Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 1, April 1989, pp. 17–25.

Classification of isolated complete intersection singularities

A J PARAMESWARAN

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400 005

MS received 5 May 1988; revised 9 August 1988

Abstract. In this article we prove that an isolated complete intersection singularity $(V, 0)$ is characterized by a module of finite length $A(V)$ (cf. §1 for definition) associated to it. The proof uses the theory of finitely determined map germs and generalises the corresponding result by Yau and Mather [4], for hypersurfaces.

Keywords. Intersection singularity; finite length; map germs.

Statement of the results

We use the following notation.
$(\mathbb{C}^m, 0)$:Germ at the origin of $\mathbb{C}^m$.

$\mathcal{O}_m$:Ring of germs of holomorphic functions on $(\mathbb{C}^m, 0)$.

$\mathfrak{M}_n$:Maximal ideal of $\mathcal{O}_m$.

Ω_m:$\mathcal{O}_m$-module of Kähler differentials of order 1

Θ_m:$\mathbb{C}$-derivations of $\mathcal{O}_m$ (tangent sheaf of $(\mathbb{C}^m, 0)$).

In [1] Le and Ramanujam had proved that the moduli algebra determines the topological type of an isolated hyper surface singularity. In [4] Mather and Yau proved that an isolated hyper surface singularity is characterized by its dimension and the moduli algebra.

Here we prove the analogous result for an isolated complete intersection singularity, i.e. an isolated complete intersection singularity, $(V, 0)$, is characterized by its dimension and the $\mathcal{O}_V$-module $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$.

After this work was done, the paper [5] was brought to my notice, in which a different method is used to give a classification of singularities of isolated type, this includes the case considered in this paper. However the present method, which follows that of Yau and Mather, involves explicit computations (see §4) and may be of independent interest in dealing with certain related problems.

Remark. Our method can be extended to yield the stronger result that if $(V, 0)$ is any analytic germ with no smooth curve contained in the singular locus, then the isomorphism class of the module $A(V)$ defined by the exact sequence (#) [not in general equal to $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$] characterises the analytic isomorphism type of V. This result is also contained in [5].

Let $f:(\mathbb{C}^m,0)\to(\mathbb{C}^k,0)$ be a holomorphic map-germ. Let $\Theta(f)$ be the pull-back of the sheaf of germs of holomorphic vector fields at the origin of $(\mathbb{C}^k,0)$ i.e. $\Theta(f) = \mathcal{O}_m \otimes_{f^*\mathcal{O}_k} \Theta_k$. $\partial f:\Theta_m \to \Theta(f)$ be the derivative map. $V=(f_1,\dots,f_k)$ be the analytic space defined by $f=0$. Set

$$A(V) = \mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$$

$$J(V) = \mathrm{Ann}_{\mathcal{O}_V} \mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V).$$

A presentation for $(V,0)\subset(\mathbb{C}^m,0)$ is a choice of generators for $I(V)$, the ideal defining $(V,0)$ in $(\mathbb{C}^m,0)$. This is equivalent to giving a map germ $f:(\mathbb{C}^m,0)\to(\mathbb{C}^k,0)$ for some k such that $\{f=0\}=(V,0)$ in $(\mathbb{C}^m,0)$. Let $(W,0)=V(g_1\dots g_k)$ be the analytic space defined by $g=0$. Assume that $(V,0)$ and $(W,0)$ are the isolated complete intersection singularities of dimension n, and $\mathcal{O}_V$ and $\mathcal{O}_W$ the corresponding analytic local rings. Then we prove the following:

Main Theorem. *Suppose there exists a* $\mathbb{C}$*-algebra isomorphism* $h\colon \mathcal{O}_V/J(V)\to\mathcal{O}_W/J(W)$ *such that* $A(V)\approx h^*A(W)$ *i.e. there exists an abelian group isomorphism* $\psi\colon A(V)\to A(W)$ *such that* $\psi(a,m)=h(a)\cdot\psi(m)$ *for all* $m\in A(V)$ *and* $\forall a\in\mathcal{O}_V/J(V)$. *Then* $\mathcal{O}_V \xrightarrow{\approx} \mathcal{O}_W$ *as* $\mathbb{C}$*-algebras.*

Notice that for any presentation of $(V,0)$ as an isolated complete intersection singularity germ $(V,0)=V(f_1\dots f_k)$ in $(\mathbb{C}^m,0)$, we have an exact sequence,

$$0\to f^*(\mathfrak{M}_k)\Theta(f)+\partial f(\Theta_m)\to\Theta(f)\to A(V)\to 0. \tag{\#}$$

Observe that $\Theta(f)$ is a free $\mathcal{O}_m$-module with basis $1\otimes(\partial/\partial t_i)$. Hence $A(V)$ can be considered as an $\mathcal{O}_{n+k}$-module. The following lemmas are standard.

