ROTATING ELLIPTIC ANALYSERS FOR THE AUTOMATIC ANALYSIS OF POLARISED LIGHT—PART I

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§ 1. INTRODUCTION

IN recent months the present authors have been interesting themselves in the study of optical phenomena associated with thin films and crystals of anisotropic metals. For the experimental investigation of these it became necessary to fabricate an automatic analyser for the analysis of polarised light. After considering some possible methods purely on an *ad hoc* basis, it became evident that a careful theoretical analysis of this problem using the concept of Poincaré sphere (1892) would prove fruitful if one has to design and construct a proper automatic analyser. When the various methods of analysing polarised light were deduced systematically using the Poincaré sphere concept it was found that some of these were already in existence while others were quite new. Even amongst these one saw that while a few were elegant from the point of view of theoretical analysis, they were hard to attain in practice. Some, on the other hand, could be made in a fairly well-equipped laboratory. It was felt worthwhile recording this theoretical approach as it proved quite illuminating to the authors from the point of view of instrumental design. It is proposed to present the practical construction of some of these analysers and the results of experimental study using these in later papers.

§ 2. THE STATEMENT OF THE PROBLEM

In the Poincaré representation (Fig. 1), a completely polarised beam is represented as a point on the surface of a sphere of unit radius. All

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possible states of polarisation can be indicated on this sphere. After Poincaré gave this elegant concept many new theorems have been deduced by several authors (Poincaré, 1892; Pancharatnam, 1956 *a*). For extensive references

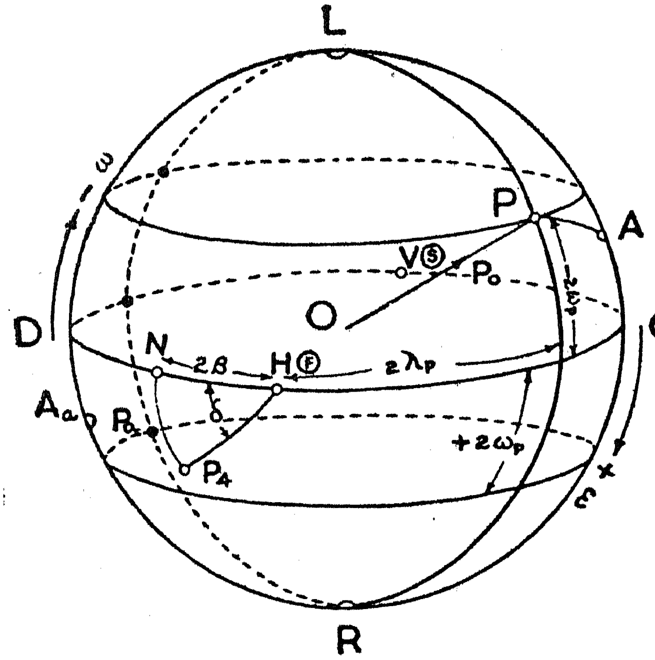

FIG. 1. The Poincaré sphere: H, C′, V, D and N denote, respectively, linearly polarised light at an azimuth 0°, 45°, 90°, —45° and (180°—β); L, R—left and right circularly polarised light; P—an elliptic vibration of azimuth λ_p and ellipticity — ω_p; P_a the state orthogonal to P; A an analyser completely transmitting the light in state A; A_a an analyser completely blocking the light in state A.

see Ramachandran and Ramaseshan, 1961. Some of the theorems which will be used in this paper are given in Appendix I.

An elliptically polarised light can be considered to be the most general state of polarisation; circular and linear states being particular cases. It is characterised by an azimuth (λ), ellipticity (ω) and the sense of rotation is given by the sign of (ω). The azimuth denotes the orientation of the major axis (*i.e.*, the angle it makes with a reference direction taken usually as the horizontal, H) and tan ω stands for the ratio of semi-minor to semi-major axis. Complete analysis of polarised light, therefore, means the determination of λ, $|\omega|$ and sign of ω. From the point of view of Poincaré representation, analysis of the state of polarization is simply finding the location of the point representing the state on the Poinçaré sphere. The point denoting the state will move on the Poincaré sphere if the state of polarisation of light changes with time. The analysis of such light beams requires continuous determination of the states of polarisation.

