Quantum Analogues of a Coherent Family of Modules at Roots of Unity : A_2, B_2

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To a given a coherent family of virtual representations of a complex semisimple Lie algebra we associated in [P] a coherent family of virtual representations of the corresponding quantum group at roots of unity [P, section 2]. This is recalled fairly explicitly in section 2 below. We also proposed a conjecture there that under some hypotheses the members of the family in a certain positive cone are actually modules (as opposed to a 'virtual' module which is in general only a difference of two modules). We verify the validity of this conjecture for A_2 and B_2. But first we recall in some length the ideas in [P] without detailed proofs.

1

Let $\underline{g}$ be a finite dimensional complex semisimple Lie algebra and let U be its universal enveloping algebra. Lusztig considered a certain $\mathbf{C}[v, v^{-1}]$ algebra $U_{\mathcal{A}}, \{\mathcal{A} = \mathbf{C}[v, v^{-1}]\}$ which is an $\mathcal{A}$-form of the 'quantum group' $U_{\mathcal{A}'}, \{\mathcal{A}' = \mathbf{C}(v),$ the field of fractions of $\mathcal{A}\}$; the latter are some Hopf-algebra deformations of U, defined by Drinfeld and Jimbo generalizing the case of $\underline{s\ell_2}$.

Let $\lambda \in \mathbf{C}^*$ and suppose that λ is a primitive $\ell - th$ root of unity where ℓ , (≥ 3) , is an odd positive integer (not divisible by 3 if G_2 is a factor of $\underline{g}$)

Let $\varphi_\lambda : \mathcal{A} \to \mathbf{C}$ be the $\mathbf{C}$-algebra homomorphism obtained by sending v to λ. The algebras $U_\lambda := U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbf{C}$, (scalar multiplication by $u \in \mathcal{A}$ in the first factor corresponds to scalar multiplication by $\varphi_\lambda(u)$ in the second factor) are called 'quantum groups at roots of unity'; these are different from those considered by Kac, Processi and De-Concini. The algebras U_λ are also Hopf algebras.

In [L1, Prop. 7.5 (a) and L2, 8.16] Lusztig defines a 'Frobenius' morphism $\psi : U_\lambda \to U$; ψ is a surjection and respects the Hopf-algebra structure.

2. Coherent family of virtual representations of U, U_λ

Let $\underline{h}$ be a Cartan subalgebra of $\underline{g}$. Let Δ be the set of roots of $\underline{g}$ with respect to $\underline{h}$ and Δ^+ a system of positive roots. Let $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be the set of simple roots in Δ^+. Let $\Lambda \subseteq \underline{h}^*(= \mathrm{Hom}_{\mathbf{C}}(\underline{h}, \mathbf{C})$) be the integral lattice defined by

$$\nu \in \Lambda \Leftrightarrow 2(\nu, \alpha)/(\alpha, \alpha) \in \mathbf{Z}, \forall \alpha \in \Delta$$

where the pairing is induced by the Killing form in the usual way.

Definition. A family of virtual (not necessarily finite dimensional) representations $\{\pi(\nu)\}_{\nu \in \Lambda}$ of U is called a *coherent family* if for every finite dimensional module F of U (in the Grothendieck group)

$$\pi(\nu) \otimes F = \sum_{\mu \in \Delta(F)} m(\mu, F)\pi(\nu + \mu)$$

where the summation is over the weights $\Delta(F)$ of F and for $\mu \in \Delta(F), m(\mu, F)$ denotes the multiplicity of μ as a weight of F.

Remark: The Grothendieck group is formed in the usual way from any subcategory of modules with finite composition series, stable under tensor products with finite dimensional modules. Depending upon the context, (see for e.g, [BV, Definition 2.2]) one often assumes extra information about the coherent family, e.g, that $\pi(\nu)$ has an infinitesimal character parametrized by the orbit of ν and also an irreducibility property for $\pi(\nu)$, when ν satisfies positivity conditions.

Interesting examples of coherent families arise by considering Harish-Chandra modules (generally infinite-dimensional) for a real form of $\underline{g}$. Given any such irreducible Harish-Chandra module there is a coherent family it belongs to (see [Vo, Theorem 7.2.7]).

Finite dimensional representations of U have been quantized by Lusztig at all U_λ. Their 'weights' can be defined as elements of Λ (see [L1, 5.2]); they admit a weight space decomposition (see [APW,9.12] and [A]). If F is an irreducible finite dimensional module for U, its quantization F' for U_λ is called a Weyl module; for $\mu \in \Lambda, \mu$ is a weight of F' iff it is a weight of F and then the multiplicity $m(\mu, F')$ equals $m(\mu, F)$. This allows us to define a coherent family $\{\overline{\pi}(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U_λ exactly as in the case of U. We will assume throughout this article, without further mention, that the U_λ modules considered are all of type 1 [L1,4.6].

Let $\rho = \frac{1}{2}\sum\limits_{\alpha\in\Delta^+} \alpha$ (half the sum of the positive roots). Recall that λ is a primitive ℓ-th root of 1. If $\nu \in \Lambda$ is dominant integral (i.e., $2(\nu,\alpha)/(\alpha,\alpha) \in \mathbf{Z}^+, \forall\alpha \in \Delta^+$), let F_ν denote the irreducible finite dimensional representation of U with highest weight ν. We have then a representation $U_\lambda \to$ End (F'_ν) of the quantum group U_λ on the corresponding Weyl module F'_ν. Recall that if $\nu = (\ell-1)\rho$, the Weyl module $F'_{(\ell-1)\rho}$ is called the 'Steinberg module'; it is irreducible (see [AW,2.2] and [A]). We let St denote the Steinberg module.

If $\pi : U \to$ End (V) is a representation of U, we define a representation $\widetilde{\pi} : U_\lambda \to$ End (V) by $\widetilde{\pi} = \pi \circ \psi$ where $\psi : U_\lambda \to U$ is the Frobenius morphism defined by Lusztig [L1, 7.5 and L2, 8.16].

Given any $\nu \in \Lambda$, we can uniquely write $\nu = \nu' + \ell\nu''$ where i) $\nu', \nu'' \in \Lambda$ and ii) $2(\nu',\alpha)/(\alpha,\alpha) \in \{0, 1, \cdots, \ell-1\}$ for every simple root α in Δ^+.

Given a coherent family $\{\pi(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U we proceed to construct a coherent family $\{\overline{\pi}(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U_λ such that for any $\nu'' \in \Lambda$

$$\overline{\pi}(\ell\nu'') = \widetilde{\pi}(\nu'') \otimes St$$

where St is the Steinberg representation of U_λ.

For this we introduce some notation mainly following [V] and [H].

Let $\delta_1, \delta_2, \cdots, \delta_n$ be the fundamental weights; i.e., $2(\delta_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$ (Kronecker delta) where $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ is the set of simple roots. Let W be the Weyl group, $W \subset$ Aut $(\underline{h}^*)$, generated by the Coxeter generators $s_i (i = 1, \cdots, n)$ defined by $s_i(\nu) = \nu - [2(\nu, \alpha_i)/(\alpha_i, \alpha_i)]\alpha_i$. For $\sigma \in W$, let $I_\sigma = \{i \mid 1 \leq i \leq n,\ \ell(\sigma s_i) < \ell(\sigma)\}$. Here $\ell(\ )$ denotes the length function on W with respect to the Coxeter generators $s_1, s_2, \cdots, s_n$. Put $\delta_\sigma = \sum\limits_{i \in I_\sigma} \delta_i$ and define $\epsilon_\sigma = \sigma(\delta_\sigma)$. Let $\mathcal{R}$ be the ring of formal integral combinations $\sum_{\eta \in \Lambda} m_\eta \, e^\eta$. Since the action of W on $\underline{h}^*$ leaves Λ stable, W obviously acts as automorphisms of the ring $\mathcal{R}$. We let $\mathcal{R}^W$ denote the subring of invariants. We now summarize some key observations of Hulsurkar in [H] which were reinforced by Verma [V].

Proposition ([H], [V])

(i) *For $\sigma \in W, -\epsilon_{\sigma\sigma_0} + \epsilon_\sigma = \sigma\rho$, where σ_0 is the unique element of W of maximum length. If $m \geq 2$ and $\tau_1, \cdots, \tau_m$ are distinct elements of W, then at least one of the elements $-\epsilon_{\tau_1\sigma_0} + \epsilon_{\tau_2}, -\epsilon_{\tau_2\sigma_0} + \epsilon_{\tau_3}, \cdots, -\epsilon_{\tau_{m-1}\sigma_0} + \epsilon_{\tau_m}, -\epsilon_{\tau_m\sigma_0} + \epsilon_{\tau_1}$ is singular. (ν is non-singular $\Leftrightarrow (\nu, \alpha) \neq 0$ for any $\alpha \in \Delta \Leftrightarrow$ "$w \in W, w\nu = \nu \Rightarrow w = 1$").*

(ii) For any $\nu \in \Lambda$, there exist unique W-invariant elements $\chi_{\nu,\tau}$, $(\tau \in W), \in \mathcal{R}^W$ such that

$$e^\nu = \sum_{\tau \in W} \chi_{\nu,\tau} \ e^{\epsilon_\tau}$$

(iii) For any $\nu \in \Lambda$, there exist unique W-invariant elements $\eta_{\nu,\tau}$, $(\tau \in W), \in \mathcal{R}^W$ such that

$$e^\nu . \chi(St) = \sum_{\tau \in W} \eta_{\nu,\tau} \ e^{\ell\epsilon_\tau}.$$

$$\eta_{0,w} = \begin{cases} \chi(St) & \textit{if } w = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

(Here $\chi(St)$, which lies in $\mathcal{R}^W$ denotes the character of the Steinberg representation.) The statement (i) is the Main Lemma of [H]. For statement (ii) see also [J, Satz 1].

For a pictorial representation of statement (iii) in the case of B_2 and A_2 see figures at the end of the next section.

