PRINCIPAL BUNDLES OVER A SMOOTH REAL PROJECTIVE CURVE OF GENUS ZERO

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ABSTRACT. Let H_0 denote the kernel of the endomorphism, defined by $z \longmapsto (z/\overline{z})^2$, of the real algebraic group given by the Weil restriction of $\mathbb{C}^*$. Let X be a nondegenerate anisotropic conic in $\mathbb{P}^2_{\mathbb{R}}$. The principal $\mathbb{C}^*$–bundle over the complexification $X_{\mathbb{C}}$, defined by the ample generator of $\mathrm{Pic}(X_{\mathbb{C}})$, gives a principal H_0–bundle F_{H_0} over X through a reduction of structure group. Given any principal G–bundle E_G over X, where G is any connected reductive linear algebraic group defined over $\mathbb{R}$, we prove that there is a homomorphism $\rho : H_0 \longrightarrow G$ such that E_G is isomorphic to the principal G–bundle obtained by extending the structure group of F_{H_0} using ρ.

The tautological line bundle over the real projective line $\mathbb{P}^1_{\mathbb{R}}$, and the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, together give a principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle F on $\mathbb{P}^1_{\mathbb{R}}$. Given any principal G–bundle E_G over $\mathbb{P}^1_{\mathbb{R}}$, where G is any connected reductive linear algebraic group defined over $\mathbb{R}$, we prove that there is a homomorphism $\rho : \mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow G$ such that E_G is isomorphic to the principal G–bundle obtained by extending the structure group of F using ρ.

1. INTRODUCTION

Let $F_{\mathbb{C}^*}$ denote the principal $\mathbb{C}^*$–bundle over the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ given by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$. So $F_{\mathbb{C}^*}$ is the complement of the zero section in the total space of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$. Let $E_{G_{\mathbb{C}}}$ be an algebraic principal $G_{\mathbb{C}}$–bundle over $\mathbb{P}^1_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is a connected reductive linear algebraic group defined over $\mathbb{C}$. A theorem due to Grothendieck says that there is a homomorphism

$$\rho : \mathbb{C}^* \longrightarrow G_{\mathbb{C}}$$

such that $E_{G_{\mathbb{C}}}$ is isomorphic to the principal $G_{\mathbb{C}}$–bundle obtained by extending the structure group of $F_{\mathbb{C}^*}$ using ρ [Gr, p. 122, Théorème 1.1]. Our aim here is to investigate the corresponding set–up for the field of real numbers.

Let X be a geometrically irreducible smooth projective curve, defined over the field of real numbers, satisfying the condition that

$$\mathrm{genus}(X) := \dim H^1(X, \mathcal{O}_X) = 0 \,.$$

This condition implies that either X is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, or it is isomorphic to the nondegenerate anisotropic conic in $\mathbb{P}^2_{\mathbb{R}}$ defined by the polynomial $x^2 + y^2 + z^2 = 0$ (see [Ha, p. 106, Exercise 4.7(e)]). This conic clearly does not have any real points. Let $X_{\mathbb{C}}$ be the base change of X to $\mathbb{C}$. So $X_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

Let T_c denote the Weil restriction of the complex torus $\mathbb{C}^*$ to the subfield $\mathbb{R}$. Therefore, T_c is a two–dimensional abelian reductive real linear algebraic group. The earlier defined principal $\mathbb{C}^*$–bundle $F_{\mathbb{C}^*}$ over $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$ gives a principal T_c–bundle F_{T_c} over X. More precisely, the Weil restriction of $F_{\mathbb{C}^*}$ is a principal T_c–bundle over the Weil restriction $\widehat{X}$ of $X_{\mathbb{C}}$, and F_{T_c} is the pull back of this principal T_c–bundle by the natural embedding of X in $\widehat{X}$.

Let H_0 be the kernel of the homomorphism $T_c \longrightarrow T_c$ defined by $z \longmapsto (z/\bar{z})^2$. Therefore, H_0 fits in a short exact sequence of groups

$$e \longrightarrow \mathbb{G}_m \longrightarrow H_0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e \,,$$

where $\mathbb{G}_m = \mathbb{R}^*$ is the multiplicative group, and the projection $H_0 \longrightarrow \mathbb{Z}/2\mathbb{Z}$ is defined by $z \longmapsto z/\bar{z}$.

The above defined principal T_c–bundle F_{T_c} admits a reduction of structure group to the subgroup $H_0 \subset T_c$, and furthermore, any two reductions of structure group of F_{T_c} to H_0 are isomorphic (see Lemma 4.6). Let

$$(1.1) \qquad\qquad\qquad\qquad F_{H_0} \subset F_{T_c}$$

be a reduction of structure group to H_0.

Our main result is the following (see Theorem 4.12):

Theorem 1.1. *Let G be a connected reductive linear algebraic group defined over $\mathbb{R}$. Let E_G be a principal G–bundle over a nondegenerate anisotropic conic X in $\mathbb{P}^2_{\mathbb{R}}$. Then there is a homomorphism*

$$\rho : H_0 \longrightarrow G$$

with the following property: The principal G–bundle E_G is isomorphic to the one obtained by extending the structure group of the principal H_0–bundle F_{H_0} in eqn. (1.1) using the homomorphism ρ.

We will now describe the corresponding result for principal bundles over $\mathbb{P}^1_{\mathbb{R}}$ proved here.

Let $F_{\mathbb{G}_m}$ denote the principal $\mathbb{G}_m$–bundle over $\mathbb{P}^1_{\mathbb{R}}$ given by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)$. Let $F_{\mathbb{Z}_2}$ denote the principal $\mathbb{Z}/2\mathbb{Z}$–bundle over $\mathbb{P}^1_{\mathbb{R}}$ defined by the natural projection $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$. Therefore, the fiber product $F_{\mathbb{G}_m} \times_X F_{\mathbb{Z}_2}$ over X is a principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle.

The following theorem is proved in Theorem 4.12.

Theorem 1.2. *Let E_G be a principal G–bundle over $\mathbb{P}^1_{\mathbb{R}}$, where G is a connected reductive linear algebraic group defined over $\mathbb{R}$. Then there is a homomorphism*

$$\rho : \mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow G$$

such that the principal G–bundle E_G is isomorphic to the one obtained by extending the structure group of the principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle $F_{\mathbb{G}_m} \times_X F_{\mathbb{Z}_2}$ using ρ.

If G is a compact real form, then there is no nontrivial homomorphism from $\mathbb{G}_m$ to G. Therefore, in that case the homomorphism ρ in Theorem 1.1 and the homomorphism ρ in Theorem 1.2 factor through $\mathbb{Z}/2\mathbb{Z}$.

2. Semistable principal bundles

Let X be a geometrically irreducible smooth projective curve defined over the field of real numbers. Let G be a connected reductive linear algebraic group defined over $\mathbb{R}$. Since G is connected, it follows that G is actually geometrically connected. Indeed, the connected component, containing the identity element, of the complexification of G is preserved by the Galois involution, thus showing that G is disconnected if its complexification is so.

A principal G–bundle E_G over X will be called *semistable* if for each proper parabolic subgroup $P \subset G$, and for each reduction of structure group

$$(2.1) \qquad\qquad E_P \subset E_G$$

of E_G to P, the following holds: for each anti–dominant character χ of P which is trivial on the center of G, the line bundle $E_P(\chi)$ on X associated to the principal P–bundle E_P for χ is of nonnegative degree. We recall that a character of P is called anti–dominant if the associated line bundle on G/P is numerically effective.

The above definition is identical to the definition in [Ra] of semistable principal bundles over a smooth complex projective curve; see [Ra, p. 131, Lemma 2.1] to compare the above definition with Definition 1.1 in [Ra, p. 129].

Remark 2.1. A couple of remarks on the above definition:

(1) A principal G–bundle E_G is semistable if and only if for each reduction E_P as in eqn. (2.1), where P is a maximal proper parabolic subgroup of G, the inequality

$$\text{degree}(\text{ad}(E_G)/\text{ad}(E_P)) \geq 0$$

holds, where $\text{ad}(E_G)$ and $\text{ad}(E_P)$ are the adjoint vector bundles of E_G and E_P respectively; see [Ra, p. 131, Lemma 2.1]. (The adjoint bundle of a principal H–bundle E_H is the vector bundle associated to E_H for the adjoint action of H on its own Lie algebra.)

(2) The group G may not have any proper parabolic subgroup. For example, if G is a compact real form of a complex reductive group, then G does not have any proper parabolic subgroup. In such a situation, the semistability condition is automatically satisfied.

Let

$$(2.2) \qquad\qquad X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$$

be the base change of X to the field $\mathbb{C}$. Let

$$G_{\mathbb{C}} = G \times_{\mathbb{R}} \mathbb{C}$$

be the base change to $\mathbb{C}$. Therefore, $G_{\mathbb{C}}$ is a connected reductive linear algebraic group defined over $\mathbb{C}$.