§ 3. BASIC PRINCIPLES OF ANALYSIS

In this section we shall discuss the various methods that may be adopted to locate the point P on the Poincaré sphere. The light pencil, which is being analysed, will be denoted by P and we shall assume, unless otherwise stated, that it is completely polarised. Its orthogonal state will be represented by P_a. The intensity, azimuth and ellipticity of P will be denoted, respectively, by I, λ_P and ω_P. I_A will stand for the intensity transmitted by analyser A.

(a) *Locating P by its Stokes parameters*

P is completely specified by the projected value of OP (Fig. 2) along three mutually perpendicular axes OX, OY and OZ which are the Stokes parameters M, C and S. Hence the experimental evaluation of M, C and S complete the determination of the position of P. This can be accomplished by measuring the intensities transmitted by the following analysers: a linear analyser set at angles 0, 45°, − 45°, 90°; a right circular analyser and a left circular analyser. If the measured intensities are, respectively, denoted by I_0, $I_{45°}$, $I_{-45°}$, $I_{90°}$, I_R and I_L then the Stokes parameters are calculated from

$$I = I_0 + I_{90°} = I_{45°} + I_{-45°} = I_L + I_R \tag{1}$$

$$M = I_0 - I_{90°} \tag{2}$$

$$C = I_{45°} - I_{-45°} = 2I_{45°} - I \tag{3}$$

$$S = I_L - I_R = 2I_L - I \tag{4}$$

λ_P and ω_P are obtained from

$$\tan 2\lambda_P = \frac{C}{M} \tag{5}$$

$$\sin 2\omega_P = \frac{S}{I} \,. \tag{6}$$

The Stokes parameters for characterising states of polarisation has found extensive application in theoretical studies. Unfortunately, this elegant idea does not appear to have been exploited much for the experimental determination. This is possibly because of the fact that the determination of Stokes parameters involves the measurement of absolute intensities which was troublesome. With the recent progress in the techniques of accurate measurement of absolute intensity, the main experimental hurdle seems to have been removed. Even preliminary experiments show that very accurate determination of the state of polarisation could be made with this technique.

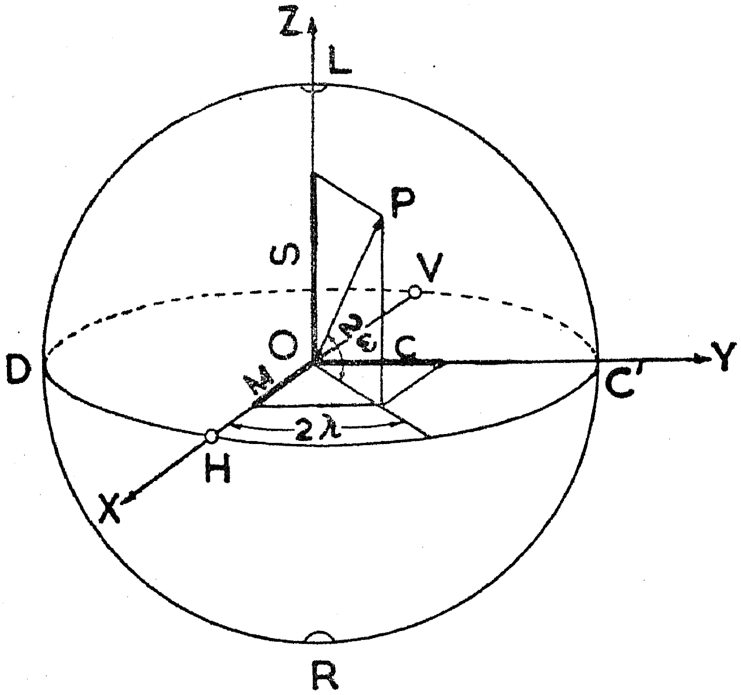

FIG. 2. The representation of the Stokes parameters of an arbitrary state P.

Simple methods for automatic recording of Stokes parameters and the accuracy attainable by this method will be discussed in Part II.