Lemma 1. If $(V,0)$ *is a complete intersection, then the minimal number of generators of* $A(V)$ *as an* $\mathcal{O}_V$*-module,* $\mu(A(V))$*, equals the embedding codimension of* V*. Thus the embedding dimension equals* $\mu(A(V))+\dim V$.

Lemma 2. Let R *and* S *be local rings which are quotients of* $\mathcal{O}_m$*. Then any isomorphism of* $\mathbb{C}$*-algebras* $Q\colon R\to S$ *can be lifted to an automorphism of* $\mathcal{O}_m$ *as a* $\mathbb{C}$*-algebra.*

COROLLARY 1

Let $(V,0)$ *and* $(W,0)$ *be as in the main theorem. Then we can embed* $(V,0)$ *and* $(W,0)$ *in* $(\mathbb{C}^{n+k},0)$ *with* $n=\dim(V,0)$ *and* $k=\mu(A(V))$*, such that* $A(V)$ *is isomorphic to* $A(W)$ *as* $\mathcal{O}_{n+k}$*-modules.*

Proof. By the hypothesis of the theorem $m\mu(A(V))=\mu(A(W))$ both have the same embedding dimension by Lemma 1. By Lemma 2, the isomorphism, $h\colon \mathcal{O}_V/J(V)\to \mathcal{O}_W/J(W)$, can be lifted to an automorphism $\tilde{h}\colon\mathcal{O}_{n+k}\hookleftarrow$. By replacing $(W,0)$ by its image i.e. $I(W)$ by h $(\mathrm{I}(W))$, we can assume that $A(V)$ is actually isomorphic to $A(W)$ as an $\mathcal{O}_{n+k}$ module.

2. The Group $\mathscr{K}$ and a theorem of Mather

In this section we recall some definitions and a theorem from [2].

Elements of $\mathscr{K}$ are pairs (h,H) of holomorphic automorphisms $h:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^{n+k},0)$ and $H:(\mathbb{C}^{n+2k},0)-(\mathbb{C}^{n+2k},0)$, such that the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{C}^{n+k},0) & \xrightarrow{i} & (\mathbb{C}^{n+2k},0) & \xrightarrow{\pi} & (\mathbb{C}^{n+k},0)\\
\downarrow h & & \downarrow H & & \downarrow h\\
(\mathbb{C}^{n+k},0) & \xrightarrow{i} & (\mathbb{C}^{n+2k},0) & \xrightarrow{\pi} & (\mathbb{C}^{n+k},0).
\end{array}$$

Here i and π are $i(z_1,\ldots,z_{n+k})=(z_1,\ldots,z_{n+k},0,\ldots,0)$ and $\pi(z_1,\ldots,z_{n+2k})=(z_1,\ldots,z_{n+k})$. We define

$$\mathscr{F}=\{f:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^k,0)/f \text{ is holomorphic}\}.$$

We define an action of $\mathscr{K}$ on $\mathscr{F}$ as follows. Let $(h,H)\in\mathscr{K}$ and $f\in\mathscr{F}$, then there exist a unique $g\in\mathscr{F}$ such that graph $g=H(\text{graph } f)$. We set $(h,H)\cdot f=g$.

Lemma 3. If f and g define n-dimensional spaces $(V,0)$ and $(W,0)$ then they are in the same $\mathscr{K}$-orbit if and only if $(V,0)$ and $(W,0)$ are biholomorphically equivalent.

Proof. Let $(h,H)\in\mathscr{K}$ be such that $(h,H)\cdot f=g$. Then $h^{-1}(W)=h^{-1}\circ i^{-1}(\text{graph } g)=i^{-1}(H^{-1}(\text{graph } g))=i^{-1}$ (graph $f)=V$. Hence h: $(\mathbb{C}^{n+k},0)\to(\mathbb{C}^k,0)$ provides a biholomorphic equivalences between $(V,0)$ and $(W,0)$.

