In the next section we will see that an assumption on X ensures that E is split if E_1 is split.

3 Reduction to the maximal torus

We continue with the notation of Section 2.

Proposition 3.1. Assume that $H^1(X, \zeta) = 0$ for all line bundles ζ on X. Let E_B be a B-bundle. Then E_B admits a reduction of the structure group to T. □

Proof. Let $R_u(B)$ denote the unipotent radical of B. Consider the exact sequence of groups

$$\{e\} \longrightarrow R_u(B) \longrightarrow B \xrightarrow{\psi} T \longrightarrow \{e\}. \tag{3.1}$$

Let $s : T \to B$ be a splitting of this exact sequence. Let $E_T := \psi_* E_B$ be the principal T-bundle obtained by extending the structure group of E_B. Note that the principal B-bundle $s_* E_T$ admits a reduction of the structure group to T. Consequently, it is enough to show that $s_* E_T$ is isomorphic to E_B.

Taking the extension of the structure group of these two B-bundles, namely E_B and $s_* E_T$, by ψ, we get the same T-bundle E_T (here we use the fact that s is a splitting). Therefore, it suffices to show that if E_B and E'_B are two B-bundles with $\psi_* E_B \cong \psi_* E'_B$, then $E_B \cong E'_B$. For this we have to understand the isomorphism classes of B-bundles which under the extension of the structure group by ψ give E_T.

We observe that T acts on the unipotent radical $R_u(B)$ by conjugation via s. For any $t \in T$ the action of t on $R_u(B)$ will be denoted as $t(u) := tut^{-1}$, where $u \in R_u(B)$. Since T is abelian, we can find a filtration of $R_u(B)$

$$R_u(B) = U_1 \supset U_2 \supset \cdots \supset U_k \supset U_{k+1} = \{e\} \tag{3.2}$$

such that U_{j+1} is normal in U_j, each U_j are invariant under the action of T and the quotients U_j/U_{j+1} are isomorphic to $\mathbb{G}_a$, the additive group $\mathbb{C}$.

Now we define twisted cohomology sets $H^1(X, U_i(E_T))$ for each i with the property that for $i = 1$, the set $H^1(X, U_1(E_T))$ parametrizes all the isomorphism classes of B-bundles that extend to E_T via ψ.

Fix a cocycle $\{t_{ij}\}$ for E_T with respect to an étale open cover $\mathcal{V} = \{V_i\}$. Let E_B be a B-bundle with $\psi_* E_B \cong E_T$, then (refining $\mathcal{V}$ if necessary) a cocycle for E_B is of the form

$$u_{ij}t_{ij} : V_i \times_X V_j \longrightarrow B, \tag{3.3}$$

where $\{t_{ij}\}$ defines the cocycle for E_T and $u_{ij} : V_i \times_X V_j \to R_u(B)$. The cocycle condition is $u_{ij}t_{ij}u_{jk}t_{jk} = u_{ik}t_{ik}$ on $V_i \times_X V_j \times_X V_k$.

Using the action of T on $R_u(B)$, we can rewrite the cocycle condition as

$$u_{ij}t_{ij}(u_{jk}) = u_{ik}. \tag{3.4}$$

These $\{u_{ij}\}$ are collectively called a twisted cocycle, and two twisted cocycles $\{u_{ij}\}$ and $\{v_{ij}\}$ are equivalent if there are morphisms $s_i : V_i \to R_u(B)$ for each index i satisfying the condition

$$s_i u_{ij} t_{ij}(s_j^{-1}) = v_{ij} \tag{3.5}$$

for each pair i, j with $V_i \times_X V_j \neq \emptyset$. Recall the notation $t(u) := tut^{-1}$. The equivalence classes of twisted cocycles form the pointed set $H^1(X, R_u(B)(E_T))$ (taken over all open covers) with distinguished element being the cocycle $\{u_{ij}\}$ with $u_{ij} = 1$ for all i and j. This pointed set is in bijective correspondence with the isomorphism classes of B-bundles that extend to E_T via ψ. Since the action of T on $R_u(B)$ preserves the filtration (3.2), this action induces actions of T on each U_m and U_m/U_{m+1}. Therefore, we can also define $H^1(X, U_m(E_T))$ and $H^1(X, U_m/U_{m+1}(E_T))$ for each m (see [4] or [7, Appendix] for more details). Note that by composing we get the morphisms of pointed sets

$$f : H^1(X, U_{m+1}(E_T)) \longrightarrow H^1(X, U_m(E_T)) \tag{3.6}$$

and $g : H^1(X, U_m(E_T)) \to H^1(X, U_m/U_{m+1}(E_T))$.