(*b*) *Determination of the state of polarisation by measuring fraction of the intensity transmitted by three analysers*

P can be located on the Poincaré sphere by measuring, on the sphere, the distance between P and three other points whose positions are known. If the position of a point A_1 and the distance $\widehat{PA_1}$ are known, then P lies on the small circle whose centre is at A_1 and radius is $\widehat{PA_1}$ (Fig. 3). If the distance $\widehat{PA_2}$ of P from a second known point A_2 is also determined, then the position of P is narrowed down to one of the two points of intersection, P and P′. The ambiguity in position can be resolved by measuring a third distance $\widehat{PA_3}$ of P from a point A_3. An important condition to be satisfied for getting only one common point of intersection for the circles is that the point A_3 should not lie on the same great circles as points A_1 and A_2. This method is very similar to the resolution of the phase problem in X-ray crystallography.

The distance $\widehat{PA_1}$ between two points (say P and A_1) on the Poincaré sphere can be experimentally found by measuring the fraction (I_{A_1}/I) of the intensity of light beam in state P transmitted by an analyser A_1 because,

$$\tfrac{1}{2}\widehat{PA_1} = \cos^{-1}\left(\sqrt{\frac{I_{A_1}}{I}}\right) \qquad (7)$$

(*see* Appendix I, Theorem 4 *b*). Therefore, by measuring the fraction transmitted through analyers A_1, A_2 and A_3, P can be located.

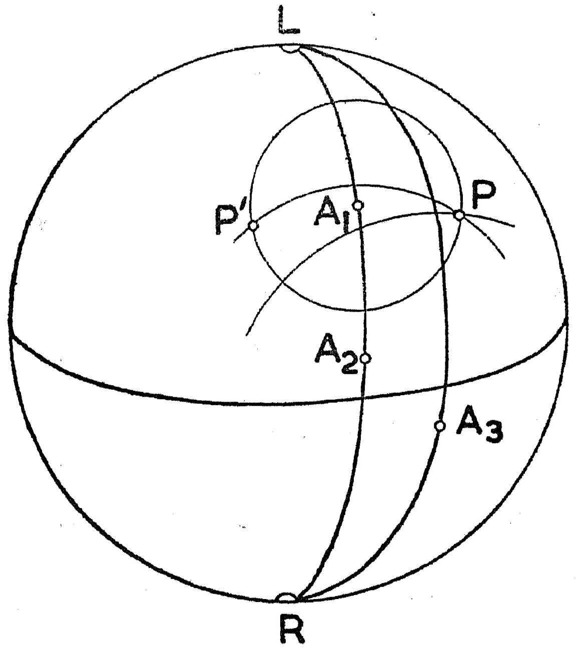

FIG. 3. Location of P by measuring distance of P from any three points A_1, A_2 and A_3 whose positions are known. For the unique determination of P the three points should not lie on the same great circle.

Two special cases can be thought of wherein the position of P is obtained by measuring the distance of P from two known points. They are:

(*a*) The known points A_1 and A_2 and the point P all lie on the same great circle, then the position of P is the point of tangential contact between the small circles with centres at A_1 and A_2 and radii $\widehat{PA_1}$ and $\widehat{PA_2}$ (Fig. 4).

(*b*) If the known positions are represented by points N_1 and N_2 on the equator then also the position of P is obtained by knowing two distances, *viz.*, $\widehat{PN_1}$ and $\widehat{PN_2}$ because the two points of intersection P and P′ of the small circles have the same λ_P and $|\omega_P|$. The sign of ω is different for P and P′.

For many applications only $|\omega|$ is required and therefore this method could be useful in such cases. This procedure is similar to the double-field analyser method of determining azimuth of polarised light (Ramachandran and Ramaseshan, 1961, page 37).

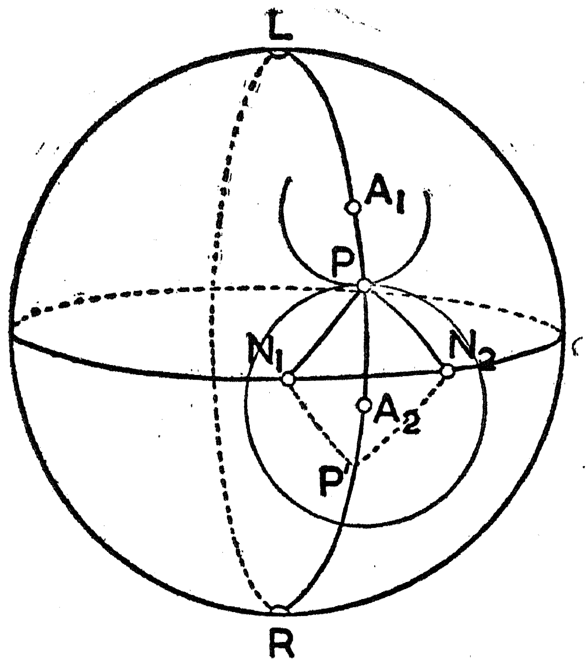

FIG. 4. Location of P by measuring two transmitted intensities.