Take a principal G–bundle E_G over X. The base change to $\mathbb{C}$

$$(2.3) \qquad\qquad E_G^{\mathbb{C}} := E_G \times_{\mathbb{R}} \mathbb{C}$$

is a principal $G_{\mathbb{C}}$–bundle over $X_{\mathbb{C}}$.

Lemma 2.2. *The principal G–bundle E_G over X is semistable if and only if the principal $G_{\mathbb{C}}$–bundle $E_G^{\mathbb{C}}$ over $X_{\mathbb{C}}$ is semistable.*

Proof. First assume E_G is not semistable. Let P be a proper parabolic subgroup of G with $E_P \subset E_G$ a reduction as in eqn. (2.1), and let χ be an anti–dominant character of P, such that

$$\mathrm{degree}(E_P(\chi)) < 0\,,$$

where $E_P(\chi)$ is the line bundle over X associated to the principal P–bundle E_P for the character χ. Let

$$E_P^{\mathbb{C}} := E_P \times_{\mathbb{R}} \mathbb{C} \subset E_G \times_{\mathbb{R}} \mathbb{C}$$

be the corresponding reduction of structure group of $E_G^{\mathbb{C}}$ to the proper parabolic subgroup $P_{\mathbb{C}} := P \times_{\mathbb{R}} \mathbb{C}$ of $G_{\mathbb{C}}$. Let $\chi_{\mathbb{C}}$ be the character of $P_{\mathbb{C}}$ given by χ. The line bundle over $X_{\mathbb{C}}$ associated to the principal $P_{\mathbb{C}}$–bundle $E_P^{\mathbb{C}}$ for the character $\chi_{\mathbb{C}}$ will be denoted by $E_P^{\mathbb{C}}(\chi_{\mathbb{C}})$. Since

$$\mathrm{degree}(E_P^{\mathbb{C}}(\chi_{\mathbb{C}})) = \mathrm{degree}(E_P(\chi))\,,$$

the pair $(E_P^{\mathbb{C}}, \chi_{\mathbb{C}})$ violates the semistability condition for $E_G^{\mathbb{C}}$. In other words, $E_G^{\mathbb{C}}$ is not semistable.

To prove the converse, assume that $E_G^{\mathbb{C}}$ is not semistable. Let

$$(2.4) \qquad\qquad E_Q \subset E_G^{\mathbb{C}}$$

be the Harder–Narasimhan reduction of $E_G^{\mathbb{C}}$; here $Q \subset G_{\mathbb{C}}$ is some proper parabolic subgroup determined uniquely by $E_G^{\mathbb{C}}$ up to an inner automorphism. Once a subgroup Q in the conjugacy class is fixed, the Harder–Narasimhan reduction E_Q is uniquely determined (see [AAB, p. 694, Theorem 1] for Harder–Narasimhan reduction).

The adjoint bundle $\mathrm{ad}(E_Q) \subset \mathrm{ad}(E_G^{\mathbb{C}})$ is one of the terms in the Harder–Narasimhan filtration of $\mathrm{ad}(E_G^{\mathbb{C}})$. More precisely, $\mathrm{ad}(E_Q)$ is the term in the Harder–Narasimhan filtration of $\mathrm{ad}(E_G^{\mathbb{C}})$ whose quotient by the previous term is of degree zero (see [AAB, p. 702, Lemma 6]). The Harder–Narasimhan filtration of $\mathrm{ad}(E_G^{\mathbb{C}})$ is the base change to $\mathbb{C}$ of the Harder–Narasimhan filtration of $\mathrm{ad}(E_G)$; this follows immediately from [La, p. 97, Proposition 3]. In particular, $\mathrm{ad}(E_Q)$ is the base change to $\mathbb{C}$ of a subbundle of $\mathrm{ad}(E_G)$.

Therefore, Q can be taken to be the base change to $\mathbb{C}$ of some parabolic subgroup of G. Once Q is the base change to $\mathbb{C}$ of a parabolic subgroup $P \subset G$, using the above mentioned fact that $\mathrm{ad}(E_Q)$ is the base change to $\mathbb{C}$ of a subbundle of $\mathrm{ad}(E_G)$ it follows that E_Q is the base change to $\mathbb{C}$ of a reduction of structure group

$$(2.5) \qquad\qquad\qquad E_P \subset E_G$$

of the principal G–bundle E_G to the subgroup P.

Let $\mathfrak{g}$ (respectively, $\mathfrak{p}$) be the Lie algebra of G (respectively, P). Let χ be the character of P defined by the one–dimensional P–module $\det \mathfrak{g}/\mathfrak{p} = \bigwedge^{\mathrm{top}} \mathfrak{g}/\mathfrak{p}$; the action of P is given by the adjoint action of P on $\mathfrak{g}$. We have

$$\mathrm{degree}(E_P(\chi)) = \mathrm{degree}(\mathrm{ad}(E_G)/\mathrm{ad}(E_P)) = \mathrm{degree}(\mathrm{ad}(E_G^{\mathbb{C}})/\mathrm{ad}(E_Q)) < 0 \,.$$

The last inequality follows from the fact that the subbundle $\mathrm{ad}(E_Q) \subset \mathrm{ad}(E_G)$ is the term in the Harder–Narasimhan filtration of $\mathrm{ad}(E_G)$ whose quotient by the previous term in the filtration has degree zero. Consequently, E_G is not semistable. This completes the proof of the lemma. $\qquad\square$

A principal $G_{\mathbb{C}}$–bundle $F_{G_{\mathbb{C}}}$ over $X_{\mathbb{C}}$ is semistable if and only if the adjoint vector bundle $\mathrm{ad}(F_{G_{\mathbb{C}}})$ is semistable [AAB, p. 698, Lemma 3]. Therefore, Lemma 2.2 has the following corollary.

Corollary 2.3. *A principal G–bundle E_G over X is semistable if and only if the adjoint vector bundle $\mathrm{ad}(E_G)$ is semistable.*

For a parabolic subgroup P of G, its Levi quotient will be denoted by $L(P)$. So $L(P)$ is the quotient of P by its unipotent radical.

Take a principal G–bundle E_G over X. Assume that E_G is not semistable. The reduction $E_P \subset E_G$ in eqn. (2.5), which is obtained from E_Q, evidently satisfies all the conditions of a Harder–Narasimhan reduction. Therefore, we have the following corollary:

Corollary 2.4. *Let E_G be a principal G–bundle over X which is not semistable. Then there is a proper parabolic subgroup $P \subset G$ and a reduction of structure group $E_P \subset E_G$ to P satisfying the following two conditions:*

(1) *The principal $L(P)$–bundle obtained by extending the structure group of E_P, using the projection of P to its Levi quotient $L(P)$, is semistable.*

(2) *For any nontrivial character χ of P which is trivial on the center of G, and which can be expressed as a nonnegative integral combination of simple roots, the line bundle $E_P(\chi)$ over X, associated to the principal P–bundle E_P for the character χ of P, has degree strictly greater than zero.*

The subgroup P is unique up to an inner automorphism, and the subbundle $\mathrm{ad}(E_P) \subset \mathrm{ad}(E_G)$ is canonical (it does not depend on the choice of P). Once P is fixed in the conjugacy class, the reduction E_P is uniquely determined.

Proof. Since the reduction E_Q in eqn. (2.4) satisfies the two conditions in Corollary 2.4 (see [AAB, p. 712, Theorem 6]), we conclude that Corollary 2.4 holds.   □

The reduction E_P in Corollary 2.4 will be called a *Harder–Narasimhan reduction* of E_G. See [AAB, p. 694, Theorem 1] for an equivalent formulation of the Harder–Narasimhan reduction.

3. REDUCTION TO TORUS OVER A NONDEGENERATE CONIC

Let X be a geometrically irreducible smooth projective curve of genus zero defined over $\mathbb{R}$. As mentioned in Section 1, either X is isomorphic to $\mathbb{P}^1_{\mathbb{R}}$ or it is isomorphic to the anisotropic conic in $\mathbb{P}^2_{\mathbb{R}}$ defined by the homogeneous polynomial $x^2 + y^2 + z^2$. As before, let G be a connected reductive linear algebraic group defined over $\mathbb{R}$.

Proposition 3.1. *Let E_G be a semistable principal G–bundle over X. Then E_G admits a reduction of structure group to a maximal torus of G.*

Proof. Let $E_G^{\mathbb{C}} := E_G \times_{\mathbb{R}} \mathbb{C}$ be the principal $G_{\mathbb{C}}$–bundle over $X_{\mathbb{C}}$ given by E_G (see eqn. (2.3)). From Lemma 2.2 it follows that $E_G^{\mathbb{C}}$ is semistable. Therefore, the adjoint vector bundle $\mathrm{ad}(E_G^{\mathbb{C}})$ is semistable [AAB, p. 698, Lemma 3].