Indication of proof (following [H] and [V]) of ii) Define an operator c : $\mathcal{R} \to \mathcal{R}$ by

$$c(e^\eta) \;=\; \frac{\sum_{\tau\in W}\,(-1)^{\ell(\tau)}e^{\tau\eta}}{\sum_{\tau\in W}\,(-1)^{\ell(\tau)}e^{\tau\rho}}.$$

The operator c is $\mathcal{R}^W$-linear.

The main idea of the proof by Hulsurkar and Verma is to solve the system of linear equations by inverting a $\mid W \mid \times \mid W \mid$ matrix $(a_{\sigma,\tau})$ where $a_{\sigma,\tau} = \; c(e^{-\epsilon_{\sigma\sigma_0}+\epsilon_\tau})$ [which is essentially guaranteed to be 'upper triangular' unipotent by i)]

To find $(\chi_{\nu,\tau})_{\tau\in W}$ which solves

$$e^\nu = \sum_\tau \; \chi_{\nu,\tau}\; e^{\epsilon_\tau}$$

multiply both sides by $e^{-\epsilon_{\sigma\sigma_0}}$ to get

$$e^{\nu-\epsilon_{\sigma\sigma_0}} = \sum_\tau \; \chi_{\nu,\tau}\; e^{-\epsilon_{\sigma\sigma_0}+\epsilon_\tau}.$$

Applying c to both sides

$$c(e^{\nu-\epsilon_{\sigma\sigma_0}}) = \sum_\tau \; \chi_{\nu,\tau}\; c(e^{-\epsilon_{\sigma\sigma_0}+\epsilon_\tau}) \qquad\qquad (\sigma\in W).$$

The left side of this system of equations is a column vector (whose $\mid W \mid$ entries belong to the ring $\mathcal{R}^W$). Multiply this column vector on the left by the $\mid W \mid \times \mid W \mid$ matrix $(\beta_{\sigma,\tau})$ (whose entries are in the same ring) which is the inverse of the matrix $(a_{\sigma,\tau})$ where $a_{\sigma,\tau} = c(e^{-\epsilon_{\sigma\sigma_0}+\epsilon_\tau})$ to solve for the unknown column vector $(\chi_{\nu,\tau})_{\tau\in W}$.

The proof of iii) is similar. In fact one can see the following :-

Let $\Phi : \mathcal{R} \to \mathcal{R}$ be the ring homomorphism defined by $\Phi(e^\theta) = e^{\ell\theta}$. Observe that

$$(\Phi\circ c(e^\eta))\chi(St) = c\circ\Phi(e^\eta) \qquad \forall\eta\in\Lambda.$$

Define a $\mid W \mid \times \mid W \mid$ matrix $(\beta'_{\sigma,\tau})$ by $\beta'_{\sigma,\tau} = \Phi(\beta_{\sigma,\tau})$, where $\beta_{\sigma,\tau}$ are as above. Then the column vector $(\eta_{\nu,\tau})_{\tau\in W}$ required in iii) is obtained by multiplying the column vector $(c(e^{\nu-\ell\epsilon_{\sigma\sigma_0}}))_{\sigma\in W}$ on the left by the matrix

$(\beta'_{\sigma,\tau})_{\sigma,\tau\in W}$.

Remark 1 : Let $\mu \in \Lambda$. Applying Proposition (iii) to $\nu + \ell\mu$ in place of ν we find that $\exists \eta_{\nu+\ell\mu,\tau} \in \mathcal{R}^W, (\tau \in W)$, such that

$$e^{\nu+\ell\mu}.\chi(St) = \sum_{\tau} \eta_{\nu+\ell\mu,\tau} e^{\ell\epsilon_\tau}.$$

Therefore

$$\begin{array}{rcl} e^{\nu}.\chi(St) & = & \sum_\tau \eta_{\nu,\tau} e^{\ell\epsilon_\tau} \\ & = & \sum_\tau \eta_{\nu+\ell\mu,\tau} e^{\ell\epsilon_\tau - \ell\mu}. \end{array}$$

We denote by $\mathcal{F}$ the Grothendieck group of formal integral combinations of finite dimensional representations of U. If $\omega \in \mathcal{F}$, the character $\chi(\omega) \in \mathcal{R}^W$ has an obvious meaning and $\chi : \mathcal{F} \to \mathcal{R}^W$ is an isomorphism. We also have to introduce the corresponding Grothendieck group $\mathcal{F}'$ for U_λ-modules. Again if $\omega \in \mathcal{F}'$, the character $\chi(\omega) \in \mathcal{R}^W$ has an obvious meaning and $\chi : \mathcal{F}' \to \mathcal{R}^W$ is an isomorphism. Sometimes, if convenient, we use the same symbol to denote an element of $\mathcal{F}'$ and its character in $\mathcal{R}^W$.

Theorem: *Suppose a coherent family $\{\pi(\nu)\}_{\nu\in\Lambda}$ of virtual representations of U is given. Given $\nu \in \Lambda$, write $\nu = \nu' + \ell\nu''$ where $\nu'' \in \Lambda$ and $2(\nu', \alpha)/(\alpha, \alpha) \in \{0, 1, \cdots, \ell - 1\}$ for each simple root α. Let $e^{\nu'}.St = \sum \eta_{\nu',\tau}.e^{\ell\epsilon_\tau}$ in the notation of Proposition iii) . Choose $\rho(\nu', \tau) \in \mathcal{F}'$ whose character is $\eta_{\nu',\tau}$. Set*

$$\overline{\pi}(\nu) = \sum_{\tau} \rho(\nu', \tau) \otimes \widetilde{\pi}(\nu'' + \epsilon_\tau)$$

(in the Grothendieck group of representations of U_λ; see Remark at the beginning of section 2). Then $\{\overline{\pi}(\nu)\}_{\nu\in\Lambda}$ is a coherent family of representations of U_λ with $\overline{\pi}(\ell\nu'') = \widetilde{\pi}(\nu'') \otimes St$.

Proof: We have $\eta_{\nu',\tau} \in \mathcal{R}^W$ and

$$e^{\nu'}.St = \sum \eta_{\nu',\tau}\ e^{\ell\epsilon_\tau}.$$

Remark 2 : By Remark 1, for any $\mu \in \Lambda$, we can also write (uniquely)

$$e^{\nu'}.St = \sum_{\tau} \eta_{\nu'+\ell\mu,\tau}\ e^{-\ell\mu+\ell\epsilon_\tau}$$

where $\eta_{\nu'+\ell\mu,\tau} \in \mathcal{R}^W$. In the course of the proof, it will be established that in the statement of the Theorem, the rightside of $\overline{\pi}(\nu)$, i.e. $\sum_\tau \rho(\nu',\tau) \otimes \tilde{\pi}(\nu'' + \epsilon_\tau)$ equals

$$\sum \rho(\nu' + \ell\mu, \tau) \otimes \tilde{\pi}(\nu'' - \mu + \epsilon_\tau)$$

where $\rho(\nu' + \ell\mu, \tau)$, $(\in \mathcal{F}')$, is chosen so as to have character $\eta_{\nu'+\ell\mu,\tau}$.

The main ingredient in the proof of the theorem is the following lemma.

Lemma: *Suppose*

$$\sum \chi_i \ e^{\ell\beta_i} = \sum \psi_j \ e^{\ell\gamma_j}$$

where χ_i , $(i = 1, \cdots, m)$, *and* ψ_j , $(j = 1, \cdots, n)$, $\in \mathcal{R}^W$ *and* $\beta_i, \gamma_j \in \Lambda$. *Assume, as in the Theorem, that* $\{\pi(\nu)\}_{\nu\in\Lambda}$ *is a coherent family of representations of* U. *Let* $\rho_i, \tau_j \in \mathcal{F}'$ *such that* $\chi(\rho_i) = \chi_i$ *and* $\chi(\tau_j) = \psi_j$. *Then,*

$$\sum \rho_i \otimes \tilde{\pi}(\beta_i) = \sum \tau_j \otimes \tilde{\pi}(\gamma_j)$$

(Both sides lie in the Grothendieck group obtained from U_λ-modules; see Remark at the beginning of section 2)

Suppose the coherent family $\pi(\nu)_{\nu\in\Lambda}$ has the property (see [BV, Definition 2.2]) that

i) $\pi(\nu)$ has infinitesimal character parametrized by the W- orbit of ν

ii) $\pi(\nu)$ is zero or irreducible when ν is dominant with respect to a fixed positive system, and $\pi(\nu) \neq 0$ if ν is dominant regular.

Then it can be expected that $\overline{\pi}(\nu)$ for dominant ν (with respect to the positive system in ii) above) is represented in the Grothendieck group by a U_λ- module (as opposed to an arbitrary element of the Grothendieck group, which in general is a virtual module, i.e., a difference of two modules). In the next section we show that this is indeed true for A_2 and B_2, using Lusztig's formula for the multiplicity of irreducibles in Weyl modules of quantum groups at roots of unity. More generally, we can also relax the conditions i) and ii) above to allow families $\pi(\nu)_{\nu\in\Lambda}$ which do not necessarily have integral infinitesimal characters.

If in the theorem we take Verma modules for the coherent family $\pi(\nu)_{\nu\in\Lambda}$ then the expression for $\overline{\pi}(\nu)$ given in the theorem can be used to deduce the multiplicities of the irreducible subquotients occuring in a composition series for the quantized Verma modules at roots of unity. The formula so obtained, of course, involves

i) the multiplicities of the irreducibles occuring in $\pi(\nu)$ for various ν and

ii) the multiplicities of irreducibles occuring in the Weyl modules for U_λ.

In addition the formula involves the knowledge of the coefficients $\eta_{\nu,\tau}$; the explicit determination of $\eta_{\nu,\tau}$ was indicated in the proof of Proposition (iii).