The proof of the proposition will be completed using the following lemma.

Lemma 3.2. The following sequence of pointed sets is exact:

$$H^1(X, U_{m+1}(E_T)) \xrightarrow{f} H^1(X, U_m(E_T)) \xrightarrow{g} H^1(X, U_m/U_{m+1}(E_T)). \tag{3.7}$$

□

Recall that this means that $\mathrm{image}(f) = g^{-1}(e)$, where e is the distinguished point in $H^1(X, U_m/U_{m+1}(E_T))$.

Proof of Lemma 3.2. It is easy to see that the composite is the constant map to e. So, we only need to verify that image$(f) \supset g^{-1}(e)$.

Let $\{u_{ij}\}$ be a twisted cocycle on an open cover $\mathcal{V} = \{V_i\}$ with values in U_m, such that $g([\{u_{ij}\}]) = e$. The element $g([\{u_{ij}\}])$ is represented by the cocycle $\{\overline{u}_{ij}\}$, where $\overline{u}_{ij}$ is obtained by composing u_{ij} with the projection morphism $U_m \mapsto U_m/U_{m+1}$. Then there are morphisms $\overline{s}_i : V_i \to U_m/U_{m+1}$ such that for each $V_i \times_X V_j$ (when nonempty) we have $\overline{s}_i \overline{u}_{ij} t_{ij}((\overline{s}_j)^{-1}) = 1$.

Since U_m and U_m/U_{m+1} as schemes are just affine spaces and the morphisms between them are projection morphisms, we can lift each of the morphisms $\overline{s}_i$ to get morphisms $s_i : V_i \to U_m$. We fix such a lifting. We define a cocycle $\{v_{ij}\}$ by $v_{ij} = s_i u_{ij} t_{ij}(s_j^{-1})$. It can be checked that this defines a cocycle which takes values in U_{m+1} and has the property that $f([\{v_{ij}\}]) = [\{u_{ij}\}]$. This completes the proof of Lemma 3.2. ■

Continuing with the proof of Proposition 3.1, to show that $H^1(X, R_u(B)(E_T)) = \{e\}$ we will inductively prove that

$$H^1(X, U_m(E_T)) = \{e\} \tag{3.8}$$

for each m. In view of the Lemma 3.2, it is enough to verify that

$$H^1(X, U_m/U_{m+1}(E_T)) = \{e\} \tag{3.9}$$

for each m. Since U_m/U_{m+1} is isomorphic to $\mathbb{G}_a$ we have $H^1(X, U_m/U_{m+1}(E_T)) \cong H^1(X, \zeta_m)$, where ζ_m is the line bundle obtained as a fiber bundle associated to E_T for the action of T on U_m/U_{m+1} (note that the action of T on U_m/U_{m+1} is linear). But by the assumption in the proposition we have $H^1(X, \zeta) = 0$ for any line bundle ζ on X. This completes the proof of the proposition. ■

Let m be an integer such that the canonical bundle K_X is isomorphic to $\xi_0^{\otimes m}$, where ξ_0 is the ample generator of $\mathrm{Pic}(X)$. The Kodaira vanishing theorem, [16, page 36, Corollary 2.32], says that $H^1(X, \xi_0^{\otimes i}) = 0$ for $i > m$. On the other hand, Serre duality gives

$$H^1(X, \xi_0^{\otimes i}) = H^{d-1}\left(X, \xi_0^{\otimes (m-i)}\right)^*. \tag{3.10}$$

So again by the Kodaira vanishing theorem, we have $H^1(X, \xi_0^{\otimes i}) = 0$ for $i < 0$. Also, the assumption $\mathrm{Pic}(X) \cong \mathbb{Z}$ ensures that $H^1(X, \mathcal{O}_X) = 0$. Therefore, the assumption in Proposition 3.1, namely $H^1(X, \zeta) = 0$ for all line bundles ζ on X, is satisfied if $m \leq 0$, that is, if X is either Fano or it has trivial canonical bundle.

