Reduction of Structure Group of Principal Bundles over a Projective Manifold with Picard Number One

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1 Introduction

Let $X$ be a connected projective manifold with $\text{Pic}(X) \cong \mathbb{Z}$. Let $G$ be a connected reductive algebraic group over $\mathbb{C}$. A principal $G$-bundle will be called split if it admits a reduction of structure group to a maximal torus of $G$. This corresponds to the usual definition of a split vector bundle as a direct sum of line bundles.

Let $B$ be a Borel subgroup of $G$ and $\rho : G \to G'$ an injective homomorphism to a connected reductive algebraic group over $\mathbb{C}$. In Theorem 2.7 we prove the following theorem.

**Theorem 1.1.** Let $E$ be a principal $G$-bundle over $X$ such that the principal $G'$-bundle $E(G') := (E \times G')/G$ obtained by extending the structure group using $\rho$ is split. Then $E$ admits a reduction of structure group to $B$. □

Furthermore, if $X$ is Fano or it has trivial canonical bundle, then we have the following stronger consequence (Theorem 3.3).

**Theorem 1.2.** For a $G$-bundle $E$ on $X$, if the $G'$-bundle $E(G')$ is split, then $E$ itself is split. □

If $W$ is a vector bundle on $\mathbb{C}P^n$, $n \geq 2$, such that the restriction $W_{|\mathbb{C}P^2}$ to a plane $\mathbb{C}P^2$ splits as a direct sum of line bundles, then $W$ is already a direct sum of line bundles [12, page 42, Theorem 2.3.2]. This result is a consequence of a splitting criterion.

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of G. Horrocks. A corollary of Theorem 1.2 (Corollary 4.1) gives the following analog for G-bundles. If \( E \) is a principal G-bundle over \( \mathbb{CP}^n \) such that the restriction \( E_{|\mathbb{CP}^2} \) is split, then \( E \) itself is split. In Lemma 2.2, we show that any split G-bundle over \( X \) admits a reduction of structure group to a one-parameter subgroup of G.

A vector bundle \( W \) on \( \mathbb{CP}^n \) is trivial if there is a point \( p \) in \( \mathbb{CP}^n \) such that the restriction of \( W \) to any line passing through \( p \) is trivial [12, page 51, Theorem 3.2.1]. In Corollary 4.2, we prove that a principal G-bundle \( E \) over \( \mathbb{CP}^n \) is trivial if and only if there is a point \( p \) in \( \mathbb{CP}^n \) such that the restriction of \( E \) to every line \( \mathbb{CP}^1 \subset \mathbb{CP}^n \) passing through \( p \) is trivial.

2 Criterion for reduction to Borel subgroup

Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \). The center of \( G \) will be denoted by \( Z(G) \). Let \( B \subset G \) be a Borel subgroup and \( T \subset B \) a maximal torus of \( G \).

Let \( X \) be a connected smooth projective variety over \( \mathbb{C} \) of dimension \( d \). We will assume that the Picard group \( \text{Pic}(X) \) is isomorphic to \( \mathbb{Z} \).

Fix an ample line bundle \( \xi \) on \( X \). The degree of a torsionfree coherent sheaf \( F \) on \( X \) is

\[
\text{deg}(F) = \int_X c_1(F) c_1(\xi)^{d-1}. \tag{2.1}
\]

Note that if \( U \) is a Zariski open subset of \( X \) such that \( \text{codim}_X(X \setminus U) \geq 2 \), then \( \text{deg}(i_* i^* F) = \text{deg}(F) \), where \( i: U \hookrightarrow X \) is the inclusion map.

Given a group homomorphism \( h: H \to G \) and a principal \( H \)-bundle \( E_H \), the extension of structure group of \( E_H \) to \( G \) is the \( G \)-bundle defined as

\[
h_* E_H := \left( E_H \times G \right)/H = E_H(G), \tag{2.2}
\]

where the quotient is taken using the diagonal action. It may be noted that even in the case \( h \) is an embedding, it can happen that \( h_* E_H \) and \( h_* E_T \), are isomorphic but \( E_H \) and \( E_T \), are not isomorphic.

If \( H \) is a closed subgroup, \( h \) is the inclusion, and \( E \) a principal \( G \)-bundle over \( X \), then a reduction of the structure group of \( E \) to \( H \) is a principal \( H \)-bundle \( E_H \) together with an isomorphism \( h_* E_H \cong E \). Equivalently, it is defined by an algebraic section of the fiber bundle

\[
E/H \to X. \tag{2.3}
\]

The quotient \( E/H \) corresponds to the restriction to \( H \) of the action of \( G \) to \( E \). If \( S \) is the
subvariety of \( E/H \) defined by the image of such a section, then we recover \( E_H \) as the subset of \( E \) given by the inverse image of \( S \) for the natural projection

\[ E \rightarrow E/H. \]

(2.4)

Note that if \( E_H \subset E \) is a reduction of the structure group of \( E \) to a subgroup \( H \), then for any \( g \in G \) the translation \( E_H g \) of \( E_H \) by \( g \) defines a reduction of the structure group of \( E \) to the conjugate \( g^{-1}Hg \) of \( H \).