(*c*) *Determination of the position of P by converting the elliptic vibration into a known state*

Suppose P is a pole of the great circle GEKF (Fig. 5) then the point P is 90° away from any point on this great circle. If an analyser is taken along this great circle then there would be no variation in the transmitted intensity as the position of the analyser is changed. The great circle GEKF is completely defined by the longitude of the point E and the angle of inclination, δ. Once these are found the position of P is determined.

P can be considered as the result of introducing a phase difference δ, between the E and F components of a linear vibration whose azimuth is equal to $\frac{1}{2}\widehat{PE}$, *i.e.*, 45° since any point on the great circle GEKF is 90° removed from P. Therefore, the situation just described can be realised by setting the axes of a variable birefringence element at $\pm \pi/4$ to the axes of the ellipse and introducing the right amount of retardation, δ. This setting converts P into circular state. This principle was first used by Kent and Lawson (1937).

Alternately, the incident light may be separated into two orthogonal linear vibrations. This is accomplished by setting the axes of a Wollaston double-image prism parallel to those of the elliptic vibration. The orientation of the ellipse is thus given by that of the axes of the double image prism and the ratio of the two intensities transmitted by the prism gives $\tan^2\omega$. Archard *et al.* (1952) used this principle for the construction of semi-automatic instrument.

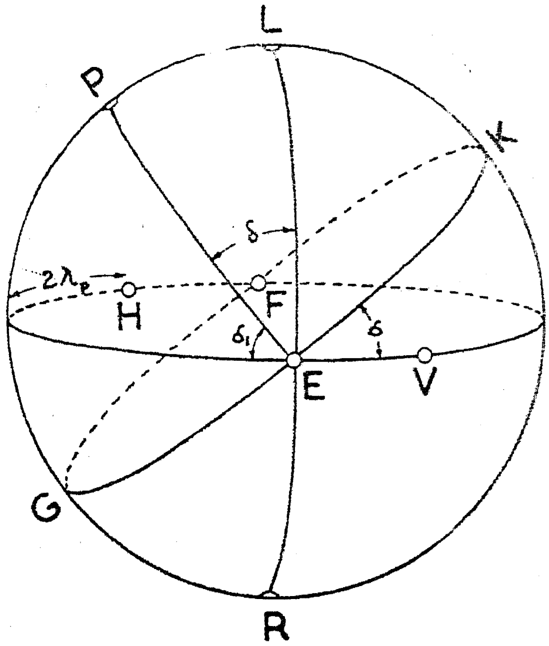

FIG. 5. Locating P by finding the great circle for which P is a pole.

(*d*) *Null method for the location of P*

In this method the state of polarisation of an elliptic analyser is changed till it becomes orthogonal to the incident elliptic vibration at which position the analyser transmits zero intensity. The intensity transmitted through the analyser is used as a guide to bring the analyser to the state P. This method has been widely used for the construction of instruments for the analysis of polarised light.

(*e*) *Location of P by interference experiments*

Pancharatnam has suggested (1956 *b*)—while developing the generalised theory of interference (1956 *a*) of two elliptic vibrations, 1 and 2, in different states P_1 and P_2—an interference method of determining distances on the Poincaré sphere. This theory leads to the result

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos c \cos \delta \tag{8}$$

where I, I_1 and I_2 are, respectively, the intensities of the resultant, beam 1 and beam 2; C stands for $\frac{1}{2}\widehat{P_1P_2}$ and δ is the general phase difference between 1 and 2. He has pointed out that cos c is equal to visibility of fringes (as defined by Michelson) formed under the condition $I_1 = I_2$ and can be determined by measuring visibility V of fringes which is given by

$$V = \frac{I_{Max.} - I_{Min.}}{I_{Max.} + I_{Min.}}. \tag{9}$$

Therefore, by interfering the incident light with a coherent beam in state A and determining V, we can find the distance of separation between P and A on the Poincaré sphere. If the experiment is repeated with analysers in two other states of polarisation, we would get the distance of P from three known points which leads to the location of P. This method does not offer any particular advantage over the methods which depend upon the measurement of absolute intensity.