Examples:$\mathbf{A_2}, \mathbf{B_2}$

$\mathbf{B_2}$

If Λ_1 is the fundamental weight which takes the value 1 on the short simple root and Λ_2 the fundamental weight which takes the value 1 on the long simple root, let us denote the weight $m\Lambda_1 + n\Lambda_2$ by the pair (m,n). The fundamental parallelopiped $\{(m,n) \mid 1 \leq m \leq \ell, 1 \leq n \leq \ell\}$ can be divided into four 'alcoves' described as follows: 'lowest' alcove $= \{(m,n) \mid 1 \leq m \leq \ell, 1 \leq n \leq \ell, 1 \leq m+n \leq \ell, 1 \leq m+2n \leq \ell\}$. Second alcove $= \{(m,n) \mid 1 \leq m \leq \ell, 1 \leq n \leq \ell, 1 \leq m+n \leq \ell, \ell \leq m+2n \leq 2\ell\}$. Third alcove $= \{(m,n) \mid 1 \leq m \leq \ell, 1 \leq n \leq \ell, \ell \leq m+n \leq 2\ell, \ell \leq m+2n \leq 2\ell\}$. 'Highest' alcove $= \{(m,n) \mid 1 \leq m \leq \ell, 1 \leq n \leq \ell, \ell \leq m+n \leq 2\ell, 2\ell \leq m+2n \leq 3\ell\}$.

If μ is a dominant integral weight let $Y(\mu)$ denote the Weyl module for U_λ (where λ is a primitive $\ell - th$ root of unity) whose character $\chi(Y(\mu))$ is given by

$$c(e^\mu) = \frac{\sum_{\tau\in W} (-1)^{\ell(\tau)} e^{\tau\mu}}{\sum_{\tau\in W} (-1)^{\ell(\tau)} e^{\tau\rho}}.$$

and let $\overline{Y}(\mu)$ denote the unique irreducible quotient of $Y(\mu)$ if the latter is nonzero and zero otherwise. (**Caution:** Elsewhere, we have used the notation $F(\mu)$ to denote the irreducible module with highest weight μ when μ is dominant integral. Thus, $Y(\mu+\rho) = F(\mu)$.) The following decomposition of the Weyl module into irreducibles is known (see [APW, section 11]).

(3.1) If (m,n) is in the lowest alcove then

i) $Y(m,n) = \overline{Y}(m,n)$

ii) $Y(m,\ell-m-n) = \overline{Y}(m,\ell-m-n) + \overline{Y}(m,n)$

iii) $Y(m+2n,\ell-m-n) = \overline{Y}(m+2n,\ell-m-n) + \overline{Y}(m,\ell-m-n)$

iv) $Y(m+2n,\ell-n) = \overline{Y}(m+2n,\ell-n) + \overline{Y}(m+2n,\ell-m-n)$

Assuming as above that (m,n) is in the lowest alcove our first task is to carry out the calculation indicated in the proof of proposition iii) in section 2 for $e^{(m,n)}.St$. We find that

(3.2)

$$\begin{aligned}
&e^{(m,n)}.St = \\
&e^{(\ell,\ell)}Y(m,n) \\
&+e^{(0,\ell)}\{Y(\ell-m,m+n) + Y(\ell-m-2n,n)\} \\
&+e^{(-\ell,\ell)}\{Y(m+2n,\ell-m-n) + Y(m,n)\} \\
&+e^{(-2\ell,\ell)}Y(\ell-m-2n,n) \\
&+e^{(0,0)}\{Y(\ell-m,\ell-n) + Y(\ell-m-2n,m+n)\} \\
&+e^{(\ell,-\ell)}Y(m,\ell-m-n) \\
&+e^{(2\ell,-\ell)}Y(\ell-m-2n,m+n) \\
&+e^{(\ell,0)}\{Y(m+2n,\ell-n) + Y(m,\ell-m-n)\}
\end{aligned}$$

See Fig.2. Let $\nu = (\ell m'' + m, \ell n'' + n)$ where $m'', n'' \in Z^+$ and m,n are as above (i.e., (m,n) is in the lowest alcove). By the theorem in section 2, the member of the coherent family (of virtual representations of U_λ) attached to $(\ell m'' + m, \ell n'' + n)$ is given by

(3.3)

$$\begin{aligned}
&\overline{\pi}(\nu)=\overline{\pi}(\ell m^{''}+m,\ell n^{''}+n)=\\
&\widetilde{\pi}(m^{''}+1,n^{''}+1)\otimes Y(m,n)\\
&+\widetilde{\pi}(m^{''},n^{''}+1)\otimes\{Y(\ell-m,m+n)+Y(\ell-m-2n,n)\}\\
&+\widetilde{\pi}(m^{''}-1,n^{''}+1)\otimes\{Y(m+2n,\ell-m-n)+Y(m,n)\}\\
&+\widetilde{\pi}(m^{''}-2,n^{''}+1)\otimes Y(\ell-m-2n,n)\\
&+\widetilde{\pi}(m^{''},n^{''})\otimes\{Y(\ell-m,\ell-n)+Y(\ell-m-2n,m+n)\}\\
&+\widetilde{\pi}(m^{''}+1,n^{''}-1)\otimes Y(m,\ell-m-n)\\
&+\widetilde{\pi}(m^{''}+2,n^{''}-1)\otimes Y(\ell-m-2n,m+n)\\
&+\widetilde{\pi}(m^{''}+1,n^{''})\otimes\{Y(m+2n,\ell-n)+Y(m,\ell-m-n)\}
\end{aligned}$$

Observe that $(m^{''}-1,n^{''}+1),(m^{''}-2,n^{''}+1),(m^{''}+1,n^{''}-1)$ and $(m^{''}+2,n^{''}-1)$ may not belong to the positive cone; consequently, $\pi(m^{''}-1,n^{''}+1),\pi(m^{''}-2,n^{''}+1),\pi(m^{''}+1,n^{''}-1)$ and $\pi(m^{''}+2,n^{''}-1)$ may only be virtual modules and not actual modules. We will see that after decomposing the Weyl modules appearing in (3.3) into irreducibles it is possible to regroup the summands in such a way that the sum of the terms in any group is an actual module. We explain this in detail below.

Notice that if (m,n) is in the lowest alcove $(\ell-m-2n,n)$ is also in the lowest alcove. Thus (3.1) implies the following decompositions.

(3.4)

i) $Y(\ell-m-2n,n)=\overline{Y}(\ell-m-2n,n)$

ii) $Y(\ell-m-2n,m+n)=\overline{Y}(\ell-m-2n,m+n)+\overline{Y}(\ell-m-2n,n)$

iii) $Y(\ell-m,m+n)=\overline{Y}(\ell-m,m+n)+\overline{Y}(\ell-m-2n,m+n)$

iv) $Y(\ell-m,\ell-n)=\overline{Y}(\ell-m,\ell-n)+\overline{Y}(\ell-m,m+n)$

Using these decompositions in 3.3, we can write

(3.5)

$$
\begin{aligned}
&\overline{\pi}(\nu)=\overline{\pi}(\ell m''+m,\ell n''+n)=\\
&\widetilde{\pi}(m''+1,n''+1)\otimes\overline{Y}(m,n)\\
&+\widetilde{\pi}(m'',n''+1)\otimes\{\overline{Y}(\ell-m,m+n)+\overline{Y}(\ell-m-2n,m+n)+\overline{Y}(\ell-m-2n,n)\}\\
&+\widetilde{\pi}(m''-1,n''+1)\otimes\{\overline{Y}(m+2n,\ell-m-n)+\overline{Y}(m,\ell-m-n)+\overline{Y}(m,n)\}\\
&+\widetilde{\pi}(m''-2,n''+1)\otimes\overline{Y}(\ell-m-2n,n)\\
&+\widetilde{\pi}(m'',n'')\otimes\{\overline{Y}(\ell-m,\ell-n)+\overline{Y}(\ell-m,m+n)+\overline{Y}(\ell-m-2n,m+n)\\
&\quad+\overline{Y}(\ell-m-2n,n)\}\\
&+\widetilde{\pi}(m''+1,n''-1)\otimes\{\overline{Y}(m,\ell-m-n)+\overline{Y}(m,n)\}\\
&+\widetilde{\pi}(m''+2,n''-1)\otimes\{\overline{Y}(\ell-m-2n,m+n)+\overline{Y}(\ell-m-2n,n)\}\\
&+\widetilde{\pi}(m''+1,n'')\otimes\{\overline{Y}(m+2n,\ell-n)+\overline{Y}(m+2n,\ell-m-n)+\overline{Y}(m,\ell-m-n)\\
&\quad+\overline{Y}(m,n)\}
\end{aligned}
$$

Now we have to group the terms so that $\overline{\pi}(\nu)$ is seen to be an actual module. First we consider the case $(m'',n'')=(0,0)$, which is in fact the most difficult case. In this case, 3.5 becomes

(3.6)

$$
\begin{aligned}
&\overline{\pi}(\nu)=\overline{\pi}(m,n)=\\
&\widetilde{\pi}(1,1)\otimes\overline{Y}(m,n)\\
&+\widetilde{\pi}(0,1)\otimes\overline{Y}(\ell-m,m+n)\\
&+\{\widetilde{\pi}(2,-1)+\widetilde{\pi}(0,1)\}\otimes\overline{Y}(\ell-m-2n,m+n)\\
&+\{\widetilde{\pi}(-2,1)+\widetilde{\pi}(0,1)+\widetilde{\pi}(2,-1)\}\otimes\overline{Y}(\ell-m-2n,n)\\
&+\{\widetilde{\pi}(-1,1)+\widetilde{\pi}(1,0)\}\otimes\overline{Y}(m+2n,\ell-m-n)\\
&+\{\widetilde{\pi}(1,-1)+\widetilde{\pi}(1,0)+\widetilde{\pi}(-1,1)\}\otimes\overline{Y}(m,\ell-m-n)\\
&+\{\widetilde{\pi}(1,-1)+\widetilde{\pi}(1,0)+\widetilde{\pi}(-1,1)\}\otimes\overline{Y}(m,n)\\
&+\widetilde{\pi}(0,0)\otimes\overline{Y}(\ell-m,\ell-n)\\
&+\widetilde{\pi}(0,0)\otimes\overline{Y}(\ell-m,m+n)\\
&+\widetilde{\pi}(0,0)\otimes\overline{Y}(\ell-m-2n,m+n)\\
&+\widetilde{\pi}(0,0)\otimes\overline{Y}(\ell-m-2n,n)\\
&+\widetilde{\pi}(1,0)\otimes\overline{Y}(m+2n,\ell-n)
\end{aligned}
$$