A principal \( G \)-bundle \( E \) over \( X \) is called semistable if for every maximal parabolic subgroup \( P \subset G \) and for every reduction of structure group \( \sigma : U \rightarrow E/P \) over some Zariski open subset \( U \) with \( \text{codim}_C(X \setminus U) \geq 2 \), the inequality

\[ \deg \sigma^*(T_{rel}) \geq 0 \]

(2.5)

is valid, where \( T_{rel} \) is the relative tangent bundle for the natural projection of \( E/P \mid U \) to \( U \). Since \( \text{Pic}(X) \cong \mathbb{Z} \), the semistability condition does not depend on the choice of the polarization \( L \).

The condition of semistability has the following equivalent reformulation [14, Lemma 2.1]. A principal \( G \)-bundle \( E \) over \( X \) is semistable if and only if for every parabolic subgroup \( P \subset G \) and every holomorphic reduction \( E_P \) of the structure group of \( E \) to \( P \) over some open subset \( U \subset X \) with \( \text{codim}_C(X \setminus U) \geq 2 \), and for every nontrivial character \( \theta : P \rightarrow \mathbb{C}^* \) which is dominant with respect to some Borel subgroup of \( G \) contained in \( P \), the inequality \( \deg(E_P(\theta)) \leq 0 \) is valid, where \( E_P(\theta) = E_P(C) \) is the line bundle \( (E_P \times C)/P \) over \( U \) associated to \( E_P \) for the character \( \theta \) of \( P \).

From the definition, it follows that a \( \text{GL}(n,\mathbb{C}) \)-bundle is semistable if and only if the \( n \) vector bundle associated to it by the standard representation is semistable. We recall that a vector bundle \( E \) is called semistable if for any coherent subsheaf \( F \) of \( E \), with \( 0 < \text{rank}(F) < \text{rank}(E) \), the inequality

\[ \frac{\deg(F)}{\text{rank}(F)} \leq \frac{\deg(E)}{\text{rank}(E)} \]

(2.6)

is valid [10].

A vector bundle is called split if it is a direct sum of line bundles, and recall from the introduction that a principal \( G \)-bundle over \( X \) is defined to be split if it admits a reduction of structure group to the maximal torus \( T \).

Remark 2.1. It is easy to see that a \( \text{GL}(n,\mathbb{C}) \)-bundle is split if and only if the vector bundle associated to it by the standard representation of \( \text{GL}(n,\mathbb{C}) \) splits as a direct sum.
of line bundles. More generally, any extension of the structure group of a split bundle is always split.

A principal $G$-bundle $E$ over $X$ is said to admit a reduction of structure group to a one-parameter subgroup of the maximal torus $T$ if there is a homomorphism

$$\gamma : \mathbb{C}^\ast \rightarrow T$$

(2.7)

and a principal $C^\ast$-bundle $E_{C^\ast}$ over $X$ such that the principal $G$-bundle obtained by extending the structure group of $E_{C^\ast}$, using the composition of $\gamma$ with the inclusion of $T$ in $G$, is isomorphic to $E$. Such a $G$-bundle is obviously split. The following lemma shows that the converse is true.

Lemma 2.2. Any principal $T$-bundle over $X$ admits a reduction of the structure group to a one-parameter subgroup. \hfill $\Box$

Proof. Since $T$ is isomorphic to a product of copies of $C^\ast$, there is a natural bijection between (isomorphism classes of) $T$-bundles and vector bundles of the form

$$\xi_1 \oplus \xi_2 \oplus \cdots \oplus \xi_r,$$

(2.8)

where $r = \dim T$ and $\xi_i$ are line bundles. Since $\text{Pic}(X) \cong \mathbb{Z}$, it is generated by a fixed line bundle $O(1)$. So we have $\xi_i \cong O(n_i)$ for some $n_i \in \mathbb{Z}$. This means that there is a reduction of the structure group of any principal $T$-bundle to $C^\ast$, with the homomorphism $\gamma : C^\ast \rightarrow T$ determined by the numbers $n_i$. More precisely, $\gamma$ is of the form $\lambda \mapsto (\lambda^{n_1}, \lambda^{n_2}, \ldots, \lambda^{n_r})$. This completes the proof of the lemma. \hfill $\Box$

Fix an injective homomorphism $\rho : G \rightarrow G'$ as in the introduction.

An important fact used in our arguments is that for any connected reductive group $G$, the center $Z(G)$ is always contained in $T$. In fact, $Z(G)$ is the intersection of all possible maximal tori of $G$.

Proposition 2.3. If the principal $G'$-bundle $E(G')$ is split, then the adjoint vector bundle $\text{ad}(E)$ is split. \hfill $\Box$

Proof. Take a faithful representation

$$\rho' : G' \rightarrow \text{GL}(V).$$

(2.9)

Let $T(V) := (E(G') \times V)/G'$ be the vector bundle associated to $E(G')$ for $\rho'$. Since $E(G')$ is the extension of $T$, the vector bundle $E(V)$ is also an extension of $T$ for $\rho' \circ \rho$. 