On the other hand, it would be worthwhile to develop a method of analysis based on the idea that the orthogonal states do not interfere and would therefore give zero visibility. This could be done by taking two coherent pencils of monochromatic light and passing one of them through the system and the other through an elliptic analyser and then combining the two beams. The elliptic analyser is changed using any one of the techniques described in Section 5 till the field is uniformly bright. When this happens the state of the analyser is antipodal to that of the incident light. The advantage of this method is that it does not require the measurement of absolute intensity.

§ 4. ANALYSIS OF PARTIALLY POLARISED LIGHT

We shall briefly indicate in this section the suitability of some of the methods for the analysis of partially polarised light, which is not cut out by any analyser in any position; the intensity transmitted by an orthogonal analyser is minimum and not zero. Null methods are, therefore, inapplicable.

The method based on the Stokes parameters (§ 3 *a*) can be used. The Stokes parameters (I, M, C, S) of a partially polarised light pencil is also given by equations 1 to 4. The essential difference between partially polarised light and completely polarised light being that in the case of partially polarised light M, C and S represent, respectively, the components of the polarised portion of the incident beam (*i.e.*, components of the Stokes vector) along the directions OX, OY and OZ and consequently,

$$\sqrt{M^2 + C^2 + S^2} < I \tag{10}$$

Therefore, it is not redundant, in the case of partially polarised light, to specify I in addition to M, C and S; though it is so in the case of completely polarised pencils.

Once the parameters I, M, C and S are known, the azimuth of the incident vibration is obtained from formula 5 and ω_p is calculated from the formula

$$\sin 2\omega_p = \frac{S}{Ip} \tag{11}$$

where Ip denotes the magnitude of the Stokes vector of the incident light beam. p is the degree of polarisation given by

$$p = \frac{\sqrt{M^2 + C^2 + S^2}}{I}. \tag{12}$$

A partially polarised beam can be looked upon as an incoherent addition of completely polarised beams (Pancharatnam, 1956 *b*, p. 403). It might, therefore, be of interest to determine the degree of coherence of partially polarised light. It can be found by splitting the beam into two coherent ones, one of them being passed through a linear analyser and the other sent through the orthogonal analyser. The beams are then made to interfere. It is so arranged that one of the beams passes through a longer path. Measurement of the Stokes parameters of the resultant beam leads to the determination of degree of coherence, γ (Pancharatnam, 1956 *b*, 5.12) because

$$\left.\begin{aligned} &\gamma = +\frac{1}{2\sqrt{I_1 I_2}} \cdot \sqrt{C^2 + S^2} \\ &\text{where} \\ &I_1 = \tfrac{1}{2}(I + C) \\ &\text{and} \\ &I_2 = \tfrac{1}{2}(I - C) \end{aligned}\right\}. \tag{13}$$

§ 5. Scanning the Poincaré Sphere

For locating automatically and continuously the point P on the Poincaré sphere, we have to make the state of an analyser travel a particular path. This can be accomplished by rotating either of the optical elements (*i.e.*, birefringent plate and linear analyser) or both of them. The state of such a rotating elliptic analyser varies with time in a difinite manner and thus scans the surface of the Poincaré sphere along a particular path. We shall in this section, using Theorem 1 (Appendix I) state the main types of scanning and the arrangement necessary for obtaining them.

(a) *Scanning along a meridian*

The combination of a stationary quarter wave plate and a rotating linear analyser results in scanning along a meridian whose longitude is equal to twice the angle between the slow axis of the birefringent plate and the reference direction, H (Fig. 6).