Therefore, Theorem 2.7, Proposition 3.1, and Lemma 2.2 combine together to give Theorem 1.2 stated in the introduction.

Theorem 3.3. Let X be a projective manifold with $\mathrm{Pic}(X) \cong \mathbb{Z}$ and X is Fano or it has trivial canonical bundle. For a G-bundle E on X, if the G'-bundle ρ_*E splits, where ρ is a faithful representation, then E admits a reduction of structure group to a one-parameter subgroup of T. □

In the final section we will give some applications for $\mathbb{CP}^n$.

4 Principal bundles over a projective space

Let $\mathbb{CP}^n$ be the projective space of all lines in $\mathbb{C}^{n+1}$. We will assume that $n \geq 2$. By $\mathbb{CP}^2 \subset \mathbb{CP}^n$ we mean a plane in $\mathbb{CP}^n$.

Corollary 4.1. A principal G-bundle E over $\mathbb{CP}^n$ admits a reduction of the structure group to a one-parameter subgroup of the maximal torus T if and only if there is a $\mathbb{CP}^2$ in $\mathbb{CP}^n$ such that the restriction of E to $\mathbb{CP}^2$ admits a reduction of the structure group to T. □

Proof. Let E be a G-bundle over $\mathbb{CP}^n$ such that the restriction $E|_{\mathbb{CP}^2}$ to a plane $\mathbb{CP}^2$ admits a reduction of the structure group to T.

Set ρ to be a faithful representation of G in $\mathrm{GL}(V)$. Let $E(V)$ denote the vector bundle over X associated to E for ρ.

Since $E(V)|_{\mathbb{CP}^2}$ admits a reduction of the structure group to T, the restriction $E(V)|_{\mathbb{CP}^2}$ of $E(V)$ to $\mathbb{CP}^2$ splits as a direct sum of line bundles. Now [12, page 42, Theorem 2.3.2] says that $E(V)$ splits. Finally, Theorem 3.3 says that E admits a reduction of the structure group to a one-parameter subgroup of T. This completes the proof. ■

Corollary 4.2. A principal G-bundle E over $\mathbb{CP}^n$ is trivial if and only if there is a point p in $\mathbb{CP}^n$ such that the restriction of E to every line $\mathbb{CP}^1 \subset \mathbb{CP}^n$ in C_p is trivial. □

Proof. As in Corollary 4.1, set ρ to be a faithful representation of G in $\mathrm{GL}(V)$.

Let E be a principal G-bundle on $\mathbb{CP}^n$ such that the restriction of E to any line $\mathbb{CP}^1 \subset \mathbb{CP}^n$ in C_p is trivial. Therefore, the restriction of $E(V)$ to any line in C_p is trivial. Now, [12, page 51, Theorem 3.2.1] says that the vector bundle $E(V)$ is trivial.

Now, Theorem 3.3 says that E admits a reduction of the structure group to a one-parameter subgroup of T. Let

$$E_{\mathbb{C}^*} \subset E \tag{4.1}$$

be a $\mathbb{C}^*$-bundle which is a reduction of the structure group of E to a one-parameter subgroup of T.

Since the restriction of E to any line l in C_p is trivial, the restriction of $E_{\mathbb{C}^*}$ to l is trivial. If ζ is a $\mathbb{C}^*$-bundle on $\mathbb{CP}^n$ satisfying the condition that its restriction to some line is trivial, then the $\mathbb{C}^*$-bundle ζ is itself trivial ($\mathbb{C}^*$-bundles correspond to line bundles). Therefore, the $\mathbb{C}^*$-bundle $E_{\mathbb{C}^*}$ is trivial. This completes the proof. ∎

In [8], Grothendieck proved that for a reductive group G, any principal G-bundle over $\mathbb{CP}^1$ admits a reduction of the structure group to a maximal torus T. Although this is not stated there, his arguments in [8, part 3 and 4] actually give the following theorem.

Theorem 4.3. Let E be a principal G-bundle on X (recall that we are always assuming $\mathrm{Pic}(X) \cong \mathbb{Z}$). Assume that

(1) if L is any line bundle on X with $\deg(L) \geq 0$, then $h^0(X, L) > 0$;

(2) the adjoint bundle $\mathrm{ad}(E)$ is a direct sum of line bundles.