Conversely suppose $h:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^{n+k},0)$ is a biholomorphic map such that $h(V)=W$. Then we define H: $(\mathbb{C}^{n+2k},0)\to(\mathbb{C}^{n+2k},0)$ as $H(z,w)=(h(z),\ U^{-1}(z),\ w)$, where $U\in GL(k,\mathcal{O}_{n+k})$ is such that $f=U(g\circ h)$, for $z\in\mathbb{C}^{n+k}$ and $w\in\mathbb{C}^k$. Then clearly $(h,H)\in\mathscr{K}$. Also,

$$H(\text{graph } f)=H(z,f(z))=(h(z),U^{-1}(z)f(z))=(h(z),g\circ h(z))$$

$$=(h(z),g(h(z))=\text{graph } g.$$

Hence $(h,H)\cdot f=g$.

Let $\mathscr{K}_r$ be the subgroup of $\mathscr{K}$ consisting of all $(h,H)\in\mathscr{K}$ such that its r-jet at 0, is the same as the r-jet of the identity. $\mathscr{K}^{(r)}=\mathscr{K}_0/\mathscr{K}_r$. Then $\mathscr{K}^{(r)}$ is a Lie group. Let $\mathscr{F}^r$ be the r-jet of elements of $\mathscr{F}$. Then $\mathscr{K}^{(r)}$ will act on $\mathscr{F}^r$ by $(h,H)^{(r)}\cdot f^{(r)}=((h,H)\cdot f)^{(r)}$.

An element $f\in\mathscr{F}$ is said to be r-determined relative to $\mathscr{K}$ if for any $g\in\mathscr{F}$ with $g^{(r)}\in\mathscr{K}^{(r)}\cdot f^{(r)}$, $g\in\mathscr{K}\cdot f$. An element of $\mathscr{F}$ is said to be finitely determined relative to $\mathscr{K}$ if it is r-determined for some $r\in\mathbb{N}$. In this situation we have

Theorem 1. ([2], Theorem 3.5): *$f\in\mathscr{F}$ is finitely determined if and only if $\Theta(f)/\partial f(\Theta_{n+k})+f^*(\mathfrak{M}_k)\Theta(f)$ is of finite $\mathbb{C}$-dimension.*

Now by (#), the function, f defining the isolated complete intersection singularity is finitely determined, because $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V,\mathcal{O}_V)$ has finite length, where V is defined by $\{f=0\}$.

3. Reduction to a special case

From now on we fix coordinates $t_1, \ldots, t_k$ on $(\mathbb{C}^k, 0)$. Let $V = V(f_1, \ldots, f_k)$ be as in the main theorem.

Lemma 4. *Let $f'_i = \sum a_{ij} f_j$ $1 \leqslant i, j \leqslant k$, be another set of generators for the ideal $I(V)$. Let $\rho_i = 1 \otimes dt_i \in \mathcal{O}_{n+k} \otimes_{f'^*\mathcal{O}_k} \Omega_k$ and $\rho'_i = 1 \otimes dt_i \in \mathcal{O}_{n+k} \otimes_{f^*\mathcal{C}_k} \Omega_k$ be the free basis for the corresponding $\mathcal{O}_{n+k}$-modules. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{O}_V \otimes (\mathcal{O}_{n+k} \otimes_{f^*\mathcal{O}_k} \Omega_k) & & \\ \downarrow & \searrow^{1 \otimes U} & \\ I/I^2 & \longrightarrow & \mathcal{O}_V \otimes (\mathcal{O}_{n+k} \otimes_{f'^*\mathcal{C}_k} \Omega_k) \end{array}$$

Here the vertical isomorphism is given by $1 \otimes \rho_i \mapsto f_i$ and the horizontal isomorphism is given by $1 \otimes \rho'_i \mapsto f_i$ and U is defined by

$$U(\rho'_i) = \sum a_{ij} \rho_j.$$

Proof. The commutativity of the diagram follows from the definition of those maps and a simple diagram chasing.

Now notice that

$$\Theta(f) = \mathrm{Hom}_{\mathcal{O}_{n+k}} (\mathcal{O}_{n+k} \otimes_{f^*\mathcal{O}_k} \Omega_k, \mathcal{O}_{n+k}) \text{ and}$$

$$\Theta(g) = \mathrm{Hom}_{\mathcal{O}_{n+k}} (\mathcal{O}_{n+k} \otimes_{g^*\mathcal{O}_k} \Omega_k, \mathcal{O}_{n+k}).$$

Hence we have the following

COROLLARY 2

U induces an isomorphism $U^t : \Theta(f) \to \Theta(f')$, which induces the identity on $A(V) = \mathrm{Ext}'_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$.