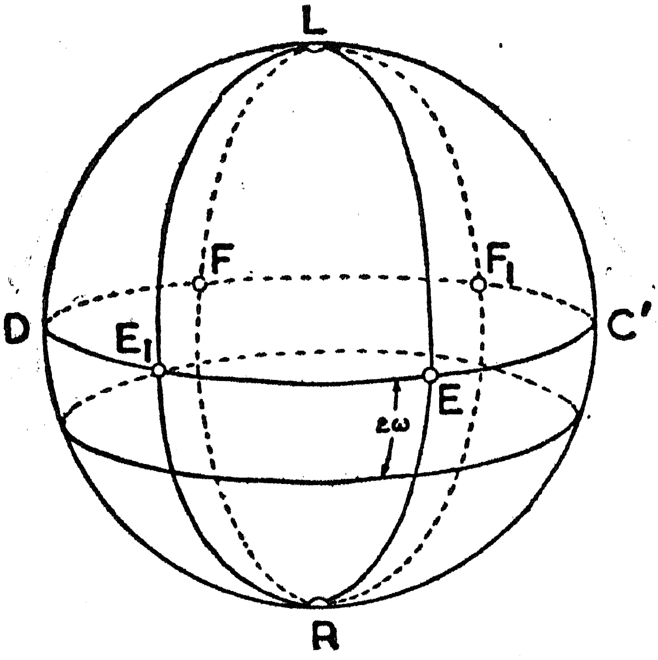

FIG. 6. Scanning the Poincaré sphere along a meridian or a latitude circle. (*a*) Stationary quarter wave plate (slow axis at E) and a linear analyser, rotating in anticlockwise direction, scanning along the meridian ERFL. By changing the slow axis to the position E_1 scanning is done along E_1RF_1L. (*b*) A linear analyser set with its vibration direction parallel to the slow axis and the two rotated at the same speed results in scanning along the equator. If the linear analyser vibration direction is inclined to the slow axis by ω then the scanning is done along the latitude circle of latitude 2ω.

(b) *Scanning along a latitude circle*

When both the elements (*i.e.*, the birefringent plate and linear analyser) are rotated with the same speed, the scanning takes place along a latitude circle; the latitude of which is equal to twice the angular separation between the slow axis of the quarter wave plate and the vibration direction of the linear element (Fig. 6).

(c) Scanning along an oblique path

If the azimuth and ellipticity of the elliptic analyser are changed continually by rotating both the elements at different speeds or the quarter wave plate alone, the analyser scans along an oblique path* (Fig. 7). The special advantage of the latter method is that very high speeds of scanning can be attained by using an electro-optic cell as a birefringent element. Fast scanning rates are necessary for the determination of rapidly changing states of polarisation for instance, the analysis of light reflected from a metal on which an oxide film is growing.

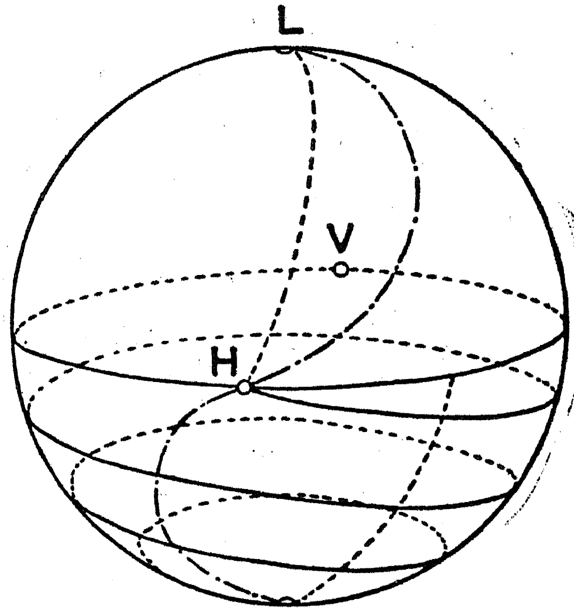

FIG. 7. Scanning the Poincaré sphere along an oblique path. ——Path traced by rotating elliptic analyser; both the elements rotating at nearly but not exactly the same speed. - · - · -Path traced by the combination of a rotating λ/4 plate and a stationary linear element. - - - -Path of a combination of a fast rotating linear analyser *plus* a very slow rotating quarter-wave plate.

(d) Scanning along a great circle of any arbitrary inclination to the equatorial circle

Rotating a linear analyser behind a variable birefringence plate leads to scanning along a great circle inclined to the equatorial circle by an angle, δ equal to the retardation introduced by the birefringent element (Fig. 8).

* The word oblique path is used here in the sense that the path described is neither a great circle nor a small circle.