Since $G_{\mathbb{C}}$ is reductive, its Lie algebra $\mathfrak{g}_{\mathbb{C}}$ as a $G_{\mathbb{C}}$–module is self–dual. Therefore, the vector bundle $\mathrm{ad}(E_G^{\mathbb{C}})$ is self–dual. In particular, we have

$$\mathrm{degree}(\mathrm{ad}(E_G^{\mathbb{C}})) = 0\,.$$

The complexification $X_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$. Any algebraic vector bundle over $\mathbb{P}^1_{\mathbb{C}}$ decomposes into a direct sum of line bundles [Gr, p. 122, Théorème 1.1]. Since $\mathrm{ad}(E_G^{\mathbb{C}})$ is semistable of degree zero, this implies that $\mathrm{ad}(E_G^{\mathbb{C}})$ is a trivial vector bundle.

Set

$$(3.1) \qquad\qquad \mathfrak{g}^0 := H^0(X_{\mathbb{C}}, \mathrm{ad}(E_G^{\mathbb{C}}))\,.$$

The Lie algebra structure of the fibers of $\mathrm{ad}(E_G^{\mathbb{C}})$ induce a Lie algebra structure on $\mathfrak{g}^0$. This Lie algebra is isomorphic to the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ because $\mathrm{ad}(E_G^{\mathbb{C}})$ is a trivial vector bundle with fibers isomorphic to $\mathfrak{g}_{\mathbb{C}}$.

Let σ be the self–map of $X_{\mathbb{C}}$ given by the action of the nontrivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Therefore, σ is an anti–holomorphic involution of $X_{\mathbb{C}}$. Let

$$(3.2) \qquad\qquad \widetilde{\sigma} : \mathrm{ad}(E_G^{\mathbb{C}}) \longrightarrow \sigma^* \overline{\mathrm{ad}(E_G^{\mathbb{C}})}$$

be the holomorphic isomorphism of vector bundles given by the action of the nontrivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Here $\overline{\mathrm{ad}(E_G^{\mathbb{C}})}$ is the smooth vector bundle over $X_{\mathbb{C}}$ whose fiber over any point $x \in X_{\mathbb{C}}$ is identified, as a real vector space, with the fiber $\mathrm{ad}(E_G^{\mathbb{C}})_x$, while this

identification is conjugate linear. The vector bundle $\sigma^*\overline{\mathrm{ad}(E_G^{\mathbb{C}})}$ has a natural holomorphic structure. We note that the composition

$$\mathrm{ad}(E_G^{\mathbb{C}}) \xrightarrow{\widetilde{\sigma}} \sigma^*\overline{\mathrm{ad}(E_G^{\mathbb{C}})} \xrightarrow{\sigma^*\overline{\widetilde{\sigma}}} \sigma^*(\overline{\sigma^*\overline{\mathrm{ad}(E_G^{\mathbb{C}})}}) = \mathrm{ad}(E_G^{\mathbb{C}})$$

is the identity automorphism of $\mathrm{ad}(E_G^{\mathbb{C}})$.

We have a conjugate linear involution

$$(3.3) \qquad \sigma^0 : \mathfrak{g}^0 = H^0(X_{\mathbb{C}}, \mathrm{ad}(E_G^{\mathbb{C}})) \longrightarrow \mathfrak{g}^0$$

defined by $\alpha \longmapsto \widetilde{\sigma}(\alpha)$, where $\mathfrak{g}^0$ is defined in eqn. (3.1) and $\widetilde{\sigma}$ is the isomorphism in eqn. (3.2). Note that using the natural $\mathbb{R}$–linear identification of $\mathrm{ad}(E_G^{\mathbb{C}})$ with $\overline{\mathrm{ad}(E_G^{\mathbb{C}})}$, the image $\widetilde{\sigma}(\alpha)$ is a smooth section of $\mathrm{ad}(E_G^{\mathbb{C}})$; this section is evidently holomorphic.

It is easy to see that σ^0 preserves the Lie algebra structure of $\mathfrak{g}^0$. The real Lie algebra given by the fixed point set

$$(3.4) \qquad (\mathfrak{g}^0)^{\sigma^0} \subset \mathfrak{g}^0$$

is isomorphic to the Lie algebra $\mathfrak{g}$ of G. Since σ^0 is a conjugate linear involution of $\mathfrak{g}^0$, the complexification $(\mathfrak{g}^0)^{\sigma^0} \bigotimes_{\mathbb{R}} \mathbb{C}$ is identified with $\mathfrak{g}^0$. In particular, the subset $(\mathfrak{g}^0)^{\sigma^0}$ is Zariski dense in $\mathfrak{g}^0$.

An element of $\mathfrak{g}^0$ is called semisimple if its adjoint action on $\mathfrak{g}^0$ is completely reducible. A semisimple element of $\mathfrak{g}^0$ is called regular if its centralizer is a Cartan subalgebra of $\mathfrak{g}^0$. The set of regular semisimple elements in $\mathfrak{g}^0$ is a Zariski open dense subset (see [Hu2, p. 28, Theorem 2.5]). We noted above that $(\mathfrak{g}^0)^{\sigma^0}$ in eqn. (3.4) is Zariski dense in $\mathfrak{g}^0$. Therefore, $(\mathfrak{g}^0)^{\sigma^0}$ contains some regular semisimple elements of $\mathfrak{g}^0$. Fix an element

$$(3.5) \qquad \omega \in (\mathfrak{g}^0)^{\sigma^0}$$

which is a regular semisimple element of $\mathfrak{g}^0$.

Since $\mathrm{ad}(E_G^{\mathbb{C}})$ is the complexification of the adjoint vector bundle $\mathrm{ad}(E_G)$, we have a canonical identification

$$(\mathfrak{g}^0)^{\sigma^0} \longrightarrow H^0(X, \mathrm{ad}(E_G)).$$

Let

$$(3.6) \qquad \omega_a \in H^0(X, \mathrm{ad}(E_G))$$

be the section corresponding to ω in eqn. (3.5). On the other hand, the two Lie algebras $(\mathfrak{g}^0)^{\sigma^0}$ and $\mathfrak{g}$ (the Lie algebra of G) are identified up to an inner conjugation. Therefore, ω gives a conjugacy class in $\mathfrak{g}$. Fix an element of the Lie algebra

$$(3.7) \qquad \omega_c \in \mathfrak{g}$$

in the conjugacy class given by ω. Let

$$(3.8) \qquad T_\omega \subset G$$

be the centralizer of ω_c for the adjoint action. Using the fact that the element $\omega \in \mathfrak{g}^0$ is regular semisimple it follows that T_ω is a maximal torus of G. More precisely, the

subgroup of a complex reductive group G' generated by the one parameter subgroup of it defined by a fixed regular semisimple element of the Lie algebra of G', together with the connected component, containing the identity element, of the center of G', is a maximal torus of G'. Since the centralizer of maximal torus T' in G' is T' itself [Hu1, p. 140, Corollary A], we conclude that T_ω is a maximal torus of G.

We recall that the adjoint vector bundle $\mathrm{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$. Let

$$q \,:\, E_G \times \mathfrak{g} \,\longrightarrow\, \mathrm{ad}(E_G)$$

be the quotient map. Let

$$p_1 \,:\, E_G \times \mathfrak{g} \,\longrightarrow\, E_G$$

be the projection to the first factor.

Finally, consider the projection

$$E_{T_\omega} \,:=\, p_1((E_G \times \{\omega_c\}) \cap q^{-1}(\omega_a(X)) \,\subset\, E_G\,,$$

where $\omega_a \,:\, X \,\longrightarrow\, \mathrm{ad}(E_G)$ is the section in eqn. (3.6), and ω_c is the element in eqn. (3.7). It is straight–forward to check that E_{T_ω} is a reduction of structure group of the principal G–bundle E_G to the subgroup T_ω in eqn. (3.8). This completes the proof of the proposition. $\square$

Let P be a parabolic subgroup of a connected reductive linear algebraic group G defined over $\mathbb{R}$. Consider the short exact sequence of groups

$$(3.9) \qquad\qquad e \,\longrightarrow\, R_u(P) \,\longrightarrow\, P \,\longrightarrow\, L(P) \,\longrightarrow\, e\,,$$

where $R_u(P)$ is the unipotent radical of P, and $L(P)$ is the Levi quotient of P. As a special case of a result of Mostow, this exact sequence is right split (see [Bo, p. 158, § 11.22]). In other words, P is a semidirect product $R_u(P) \rtimes L(P)$ of the unipotent radical $R_u(P)$ and the Levi quotient $L(P)$. Hence there is a subgroup of P that projects isomorphically onto $L(P)$. The result of Mostow also says that any two such subgroups of P are conjugate [Bo, p. 158, § 11.22]. Fix a subgroup of P that projects isomorphically onto $L(P)$. This subgroup will also be denoted by $L(P)$; this subgroup of P will be called the Levi subgroup of P.

Proposition 3.2. *Let E_P be a principal P–bundle over the curve X of genus zero that satisfies the following condition: There is a principal G–bundle over X whose Harder–Narasimhan reduction, described in Corollary 2.4, is E_P. Then E_P admits a reduction of structure group to the Levi subgroup $L(P) \subset P$.*

Proof. Let $\mathrm{Ad}(E_P) \,:=\, E_P \times_P P$ denote the adjoint bundle of E_P. We recall that $\mathrm{Ad}(E_P)$ is the fiber bundle over X associated to the principal P–bundle E_P for the adjoint action of P on itself. Therefore, $\mathrm{Ad}(E_P)$ is a quotient of $E_P \times P$, and it is a group–scheme over X. The Lie algebra bundle for $\mathrm{Ad}(E_P)$ is the adjoint vector bundle $\mathrm{ad}(E_P)$.