Lemma 5. *Let $(V, 0)$ and $(W, 0)$ in $(\mathbb{C}^{n+k}, 0)$ be as in Corollary 1. Then any given presentation $g = (g_1, \ldots, g_k)$ of $(W, 0) \subset (\mathbb{C}^{n+k}, 0)$ we can choose a presentation $f = (f_1, \ldots, f_k)$ of $(V, 0) \subset (\mathbb{C}^{n+k}, 0)$ such that the isomorphism $A(V) \xrightarrow{\sim} A(W)$ of Corollary 1 can be lifted to an isomorphism $\Theta(f) \to \Theta(g)$ such that*

$$\frac{\partial}{\partial t_i} \circ f \mapsto \frac{\partial}{\partial t_i} \circ g.$$

Proof. By Nakayama's lemma we can get a lift $U : \Theta(f) \to \Theta(g)$, of $A(V) \to A(W)$. Let $U = (a_{ij})$ be an invertible $k \times k$ matrix with entries in $\mathcal{O}_{n+k}$ such that $U^t = \tilde{U}$. Here $\tilde{U}$ is thought of as a matrix with respect to the basis of $\Theta(f)$ and $\Theta(g)$ given by $(\partial/\partial t_i) \circ f$ and $(\partial/\partial t_i) \circ g$ respectively. Then by, Corollary 2, if we change the generators of $I(V)$ by $f'_i = \sum a_{ij} f_j$, then $U^{t^{-1}} : \Theta(f') \to \Theta(f)$ is an isomorphism in inducing identity on $A(V)$. Hence if we replace f by f', the isomorphism $\Theta(f') \to \Theta(g)$ has the required property.

4. Some local analytic computations

In this section we prove Proposition 6, which is the main technical step in the proof of the main theorem. From now on, we fix the presentations f, g for $(V,0)$; $(W,0)$, respectively, given by Lemma 5. Let $e_i = (\partial/\partial t_i) f$ and $e_i' = (\partial/\partial t_i) g$ be the natural basis vectors for $\Theta(f)$ and $\Theta(g)$ respectively. Then under the identification of $\Theta(f)$ and $\Theta(g)$ with $\mathcal{C}^k_{n+k}$ given by the basis e_i and e_i', the following equality

$$\partial f(\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f) = \partial g(\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g) \tag{$*$}$$

obtained from (#) of §1 takes the following explicit form (once we fix coordinates $z_1,\ldots,z_{n+k}$ on $(\mathbb{C}^{n+k},0)$): The L.H.S. (left-hand side) of $(*)$ is generated by the vectors

$$\left(\frac{\partial f_1}{\partial z_i},\ldots,\frac{\partial f_k}{\partial z_i}\right)\quad i=1,\ldots,n+k,\text{ and }(f_i,0\ldots0),$$

$(0f_i,0\ldots0),\ldots,(0,0,\ldots,f_i)i=1,\ldots,k$. Similarly the R.H.S. (right-hand side) is generated by $((\partial g_1/\partial z_i),\ldots,(\partial g_k/\partial z_i))$, and

$$(g_j,0\ldots0),\ldots,(0,\ldots,0,g_j)\ 1\leqslant i\leqslant n+k$$
$$1\leqslant j\leqslant k.$$

In this situation we have the following:

PROPOSITION 6

$\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f) = \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$.

Proof. Since the proofs are identical we prove only one inclusion,

L.H.S. $\subset$ R.H.S.

Step 1. $\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) \subset \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$.

Proof. Notice that element of $\partial f(\mathfrak{M}_{n+k}\Theta_{n+k})$ are generated by

$$z_i\left(\frac{\partial f_1}{\partial z_j},\ldots,\frac{\partial f_k}{\partial z_j}\right),\quad 1\leqslant i,j\leqslant n+k.$$

Since by $(*)$ $((\partial f_1/\partial z_j),\ldots,(\partial f_k/\partial z_j))\in\partial g(\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$ and ∂g is $\mathcal{C}_{n+k}$ linear, we have

$$z_i\left(\frac{\partial f_1}{\partial z_j},\ldots,\frac{\partial f_k}{\partial z_j}\right)\in\partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$$

Step 2. $f^*(\mathfrak{M}_k)\Theta(f) \subset \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$.