Since $\rho' \circ \rho$ is a faithful representation of $G$, and $G$ is reductive, any irreducible $G$-module is a direct summand of $E(V)^{\otimes i} \otimes (E(V)^*)^{\otimes i'}$ for some $i, i' \geq 0$ [5, page 40, Proposition 3.1(a)]. In other words, any $G$-module is a direct summand of a finite sum of the type

$$\bigoplus_j \left( E(V)^{\otimes i_j} \otimes (E(V)^*)^{\otimes i'_j} \right).$$

(2.10)

In particular, the adjoint bundle $\text{ad}(E)$ is a direct summand of a direct sum of vector bundles of this type.

Since $E(G')$ is split, from Remark 2.1 we know that $E(V)$ is a direct sum of line bundles. Therefore, any $E(V)^{\otimes i} \otimes (E(V)^*)^{\otimes i'}$ is also a direct sum of line bundles.

We will show that a direct summand of a split vector bundle over a compact connected projective manifold is split. Let $W$ be a split vector bundle and $W = W^1 \oplus W^2$ any decomposition into a direct sum of vector bundles. Let $W^i = \bigoplus_{j \in I^i} V^i_j$, $i = 1, 2$, be the decomposition into indecomposable vector bundles. The existence of such a decomposition is ensured by [2, page 315, Lemma 9]. Consequently, $(\bigoplus V^1_j) \oplus (\bigoplus V^2_j)$ is a decomposition of $W$ into indecomposable vector bundles. A theorem of Atiyah [2, page 315, Theorem 3] says that any given vector bundle over a connected complex projective manifold can be uniquely decomposed as a direct sum of indecomposable vector bundles (unique up to reordering the summation). Now, since $W$ decomposes as a direct sum of line bundles, it follows immediately that each $V^i_j$ is a line bundle.

We have shown that $\text{ad}(E)$ is a direct summand of a split vector bundle. Consequently, $\text{ad}(E)$ is split. This completes the proof of the proposition. ■

**Lemma 2.4.** Let $E$ be a semistable $G$-bundle over $X$ such that the vector bundle $\text{ad}(E)$ is split. Then $E$ admits a reduction of the structure group to $Z(G)$. In particular, $E$ is split.

Proof. Since $E$ is semistable hence $\text{ad}(E)$ is semistable.

Now, since $\text{Pic}(X) \cong \mathbb{Z}$, the semistability condition of $\text{ad}(E)$ ensures that $\text{ad}(E)$ is of the form

$$\text{ad}(E) \cong \zeta^{\otimes N},$$

(2.11)

where $\zeta$ is a line bundle over $X$ and $N = \text{rank}(\text{ad}(E))$. Since $\text{Pic}(X) \cong \mathbb{Z}$ and the degree of $\text{ad}(E)$ is zero, it follows immediately that the line bundle $\zeta$ is trivial and hence $\text{ad}(E)$ is a trivial vector bundle. Since $\text{ad}(E)$ is isomorphic to the trivial vector bundle, any section of $\text{ad}(E)$ over $X$ is a constant section. So if $s_1$ and $s_2$ are two sections of $\text{ad}(E)$ over $X$, then their bracket $[s_1, s_2]$ is also constant.
Consequently, \( \text{ad}(E) \) is trivial as a Lie algebra bundle. Let \( \text{Aut}(g) \) denote the group of Lie algebra automorphisms of \( g \). The triviality of \( \text{ad}(E) \) as a Lie algebra bundle implies that the principal bundle \( E(\text{Aut}(g)) \) associated to the adjoint representation \( G \to \text{Aut}(g) \) is trivial. The adjoint representation factors as follows:

\[
G \to G/Z \xrightarrow{j} \text{Aut}(g).
\]

(2.12)

The group \( G/Z \) is the connected component of the identity of \( \text{Aut}(g) \) and \( j \) is the inclusion, hence we have a short exact sequence of groups

\[
(c) \to G/Z \xrightarrow{j} \text{Aut}(g) \to F \to (c),
\]

(2.13)

where \( F \) is a discrete group. This gives an exact sequence of pointed sets (see [6, page 153, Section 5], [15, Proposition 11], or [11, page 122, Proposition III.4.5])

\[
H^0(X, F) \xrightarrow{j_*} H^1(X, G/Z) \xrightarrow{j^*} H^1(X, \text{Aut}(g)).
\]

(2.14)

This means that \( (j_*)^{-1}(c) = \text{image}(\delta) \), where \( c \) is the point corresponding to the trivial bundle. Let \( E(G/Z) \) be the principal bundle associated to \( G \to G/Z \). It gives a point \( [E(G/Z)] \) in \( H^1(X, G/Z) \), whose image under \( j \), is equal to \( [E(\text{Aut}(g))] \). Since this principal bundle is trivial, by exactness we know that \( [E(G/Z)] \) is in the image of \( \delta \). Recall that \( \delta(c) \) is defined as the point corresponding to the principal \( G/Z \)-bundle \( E' \) given by the Cartesian diagram

\[
\begin{array}{ccc}
E' & \to & X \times \text{Aut}(g) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\sigma} & X \times F.
\end{array}
\]

(2.15)

Since \( F \) is discrete, \( \sigma \) has to be constant, and then, for any \( \sigma \), we have that \( \delta(\sigma) \) corresponds to the trivial bundle, so we conclude that \( E(G/Z) \) is trivial.