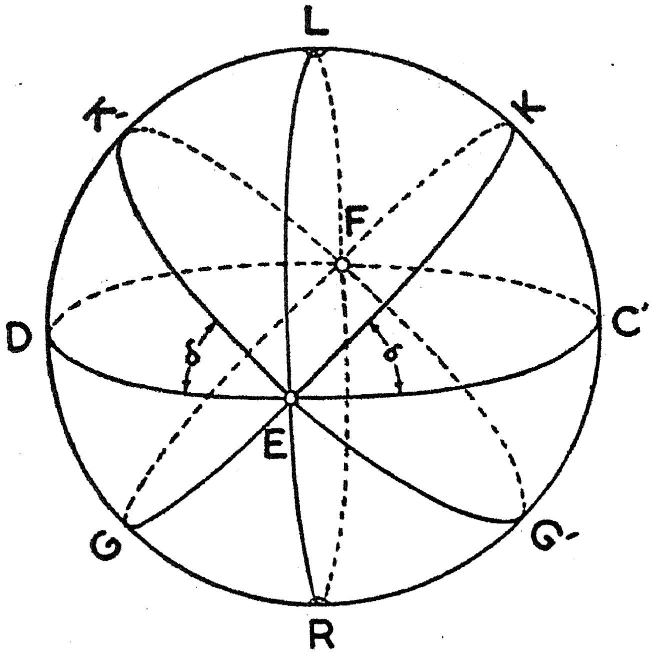

FIG. 8. Scanning the Poincaré sphere along a great circle of any arbitrary inclination. A stationary variable birefringence element (its slow axis at E) and a counterclockwise rotating linear element trace the path EC'FD when $\delta = 0$, EG'FK' when $\delta = \delta$, ERFL when $\delta = \pi/2$. If the fast axis of the birefringent element is at E then the path traced by the elliptic analyser (linear analyser rotated in anticlockwise direction) is EC'FD when $\delta = 0$, EKFG when $\delta = \delta$ and ELFR when $\delta = \pi/2$.

§ 6. AUTOMATION AND RECORDING

Broadly speaking there are two ways of using the rotating elliptic analyser for automatic analysis of polarised light.

(*a*) It can be made to scan along a definite path and the position of P is obtained from the intensities transmitted by it at predetermined positions.

(*b*) It can be made to "hunt" the point P_a by making the output of the photocell receiving the transmitted light control and guide the movement of the elliptic analyser. When the state of the rotating elliptic analyser is the same as P_a, the movement of the elliptic analyser can be frozen and its state recorded.

The speed of analysis depends upon the rate at which the Poincaré sphere is scanned. Rotating analysers made from a mica plate and a polaroid or nicol as a linear element can be rotated by using synchronous motors and beat frequency oscillators. As these are mechanically rotated it would not be possible to achieve speeds greater than 30 to 40 r.p.s. corresponding to a scanning speed of 60 to 80 times a second. On the other hand, if an electro-optic cell with a liquid or crystal is used as a birefringent element, the field causing the birefringence could be rotated with extremely high frequencies. Therefore, scanning along an oblique path can be done at very high speeds, thus making continuous analysis of rapidly changing states of polarisation feasible. It must be remarked that since one cannot think of a simple linear analyser which is based on electro-optic principle one cannot achieve very high scanning rates with rotating analysers in which both the elements or the linear element alone is rotated.

Some of the methods discussed in Section 3 depend upon measurement of absolute intensity which is best done with a photomultiplier tube and modulated light. Electro-optic effect and Faraday rotation have been used for modulating light (Takasaki, 1962; William and Weingart, 1964). It can also be done by using an *ac* arc source run by a stabilised power supply (Sivaramakrishnan, 1956; Ingersoll and Liebenberg, 1954).

In the visual measurement, higher accuracy is realised by matching two intensities for equality using a half-shade. In the photo-electric method also the half shade principle would prove useful, because such a comparison gives directly the error signal required for driving the servo-mechanism. Therefore, wherever possible it is preferable to employ two rotating analysers and use the half-shade principle. Since a photocell, unlike the human eye, is sensitive even at higher levels of illumination the two analysers may be separated by an angle that gives maximum sensitivity.

The methods based on intensity measurement can be accomplished in two ways. One may either record the state of the analyser when the transmitted intensity attains a predetermined value or record the transmitted intensity corresponding to predetermined state of the analyser. Although the former method is more accurate, it is much more difficult to record angles than to record intensity.