For the following it may be profitable to refer to Fig.2. There are lot of things to observe about the grouping in the right side of 3.6. Observe

$$\pi(-1,1)+\pi(1,0)=pr_{(-1,1)}\{\pi(0,1)\otimes F(1,0)\}$$

where $F(1,0)$ is the irreducible finite dimensional module with highest weight $(1,0)$ and $pr_{(,)}$ means 'projection to the direct summand in the tensor product with infinitesimal character corresponding to (,)'. Hence recalling our assumption 3.0, we conclude that $\pi(-1,1)+\pi(1,0)$ is an actual module as $(0,1)$ is in the positive cone. Hence, in the right side of 3.6, $\{\widetilde{\pi}(-1,1)+\widetilde{\pi}(1,0)\}\otimes\overline{Y}(m+2n,\ell-m-n)$ is an actual module. Observe

$$\pi(2,-1)+\pi(0,1)=pr_{(2,-1)}\{\pi(1,0)\otimes F(1,0)\}$$

Hence, $\pi(2,-1)+\pi(0,1)$ is an actual module as $(0,1)$ is in the positive cone. Hence, in the right side of 3.6, $\{\widetilde{\pi}(2,-1)+\widetilde{\pi}(0,1)\}\otimes\overline{Y}(\ell-m-2n,m+n)$ is an actual module.

Next, we make the following observations:

$F(1,0)\otimes F(1,0)$ decomposes as the direct sum of $\Lambda^2F(1,0)$ and $S^2F(1,0)$, the second exterier and second symmetric power respectively. The latter is irreducible and is isomorphic to F(2,0); the former has an invariant given by the invariant alternating form in F(1,0) - in fact, $\Lambda^2F(1,0)=$ F(0,0) + F(0,1). The weight (0,0) occurs with multiplicity 1 in F(0,1) and with multiplicity 2 in F(2,0).

Using this information, we wish to assert that

$\pi(2,-1)+\pi(0,1)+\pi(-2,1)$

is an actual module. It is much easier to see that $\pi(2,-1)+2\pi(0,1)+\pi(-2,1)$ is an actual module, as the latter equals $pr_{(0,1)}[\pi(0,1)\otimes F(2,0)]$. Call this last module V (we would tacitly use the identification of F(2,0) as the second symmetric power of F(1,0)). If we show that $\pi(0,1)$is a submodule of V our assertion would follow. We show this now. Let T $=pr_{(1,1)}[\pi(0,1)\otimes F(1,0)]$. Then T $=\pi(1,1)+\pi(-1,2)$. Let W $=pr_{(0,1)}[T\otimes F(1,0)]$. Then W $=2\pi(0,1)$. If $\pi(0,1)$ is non zero, regard the last identity as giving a submodule W (isomorphic to two copies of $\pi(0,1)$) of T $\otimes$F(1,0). The modules W and T$\otimes F(1,0)$ are submodules of $\pi(0,1)\otimes F(1,0)\otimes F(1,0)$. Our assertion made a while ago would be established if we show that under the projection of $\pi(0,1)\otimes F(1,0)\otimes F(1,0)$ into $\pi(0,1)\otimes S^2F(1,0)$ the image of W is non-zero.(This would give the required non-zero submodule of V.) To show this we just observe that W does not contain $\pi(0,1)\otimes$F(0,0) where F(0,0) is identified with the space of invariants in Λ^2F(1,0). The point is that $pr_{(0,1)}[\pi(0,1)\otimes F(1,0)\otimes F(1,0)]$ equals $4\pi(0,1)+\pi(2,-1)+\pi(-2,1)$, while $pr_{(0,1)}[\pi(0,1)\otimes\Lambda^2F(1,0)]$ and $pr_{(0,1)}[\pi(0,1)\otimes S^2F(1,0)]$ are equal to $2\pi(0,1)$ and $2\pi(0,1)+$

$\pi(2,-1)+\pi(-2,1)$ respectively. Thus our assertion that $\pi(2,-1)+\pi(0,1)+\pi(-2,1)$ is an actual module is completely proved. Hence in the right side of 3.6, $\{\tilde{\pi}(-2,1)+\tilde{\pi}(0,1)+\tilde{\pi}(2,-1)\}\otimes\overline{Y}(\ell-m-2n,n)$ is an actual module.

Next observe that $pr_{(1,0)}[\pi(1,0)\otimes F(0,1)]=\pi(1,-1)+\pi(1,0)+\pi(-1,1)$. Hence, the right side of the above equation is an actual module. Hence, in the right side of 3.6,$\{\tilde{\pi}(1,-1)+\tilde{\pi}(1,0)+\tilde{\pi}(-1,1)\}\otimes\overline{Y}(m,\ell-m-n)$ and $\{\tilde{\pi}(1,-1)+\tilde{\pi}(1,0)+\tilde{\pi}(-1,1)\}\otimes\overline{Y}(m,n)$ are both actual modules.

Thus, though $\overline{\pi}(\nu)$ is apriori only a virtual module, from the right side of 3.6 one sees that $\overline{\pi}(\nu)$ is indeed represented in the Grothendieck group by an actual module for$\nu=(m,n)$ in the lowest alcove. When $\nu=(\ell m''+m,\ell n''+n)$ where $m'',n''\in Z^+$ and m,n are as above (i.e., (m,n) is in the lowest alcove) and $(m'',n'')\neq(0,0)$ the proof that $\overline{\pi}(\nu)$ is an actual module is along similar lines. We have given in the following pages details for (m,n) in the second alcove and (m'',n'') arbitrary.

Next we suppose that (m,n) is in the second alcove; recall that this means that $m,n,m+n$ lie between 0 and ℓ, while $m+2n$ lies between ℓ and 2ℓ. Let $m_0=m$ and $n_0=\ell-m-n$; then one sees that (m_0,n_0) lies in the lowest alcove. Also, note $(m,n)=(m_0,\ell-m_0-n_0)$. From 3.1,

(3.7)

i) $Y(m_0,n_0)=\overline{Y}(m_0,n_0)$

ii) $Y(m_0,\ell-m_0-n_0)=\overline{Y}(m_0,\ell-m_0-n_0)+\overline{Y}(m_0,n_0)$

iii) $Y(m_0+2n_0,\ell-m_0-n_0)=\overline{Y}(m_0+2n_0,\ell-m_0-n_0)+\overline{Y}(m_0,\ell-m_0-n_0)$

iv) $Y(m_0+2n_0,\ell-n_0)=\overline{Y}(m_0+2n_0,\ell-n_0)+\overline{Y}(m_0+2n_0,\ell-m_0-n_0)$

To begin we need an expression for $e^{(m,n)}.St$ - either following the method indicated in the proof of proposition (iii) in section 2, or any equivalent expression (see Lemma after remark 2, section 2). We will use the following expression. See Fig.3.

(3.8)

$$
\begin{aligned}
&e^{(m,n)}.St = e^{(m_0,\ell-m_0-n_0)}.St\\
&e^{(\ell,-\ell)}Y(m_0,n_0)\\
&+e^{(0,0)}\{Y(\ell-m_0,m_0+n_0)+Y(\ell-m_0-2n_0,n_0)\}\\
&+e^{(-\ell,\ell)}\{Y(m_0+2n_0,\ell-m_0-n_0)+Y(m_0,n_0)\}\\
&+e^{(-2\ell,2\ell)}Y(\ell-m_0-2n_0,n_0)\\
&+e^{(0,\ell)}\{Y(\ell-m_0,\ell-n_0)+Y(\ell-m_0-2n_0,m_0+n_0)\}\\
&+e^{(\ell,\ell)}Y(m_0,\ell-m_0-n_0)\\
&+e^{(2\ell,0)}Y(\ell-m_0-2n_0,m_0+n_0)\\
&+e^{(\ell,0)}\{Y(m_0+2n_0,\ell-n_0)+Y(m_0,\ell-m_0-n_0)\}
\end{aligned}
$$

Suppose $\nu = (\ell m'' + m, \ell n'' + n)$ where $m'', n'' \in Z^+$ and m, n are as above (i.e., (m,n) is in the second alcove). By the theorem in section 2, the member of the coherent family (of virtual representations of U_λ) attached to $(\ell m'' + m, \ell n'' + n)$ is given by

(3.9)

$$
\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m''+m,\ell n''+n) =\\
&\widetilde{\pi}(m''+1,n''-1)Y(m_0,n_0)\\
&+\widetilde{\pi}(m'',n'')\{Y(\ell-m_0,m_0+n_0)+Y(\ell-m_0-2n_0,n_0)\}\\
&+\widetilde{\pi}(m''-1,n''+1)\{Y(m_0+2n_0,\ell-m_0-n_0)+Y(m_0,n_0)\}\\
&+\widetilde{\pi}(m''-2,n''+2)Y(\ell-m_0-2n_0,n_0)\\
&+\widetilde{\pi}(m'',n''+1)\{Y(\ell-m_0,\ell-n_0)+Y(\ell-m_0-2n_0,m_0+n_0)\}\\
&+\widetilde{\pi}(m''+1,n''+1)Y(m_0,\ell-m_0-n_0)\\
&+\widetilde{\pi}(m''+2,n'')Y(\ell-m_0-2n_0,m_0+n_0)\\
&+\widetilde{\pi}(m''+1,n'')\{Y(m_0+2n_0,\ell-n_0)+Y(m_0,\ell-m_0-n_0)\}
\end{aligned}
$$

Using the decomposition of the modules Y(,) appearing in 3.9 into irreducibles $\overline{Y}(,)$, we obtain similar to 3.5 the following:

(3.10)

$$
\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m'' + m, \ell n'' + n) = \\
&\widetilde{\pi}(m''+1, n''-1) \otimes \overline{Y}(m_0, n_0) \\
&+\widetilde{\pi}(m'', n'') \otimes \{\overline{Y}(\ell - m_0, m_0 + n_0) + \overline{Y}(\ell - m_0 - 2n_0, m_0 + n_0) \\
&\quad + \overline{Y}(\ell - m_0 - 2n_0, n_0)\} \\
&+\widetilde{\pi}(m''-1, n''+1) \otimes \{\overline{Y}(m_0 + 2n_0, \ell - m_0 - n_0) + \overline{Y}(m_0, \ell - m_0 - n_0) \\
&\quad + \overline{Y}(m_0, n_0)\} \\
&+\widetilde{\pi}(m''-2, n''+2) \otimes \overline{Y}(\ell - m_0 - 2n_0, n_0) \\
&+\widetilde{\pi}(m'', n''+1) \otimes \{\overline{Y}(\ell - m_0, \ell - n_0) + \overline{Y}(\ell - m_0, m_0 + n_0) \\
&\quad + \overline{Y}(\ell - m_0 - 2n_0, m_0 + n_0) + \overline{Y}(\ell - m_0 - 2n_0, n_0)\} \\
&+\widetilde{\pi}(m''+1, n''+1) \otimes \{\overline{Y}(m_0, \ell - m_0 - n_0) + \overline{Y}(m_0, n_0)\} \\
&+\widetilde{\pi}(m''+2, n'') \otimes \{\overline{Y}(\ell - m_0 - 2n_0, m_0 + n_0) + \overline{Y}(\ell - m_0 - 2n_0, n_0)\} \\
&+\widetilde{\pi}(m''+1, n'') \otimes \{\overline{Y}(m_0 + 2n_0, \ell - n_0) + \overline{Y}(m_0 + 2n_0, \ell - m_0 - n_0) \\
&\quad + \overline{Y}(m_0, \ell - m_0 - n_0) + \overline{Y}(m_0, n_0)\}
\end{aligned}
$$

If $2 \leq m''$ and $1 \leq n''$, $\pi(m''+1, n''-1), \pi(m'', n''), \pi(m''-1, n''+1), \pi(m''-2, n''+2), \pi(m'', n''+1), \pi(m''+1, n''+1), \pi(m''+2, n'')$ and $\pi(m''+1, n'')$ are members of the coherent family whose parameters are in the positive cone. Hence,the corresponding virtual modules are actual modules. For other choices of (m'', n''), we need to regroup the terms on the right side of 3.10, similar to 3.6.

Case 1-a). Suppose $n'' = 0$ and $2 \leq m''$.

Then,$\pi(m''+1, -1) + \pi(m''-1, 1) = pr_{(m''+1,-1)}\{\pi(m'', 0) \otimes F(1,0)\}$ Thus, $\{\overline{\pi}(m''+1, -1) + \overline{\pi}(m''-1, 1)\} \otimes \overline{Y}(m_0, n_0)$ is an actual module. This is the only grouping needed in this case. All the other terms in 3.10 are seen to involve $\overline{\pi}(,)$ for a parameter in the positive cone.

Case 1-b). $n'' = 0$ and $m'' = 1$.

Observe $\pi(-1, 2) + \pi(1, 1) = pr_{(-1,2)}\{\pi(0, 1) \otimes F(1, 0)\}$. As already noted in case 1-a, $\pi(2, -1) + \pi(0, 1) = pr_{(2,-1)}\{\pi(1, 0) \otimes F(1, 0)\}$ Thus, in the right side of 3.10, $\{\overline{\pi}(-1, 2) + \overline{\pi}(1, 1)\} \otimes \overline{Y}(\ell - m_0 - 2n_0, n_0)$ and $\{\overline{\pi}(2, -1) + \overline{\pi}(0, 1)\} \otimes \overline{Y}(m_0, n_0)$ are both actual modules. No other grouping is needed in 3.10 in this case.

Case 1-c). $n'' = 0$ and $m'' = 0$.

3.10 becomes

(3.11)

$$
\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(m,n) = \\
&\tilde{\pi}(1,-1) \otimes \overline{Y}(m_0,n_0) \\
&+\tilde{\pi}(0,0) \otimes \{\overline{Y}(\ell-m_0,m_0+n_0) + \overline{Y}(\ell-m_0-2n_0,m_0+n_0) \\
&\quad + \overline{Y}(\ell-m_0-2n_0,n_0)\} \\
&+\tilde{\pi}(-1,1) \otimes \{\overline{Y}(m_0+2n_0,\ell-m_0-n_0) + \overline{Y}(m_0,\ell-m_0-n_0) + \overline{Y}(m_0,n_0)\} \\
&+\tilde{\pi}(-2,2) \otimes \overline{Y}(\ell-m_0-2n_0,n_0) \\
&+\tilde{\pi}(0,1) \otimes \{\overline{Y}(\ell-m_0,\ell-n_0) + \overline{Y}(\ell-m_0,m_0+n_0) \\
&\quad + \overline{Y}(\ell-m_0-2n_0,m_0+n_0) + \overline{Y}(\ell-m_0-2n_0,n_0)\} \\
&+\tilde{\pi}(1,1) \otimes \{\overline{Y}(m_0,\ell-m_0-n_0) + \overline{Y}(m_0,n_0)\} \\
&+\tilde{\pi}(2,0) \otimes \{\overline{Y}(\ell-m_0-2n_0,m_0+n_0) + \overline{Y}(\ell-m_0-2n_0,n_0)\} \\
&+\tilde{\pi}(1,0) \otimes \{\overline{Y}(m_0+2n_0,\ell-n_0) + \overline{Y}(m_0+2n_0,\ell-m_0-n_0) \\
&\quad + \overline{Y}(m_0,\ell-m_0-n_0) + \overline{Y}(m_0,n_0)\}
\end{aligned}
$$

Regrouping the terms on the right side we rewrite 3.11 as follows:

(3.12)

$$
\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(m,n) = \\
&\{\tilde{\pi}(1,-1) + \tilde{\pi}(1,0) + \tilde{\pi}(-1,1)\} \otimes \overline{Y}(m_0,n_0) \\
&+\tilde{\pi}(0,0) \otimes \{\overline{Y}(\ell-m_0,m_0+n_0) + \overline{Y}(\ell-m_0-2n_0,m_0+n_0) \\
&\quad + \overline{Y}(\ell-m_0-2n_0,n_0)\} \\
&+\{\tilde{\pi}(-1,1) + \tilde{\pi}(1,0)\} \otimes \{\overline{Y}(m_0+2n_0,\ell-m_0-n_0) + \overline{Y}(m_0,\ell-m_0-n_0)\} \\
&+\{\tilde{\pi}(-2,2) + \tilde{\pi}(2,0)\} \otimes \overline{Y}(\ell-m_0-2n_0,n_0) \\
&+\tilde{\pi}(0,1) \otimes \{\overline{Y}(\ell-m_0,\ell-n_0) + \overline{Y}(\ell-m_0,m_0+n_0) \\
&\quad + \overline{Y}(\ell-m_0-2n_0,m_0+n_0) + \overline{Y}(\ell-m_0-2n_0,n_0)\} \\
&+\tilde{\pi}(1,1) \otimes \{\overline{Y}(m_0,\ell-m_0-n_0) + \overline{Y}(m_0,n_0)\} \\
&+\tilde{\pi}(2,0) \otimes \overline{Y}(\ell-m_0-2n_0,m_0+n_0) \\
&+\tilde{\pi}(1,0) \otimes \overline{Y}(m_0+2n_0,\ell-n_0)
\end{aligned}
$$

We observe that $pr_{(1,-1)}\{\pi(1,0)\otimes F(0,1)\} = \pi(1,-1)+\pi(1,0)+\pi(-1,1)$ and $pr_{(-1,1)}\{\pi(0,1) \otimes F(1,0)\} = \pi(-1,1) + \pi(1,0)$ and $pr_{(-2,2)}\{\pi(0,1) \otimes F(0,1)\} = \pi(-2,2) + \pi(2,0)$. (It may be useful to refer to Fig.3.) ¿From this we see that $\overline{\pi}(0,0)$ is an actual module.

Case 2-a). Suppose $0 \leq n''$ and $2 \leq m''$.

This is already taken care of : see the paragraph before Case 1-a).

Case 2-b). $0 \leq n''$ and m'' =1.

In this case, $(m^{''}-2,n^{''}+2)$ is not in the positive cone but occurs in the right side of 3.10; it suffices to observe that since $pr_{(m^{''}-2,n^{''}+2)}\{\pi(m^{''}-1,n^{''}+1)\otimes F(1,0)\}=\pi(m^{''}-2,n^{''}+2)+\pi(m^{''},n^{''}+1)$, one can conclude $\{\widetilde{\pi}(m^{''}-2,n^{''}+2)+\widetilde{\pi}(m^{''},n^{''}+1)\}\otimes\overline{Y}(\ell-m_0-2n_0,n_0)$ is an actual module. All the other terms in the right side of 3.10 contribute actual modules as the remaining $\widetilde{\pi}(,)$ involve parameters in the positive cone.

Case 2-c). $0\leq n^{''}$ and $m^{''}=0$.

The parameters $(,)$ which are not in the positve cone but for which $\widetilde{\pi}(,)$ occur in the right side of 3.10 are $(-1,n^{''}+1)$ and $(-2,n^{''}+2)$. They can be paired with other $\widetilde{\pi}(,)$ which also occur in 3.10. As $\pi(-1,n^{''}+1)+\pi(1,n^{''})=pr_{(-1,n^{''}+1)}\{\pi(0,n^{''})\otimes F(1,0)\}$ and $\pi(-2,n^{''}+2)+\pi(2,n^{''})=pr_{(-2,n^{''}+2)}\{\pi(0,n^{''})\otimes F(2,0)\}$ regrouping the right side of 3.10 presents no difficulty.

Next we suppose that (m,n) is in the third alcove; recall that this means that m,n lie between 0 and ℓ, while $m+n,m+2n$ lie between ℓ and 2ℓ. Let $m_0=2\ell-m-2n$ and $n_0=m+n-\ell$; then one sees that (m_0,n_0) lies in the lowest alcove. Also, note $(m,n)=(m_0+2n_0,\ell-m_0-n_0)$.