Proof. Here $f^*(\mathfrak{M}_k)\Theta(f)$ is generated by $(f_i,0\ldots0),\ldots,(0,0\ldots f_i)$ $1\leqslant i\leqslant k$. So we have to prove that all these k^2 elements belong to the R.H.S. Again since the argument is identical, we prove that $(f_1,0,\ldots,0)\in$R.H.S. By $(*)$ $(f_1,0\ldots0)\in\partial g(\Theta_{n+k})+g^*(\mathfrak{M}_k)\Theta(g)$ i.e. $(f_1,0\ldots0)=v+\partial g(\eta)$ for some $v\in g^*(\mathfrak{M}_k)\Theta(g)$ and $\eta\in\Theta_{n+k}$.

Key Lemma. $\eta\in\mathfrak{M}_{n+k}\Theta_{n+k}$.

Proof. Suppose $\eta(0)\neq 0$. Then we can choose a coordinate system $z_1,\ldots,z_{n+k}$ in $\mathbb{C}^{n+k}$ such that $\eta=(\partial/\partial z_1)$. Then the above equation will become

$$(f_1,0,\ldots,0)=v+\partial g\left(\frac{\partial}{\partial z_1}\right)$$

i.e. by Lemma 5, we have the following equations;

$$\begin{aligned}
&(1)\ f_1=a_1^{(1)}g_1+a_2^{(1)}g_2+\cdots+a_k^{(1)}g_k+\frac{\partial g_1}{\partial z_1}\\
&(2)\ \ 0=a_1^{(2)}g_1+a_2^{(2)}g_2+\cdots+a_k^{(2)}g_k+\frac{\partial g_2}{\partial z_1} \qquad (**)\\
&(k)\ \ 0=a_1^{(k)}g_1+a_2^{(k)}g_2+\cdots+a_k^{(k)}g_k+\frac{\partial g_k}{\partial z_1}
\end{aligned}$$

$$a_i^{(j)}\in\mathcal{O}_{n+k}.$$

Also by $(*)$ $(\partial f/\partial z_j)\in\partial g\ (\Theta_{n+k})+g^*(\mathfrak{M}_k)\Theta(g)$. Looking at the first component (coefficients of $e_1=e_1'$) of $\partial f(\partial/\partial z_j)$, we obtain

$$\frac{\partial f_1}{\partial z_j}=b_1g_1+\cdots+b_kg_k+\sum_{i=1}^{n+k}\frac{\partial g_1}{\partial z_i}\xi_i \qquad (***)$$

DEFINITION

For any $h\in\mathcal{O}_{n+k}$, let $(h)_\nu$ denote the νth-order homogeneous component of h with respect to the coordinate system chosen above. Let $o(h)$ be the order of h. Let $\deg_{z_1}(h)_\nu$ be the degree of $(h)_\nu$ as a polynomial in z_1. Let $m=\mathrm{Min}_{1\leqslant i,j\leqslant k}\{o(f_i),o(g_j)\}$. Then by the assumption that k is the embedding codimension for both $(V,0)$ and $(W,0)$, we have $m\geqslant 2$. Notice that

$$(f_1)_{m-1}=(g_1)_{m-1}=\cdots=(g_k)_{m-1}=0.$$

Claim (i). $(\partial g_r/\partial z_1)_{m-1}=0$ i.e. $\deg_{z_1}(g_r)_m=0$ for $r=1,\ldots,k$.

Proof. By (r) of $(**)$, $(\partial g_r/\partial z_1)_{m-1}=0$ hence $\deg_{z_1}(g_r)_m=0$.

Claim (ii). $\deg_{z_1}(f_1)_m=0$.

Proof. For it is enough to check that $\deg_{z_1}(\partial f_1/\partial z_j)_{m-1}=0$, for every j, because $m\geqslant 2$. But then by looking at $(***)$ we find that

$$\left(\frac{\partial f_1}{\partial z_j}\right)_{m-1} = (b_1 g_1)_{m-1} + \cdots + (b_k g_k)_{m-1} + \left(\sum \frac{\partial g_1}{\partial z_i} \xi_i\right)_{m-1}$$
$$= \sum \left(\frac{\partial g_1}{\partial z_i}\right)_{m-1} \xi_i(0).$$

Now since $\deg_{z_1}(g_1)_m = 0$, we have $\deg_{z_1}(\partial g_i/\partial z_i)_{m-1} = 0$ and hence $\deg_{z_1}(\partial f_1/\partial z_j)_{m-1} = 0$. Hence $\deg_{z_1}(f_1)_m = 0$. Now we prove the following assertions by induction on ν. The argument above yields the case $\nu = 0$, to begin the induction.