Now, consider the exact sequence of pointed sets

\[
H^1(X, Z) \xrightarrow{\iota_*} H^1(X, G) \to H^1(X, G/Z).
\]

(2.16)

Since the point \( [E] \in H^1(X, G) \) maps to the trivial element \( [E(G/Z)] \), there is a principal \( Z \)-bundle \( E_Z \) such that \( \iota_* E_Z \cong E \). In other words, \( E_Z \) is a reduction of structure group of \( E \) to \( Z \) and the proof is complete.

Now we are in a position to prove the main result of this section.
Proposition 2.5. Let \( E \) be a nonsemistable \( G \)-bundle such that the vector bundle \( \text{ad}(E) \) is split. Then \( E \) admits a reduction of the structure group to a Borel subgroup of \( G \). \( \square \)

Proof. A nonsemistable \( G \)-bundle \( F \) admits a canonical reduction of structure group [1, Theorem 1.1], [3], which for the case of vector bundles is the usual Harder-Narasimhan filtration [10]. Consider the adjoint vector bundle \( \text{ad}(F) \). Since \( F \) is not semistable, the vector bundle \( \text{ad}(F) \) is not semistable (if \( \text{ad}(F) \) is semistable, then \( F \) is semistable [13, Theorem 3.18]). Consider the Harder-Narasimhan filtration of \( \text{ad}(F) \). Since \( G \) is reductive, the Lie algebra \( g \) admits a nondegenerate symmetric bilinear form invariant under the adjoint action of \( G \). Such a form induces a nondegenerate symmetric bilinear form on \( \text{ad}(F) \). Since \( \text{ad}(F)^* \cong \text{ad}(F) \), the Harder-Narasimhan filtration is of the form

\[
\emptyset = W_{-1} \subset W_{-2} \subset \cdots \subset W_{-l} \subset W_{-l+1} \subset \cdots \subset W_0 \subset \cdots \subset W_{l-1} \subset W_l = \text{ad}(F),
\]

where \( W_i \) coincides with \( W_i^\perp \) with respect to the nondegenerate symmetric bilinear form. The canonical reduction \( F_P \subset F \) over a Zariski open subset \( U \) of \( X \) with \( \text{codim}_C(X \setminus U) \geq 2 \) is determined by the condition that the adjoint bundle \( \text{ad}(F_P) \) coincides with \( W_0 \). The open subset \( U \) is the one over which each \( W_i \) is a subbundle of \( \text{ad}(F) \). It turns out that \( W_{-1} \) coincides with the vector bundle \( E_P(n) \) associated to \( E_P \) for the adjoint action of \( P \) on the nilpotent radical \( n \) of the Lie algebra of \( P \).

Let \( E_L = (\psi_P)^* E_P \) be the extension of the structure group by \( \psi_P \). Its adjoint bundle \( E_L(l) \) will be denoted by \( \text{ad}(E_L) \). From the construction of canonical reduction it follows that \( \text{ad}(E_L) \) is semistable. In fact, \( \text{ad}(E_L) \) coincides with \( W_0/W_{-1} \), if \( W_0 \) is the Harder-Narasimhan filtration of \( \text{ad}(E) \). Therefore, the \( L \)-bundle \( E_L \) is semistable [13, Theorem 3.18].

Since \( \text{ad}(E) \) is a direct sum of line bundles, each quotient \( W_i/W_{i-1} \) is naturally a direct summand of \( \text{ad}(E) \), where \( W_i \) is the Harder-Narasimhan filtration of \( \text{ad}(E) \).
(This follows from the construction of Harder-Narasimhan filtration of a direct summand of line bundles.) Consequently, \( \text{ad}(E_L) \) is a direct summand of \( \text{ad}(E) \). We saw in the proof of Proposition 2.3 that a direct summand of a split vector bundle is again split. Therefore, \( \text{ad}(E_L) \) is a split vector bundle.

Since \( E_L \) is semistable and \( \text{ad}(E_L) \) is split, from Proposition 2.3 and Lemma 2.4 it follows that the \( L \)-bundle \( E_L \) admits a reduction of the structure group to its center hence to the maximal torus \( T(L) \) of \( L \).

Let \( E_{T(L)} \subset E_L \) be a \( T(L) \)-bundle giving a reduction of the structure group of \( E_L \) to \( T(L) \).