We begin with the expression

(3.13)

$$\begin{aligned}
&e^{(m,n)}.St=e^{(m_0+2n_0,\ell-m_0-n_0)}.St\\
&e^{(3\ell,-\ell)}Y(m_0,n_0)\\
&+e^{(2\ell,0)}\{Y(\ell-m_0,m_0+n_0)+Y(\ell-m_0-2n_0,n_0)\}\\
&+e^{(\ell,\ell)}\{Y(m_0+2n_0,\ell-m_0-n_0)+Y(m_0,n_0)\}\\
&+e^{(0,2\ell)}Y(\ell-m_0-2n_0,n_0)\\
&+e^{(0,\ell)}\{Y(\ell-m_0,\ell-n_0)+Y(\ell-m_0-2n_0,m_0+n_0)\}\\
&+e^{(-\ell,\ell)}Y(m_0,\ell-m_0-n_0)\\
&+e^{(0,0)}Y(\ell-m_0-2n_0,m_0+n_0)\\
&+e^{(\ell,0)}\{Y(m_0+2n_0,\ell-n_0)+Y(m_0,\ell-m_0-n_0)\}
\end{aligned}$$

(See Fig.4).

Suppose $\nu=(\ell m^{''}+m,\ell n^{''}+n)$ where $m^{''},n^{''}\in Z^{+}$ and m,n are as above (i.e., (m,n) is in the third alcove). By the theorem in section 2, the member of the coherent family (of virtual representations of U_λ) attached to $(\ell m^{''}+m,\ell n^{''}+n)$ is given by

(3.14)

$$
\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m^{''} + m, \ell n^{''} + n) = \\
&\tilde{\pi}(m^{''} + 3, n^{''} - 1)Y(m_0, n_0) \\
&+\tilde{\pi}(m^{''} + 2, n^{''})\{Y(\ell - m_0, m_0 + n_0) + Y(\ell - m_0 - 2n_0, n_0)\} \\
&+\tilde{\pi}(m^{''} + 1, n^{''} + 1)\{Y(m_0 + 2n_0, \ell - m_0 - n_0) + Y(m_0, n_0)\} \\
&+\tilde{\pi}(m^{''}, n^{''} + 2)Y(\ell - m_0 - 2n_0, n_0) \\
&+\tilde{\pi}(m^{''}, n^{''} + 1)\{Y(\ell - m_0, \ell - n_0) + Y(\ell - m_0 - 2n_0, m_0 + n_0)\} \\
&+\tilde{\pi}(m^{''} - 1, n^{''} + 1)Y(m_0, \ell - m_0 - n_0) \\
&+\tilde{\pi}(m^{''}, n^{''})Y(\ell - m_0 - 2n_0, m_0 + n_0) \\
&+\tilde{\pi}(m^{''} + 1, n^{''})\{Y(m_0 + 2n_0, \ell - n_0) + Y(m_0, \ell - m_0 - n_0)\}
\end{aligned}
$$

The details of the proof that $\overline{\pi}(\ell m^{''} + m, \ell n^{''} + n)$ is an actual module are similar to what we saw earlier in the case of the second alcove - namely, though certain terms $\tilde{\pi}(,)$ which appear on the right side of 3.14 may, apriori, be only virtual modules they could always be paired with other terms $\tilde{\pi}(,)$ such that the sum is an actual module. Occasionally, one may require to regroup the original troublesome $\tilde{\pi}(,)$ with more than one term. Such a pairing is invariably possible, by appealing, if necessary,to the decomposition of the modules $Y(,)$ into irreducibles $\overline{Y}(,)$. In the present case, $(m^{''}+3, n^{''}-1)$ is not in the positive cone if $n^{''} = 0$. Likewise, $(m^{''} - 1, n^{''} + 1)$ is not in the positive cone if $m^{''} = 0$. If $n^{''} = 0$, $\pi(m^{''} + 3, n^{''} - 1) + \pi(m^{''} + 1, n^{''} + 1) = pr_{(m^{''}+3, n^{''}-1)}\{\pi(m^{''} + 2, n^{''}) \otimes F(1, 0)\}$ As $(m^{''} + 2, n^{''})$ is in the positve cone, the required pairing is achieved. Likewise, when $m^{''} = 0$, it suffices to observe, $\pi(m^{''} - 1, n^{''} + 1) + \pi(m^{''} + 1, n^{''}) = pr_{(m^{''}-1, n^{''}+1)}\{\pi(m^{''}, n^{''} + 1) \otimes F(1, 0)\}$.

Finally, we consider the case when (m, n) lies in the 'highest' alcove. In otherwords, m, n, lie between 0 and ℓ, $m+n$ lies between ℓ and 2ℓ and $m+2n$ lies between 2ℓ and 3ℓ. Let $m_0 = \ell - m$ and $n_0 = \ell - n$. Then, (m_0, n_0) lies in the lowest alcove. We use the expression,

(3.15)

$$
\begin{aligned}
&e^{(m,n)}.St = e^{(\ell-m_0,\ell-n_0)}.St\\
&e^{(0,0)}Y(m_0,n_0)\\
&+e^{(\ell,0)}\{Y(\ell-m_0,m_0+n_0)+Y(\ell-m_0-2n_0,n_0)\}\\
&+e^{(2\ell,0)}\{Y(m_0+2n_0,\ell-m_0-n_0)+Y(m_0,n_0)\}\\
&+e^{(3\ell,0)}Y(\ell-m_0-2n_0,n_0)\\
&+e^{(\ell,\ell)}\{Y(\ell-m_0,\ell-n_0)+Y(\ell-m_0-2n_0,m_0+n_0)\}\\
&+e^{(0,2\ell)}Y(m_0,\ell-m_0-n_0)\\
&+e^{(-\ell,2\ell)}Y(\ell-m_0-2n_0,m_0+n_0)\\
&+e^{(0,\ell)}\{Y(m_0+2n_0,\ell-n_0)+Y(m_0,\ell-m_0-n_0)\}
\end{aligned}
$$

(See Fig.5).

Suppose $\nu = (\ell m'' + m, \ell n'' + n)$ where $m'', n'' \in Z^+$ and m, n are as above (i.e., (m,n) is in the highest alcove). By the theorem in section 2, the member of the coherent family (of virtual representations of U_λ) attached to $(\ell m'' + m, \ell n'' + n)$ is given by

(3.16)

$$
\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m''+m,\ell n''+n) =\\
&\widetilde{\pi}(m'',n'')Y(m_0,n_0)\\
&+\widetilde{\pi}(m''+1,n'')\{Y(\ell-m_0,m_0+n_0)+Y(\ell-m_0-2n_0,n_0)\}\\
&+\widetilde{\pi}(m''+2,n'')\{Y(m_0+2n_0,\ell-m_0-n_0)+Y(m_0,n_0)\}\\
&+\widetilde{\pi}(m''+3,n'')Y(\ell-m_0-2n_0,n_0)\\
&+\widetilde{\pi}(m''+1,n''+1)\{Y(\ell-m_0,\ell-n_0)+Y(\ell-m_0-2n_0,m_0+n_0)\}\\
&+\widetilde{\pi}(m'',n''+2)Y(m_0,\ell-m_0-n_0)\\
&+\widetilde{\pi}(m''-1,n''+2)Y(\ell-m_0-2n_0,m_0+n_0)\\
&+\widetilde{\pi}(m'',n''+1)\{Y(m_0+2n_0,\ell-n_0)+Y(m_0,\ell-m_0-n_0)\}
\end{aligned}
$$

$(m''-1, n''+2)$ is not in the positive cone if $m'' = 0$. In this case, one can however see that $\pi(m''-1,n''+2)+\pi(m''+1,n''+1)$ is an actual module as $\pi(m''-1,n''+2)+\pi(m''+1,n''+1) = pr_{(m''-1,n''+2)}\{\pi(m'',n''+1)\otimes F(1,0)\}$. This leads to the proof that $\overline{\pi}(\ell m''+m,\ell n''+n)$ is an actual module in this case.