Assertion $(i)_\nu$. $\deg_{z_1}(g_1)_{m+\nu}, \ldots, \deg_{z_1}(g_k)_{m+\nu} \leqslant \nu$. i.e. none of these homogeneous polynomials contain monomials of degree bigger than ν, in z_1.

Assertion $(ii)_\nu$. $\deg_{z_1}(f_1)_{m+\nu} \leqslant \nu$.

Proof of $(i)_\nu$. We assume $(\mathrm{i})_\mu$ and $(\mathrm{ii})_\mu$ for all $\mu < \nu$. i.e. $\deg_{z_1}(g_i)_{m+\mu}$, $\deg_{z_1}(f_1)_{m+\mu} \leqslant \mu$ for all $\mu < \nu$. Let $D^\mu = (\partial^\mu/\partial z_1^\mu)$. Then $o(D^\mu g_i) \geqslant m$ and $o(D^\mu f_1) \geqslant m$ for every $\mu \leqslant \nu$. We prove that $\deg_{z_1}(D^\nu g_i)_m = 0$. Now apply D^ν to both sides of (1) of (**),

$$D^\nu f_1 = D^\nu(a_1^{(1)} g_1) + \cdots + D^\nu(a_k^{(1)} g_k) + D^\nu \frac{\partial g_1}{\partial z_1}.$$

Since

$$D^\nu(a_i^{(1)} g_i) = \sum \binom{\nu}{r} D^r a_i^{(1)} D^{\nu-r} g_i$$

has order $\geqslant m$, we have

$$0 = (D^\nu f_1)_{m-1} = \left(D^\nu \frac{\partial g_1}{\partial z_1}\right)_{m-1} = \frac{\partial}{\partial z_1}(D^\nu g_1)_m.$$

Hence

$$\deg_{z_1}(D^\nu g_1)_m = 0 \text{ i.e., } \deg_{z_1}(g_1)_{m+\nu} \leqslant \nu.$$

Similarly by applying D^ν to both sides of the other equations of (**) and comparing the $(m-1)$th order terms on both sides we obtain

$$0 = D^\nu\left(\frac{\partial g_i}{\partial z_1}\right)_{m-1} = \frac{\partial}{\partial z_1}(D^\nu g_i)_m.$$

Hence $\deg_{z_1}(D^\nu g_i) = 0$ i.e. $\deg_{z_1}(g_i)_{m+\nu} \leqslant \nu$. We deduce that $\deg_{z_1} D^\nu(\partial g_1/\partial z_1)_{m-1} = 0$.

Proof of $(ii)_\nu$. Apply D^ν to the equation (***),

$$D^\nu \frac{\partial f_1}{\partial z_j} = \frac{\partial}{\partial z_j}(D^\nu f_1) = D^\nu(b_1 g_1) + \cdots + D^\nu(b_k g_k) + D^\nu \sum \frac{\partial g_1}{\partial z_i} \xi_i.$$

Notice that $o(D^\nu(b_i g_i)) \geqslant m$. So again by comparing the terms of order $m-1$, we get

$$\left(\frac{\partial}{\partial z_j} D^\nu f_1\right)_{m-1} = \left(\sum D^\nu\left(\frac{\partial g_1}{\partial z_i} \xi_i\right)\right)_{m-1}$$

$$= \left(\sum\sum \binom{\mu}{r} D^r \frac{\partial g_1}{\partial z_i} D^{\nu - r} \xi_i \right)_{m-1}$$

$$= \sum\sum \binom{\nu}{r} \left(D^r \frac{\partial g_1}{\partial z_i} \right)_{m-1} D^{\nu - r} \xi_i(0).$$

Since $\deg_{z_1} D^r(\partial g_i/\partial z_i) = 0$, for every $r \leqslant \nu$, we conclude that $\deg_{z_1} ((\partial/\partial z_j) D^\nu f_1)_{m-1} = 0$. Hence $\deg_{z_1}(f_1)_{m+\nu} \leqslant \nu$. This proves (ii)$_\nu$. Since we have already checked the induction hypothesis for $\nu = 0$, we get, $\deg_{z_1}(g_i)_{m+\nu} \leqslant \nu$ for all ν.