Let

(3.10) $$E_P(R_u(P)) := E_P \times_P R_u(P) \subset E_P \times_P P =: \mathrm{Ad}(E_P)$$

be the subgroup–scheme over X given by the P invariant subgroup $R_u(P) \subset P$. We note that $E_P(R_u(P))$ is associated to the principal P–bundle E_P for the adjoint action of P on $R_u(P)$.

The group–scheme $\mathrm{Ad}(E_P)$ has a natural action on the principal P–bundle E_P. Indeed, the map

$$(E_P \times P) \times E_P \longrightarrow P$$

defined by $((z,p), zp') \longmapsto zpp'$, where $z \in E_P$ and $p, p' \in P$, descends to a map

$$((E_P \times P)/P) \times E_P = \mathrm{Ad}(E_P) \times E_P \longrightarrow P$$

giving the action of $\mathrm{Ad}(E_P)$ on E_P. This action of $\mathrm{Ad}(E_P)$ on E_P clearly commutes with the right action of P on E_P.

The above action of $\mathrm{Ad}(E_P)$ on E_P gives an action of $\mathrm{Ad}(E_P)$ on any fiber bundle over X associated to E_P. To see this induced action, let Y be a variety on which P acts on the left, and let

$$E_P(Y) := E_P \times_P Y$$

be the fiber bundle over X associated to E_P for its action on Y. Then the map

$$\mathrm{Ad}(E_P) \times (E_P \times Y) \longrightarrow E_P \times Y$$

defined by $(g, (z,y)) \longmapsto (gz, y)$ descends to an action of $\mathrm{Ad}(E_P)$ on the quotient space $E_P(Y)$ of $E_P \times Y$.

Consider the quotient space

$$E_P(P/L(P)) := E_P/L(P)$$

which is a fiber bundle over X with $P/L(P)$ as the fiber; here $L(P)$ is the Levi sub-group of P. We note that $E_P(P/L(P))$ is the fiber bundle associated to E_P for the left–translation action of P on $P/L(P)$. Restricting the action of $\mathrm{Ad}(E_G)$ on $E_P(P/L(P))$ to the subgroup–scheme $E_P(R_u(P))$ (defined in eqn. (3.10)) we get an action of $E_P(R_u(P))$ on $E_P(P/L(P))$. Using the fact that the subgroup $L(P)$ of P projects isomorphically onto the quotient $P/R_u(P)$ (see eqn. (3.9)) it follows that $E_P(P/L(P))$ is a torsor for the group–scheme $E_P(R_u(P))$. In other words, the group–scheme $E_P(R_u(P))$ acts freely transitively on the fiber bundle $E_P(P/L(P))$.

Isomorphism classes of torsors for $E_P(R_u(P))$ are parametrized by $H^1(X, E_P(R_u(P)))$. Since E_P is Zariski locally trivial, we may use Zariski topology. Let

(3.11) $$\theta \in H^1(X, E_P(R_u(P)))$$

be the element corresponding to the above torsor $E_P(P/L(P))$.

The Lie algebra of the unipotent radical $R_u(P)$ of P will be denoted by $R_n(\mathfrak{p})$. So $R_n(\mathfrak{p})$ is the nilpotent radical of the Lie algebra of P. We note that the adjoint action P

on its own Lie algebra leaves the subalgebra $R_n(\mathfrak{p})$ invariant, just as the adjoint action of P on itself leaves the subgroup $R_u(P)$ invariant. Let $E_P(R_n(\mathfrak{p}))$ be the vector bundle over X associated to the principal P–bundle E_P for the P–module $R_n(\mathfrak{p})$. Since the adjoint action of P on $R_n(\mathfrak{p})$ preserves its Lie algebra structure, it follows that $E_P(R_n(\mathfrak{p}))$ is a bundle of Lie algebras over X. The Lie algebra bundle for the group–scheme $E_P(R_u(P))$ is evidently identified with $E_P(R_n(\mathfrak{p}))$.

Let

$$(3.12) \qquad 0 = E_{-\ell-1} \subset E_{-\ell} \subset E_{-\ell+1} \subset \cdots \subset E_{-2} \subset E_{-1} = E_P(R_n(\mathfrak{p}))$$

be the Harder–Narasimhan filtration of the vector bundle. We know that

$$(3.13) \qquad\qquad\qquad [E_{-j}, E_{-1}] \subset E_{-j-1}$$

for all $j \in [1, \ell]$; see [AAB, p. 699, (2)]. From eqn. (3.13) we conclude the following:

- Each E_{-j} is a bundle of ideals in the Lie algebra bundle $E_P(R_n(\mathfrak{p}))$.
- For each $j \in [1, \ell]$, the quotient Lie algebra bundle E_{-j}/E_{-j-1} is abelian.

Given a unipotent linear algebraic group U, if its Lie algebra $\mathfrak{u}$ is abelian, then the exponential map

$$\exp : \mathfrak{u} \longrightarrow U$$

is an isomorphism of algebraic groups. Hence the filtration in eqn. (3.12) gives a filtration of normal subgroup-schemes of $E_P(R_u(P))$

$$(3.14) \qquad e_X = \mathcal{G}_{-\ell-1} \subset \mathcal{G}_{-\ell} \subset \mathcal{G}_{-\ell+1} \subset \cdots \subset \mathcal{G}_{-2} \subset \mathcal{G}_{-1} = E_P(R_u(P))$$

such that

$$(3.15) \qquad\qquad\qquad \mathcal{G}_{-j}/\mathcal{G}_{-j-1} = E_{-j}/E_{-j-1}$$

as group–schemes for all $j \in [1, \ell]$; here e_X is the group–scheme over X for the trivial group.

For each $j \in [1, \ell]$, the quotient E_{-j}/E_{-j-1} is a semistable vector bundle over X of positive degree. (We recall that $\mathrm{ad}(E_P)$ is the term in the Harder–Narasimhan filtration of $\mathrm{ad}(E_G)$ whose quotient by the previous term is of degree zero.) Since $(E_{-j}/E_{-j-1})^* \bigotimes K_X$ is a semistable vector bundle of negative degree, where K_X is the canonical line bundle of X (its degree is -2), using Serre duality we have

$$H^1(X, E_{-j}/E_{-j-1}) = H^0(X, (E_{-j}/E_{-j-1})^* \otimes K_X)^* = 0.$$

Therefore, using eqn. (3.15) we have $H^1(X, \mathcal{G}_{-j}/\mathcal{G}_{-j-1}) = 0$ for all $j \in [1, \ell]$. Now using the filtration in eqn. (3.14) it follows that

$$H^1(X, E_P(R_u(P))) = 0.$$

To prove this note that since $H^1(X, \mathcal{G}_{-j}/\mathcal{G}_{-j-1}) = 0$, in order to show that

$$H^1(X, \mathcal{G}_{-j}) = 0$$

it suffices to prove that $H^1(X, \mathcal{G}_{-j-1}) = 0$. In particular, the element θ in eqn. (3.11) vanishes. Hence $E_P(P/L(P)) = E_P/L(P)$ is a trivial $E_P(R_u(P))$–torsor. In particular, the fiber bundle $E_P/L(P)$ over X admits a section.

Any section $\sigma : X \longrightarrow E_P/L(P)$ of the fiber bundle $E_P/L(P)$ gives a reduction of structure group of E_P to $L(P)$. Indeed, the inverse image of the subvariety $\sigma(X) \subset E_P/L(P)$ for the quotient map $E_P \longrightarrow E_P/L(P)$ is a reduction of structure group of E_P to $L(P)$. This completes the proof of the proposition. $\square$

Proposition 3.1 and Proposition 3.2 together have the following corollary.

Corollary 3.3. *Let E_G be a principal G–bundle over the curve X of genus zero, where G is a connected reductive linear algebraic group over $\mathbb{R}$. Then E_G admits a reduction of structure group to a maximal torus of G.*

Proof. If E_G is semistable, then it follows from Proposition 3.1.

Now assume that E_G is not semistable. Let E_P be the Harder–Narasimhan reduction of E_G (see Corollary 2.4). From Proposition 3.2 we know that E_P admits a reduction of structure group $E_{L(P)} \subset E_P$ to the Levi subgroup $L(P)$ of P. Hence E_G admits a reduction of structure group to $L(P)$. The principal $L(P)$–bundle obtained by extending the structure group of E_P using the projection of P to $L(P) = P/R_u(P)$ is semistable (see Corollary 2.4). Since the principal $L(P)$–bundle $E_{L(P)}$ is identified with this principal $L(P)$–bundle obtained by extending the structure group of E_P, we conclude that $E_{L(P)}$ is semistable.