For recording we can use either a pen recorder or an oscilloscope. A pen recorder can be used in conjunction with devices which make use of mechanically rotating elliptic analysers for recording stationary states of polarisation or those whose variation is so slow that one can consider it to be constant for a second. Oscilloscopic recording can be used for the rapid analysis of polarised light using a rotating analyser comprising of a stationary linear element and an electro-optic cell in which the field is rotated.

§ 7. Summary

An elliptic analyser, consisting of a birefringent element and a linear element, can be converted into a rotating elliptic analyser by rotating one

or both the elements. The Poincaré sphere could be scanned, with such a device, along a meridian, a latitude circle, a great circle of any arbitrary inclination to the equator or any oblique path. Continuous analysis of polarised light can be accomplished by using such an analyser. The principles of some of the possible methods of analysis are presented. The problem of analysis of partially polarised light is also briefly discussed. The speed of analysis is important in analysing changing states of polarisation and depends on the speed of rotation of the elliptic analyser. It is pointed out here that fast rates of scanning are possible by using an electro-optic cell as a birefringent element and rotating the field that causes birefringence.

§ 8. ACKNOWLEDGEMENT

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APPENDIX I

1. A completely polarised light beam in any arbitrary state of polarisation characterised by an azimuth, λ_P and ellipticity $-\omega_P$ is represented by a point P—whose longitude is $2\lambda_P$ and latitude is $-2\omega_P$—on the Poincaré's sphere. Positive values of ω (lower hemisphere) correspond to left rotating ellipses and negative values of ω (upper hemisphere) to right rotating ellipses (Fig. 1).

2. The effect of passage of plane polarised light—with any arbitrary azimuth N—through a birefringent medium brings the state from N to P_4 by introducing a retardation δ which makes state V lag behind the state H. The point P_4 can be located by rotating the sphere, through an angle, δ, in anticlockwise direction, around an axis passing through the point denoting the faster state (HⒻ in Fig. 1). The same result can be derived by a clockwise rotation of the sphere around an axis passing the slower state, VⓈ.

3. An analyser is said to be in state P if it completely transmits light in state P. Hence, a point P on the Poincaré sphere denotes either a polariser producing light in state P or an analyser completely transmitting P.

4. A beam of light, of intensity I, in state P can be resolved into any two orthogonal states A and A_a; the intensities of the decomposed beams being $I \cos^2 \frac{1}{2} \widehat{PA}$ (for the " A-component ") and $I \sin^2 \frac{1}{2} \widehat{PA}$ (for the "A_a-component"); where $\widehat{PA}$ is the length of the great circular arc connecting the points P and A.

Corollary.—(*a*) The fraction of the intensity—in state P transmitted by an analyser A is $\cos^2 \frac{1}{2} \widehat{PA}$.

(*b*) Angular separation between any two points, P and A, on the Poincaré sphere can be determined by measuring the fraction of the intensity of P transmitted by an analyser whose state is represented by the point A.

The important expression

$$\frac{1}{2} PA = \cos^{-1} \sqrt{\frac{I_A}{I}}$$

(where I_A is the intensity transmitted by A and I is the total intensity) provides a link between the measurable quantity I_A/I and Poincaré representation.

This expression is not valid for partially polarised light. For such a light the expression takes the form

$$\tfrac{1}{2}\widehat{PA} = \cos^{-1}\sqrt{\frac{I_A - \frac{1}{2}(1-p)\,I}{Ip}}.$$

where p is the degree of polarisation. Therefore, for determining the angular separation PA in the case of partially polarised light I, I_A and p have to be measured.

5. A beam consisting of a fraction p of completely polarised light in state P and a fraction $(1-p)$ of unpolarised light is represented by a point P_0 inside the sphere such that OP_0 is equal to p. OP_0 is called the Poincaré vector. The vector Ip whose magnitude gives the intensity of polarised part of the light beam is known as the Stokes vector.

6. The completely polarised part of the beam can be resolved along three mutually perpendicular directions, OX, OY and OZ which are denoted by OH, OC′ and OL (Fig. 2). The resolved components along the three directions, respectively, are M, C and S which together with the total intensity I of the beam are called the Stokes parameters of the incident light. The Poincaré vector OP is unity if the beam is completely polarised.

7. Coherent addition of two completely polarised beams 1 and 2 in states P_1 and P_2 lead to a resultant in state P_3. The intensity of the same is given by equation 8.