Now since $m \geqslant 2$, every monomial of g_i is of order $\geqslant 2$ in the variables $z_2, \ldots, z_{n+k}$. Hence $g_i(z_1, 0, \ldots, 0) = 0$ and $(\partial g_i/\partial z_j)$ $(z_1, 0, \ldots, 0) = 0$ for every i and j. Hence the Jacobian matrix $(\partial g_i/\partial z_j)$ is identically zero along the z_1-axis, which implies that z_1-axis is contained in the singular locus of $\{g = 0\}$. This contradicts the assumption that g defines an isolated singularity. This contracdiction was due to the assumption that $\eta(0) \neq 0$. Hence $\eta(0) = 0$ i.e. $\eta \in \mathfrak{M}_{n+k}\Theta_{n+k}$.

5. Proof of the Main Theorem

We fix presentations $\{f = 0\}$ and $\{g = 0\}$ for $(V, 0)$ and $(W, 0)$ as in Lemma 5. Then by Theorem 1, f and g are finitely determined. By Lemma 3 f and g are biholomorphically equivalent if and only if they are in the same $\mathscr{K}$-orbit. But to prove f and g are in the same $\mathscr{K}$-orbit it is enough to prove $f^{(l)}$ and $g^{(l)}$ are in the same $\mathscr{K}^{(l)}$-orbit, for every l. Now fix an $l \in \mathbb{N}$. Note that $\mathscr{F}^{(l)}$ can be given a global coordinate system so that it has the structure of a complex affine space. In his paper [2]. Mather defines a projection, π^l: $\mathfrak{M}_{n+k}\Theta(h) \to T_{h^{(l)}}\mathscr{F}^{(l)}$, for any $h \in \mathscr{F}$. Here $T_{h^{(l)}}\mathscr{F}^{(l)}$ is the tangent space to $\mathscr{F}^{(l)}$ at the l-jet $h^{(l)}$ of h. In our context, if we identify $T_{h^{(l)}}\mathscr{F}^{(l)}$ with $\mathscr{F}^{(l)}$ (using the affine structure), and $\Theta(h)$ with $\mathscr{O}^k_{n+k}$ using the basis $(\partial/\partial t_i) \circ h$, then for any $\eta = (\eta_1, \ldots, \eta_k) \in \mathfrak{M}_{n+k}\Theta(h)$ $(\approx \mathfrak{M}_{n+k}\mathscr{O}^k_{n+k})$, $\pi^l(\eta) = \eta^{(l)}$. We may think of η as an element of $\mathscr{F} \cdot \mathscr{F}^{(l)}$ has a natural structure as an $\mathscr{O}_{n+k}$ module; note that π^l is then $\mathscr{O}_{n+k}$-linear. Mather also proves the following:

Theorem 2. ([2], Proposition 7.4).

$$T_{h^{(l)}}(\mathscr{K}^{(l)} \cdot h^{(l)}) = \pi^l(\partial h(\mathscr{M}_{n+k}\Theta_{n+k}) + h^*(\mathfrak{M}_k)\Theta(h)).$$

Here $T_h(l)$ $\mathscr{K}^{(l)} \cdot h^{(l)}$ denotes the tangent space to the Space to the $\mathscr{K}^{(l)}$-orbit of $h^{(l)}$ at the point $h^{(l)} \in \mathscr{F}^{(l)}$.

Note that by Theorem 2, $T_h(l)\mathscr{K}^{(l)}h^{(l)}$ is an $\mathscr{O}_{n+k}$-submodule of $T_h(l)$ $\mathscr{F}^{(l)}$ and it is generated as a module by the elements of the form $(z_j(\partial h_1/\partial z_i), \ldots, z_j\ (\partial h_k/\partial z_i)^{(l)}$ and $(h_i, 0 \ldots 0)^{(l)}, \ldots, (0, 0, \ldots, h_i)^{(l)}$. We denote these generators by

$$\rho_1(h), \ldots, \rho_N(h), \text{ where } N = k^2 + (n+k)^2.$$

If $f^{(l)} \neq g^{(l)}$, then consider the complex line L joining $f^{(l)}$ and $g^{(l)}$. Define

$$\begin{aligned} L_0 &= \{h^{(l)} \in L / \pi^l(\partial h(\mathfrak{M}_{n+k}\Theta_{n+k}) + h^*(\mathfrak{M}_k)\Theta(h)) \\ &= \pi^l(\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f))\} \\ &= \{h^{(l)} \in L / T_{h^{(l)}}(\mathscr{K}^{(l)} \cdot h^{(l)}) = T_f(l)(\mathscr{K}^{(l)} \cdot h^{(l)})\}. \end{aligned}$$