Therefore, using Proposition 3.1 we now conclude that the principal G–bundle E_G admits a reduction of structure group to a maximal torus of the Levi subgroup $L(P)$. A maximal torus of $L(P)$ is also a maximal torus of G. This completes the proof of the corollary. $\square$

Consider the base change $X_{\mathbb{C}}$ in eqn. (2.2) of X to $\mathbb{C}$. Therefore, $X_{\mathbb{C}}$ is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

Definition 3.4. The natural projection

$$f : X_{\mathbb{C}} \longrightarrow X$$

defines a principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X. This principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X will be denoted by $F_{\mathbb{Z}_2}$.

We note that $F_{\mathbb{Z}_2}$ is a nontrivial principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X. To see this observe that $X_{\mathbb{C}}$ is irreducible.

Lemma 3.5. *Let Γ be a finite group and E_Γ a principal Γ–bundle over the real projective curve X of genus zero. Then there is a homomorphism*

$$\rho : \mathbb{Z}/2\mathbb{Z} \longrightarrow \Gamma$$

such that the principal Γ*–bundle* E_Γ *is isomorphic to the one obtained by extending the structure group of the principal* $\mathbb{Z}/2\mathbb{Z}$*–bundle* $F_{\mathbb{Z}_2}$ *in Definition 3.4 using* ρ.

Proof. Consider the short exact sequence of étale fundamental groups

$$e \longrightarrow \pi_1(X_{\mathbb{C}}, x) \longrightarrow \pi_1(X, x) \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z} \longrightarrow e \,,$$

where x is a point of $X_{\mathbb{C}}$ (see [Mu, p. 153, Theorem (8.1.1)]). Using it and the fact that $\pi_1(X_{\mathbb{C}}, x) = e$ it follows that $\pi_1(X, x) = \mathbb{Z}/2\mathbb{Z}$. We noted earlier that the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ over X in Definition 3.4 is nontrivial. Therefore, all principal bundles over X with a finite group as the structure group are obtained from $F_{\mathbb{Z}_2}$ by extension of structure group. This completes the proof of the lemma. $\square$

4. Reduction of a torus bundle over a conic

Let $T_s = \mathbb{G}_m$ be the split torus of dimension one defined over $\mathbb{R}$. The anisotropic torus of dimension one defined over $\mathbb{R}$ will be denoted by T_a (see [Bo, p. 121, § 8.16]). So the base change of T_a to $\mathbb{C}$ is isomorphic to $\mathbb{C}^*$, and the action of the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{C}^*$ is the involution defined by

$$(4.1) \qquad\qquad\qquad\qquad z \longmapsto 1/\overline{z} \,.$$

Let T_c denote the Weil restriction of the complex torus $\mathbb{C}^*$ to the subfield of real numbers. Therefore, T_c is a quotient of $T_s \times T_a$ by $\mathbb{Z}/2\mathbb{Z}$. It can be shown that $T_c \neq T_s \times T_a$. To prove this we note that any torus T' has a unique maximal anisotropic subtorus T'_a and a unique maximal split subtorus T'_d; the two subgroups T'_a and T'_d generate T', while $T'_a \bigcap T'_d$ is a finite group [Bo, p. 121, Proposition]. In the case of $T' = T_c$, we have $T'_a \bigcap T'_d = \mathbb{Z}/2\mathbb{Z}$. Hence $T_c \neq T_s \times T_a$.

The following lemma is well known (see [PR, p. 76, line 8] for proof):

Lemma 4.1. *Let* T *be an algebraic torus defined over* $\mathbb{R}$. *Then* T *is isomorphic to the Cartesian product* $(T_s)^{n_1} \times (T_a)^{n_2} \times (T_c)^{n_3}$ *for some nonnegative integers* n_1, n_2 *and* n_3.

As in Section 3, we will denote by X a geometrically irreducible smooth projective curve of genus zero defined over $\mathbb{R}$. We will construct a nontrivial principal T_a–bundle over X.

Let

$$(4.2) \qquad\qquad\qquad\qquad h_0 \,:\, \mathbb{Z}/2\mathbb{Z} \longrightarrow T_a$$

be the injective homomorphism whose image is $\{\pm 1\}$. Let F_{T_a} be the principal T_a–bundle over X obtained by extending the structure group of the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ using the homomorphism h_0 in eqn. (4.2); see Definition 3.4 for $F_{\mathbb{Z}_2}$.

Proposition 4.2. *The above defined principal* T_a*–bundle* F_{T_a} *over the genus zero curve* X *is nontrivial. Any nontrivial principal* T_a*–bundle over* X *is isomorphic to* F_{T_a}.

Proof. Take any principal T_a–bundle E_{T_a} over X. Let

$$E_{T_a}^{\mathbb{C}} := E_{T_a} \times_{\mathbb{R}} \mathbb{C}$$

be the base change to $\mathbb{C}$. Therefore, $E_{T_a}^{\mathbb{C}}$ is a principal

$$(T_a)_{\mathbb{C}} := T_a \times_{\mathbb{R}} \mathbb{C} = \mathbb{C}^*$$

bundle over $X_{\mathbb{C}} = X \times_{\mathbb{R}} \mathbb{C}$ (see eqn. (2.2)). Let

(4.3) $$\sigma : X_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$$

be the action of the nontrivial element in $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$. Therefore, σ is an anti–holomorphic involution of $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$.

Let L denote the algebraic line bundle over $X_{\mathbb{C}}$ defined by the principal $\mathbb{C}^*$–bundle $E_{T_a}^{\mathbb{C}}$. The dual line bundle of L will be denoted by L^*. Let $\overline{L^*}$ be the real analytic complex line bundle over $X_{\mathbb{C}}$ whose underlying real vector bundle of rank two is identified with that of L^*, while the complex structure of the fibers of $\overline{L^*}$ are conjugate to the complex structure of the fibers of L. In other words, the identification of L^* with $\overline{L^*}$ is fiberwise conjugate linear. The pull back $\sigma^* \overline{L^*}$ has a natural structure of a complex algebraic line bundle over $X_{\mathbb{C}}$. Since L is given by the complexification of a principal T_a–bundle over X, there is a holomorphic isomorphism of line bundles

(4.4) $$\eta : L \longrightarrow \sigma^* \overline{L^*}$$

satisfying a condition which we will describe. Let

$$\sigma^* \overline{\eta^*} : \sigma^* \overline{L^*} \longrightarrow \sigma^* \overline{(\sigma^* \overline{L^*})^*} = L$$

be the holomorphic isomorphism given by η. The condition on η says that the composition

(4.5) $$\sigma^* \overline{\eta^*} \circ \eta = \mathrm{Id}_L \, .$$

This condition follows from the fact that the Galois involution of the complexification $(T_a)_{\mathbb{C}}$ is the one in eqn. (4.1).

Conversely, any pair (L, η), where

- L is a holomorphic line bundle over $X_{\mathbb{C}}$, and
- $\eta : L \longrightarrow \sigma^* \overline{L^*}$ is a holomorphic isomorphism of line bundles satisfying the identity in eqn. (4.5),

define a principal T_a–bundle over X. Furthermore, isomorphisms between two principal T_a–bundles over X are parametrized by isomorphisms between the corresponding pairs.

Take any pair (L, η) satisfying the above conditions. For a topological complex vector bundle V over $X_{\mathbb{C}}$ we have

$$\mathrm{degree}(V^*) = \mathrm{degree}(\sigma^* V) = \mathrm{degree}(\overline{V}) \quad \text{and} \quad \mathrm{degree}(V^*) = -\mathrm{degree}(V) \, .$$

Hence $\mathrm{degree}(\sigma^* \overline{L^*}) = -\mathrm{degree}(L)$. Therefore, the existence of the isomorphism $\eta : L \longrightarrow \sigma^* \overline{L^*}$ implies that $\mathrm{degree}(L) = 0$. Thus L is a trivial line bundle over $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$.

Fix a trivialization of L. This trivialization gives a trivialization of the dual line bundle L^*, hence we also have a trivialization of $\sigma^*\overline{L^*}$. Using these trivializations of L and $\sigma^*\overline{L^*}$, the isomorphism η corresponds to multiplication by some nonzero complex number $\lambda \in \mathbb{C}^*$. It is straight–forward to check that the isomorphism $\sigma^*\overline{\eta^*}$ in eqn. (4.5) is given by multiplication with $1/\overline{\lambda}$ with respect to the trivializations. Therefore, from the given condition that η satisfies the identity in eqn. (4.5) we conclude that $\lambda \in \mathbb{R}^*$. By altering trivialization we see that $\lambda = \pm 1$.

The case of $\lambda = 1$ corresponds to the trivial principal T_a–bundle over X. Therefore, there is at most one nontrivial principal T_a–bundle over X up to an isomorphism.

Consider the short exact sequence of algebraic groups

$$(4.6) \qquad e \longrightarrow \mathbb{Z}/2\mathbb{Z} = \pm 1 \hookrightarrow T_a \xrightarrow{\psi} T_a \longrightarrow e \,,$$

where ψ is defined by $g \longmapsto g^2$. Take any principal T_a–bundle E_{T_a} over X. Let E'_{T_a} be the principal T_a–bundle over X obtained by extending the structure group of E_{T_a} using the homomorphism ψ in eqn. (4.6).