Let \( \Psi : E_P \to E_L \) denote the projection induced by the natural projection \( \psi_P \) of \( P \) to \( L \). The inverse image

\[
\Psi^{-1}(E_{T(L)}) \subset E_P
\]

(2.19)
defines a reduction of the structure group of the \( P \)-bundle \( E_P \) to the subgroup \( \psi_P^{-1}(T(L)) \subset P \). But \( \psi_P^{-1}(T(L)) \) lies in a Borel subgroup of \( G \) contained in \( P \). This completes the proof.

Remark 2.6. In the proof of Proposition 2.5, it is possible to directly prove that the bundle \( \text{ad}(E_L) \) is trivial from the description of the canonical filtration of the \( G \)-bundle \( E \) in terms of the canonical filtration of \( \text{ad}(E) \).

Combining all these results, we obtain Theorem 1.1 stated in the introduction.

**Theorem 2.7.** Let \( E \) be a principal \( G \)-bundle over \( X \) such that the principal \( G' \)-bundle \( E(G') := (E \times G)/G \) obtained by extending the structure group using \( \rho \) is split. Then \( E \) admits a reduction of structure group to \( B \).

Proof. By Proposition 2.3, \( \text{ad}(E) \) is split. If \( E \) is semistable, then by Lemma 2.4 it is split (in particular, it has a reduction to a Borel subgroup). If \( E \) is not semistable, then we apply Proposition 2.5.

Remark 2.8. It is not true in general that if \( \text{ad}(E) \) is split then \( E \) is split. Take two elements \( g_1, g_2 \in \text{SU}(2) \) such that \( g_1 g_2^{-1} g_2^{-1} g_1^{-1} = -I \). Let \( Y \) be a Riemann surface of genus two. Let \( \Gamma \) denote the free group generated by \( a_1, b_1, a_2, b_2 \). Take a standard presentation of the fundamental group \( \pi_1(Y) \) as the quotient of \( \Gamma \) by the normal subgroup generated by \( a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \). Consider the homomorphism

\[
\beta : \pi_1(Y) \to \text{SU}(2)
\]

(2.20)
defined by $\beta(a_i) = g_i$ and $\beta(b_i) = 1$, $i = 1, 2$. Since $g_1$ and $g_2$ do not commute, $\beta$ defines an irreducible unitary representation. Therefore, the rank two vector bundle $V$ on $Y$ associated to this representation is stable. In particular, $V$ is indecomposable. Since the vector bundle $V$ is indecomposable, the principal $SL(2, \mathbb{C})$-bundle defined by $V$ is not split. But the representation of $\pi_1(Y)$ on the Lie algebra $su(2)$ defined by $\beta$ is reducible.

Indeed, as the joint action of $-1 \in SU(2)$ on the Lie algebra $su(2)$ is trivial, the joint action of $g_1$ and $g_2$ commutes. Therefore, the joint vector bundle $ad(V)$ defined by the sheaf of trace zero endomorphisms of $V$ decomposes as a direct sum of line bundles.

In the next section we will see that an assumption on $X$ ensures that $E$ is split if $E_1$ is split.

3 Reduction to the maximal torus

We continue with the notation of Section 2.

Proposition 3.1. Assume that $H^1(X, \zeta) = 0$ for all line bundles $\zeta$ on $X$. Let $E_B$ be a $B$-bundle. Then $E_B$ admits a reduction of the structure group to $T$. □

Proof. Let $R_u(B)$ denote the unipotent radical of $B$. Consider the exact sequence of groups

$$[e] \to R_u(B) \to B \to T \to [e].$$

(3.1)

Let $s : T \to B$ be a splitting of this exact sequence. Let $E_T := \psi_*E_B$ be the principal $T$-bundle obtained by extending the structure group of $E_B$. Note that the principal $B$-bundle $s_*E_T$ admits a reduction of the structure group to $T$. Consequently, it is enough to show that $s_*E_T$ is isomorphic to $E_B$.

Taking the extension of the structure group of these two $B$-bundles, namely $E_B$ and $s_*E_T$, by $\psi$, we get the same $T$-bundle $E_T$ (here we use the fact that $s$ is a splitting). Therefore, it suffices to show that if $E_B$ and $E_T'$ are two $B$-bundles with $\psi_*E_B \cong \psi_*E_T'$, then $E_B \cong E_T'$. For this we have to understand the isomorphism classes of $B$-bundles which under the extension of the structure group by $\psi$ give $E_T$.

We observe that $T$ acts on the unipotent radical $R_u(B)$ by conjugation via $s$. For any $t \in T$ the action of $t$ on $R_u(B)$ will be denoted as $t(u) := tut^{-1}$, where $u \in R_u(B)$.

Since $T$ is abelian, we can find a filtration of $R_u(B)$

$$R_u(B) = U_1 \supset U_2 \supset \cdots \supset U_k \supset U_{k+1} = [e]$$

(3.2)

such that $U_{i+1}$ is normal in $U_i$, each $U_i$ are invariant under the action of $T$ and the quotients $U_i/U_{i+1}$ are isomorphic to $G_\mu$, the additive group $\mathbb{C}$. 