Then L_0 has the following properties:

(1) $g^{(l)} \in L_0$ by Proposition 6.
(2) $h^{(l)} \in L$, $h^{(l)} = (1-t)f^{(l)} + tg^{(l)}$, then $\rho_i(h) = (1-t)\rho_i(f) + t\rho_i(g)$. Hence $T_{h^{(l)}}(\mathscr{K}^{(l)} \cdot h^{(l)}) \subset T_{f^{(l)}}(\mathscr{K}^{(l)} \cdot f^{(l)})$.
(3) $\{\rho_1(f), \ldots, \rho_N(f)\}$ and $\{\rho_1(g), \ldots, \rho_N(g)\}$ generate the same $\mathscr{O}_{n+k}$-submodule in $\mathscr{F}^{(l)}$, hence for all but a finite set of $t \in \mathbb{C}$, $\{(1-t)\rho_i(f) + t\rho_i(g)\}$ will generate the same submodule.
(4) By (2) and (3), L_0 is connected, since L_0 is $\mathbb{C}$ with at most finitely many points deleted.

Now we prove that L_0 is contained in a single orbit of $\mathscr{K}^{(l)}$. For this we need the following theorem,

Theorem 3 ([3], Lemma 3.1). *Let α: $G \times U \to U$ be a C^∞ action of a Lie group G on a C^∞ manifold U, $V \subset U$ a connected submanifold of U. Then V is contained in a single orbit of α if and only if*

(i) $T_v V \subset T_v G \cdot v$
(ii) $\dim T_v G \cdot v$ *is independent of* $v \in V$.

In our context we take $G = \mathscr{K}^{(l)}$, $U = \mathscr{F}^{(l)}$, $V = L_0$ and check the conditions (i) and (ii) of Theorem 3. Notice that $T_{h^{(l)}} L_0$ is generated by $(f-g)^{(l)}$, for any $h^{(l)} \in L_0$. By Proposition 6, $f - g \in \partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f)$. Hence

$$
\begin{aligned}
(f-g)^{(l)} &\in \pi^l(\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f)) \\
&= \pi^l(\partial h(\mathfrak{M}_{n+k}\Theta_{n+k}) + h^*(\mathfrak{M}_k)\Theta(h)) \\
&= T_{h^{(l)}}\mathscr{K}^{(l)} \cdot h^{(l)}.
\end{aligned}
$$

Hence

$$T_{h^{(l)}} L_0 \subset T_{h^{(l)}}\mathscr{K}^{(l)} \cdot h^{(l)} \quad \forall h^{(l)} \in L_0.$$

Since $T_{h^{(l)}}\mathscr{K}^{(l)} \cdot h^{(l)} = T_{f^{(l)}}\mathscr{K}^{(l)} \cdot f^{(l)}$, $\dim T_{h^{(l)}}\mathscr{K}^{(l)}$ is independent of $h^{(l)} \in L_0$. Hence (i) and (ii) are satisfied, and so L_0 is contained in a single orbit of $\mathscr{K}^{(l)}$. In particular $f^{(l)} \in \mathscr{K}^{(l)} \cdot g^{(l)}$. Hence $f \in \mathscr{K} \cdot g$. This completes the proof of the main theorem.

Acknowledgements

I thank V Srinivas for suggesting this problem, and also various modifications in the proofs. I thank N Mohan Kumar, for stimulating discussions, and for suggesting the final statement of the main theorem.

References

[1] Le D T and Ramanujam C P, The invariance of Milnor's number implies the invariance of the topological type, *Am. J. Math.* **98** (1976) 67–78

[2] Mather J N, Stability of C^∞ mappings III, *Publ. Math. IHES* **35** (1968) 127–156

[3] Mather J N, Stability of C^∞ mappings IV, *Publ. Math. IHES* **37** (1970) 223–248

[4] Mather J N and Yau S S T, Classification of hyper surface singularities by their moduli algebras, *Invent. Math.* **69** (1982) 243–251

[5] Gaffney T and Hauser H, Characterizing singularities of varieties and of mappings. *Invent. Math.* **81** (1985) 427–447