From our earlier observation that E_{T_a} corresponds to either $\lambda = 1$ or $\lambda = -1$ it follows immediately that E'_{T_a} corresponds to 1. Hence E'_{T_a} is a trivial principal T_a–bundle. Now using the short exact sequence in eqn. (4.6) we conclude that isomorphism classes of principal T_a–bundles over X are parametrized by isomorphism classes of principal $\mathbb{Z}/2\mathbb{Z}$–bundles over X; a principal $\mathbb{Z}/2\mathbb{Z}$–bundle gives a principal T_a–bundle by extending the structure group using the inclusion homomorphism in eqn. (4.6).

Since the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ in Definition 3.4 is nontrivial, we conclude that the principal T_a–bundle F_{T_a} over X given by it using the homomorphism h_0 is nontrivial. This completes the proof of the proposition. $\square$

Remark 4.3. A couple of remarks on the proof of Proposition 4.2:

- Let L denote the trivial complex line bundle over $X_\mathbb{C}$ equipped with a trivialization. So $\sigma^*\overline{L^*}$ is equipped with an induced trivialization. As we saw in the proof of Proposition 4.2, any isomorphism as in eqn. (4.4) satisfying the identity in eqn. (4.5) corresponds to multiplication by some $\lambda \in \{\pm 1\}$ with respect to the trivializations of L and $\sigma^*\overline{L^*}$. It is easy to see directly that the two pairs $(\mathcal{O}_{X_\mathbb{C}}, 1)$ and $(\mathcal{O}_{X_\mathbb{C}}, -1)$ are not isomorphic. Indeed, finding an isomorphism between them would amount to finding a $\mu \in \mathbb{C}^*$ such that $\mu = -1/\overline{\mu}$, which is impossible.
- It may also be pointed out that real algebraic line bundles over X are in natural bijective correspondence with pairs of the form (ζ, β), where ζ is a holomorphic line bundle over $X_\mathbb{C}$, and

$$\beta : \zeta \longrightarrow \sigma^*\overline{\zeta}$$

is a holomorphic isomorphism of line bundles, such that the composition

$$\zeta \xrightarrow{\beta} \sigma^*\overline{\zeta} \xrightarrow{\sigma^*\overline{\beta}} \sigma^*(\overline{\sigma^*\overline{\zeta}}) = \zeta$$

is the identity map of ζ.

Proposition 4.2 has the following corollary:

Corollary 4.4. *Any principal T_a–bundle over X admits a reduction of structure group to the subgroup $\{\pm 1\} \subset T_a$.*

Consider the split torus $T_s = \mathbb{G}_m$. Principal T_s–bundles over X are in bijective correspondence with the real algebraic line bundles over X. Given a principal T_s–bundle, the corresponding line bundle is the one associated to it by the standard action of $\mathbb{G}_m$ on $\mathbb{R}$. Conversely, given a line bundle ξ over X, the complement of the zero section in the total space of ξ is a principal T_s–bundle.

The following proposition is easy to prove (it follows from [BN, p. 1208, Theorem 1.1] in the case of anisotropic conic, and it follows from [BN, p. 1210, Proposition 3.1] in the case of projective line).

Proposition 4.5. *Assume that X is isomorphic to a nondegenerate anisotropic conic in $\mathbb{P}^2_{\mathbb{R}}$. The group of line bundles over X is isomorphic to $\mathbb{Z}$, and it is generated by the tangent bundle TX.*

The group of line bundles over $\mathbb{P}^1_{\mathbb{R}}$ is isomorphic to $\mathbb{Z}$, and it is generated by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)$.

We will now consider the principal T_c–bundles over X, where T_c is the two–dimensional torus defined earlier (see Lemma 4.1). First observe that there is a short exact sequence of algebraic groups

$$(4.7) \qquad e \longrightarrow T_s \hookrightarrow T_c \xrightarrow{\phi} T_a \longrightarrow e \,,$$

where the injective homomorphism is given by the inclusion of $\mathbb{R}^*$ in $\mathbb{C}^*$, and the projection ϕ is defined by $z \longmapsto z/\bar{z}$. Consider the subgroup

$$(4.8) \qquad \mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \subset T_a \,.$$

Let

$$(4.9) \qquad H_0 := \phi^{-1}(\mathbb{Z}/2\mathbb{Z}) \subset T_c$$

be the closed subgroup, where ϕ is the projection in eqn. (4.7). Therefore, from eqn. (4.7) we have the exact sequence

$$(4.10) \qquad e \longrightarrow T_s \hookrightarrow H_0 \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \longrightarrow e \,.$$

Lemma 4.6. *Any principal T_c–bundle over the genus zero curve X admits a reduction of structure group to the subgroup H_0 defined in eqn. (4.9).*

Let E_{T_c} be a principal T_c–bundle over X, and let $E^1_{H_0} \subset E_{T_c}$ and $E^2_{H_0} \subset E_{T_c}$ be reductions of structure group of E_{T_c} to the subgroup H_0. Then the two principal H_0–bundles $E^1_{H_0}$ and $E^2_{H_0}$ over X are isomorphic.

Proof. Let E_{T_c} be a principal T_c–bundle over X. Let E_{T_a} be the principal T_a–bundle obtained by extending the structure group of E_{T_c} using the projection ϕ in eqn. (4.7). From Corollary 4.4 we know that E_{T_a} admits a reduction of structure group to the subgroup $\mathbb{Z}/2\mathbb{Z}$ in eqn. (4.8).

Let $E_{\mathbb{Z}/2\mathbb{Z}} \subset E_{T_a}$ be a reduction of structure group of E_{T_a} to $\mathbb{Z}/2\mathbb{Z}$. Now the inverse image

$$q^{-1}(E_{\mathbb{Z}/2\mathbb{Z}}) \subset E_{T_c},$$

where $q : E_{T_c} \longrightarrow E_{T_a}$ is the natural projection, is a reduction of structure group of E_{T_c} to the subgroup $H_0 \subset T_c$.

We will now show that any two reductions of structure group of E_{T_c} to H_0 are isomorphic.

Let $E_{H_0}^1$ and $E_{H_0}^2$ be two reductions of structure group of E_{T_c} to H_0. For $i = 1, 2$, let $E_{\mathbb{Z}/2\mathbb{Z}}^i$ be the principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X obtained from $E_{H_0}^i$ by extending the structure group using the projection ϕ in eqn. (4.10). Therefore, $E_{\mathbb{Z}/2\mathbb{Z}}^i$ is a reduction of structure group of the earlier defined principal T_a–bundle E_{T_a} to the subgroup $\mathbb{Z}/2\mathbb{Z}$.

To prove that $E_{H_0}^1$ and $E_{H_0}^2$ are isomorphic it suffice to produce a real point $t \in T_a$ such that

(4.11)
$$E_{\mathbb{Z}/2\mathbb{Z}}^2 = E_{\mathbb{Z}/2\mathbb{Z}}^1 t \subset E_{T_a}.$$

Indeed, a lift $\widetilde{t} \in \phi^{-1}(t) \subset T_c$ of any t satisfying this condition, where ϕ is the projection in eqn. (4.7), has the property that

$$E_{H_0}^2 = E_{H_0}^1 \widetilde{t} \subset E_{T_c},$$

which means that the automorphism of the principal T_c–bundle E_{T_c} defined by multiplication with $\widetilde{t}$ is an isomorphism of $E_{H_0}^2$ with $E_{H_0}^1$. We note that the projection ϕ in eqn. (4.7) is surjective on the real points; therefore, any real point $t \in T_a$ can be lifted to a real point of T_c.

The principal T_a–bundle obtained by extending the structure group of E_{T_a} using the homomorphism ψ in eqn. (4.6) is trivial. Therefore, any reduction of structure group of the principal T_a–bundle E_{T_a} to the subgroup $\mathbb{Z}/2\mathbb{Z}$ is given by a morphism from X to the quotient space $T_a/(\mathbb{Z}/2\mathbb{Z}) \cong T_a$. We note that there are no nonconstant maps from the irreducible projective variety X to the affine variety $T_a/(\mathbb{Z}/2\mathbb{Z})$.

Since both $E_{\mathbb{Z}/2\mathbb{Z}}^1$ and $E_{\mathbb{Z}/2\mathbb{Z}}^2$ are reductions of structure group of E_{T_a} to $\mathbb{Z}/2\mathbb{Z}$, there is a real point $t_0 \in T_a/(\mathbb{Z}/2\mathbb{Z})$ such that

$$z_2 t_0 = z_1,$$

where $z_i \in T_a/(\mathbb{Z}/2\mathbb{Z})$, $i = 1, 2$, is the image of the constant function

$$X \longrightarrow T_a/(\mathbb{Z}/2\mathbb{Z})$$

that gives the reduction $E_{\mathbb{Z}/2\mathbb{Z}}^i$. We note that the homomorphism ψ in eqn. (4.6) is surjective on the real points. Therefore, there is a real point $t \in T_a$ such that $\psi(t) = t_0$.