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Now we define twisted cohomology sets $H^i(X, U_i(E_i))$ for each $i$ with the property that for $i = 1$, the set $H^1(X, U_1(E_1))$ parametrizes all the isomorphism classes of $B$-bundles that extend to $E$ via $\psi$.

Fix a cocycle $(t_{ij})$ for $E$ with respect to an étale open cover $\mathcal{V} = \{V_i\}$. Let $E_0$ be a $B$-bundle with $\psi, E_0 \rightarrow \Gamma$, then (refining $\mathcal{V}$ if necessary) a cocycle for $E_0$ is of the form

$$u_{ij}t_{ij} : V_i \times_X V_j \rightarrow B,$$

where $(t_{ij})$ defines the cocycle for $E$ and $u_{ij} : V_i \times_X V_j \rightarrow R_0(B)$. The cocycle condition is $u_{ij}t_{ij}u_{jk}t_{kl} = u_{ik}t_{kl}$ on $V_i \times_X V_j \times_X V_k$.

Using the action of $T$ on $R_0(B)$, we can rewrite the cocycle condition as

$$u_{ij}t_{ij}(u_{ik}) = u_{ik}$$

These $[u_{ij}]$ are collectively called a twisted cocycle, and two twisted cocycles $[u_{ij}]$ and $[v_{ij}]$ are equivalent if there are morphisms $s_i : V_i \rightarrow R_0(B)$ for each index $i$ satisfying the condition

$$s_iu_{ij}t_{ij}(s_i^{-1}) = v_{ij}$$

for each pair $i, j$ with $V_i \times_X V_j \neq \emptyset$. Recall the notation $t(u) := tut^{-1}$. The equivalence classes of twisted cocycles form the pointed set $H^1(X, R_0(B)(\mathcal{E}))$ (taken over all open covers) with distinguished element being the cocycle $[u_{ij}]$ with $u_{ij} = 1$ for all $i$ and $j$. This pointed set is in bijective correspondence with the isomorphism classes of $B$-bundles that extend to $E$ via $\psi$. Since the action of $T$ on $R_0(B)$ preserves the filtration (3.2), this action induces actions of $T$ on each $U_m$ and $U_{m}/U_{m+1}$. Therefore, we can also define $H^1(X, U_m(E))$ and $H^1(X, U_m/U_{m+1}(E))$ for each $m$ (see [4] or [7, Appendix] for more details). Note that by composing we get the morphisms of pointed sets

$$f : H^1(X, U_{m+1}(E)) \rightarrow H^1(X, U_m(E))$$

and $g : H^1(X, U_m(E)) \rightarrow H^1(X, U_{m}/U_{m+1}(E))$.

The proof of the proposition will be completed using the following lemma.

**Lemma 3.2.** The following sequence of pointed sets is exact:

$$H^1(X, U_{m+1}(E)) \rightarrow H^1(X, U_m(E)) \rightarrow H^1(X, U_m/U_{m+1}(E)).$$

Recall that this means that $\text{image}(f) = g^{-1}(1)$, where $1$ is the distinguished point in $H^1(X, U_m/U_{m+1}(E))$. 

□
Proof of Lemma 3.2. It is easy to see that the composite is the constant map to $e$. So, we only need to verify that $\text{image}(f) \supseteq g^{-1}(e)$.

Let $\{u_i\}$ be a twisted cocycle on an open cover $\mathcal{V} = \{V_i\}$ with values in $U_m$, such that $g(\{u_i\}) = e$. The element $g(\{u_i\})$ is represented by the cocycle $\{\eta_i\}$, where $\eta_i$ is obtained by composing $u_i$ with the projection morphism $U_m \to U_m/U_{m+1}$. Then there are morphisms $\pi_i : V_i \to U_m/U_{m+1}$ such that for each $V_i \times_X V_j$ (when nonempty) we have $\pi_i \pi_j^{-1}(\eta_i^* \pi_j^{-1}) = 1$.

Since $U_m$ and $U_m/U_{m+1}$ as schemes are just affine spaces and the morphisms between them are projection morphisms, we can lift each of the morphisms $\pi_i$ to get morphisms $s_i : V_i \to U_m$. We fix such a lifting. We define a cocycle $\{v_{ij}\}$ by $v_{ij} = s_i u_i s_j (s_j^{-1})$. It can be checked that this defines a cocycle which takes values in $U_{m+1}$ and has the property that $f(\{v_{ij}\}) = \{u_i\}$. This completes the proof of Lemma 3.2.

Continuing with the proof of Proposition 3.1, to show that $H^1(X, K_u(B)(E_1)) = \{e\}$ we will inductively prove that

$$H^1(X, U_m(E_1)) = \{e\}$$

(3.8)

for each $m$. In view of the Lemma 3.2, it is enough to verify that

$$H^1(X, U_m/U_{m+1}(E_1)) = \{e\}$$

(3.9)

for each $m$. Since $U_m/U_{m+1}$ is isomorphic to $G_m$, we have $H^1(X, U_m/U_{m+1}(E_1)) \cong H^1(X, \xi_m)$, where $\xi_m$ is the line bundle obtained as a fiber bundle associated to $E_1$ for the action of $T$ on $U_m/U_{m+1}$ (note that the action of $T$ on $U_m/U_{m+1}$ is linear). But by the assumption in the proposition we have $H^1(X, \xi) = \{0\}$ for any line bundle $\xi$ on $X$.