Then E admits a reduction of the structure group to the normalizer N of a maximal torus T. □

Proof. For the convenience of the reader, we will give the proof. Recall that an element $v \in \mathfrak{g}$ is *regular semisimple* (or just *regular* for short) if the centralizer of v is a Cartan subgroup. For any $v \in \mathfrak{g}$, let $\mathrm{ad}(v)$ denote the adjoint action of v on the Lie algebra $\mathfrak{g}$. Consider the characteristic polynomial

$$\det\left(t - \mathrm{ad}(v)\right) = \sum_{i=0}^{\dim G} a_i(v) t^i. \tag{4.2}$$

The element v is regular semisimple if and only if we have $a_{\mathrm{rank}(G)}(v) \neq 0$ [9, page 192, (v)].

Let s be a global section of the adjoint bundle. If s is regular at a point $x_0 \in X$, then it is regular for all points $x \in X$. Indeed, the coefficients of the characteristic polynomial $\sum a_i(\mathrm{ad}\, s(x))t^i$ are holomorphic functions on X, hence constant. Consequently, the assertion follows from the above criterion for regularity.

Now we will show that if $\mathrm{ad}(E)$ has a section s with $s(x)$ regular, then the structure group of E admits a reduction to the normalizer N of a maximal torus T of G.

Let $h(x) \in \mathrm{ad}(E)_x$ be the centralizer of $s(x)$. Since $s(x)$ is regular, $h(x)$ is a Cartan algebra. Note that G/N is the space of Cartan subalgebras of $\mathfrak{g}$, hence $h(x)$ gives an element of $E(G/N)_x$. Therefore, the section s gives a reduction of the structure group to N.

In view of the above observation, to prove the theorem, it is enough to find a section s of the adjoint bundle, with $s(x)$ regular at some point $x \in X$. Since G is reductive, the Lie algebra is a direct sum of the center and the semisimple part

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'. \tag{4.3}$$

We have $E(\mathfrak{g}) = E(\mathfrak{z}) \oplus E(\mathfrak{g}')$. If an element $a \in \mathfrak{g}'$ is regular in $\mathfrak{g}'$, then it is also regular in $\mathfrak{g}$, so we can assume that $\mathfrak{z}$ is trivial, that is, the group is semisimple.

By hypothesis, $\mathrm{ad}(E)$ is a direct sum of line bundles L_i. Let

$$E_k = \bigoplus_{\deg L_i \geq k} L_i \subset \mathrm{ad}(E). \tag{4.4}$$

Fix an isomorphism of $\mathfrak{g}$ with the fiber of $\mathrm{ad}(E)$ over $x \in X$, and let $\mathfrak{g}_k$ be the fiber of E_k over x. It is easy to check that $[E_k, E_{k'}] \subset E_{k+k'}$. In particular, $\mathfrak{g}_1$ is a subalgebra, and if $Y \in \mathfrak{g}_1$, then $\mathrm{ad}\, Y$ is nilpotent. Since we are assuming that G is semisimple, using the Killing form, $\mathrm{ad}(E)$ becomes an orthogonal bundle (i.e., the fibers are equipped with a nondegenerate symmetric bilinear form). For this orthogonal structure we have $(E_1)^\perp = E_0$. By [8, Lemme 4.2]

$$(\mathfrak{g}_1)^\perp = \mathfrak{g}_0 \supset \mathfrak{R} \supset \mathfrak{h}, \tag{4.5}$$

where $\mathfrak{R}$ is a maximal solvable algebra and $\mathfrak{h}$ is a Cartan subalgebra. Since $\mathfrak{h}$ has regular elements, it follows that there is a regular element $a \in \mathfrak{g}_0$. Now, E_0 is a direct sum of line bundles of nonnegative degree, hence by hypothesis there is a section s of $E_0 \subset \mathrm{ad}(E)$ such that $s(x) = a$. This completes the proof of the theorem. ■

If we further assume that X is simply connected, then E admits a reduction to a maximal torus T. This is because the normalizer $N(T)$ of a maximal torus T contains the maximal torus as a finite index subgroup hence any $N(T)$-bundle gives rise to a finite cover of X and since X is simply connected, this cover is trivial hence giving a reduction of structure group to the maximal torus, and hence, by Lemma 2.2, to a one-parameter subgroup (compare with Theorem 3.3).

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