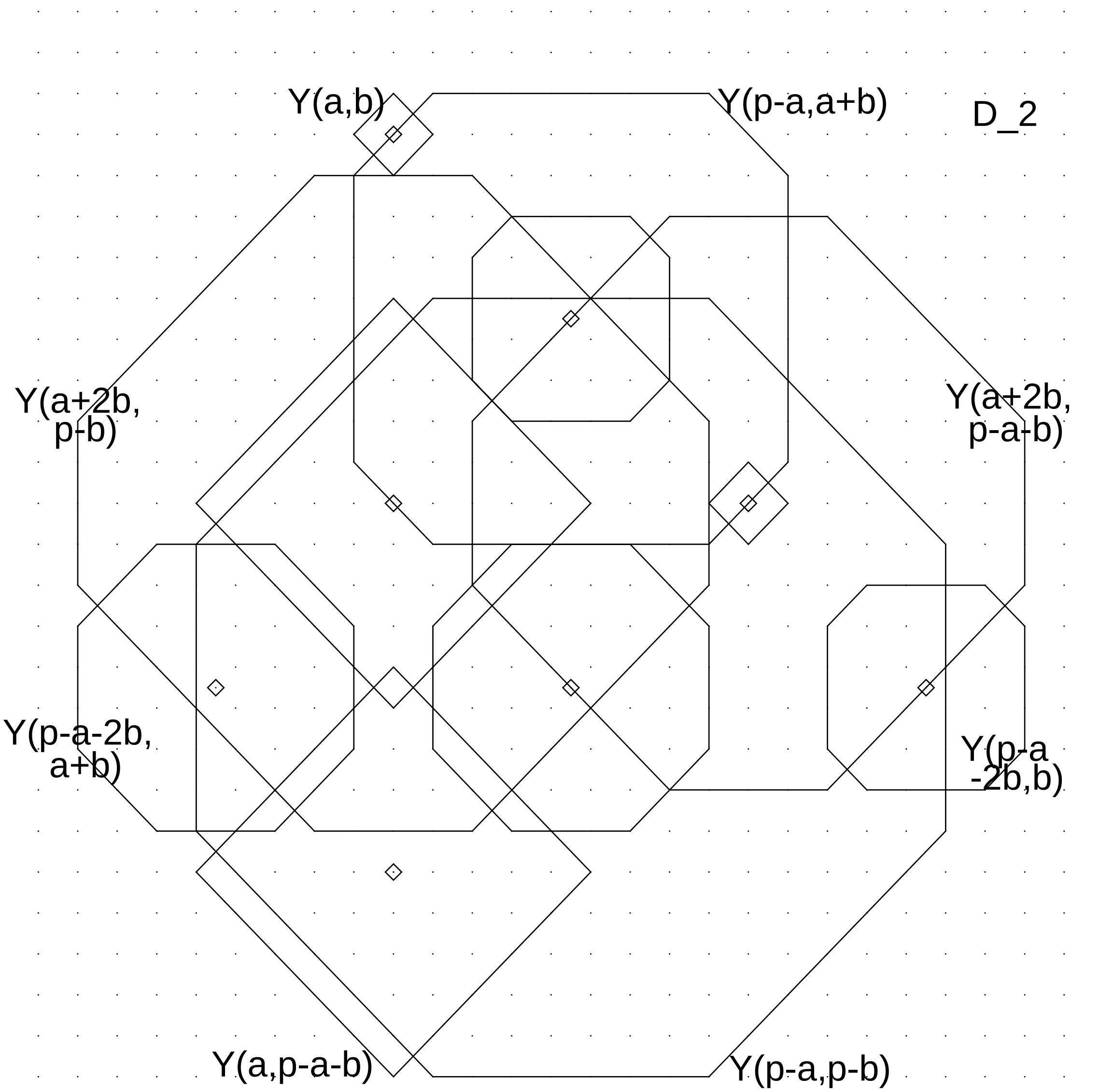


Fig.1. p=\ell, (a,b) \in lowest alcove. Octagons represent Weyl modules with center(◇)shifted. Outer octagon represents St. Shift is determined by choice of origin and positive system. See figs.2,3,4,5.

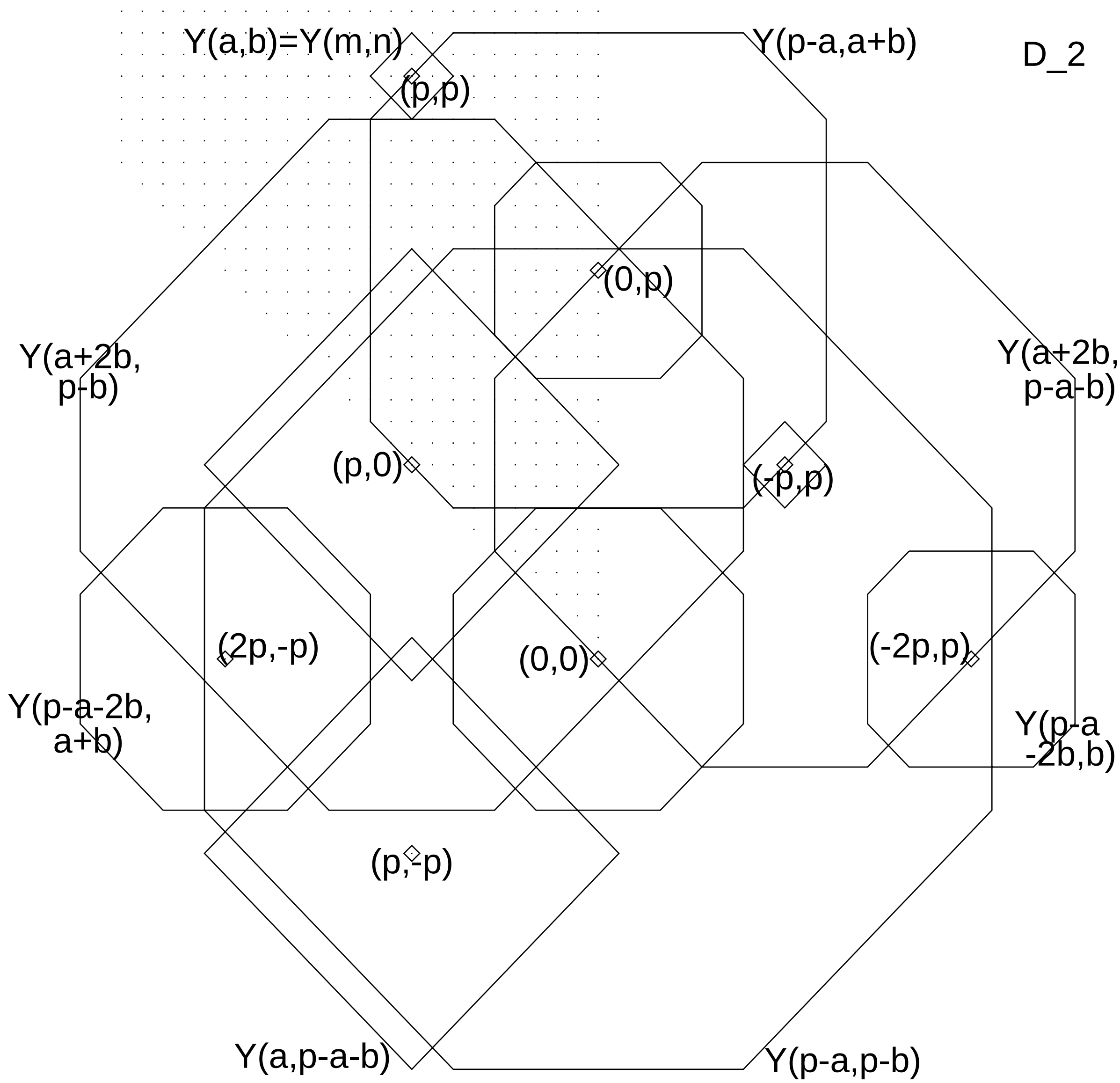


Fig.2. See Eqns 3.2,3.3. p=\ell, (a,b)=(m,n) \in lowest alcove. (m'',n'')=(0,0). Shaded region is positive chamber

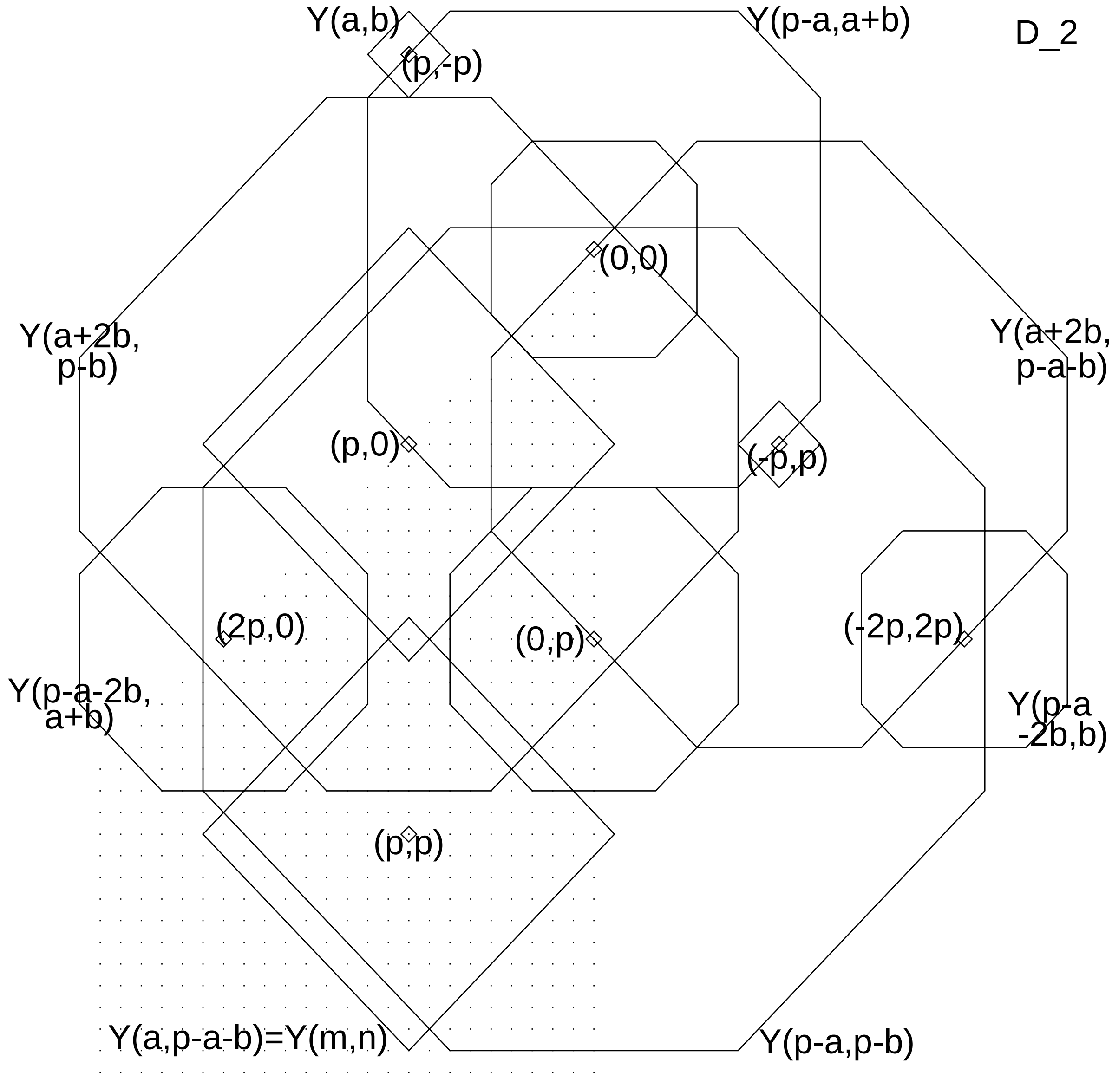


Fig.3. See Eqns 3.8,3.9. p=\ell, (a,b)=(m_0,n_0),
(m'',n'')=(0,0). (m,n) \in second alcove.
Shaded region is positive chamber