This completes the proof of the proposition.

Let $m$ be an integer such that the canonical bundle $K_X$ is isomorphic to $E_m^\otimes m$, where $E_0$ is the ample generator of Pic($X$). The Kodaira vanishing theorem [16, page 36, Corollary 2.32] says that $H^i(X, E_m^\otimes i) = \{0\}$ for $i > m$. On the other hand, Serre duality gives

$$H^1(X, E_0^\otimes 1) = H^{d-1}(X, E_0^{\otimes (m-1)})^*$$

(3.10)

So again by the Kodaira vanishing theorem, we have $H^1(X, E_0^\otimes i) = \{0\}$ for $i < 0$. Also, the assumption Pic($X$) $\cong \mathbb{Z}$ ensures that $H^1(X, \mathcal{O}_X) = \{0\}$. Therefore, the assumption in Proposition 3.1, namely $H^1(X, \xi) = \{0\}$ for all line bundles $\xi$ on $X$, is satisfied if $m \leq 0$, that is, if $X$ is either Fano or it has trivial canonical bundle.

Therefore, Theorem 2.7, Proposition 3.1, and Lemma 2.2 combine together to give Theorem 1.2 stated in the introduction.
Theorem 3.3. Let $X$ be a projective manifold with $\text{Pic}(X) \cong \mathbb{Z}$ and $X$ is Fano or it has trivial canonical bundle. For a $G$-bundle $E$ on $X$, if the $G'$-bundle $\rho^*E$ splits, where $\rho$ is a faithful representation, then $E$ admits a reduction of structure group to a one-parameter subgroup of $T$. □

4 Principal bundles over a projective space

Let $\mathbb{C}P^n$ be the projective space of all lines in $\mathbb{C}^{n+1}$. We will assume that $n \geq 2$. By $\mathbb{C}P^2 \subset \mathbb{C}P^n$ we mean a plane in $\mathbb{C}P^n$.

Corollary 4.1. A principal $G$-bundle $E$ over $\mathbb{C}P^n$ admits a reduction of the structure group to a one-parameter subgroup of the maximal torus $T$ if and only if there is a $\mathbb{C}P^2$ in $\mathbb{C}P^n$ such that the restriction of $E$ to $\mathbb{C}P^2$ admits a reduction of the structure group to $T$. □

Proof. Let $E$ be a $G$-bundle over $\mathbb{C}P^n$ such that the restriction $E|_{\mathbb{C}P^2}$ to a plane $\mathbb{C}P^2$ admits a reduction of the structure group to $T$.

Set $\rho$ to be a faithful representation of $G$ in $\text{GL}(V)$. Let $I(V)$ denote the vector bundle over $X$ associated to $E$ for $\rho$.

Since $E(V)|_{\mathbb{C}P^2}$ admits a reduction of the structure group to $T$, the restriction $I(V)|_{\mathbb{C}P^2}$ of $I(V)$ to $\mathbb{C}P^2$ splits as a direct sum of line bundles. Now [12, page 42, Theorem 2.3.2] says that $I(V)$ splits. Finally, Theorem 3.3 says that $E$ admits a reduction of the structure group to a one-parameter subgroup of $T$. This completes the proof. □

Corollary 4.2. A principal $G$-bundle $E$ over $\mathbb{C}P^n$ is trivial if and only if there is a point $p$ in $\mathbb{C}P^n$ such that the restriction of $E$ to every line $\mathbb{C}P^1 \subset \mathbb{C}P^n$ in $\mathbb{C}P^1$ is trivial. □

Proof. As in Corollary 4.1, set $\rho$ to be a faithful representation of $G$ in $\text{GL}(V)$.

Let $E$ be a principal $G$-bundle on $\mathbb{C}P^n$ such that the restriction of $E$ to any line $\mathbb{C}P^1 \subset \mathbb{C}P^n$ in $\mathbb{C}P^1$ is trivial. Therefore, the restriction of $E(V)$ to any line in $\mathbb{C}P^1$ is trivial. Now, [12, page 51, Theorem 3.2.1] says that the vector bundle $E(V)$ is trivial.

Now, Theorem 3.3 says that $E$ admits a reduction of the structure group to a one-parameter subgroup of $T$. Let

$$E_{C^*} \subset E \quad (4.1)$$

be a $C^*$-bundle which is a reduction of the structure group of $E$ to a one-parameter subgroup of $T$. 
Since the restriction of $E$ to any line $l$ in $\mathbb{CP}^n$ is trivial, the restriction of $E_{C^\ast}$ to $l$ is trivial. If $\xi$ is a $C^\ast$-bundle on $\mathbb{CP}^n$ satisfying the condition that its restriction to some line is trivial, then the $C^\ast$-bundle $\xi$ is itself trivial ($C^\ast$-bundles correspond to line bundles). Therefore, the $C^\ast$-bundle $E_{C^\ast}$ is trivial. This completes the proof. ■

In [8], Grothendieck proved that for a reductive group $G$, any principal $G$-bundle over $\mathbb{CP}^1$ admits a reduction of the structure group to a maximal torus $T$. Although this is not stated there, his arguments in [8, part 3 and 4] actually give the following theorem.