It is now straight–forward to check that this t satisfies the condition in eqn. (4.11). This completes the proof of the lemma. $\square$

Proposition 4.7. *Let E_{T_c} be a principal T_c–bundle over the real projective line $\mathbb{P}^1_{\mathbb{R}}$. Then there is a principal T_s–bundle E_{T_s} over $\mathbb{P}^1_{\mathbb{R}}$ such that E_{T_c} is obtained from E_{T_s} by extending the structure group using the inclusion homomorphism in eqn. (4.7).*

Proof. Using Lemma 4.6 we know that there is a reduction of structure group

$$(4.12) \qquad\qquad E_{H_0} \subset E_{T_c}$$

to the subgroup H_0 defined in eqn. (4.9).

Let

$$(4.13) \qquad\qquad \widetilde{\varphi} : T_c \longrightarrow \mathrm{GL}(2, \mathbb{R})$$

be the embedding defined by

$$a + \sqrt{-1}b \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} .$$

Therefore, $\widetilde{\varphi}$ corresponds to multiplication of $\mathbb{C} = \mathbb{R}^2$ by $\mathbb{C}^*$. The restriction of $\widetilde{\varphi}$ to the subgroup $H_0 \subset T_c$ will be denoted by φ.

Consider the adjoint action of H_0 on the Lie algebra $\mathrm{M}(2, \mathbb{R})$ of $\mathrm{GL}(2, \mathbb{R})$ constructed using φ. The H_0–module $\mathrm{M}(2, \mathbb{R})$ decomposes into a direct sum of one–dimensional H_0–modules. Indeed, the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generate the four one–dimensional H_0–modules. The adjoint action of H_0 on $\mathrm{M}(2, \mathbb{R})$ factors through the quotient $\mathbb{Z}/2\mathbb{Z}$ in eqn. (4.10) because the subgroup $T_s \subset H_0$ acts trivially on $\mathrm{M}(2, \mathbb{R})$. Therefore, for any one–dimensional submodule L of $\mathrm{M}(2, \mathbb{R})$, the action of H_0 on $L \bigotimes L$ is trivial.

Consider the action of H_0 on $\mathbb{R}^2$ constructed using the homomorphism φ and the standard representation of $\mathrm{GL}(2, \mathbb{R})$. Let V denote the vector bundle over $\mathbb{P}^1_{\mathbb{R}}$ associated to the principal H_0–bundle E_{H_0} in eqn. (4.12) for this H_0–module $\mathbb{R}^2$. The vector bundle $\mathrm{End}(V)$ is identified with the vector bundle associated to the principal H_0–bundle E_{H_0} for the H_0–module $\mathrm{M}(2, \mathbb{R})$. Since $\mathrm{M}(2, \mathbb{R})$ decomposes into a direct sum of one–dimensional H_0–modules of order two, it follows that the vector bundle $\mathrm{End}(V)$ decomposes into a direct sum of line bundles of order two. Now we conclude that $\mathrm{End}(V)$ is a trivial vector bundle because any line bundle over $\mathbb{P}^1_{\mathbb{R}}$ of order two is trivial (see Proposition 4.5).

Since $\mathrm{End}(V)$ is a trivial vector bundle, using the second part of Proposition 4.5 it follows that

$$V \cong \mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(\ell)^{\oplus 2}$$

for some $\ell \in \mathbb{Z}$.

Let F_{T_s} be the principal T_s–bundle over $\mathbb{P}^1_{\mathbb{R}}$ given by the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(1)$. Let $F^\ell_{H_0}$ be the principal H_0–bundle over $\mathbb{P}^1_{\mathbb{R}}$ obtained by extending the structure group of F_{T_s} using the homomorphism

$$T_s \longrightarrow T_a \subset H_0$$

defined by $g \longmapsto g^\ell$ (see eqn. (4.10)). The proof of the proposition will be completed by showing that the two principal H_0–bundles E_{H_0} and $F^\ell_{H_0}$ are isomorphic.

For that purpose, consider the short exact sequence of algebraic groups

$$(4.14) \qquad\qquad e \longrightarrow \mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \longrightarrow H_0 \overset{\nu}{\longrightarrow} T_s \longrightarrow e \,,$$

where ν is the norm map defined by $z \longmapsto z\overline{z}$. Both the principal H_0–bundles E_{H_0} and $F^\ell_{H_0}$ have the property that the principal T_s–bundle obtained by extending the structure group using the homomorphism ν in eqn. (4.14) coincides with the principal T_s–bundle over $\mathbb{P}^1_{\mathbb{R}}$ given by the line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{R}}}(2\ell)$. Therefore, to prove that E_{H_0} and $F^\ell_{H_0}$ are isomorphic it suffices to show that for any principal $\mathbb{Z}/2\mathbb{Z}$–bundle $E'_{\mathbb{Z}/2\mathbb{Z}}$ over $\mathbb{P}^1_{\mathbb{R}}$, the principal H_0–bundle E'_{H_0} obtained by extending its structure group using the homomorphism in eqn. (4.14) is trivial.

The inclusion $\mathbb{Z}/2\mathbb{Z} \hookrightarrow H_0$ in eqn. (4.14) factors through the subgroup $T_s \subset H_0$ in eqn. (4.10). Therefore, to prove that E'_{H_0} is a trivial it is enough to show that that the principal T_s–bundle E'_{T_s} obtained by extending the structure group of the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $E'_{\mathbb{Z}/2\mathbb{Z}}$ is trivial.

Any finite order line bundle over $\mathbb{P}^1_{\mathbb{R}}$ is trivial (see Proposition 4.5). In particular, the order two line bundle given by the principal T_s–bundle E'_{T_s} is trivial. Hence the principal H_0–bundle E'_{H_0} is trivial. This completes the proof of the proposition. $\qquad\square$

Let X be a nondegenerate anisotropic conic in $\mathbb{P}^2_{\mathbb{R}}$. Before classifying principal T_c–bundles over X, we will construct a certain principal T_c–bundles over X.

Consider the ample generator of $\mathrm{Pic}(X_{\mathbb{C}})$ (see eqn. (2.2)). This is the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ over $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$. Let $E^0_{\mathbb{C}^*}$ be the principal $\mathbb{C}^*$–bundle over $X_{\mathbb{C}}$ given by this line bundle. The Weil restriction of $E^0_{\mathbb{C}^*}$ to the subfield $\mathbb{R} \subset \mathbb{C}$ is a principal T_c–bundle over the Weil restriction $\widetilde{X}$ of $X_{\mathbb{C}}$. (Since T_c is the Weil restriction of $\mathbb{C}^*$, it follows immediately that the Weil restriction of $E^0_{\mathbb{C}^*}$ is a principal T_c–bundle.) This principal T_c–bundle over $\widetilde{X}$ will be denoted by $E^0_{T_c}$

Definition 4.8. Let F_{T_c} be the principal T_c–bundle over X obtained by pulling back the above principal T_c–bundle $E^0_{T_c}$ over the Weil restriction $\widetilde{X}$ using the canonical inclusion map

$$(4.15) \qquad\qquad\qquad\qquad X \hookrightarrow \widetilde{X} \,.$$

Fix a reduction of structure group

$$F_{H_0} \subset F_{T_c}$$

to the subgroup H_0 of T_c. (From Lemma 4.6 we know that such a reduction of structure group exists, and any two reductions are isomorphic.)

Proposition 4.9. *Any real algebraic line bundle over the nondegenerate anisotropic conic X is obtained by extending the structure group of the principal H_0–bundle F_{H_0} (see Definition 4.8) using some character of H_0.*

The principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X obtained by extending the structure group of F_{H_0} using the projection ϕ in eqn. (4.10) is isomorphic to the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ in Definition 3.4.

Proof. Let $F_{\mathrm{GL}(2,\mathbb{R})}$ be the principal $\mathrm{GL}(2,\mathbb{R})$–bundle over X obtained by extending the structure group of the principal T_c–bundle F_{T_c} (see Definition 4.8) using the homomorphism $\widetilde{\varphi}$ in eqn. (4.13). Let V denote the vector bundle of rank two over X associated to $F_{\mathrm{GL}(2,\mathbb{R})}$ for the standard representation of $\mathrm{GL}(2,\mathbb{R})$. Let

$$V_{\mathbb{C}} \, := \, V \otimes_{\mathbb{R}} \mathbb{C}$$

be the vector bundle of rank two over the complexification $X_{\mathbb{C}}$ obtained from V by changing the base field to $\mathbb{C}$. From the construction of F_{T_c} it follows immediately that the vector bundle $V_{\mathbb{C}}$ is isomorphic to the direct sum $\zeta \bigoplus \sigma^* \overline{\zeta}$ on $X_{\mathbb{C}}$, where ζ is the ample generator of $\mathrm{Pic}(X_{\mathbb{C}})$, and σ is the anti–holomorphic involution of $X_{\mathbb{C}}$ given by the action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on the complexification $X_{\mathbb{C}}$.