**Theorem 4.3.** Let $E$ be a principal $G$-bundle on $X$ (recall that we are always assuming $\text{Pic}(X) \cong \mathbb{Z}$). Assume that

1. if $L$ is any line bundle on $X$ with $\deg(L) \geq 0$, then $h^0(X,L) > 0$;
2. the adjoint bundle $\text{ad}(E)$ is a direct sum of line bundles.

Then $E$ admits a reduction of the structure group to the normalizer $N$ of a maximal torus $T$. □

**Proof.** For the convenience of the reader, we will give the proof. Recall that an element $v \in g$ is regular semisimple (or just regular for short) if the centralizer of $v$ is a Cartan subgroup. For any $v \in g$, let $\text{ad}(v)$ denote the adjoint action of $v$ on the Lie algebra $g$. Consider the characteristic polynomial

$$\det(1 - \text{ad}(v)) = \sum_{i=0}^{\dim G} a_i(v) t^i.$$  \tag{4.2}

The element $v$ is regular semisimple if and only if we have $a_{\text{rank}(G)}(v) \neq 0$ [9, page 192, (v)].

Let $s$ be a global section of the adjoint bundle. If $s$ is regular at a point $x_0 \in X$, then it is regular for all points $x \in X$. Indeed, the coefficients of the characteristic polynomial $\sum a_i(\text{ad}(s(x))) t^i$ are holomorphic functions on $X$, hence constant. Consequently, the assertion follows from the above criterion for regularity.

Now we will show that if $\text{ad}(E)$ has a section $s$ with $s(x)$ regular, then the structure group of $E$ admits a reduction to the normalizer $N$ of a maximal torus $T$ of $G$.

Let $h(x) \in \text{ad}(E)_x$ be the centralizer of $s(x)$. Since $s(x)$ is regular, $h(x)$ is a Cartan algebra. Note that $G/N$ is the space of Cartan subalgebras of $g$, hence $h(x)$ gives an element of $E(G/N)_x$. Therefore, the section $s$ gives a reduction of the structure group to $N$.

In view of the above observation, to prove the theorem, it is enough to find a section $s$ of the adjoint bundle, with $s(x)$ regular at some point $x \in X$. Since $G$ is reductive, the Lie algebra is a direct sum of the center and the semisimple part

$$g = z \oplus g' \oplus g''.$$  \tag{4.3}
We have $E(g) = E(\mathfrak{g}) \oplus E(g')$. If an element $a \in g'$ is regular in $g'$, then it is also regular in $g$, so we can assume that $\mathfrak{g}$ is trivial, that is, the group is semisimple.

By hypothesis, $\text{ad}(E)$ is a direct sum of line bundles $L_i$. Let

$$E_k = \bigoplus_{\deg L_i \geq k} L_i \subset \text{ad}(E).$$

(4.4)

Fix an isomorphism of $\mathfrak{g}$ with the fiber of $\text{ad}(E)$ over $x \in X$, and let $\mathfrak{g}_k$ be the fiber of $E_k$ over $x$. It is easy to check that $[E_k, E_k] \subset E_{k+k'}$. In particular, $g_1$ is a subalgebra, and if $y \in g_1$, then $\text{ad}y$ is nilpotent. Since we are assuming that $G$ is semisimple, using the Killing form, $\text{ad}(E)$ becomes an orthogonal bundle (i.e., the fibers are equipped with a nondegenerate symmetric bilinear form). For this orthogonal structure we have $(E_1)^\perp = E_0$. By [8, Lemme 4.2]

$$(g_1)^\perp = \mathfrak{g}_0 \supset \mathfrak{h},$$

(4.5)

where $\mathfrak{h}$ is a maximal solvable algebra and $\mathfrak{h}$ is a Cartan subalgebra. Since $\mathfrak{h}$ has regular elements, it follows that there is a regular element $a \in \mathfrak{g}_0$. Now, $E_0$ is a direct sum of line bundles of nonnegative degree, hence by hypothesis there is a section $s$ of $E_0 \subset \text{ad}(E)$ such that $s(x) = a$. This completes the proof of the theorem.

If we further assume that $X$ is simply connected, then $E$ admits a reduction to a maximal torus $T$. This is because the normalizer $N(T)$ of a maximal torus $T$ contains the maximal torus as a finite index subgroup hence any $N(T)$-bundle gives rise to a finite cover of $X$ and since $X$ is simply connected, this cover is trivial hence giving a reduction of structure group to the maximal torus, and hence, by Lemma 2.2, to a one-parameter subgroup (compare with Theorem 3.3).

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References


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