Since $\mathrm{degree}(\sigma^* \overline{\zeta}) = \mathrm{degree}(\zeta)$, and $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}$, we conclude that

$$\zeta \oplus \sigma^* \overline{\zeta} \, = \, \zeta \oplus \zeta \, = \, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1) \oplus \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$$

on $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$. Therefore,

(4.16)
$$V_{\mathbb{C}} \, = \, \mathcal{O}_{X_{\mathbb{C}}}(1) \oplus \mathcal{O}_{X_{\mathbb{C}}}(1)$$

on $X_{\mathbb{C}} = \mathbb{P}^1_{\mathbb{C}}$.

We will show that the vector bundle V on X is indecomposable. To prove this, assume that V decomposes as

(4.17)
$$V \, = \, \xi_1 \oplus \xi_2$$

into a direct sum of line bundles. We know that both ξ_1 and ξ_2 are tensor powers of the tangent bundle TX (see Proposition 4.5). In particular, both $\mathrm{degree}(\xi_1)$ and $\mathrm{degree}(\xi_2)$ are even integers. For $i = 1, 2$, let

$$(\xi_i)_{\mathbb{C}} \, := \, \xi_i \otimes_{\mathbb{R}} \mathbb{C}$$

be the complex algebraic line bundle over $X_{\mathbb{C}}$ given by ξ_i. The decomposition in eqn. (4.17) gives a decomposition

$$V_{\mathbb{C}} \, = \, (\xi_1)_{\mathbb{C}} \oplus (\xi_2)_{\mathbb{C}} \,.$$

Comparing this decomposition with the one in eqn. (4.16), and using the Atiyah–Krull–Schmidt theorem on the uniqueness of the decomposition type of a vector bundle (see [At,

p. 315, Theorem 3]), we conclude that

$$\mathrm{degree}((\xi_i) \,=\, \mathrm{degree}((\xi_i)_{\mathbb{C}}) \,=\, 1\,.$$

This contradicts the earlier observation that the degree of ξ_i is even. Therefore, we now conclude that the vector bundle V is indecomposable.

Since V is indecomposable, it can be shown that the principal $\mathrm{GL}(2,\mathbb{R})$–bundle $F_{\mathrm{GL}(2,\mathbb{R})}$ does not admit any reduction of structure group to $T_s \,=\, \mathbb{G}_m$. Indeed, any finite dimensional $\mathbb{G}_m$–module splits into a direct sum of one–dimensional modules, hence V will decompose into a direct sum of line bundles if it is associated to a principal T_s–bundle. Therefore, the principal H_0–bundle F_{H_0} does not admit any reduction of structure group to the subgroup $T_s \,\subset\, H_0$ in eqn. (4.10). This implies that the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{H_0}(\mathbb{Z}/2\mathbb{Z})$ over X obtained by extending the structure group of F_{H_0} using the projection in eqn. (4.10) is nontrivial. On the other hand, the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{\mathbb{Z}_2}$ (see Definition 3.4) is the unique nontrivial principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X up to an isomorphism; this follows from Lemma 3.5. Therefore, the principal $\mathbb{Z}/2\mathbb{Z}$–bundle $F_{H_0}(\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $F_{\mathbb{Z}_2}$.

We will now show that all the line bundles over X are associated to F_{H_0}. Since the group of all line bundles over X is generated by the tangent bundle (see Proposition 4.5), it suffices to show that the tangent bundle TX is associated to F_{H_0} by some character of H_0.

Consider the character χ of H_0 defined by $g \,\longmapsto\, \det \widetilde{\varphi}(g) \,\in\, \mathbb{G}_m$, where $\widetilde{\varphi}$ is the homomorphism defined in eqn. (4.13). This character coincides with the homomorphism ν in eqn. (4.14). It is easy to see that the line bundle over X associated to F_{H_0} for this character χ has degree two. Therefore, this associated line bundle is isomorphic to TX. This completes the proof of the proposition. $\qquad\square$

Remark 4.10. The vector bundle $V_{\mathbb{R}}$ in [BN, p. 1213] is isomorphic to the vector bundle V^* in the proof of Proposition 4.9. This follows immediately from the classification of vector bundles on X given in [BN, p. 1208, Theorem 1.1].

Proposition 4.11. *Let E_{T_c} be a principal T_c–bundle over the nondegenerate anisotropic conic X. Then there is a homomorphism*

$$\beta \,:\, H_0 \,\longrightarrow\, T_c$$

with the following property: The principal T_c–bundle over X obtained by extending the structure group of the principal H_0–bundle F_{H_0} (see Definition 4.8) using β is isomorphic to E_{T_c}.

Proof. Fix a reduction of structure group

$$E_{H_0} \,\subset\, E_{T_c}$$

to the subgroup $H_0 \subset T_c$ (see Lemma 4.6). To prove the proposition it suffices to construct a homomorphism

$$(4.18) \qquad\qquad h \,:\, H_0 \,\longrightarrow\, H_0$$

such that the principal H_0–bundle E_{H_0} is isomorphic to the one obtained by extending the structure group of the principal H_0–bundle F_{H_0} using the homomorphism h.

Let $E_{\mathbb{Z}/2\mathbb{Z}}$ be the principal $\mathbb{Z}/2\mathbb{Z}$–bundle over X obtained by extending the structure group of E_{H_0} using the projection ϕ in eqn. (4.10). We first assume that $E_{\mathbb{Z}/2\mathbb{Z}}$ is a trivial principal $\mathbb{Z}/2\mathbb{Z}$–bundle. Then using the short exact sequence in eqn. (4.10) it follows that E_{H_0} admits a reduction of structure group to the subgroup $T_s \subset H_0$. From the first part of Proposition 4.9 it follows immediately that there is homomorphism h as in eqn. (4.18) which factors through the subgroup T_s and satisfies the condition that E_{H_0} is isomorphic to the principal H_0–bundle obtained by extending the structure group of F_{H_0} using h.

Now assume that the above principal $\mathbb{Z}/2\mathbb{Z}$–bundle $E_{\mathbb{Z}/2\mathbb{Z}}$ is nontrivial. We note that $F_{\mathbb{Z}_2}$ is the unique nontrivial principal $\mathbb{Z}/2\mathbb{Z}$–bundle on X (see Lemma 3.5), and furthermore, $F_{\mathbb{Z}_2}$ is the extension of structure group of F_{H_0} by ϕ (see the second part of Proposition 4.9). Therefore, using the exact sequence in eqn. (4.10) we conclude that the two principal H_0–bundles E_{H_0} and F_{H_0} differ by a principal T_s–bundle. This means that there is a principal T_s–bundle E'_{T_s} over X such that the extension of structure group of the principal $T_s \times H_0$–bundle $E'_{T_s} \times_X F_{H_0}$ over X using the multiplication homomorphism

$$T_s \times H_0 \,\longrightarrow\, H_0$$

is isomorphic to E_{H_0}. Now the proof of the proposition is completed using the first part of Proposition 4.9. $\qquad\square$

Theorem 4.12. *Let G be any connected reductive linear algebraic group defined over $\mathbb{R}$.*

Let E_G be a principal G–bundle over a nondegenerate anisotropic conic X. Then there is a homomorphism

$$\rho \,:\, H_0 \,\longrightarrow\, G$$

such that E_G is isomorphic to the principal G–bundle obtained by extending the structure group of the principal H_0–bundle F_{H_0} using ρ (see Definition 4.8 for F_{H_0}).

Let $F_{\mathbb{G}_m}$ denote the principal $\mathbb{G}_m$–bundle over $\mathbb{P}^1_\mathbb{R}$ corresponding to the tautological line bundle $\mathcal{O}_{\mathbb{P}^1_\mathbb{R}}(1)$. Let $F_{\mathbb{G}_m} \times_X F_{\mathbb{Z}_2}$ be the principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle over X given by the fiber product (see Definition 3.4 for $F_{\mathbb{Z}_2}$). Given any principal G–bundle E_G over $\mathbb{P}^1_\mathbb{R}$, there is a homomorphism

$$\rho \,:\, \mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z}) \,\longrightarrow\, G$$

such that E_G is isomorphic to the principal G–bundle obtained by extending the structure group of the principal $\mathbb{G}_m \times (\mathbb{Z}/2\mathbb{Z})$–bundle $F_{\mathbb{G}_m} \times_X F_{\mathbb{Z}_2}$ using ρ.

Proof. In view of Corollary 3.3, we may assume that G is a torus.

In view of Lemma 4.1, it suffices to treat the three cases T_s, T_a and T_c.

Using the second part of Proposition 4.5 and the first part of Proposition 4.9 it follows that the theorem is valid for $G = T_s$. The second part of Proposition 4.9 implies that the theorem is valid for $G = T_a$.

For the anisotropic conic X, from Proposition 4.11 it follows that the statement in the theorem is valid for $G = T_c$. For $\mathbb{P}^1_{\mathbb{R}}$, the case of T_c follows from Proposition 4.7 combined with the fact that the theorem is valid for T_s. This completes the proof of the theorem. $\square$

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