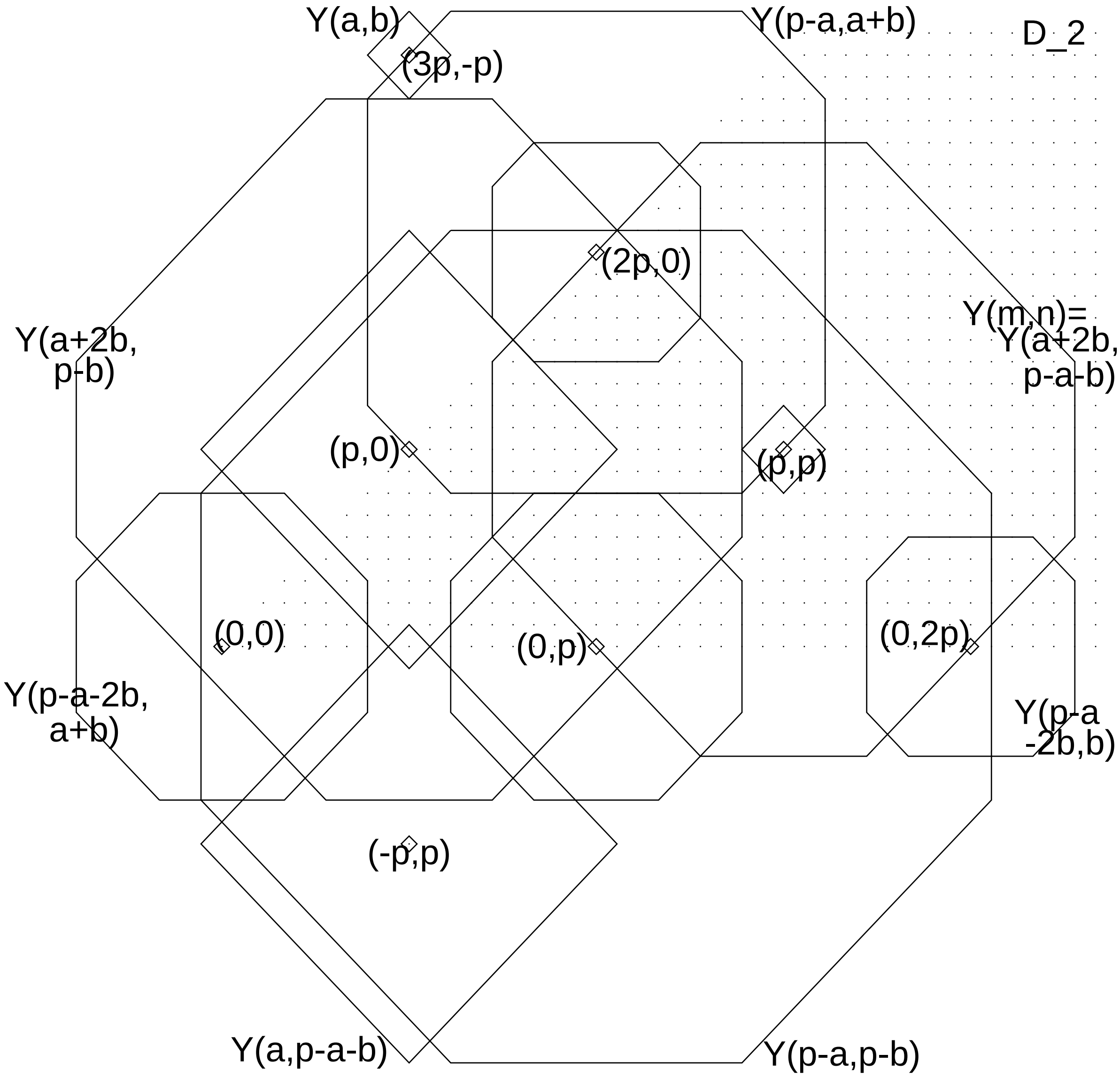


Fig.4. See Eqns 3.13,3.14. p=\ell, (a,b)=(m_0,n_0), (m'',n'')=(0,0). (m,n) \in third alcove. Shaded region is positive chamber

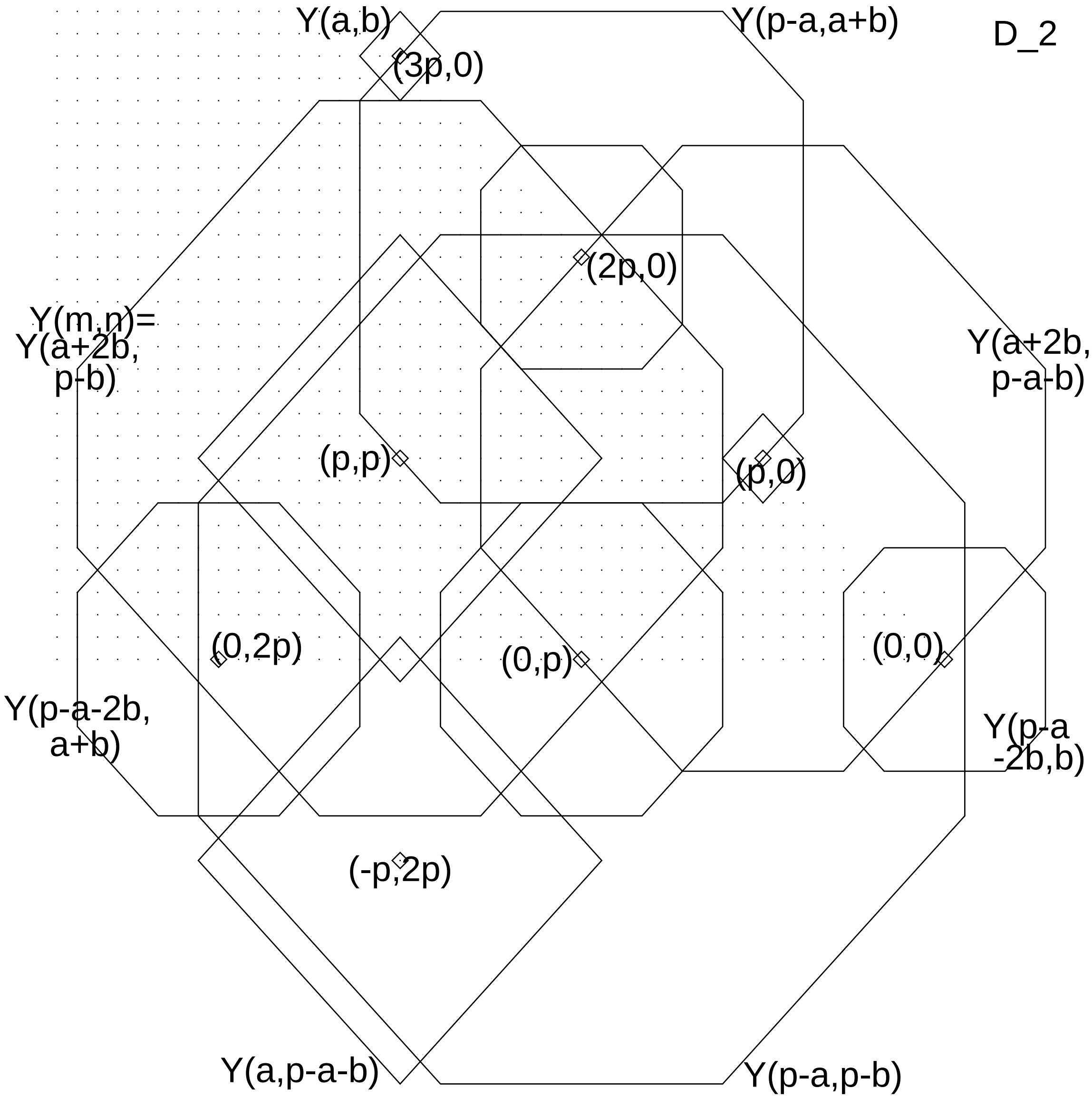


Fig.5. See Eqns 3.15,3.16. p=\ell, (a,b)=(m_0,n_0).
(m,n) \in highest alcove. (m’’,n’’)=(0,0).
Shaded region is positive chamber.

Finally, we remark that when the members of the coherent family $\pi()$ have nonintegral infinitesimal character, no additional problems are encountered. This remark applies also in the case of $\mathbf{A_2}$ which is taken up next. Thus, for $\mathbf{B_2}$ and $\mathbf{A_2}$ (and of course $\mathbf{A_1}$) we have the following result.

Theorem: *Suppose a coherent family $\{\pi(\xi+\nu)\}_{\nu\in\Lambda}$ of virtual representations of U is given ($\xi \in \underline{h}^*$) having the property (see [BV, Definition 2.2]) that*

i) $\pi(\xi+\nu)$ *has infinitesimal character parametrized by the W- orbit of* $\xi+\nu$

ii) $\pi(\xi+\nu)$ *is zero or irreducible when* $\xi+\nu$ *is dominant with respect to a fixed positive system* Ψ, *(i.e., for* $\alpha \in \Psi, 2(\xi+\nu,\alpha)/(\alpha,\alpha) \in \mathbf{Z}, \Leftrightarrow 2(\xi+\nu,\alpha)/(\alpha,\alpha) \in \mathbf{Z}^+$) *and* $\pi(\xi+\nu) \neq 0$ *if* $\xi+\nu$ *is dominant regular.*

Given $\nu \in \Lambda$, *write* $\nu = \nu' + \ell\nu''$ *where* $\nu'' \in \Lambda$ *and* $2(\nu',\alpha)/(\alpha,\alpha) \in \{0,1,\cdots,\ell-1\}$ *for each simple root* α. *Let* $e^{\nu'}.St = \sum \eta_{\nu',\tau}.e^{\ell\epsilon_\tau}$ *in the notation of Proposition (iii) . Choose* $\rho(\nu',\tau) \in \mathcal{F}'$ *whose character is* $\eta_{\nu',\tau}$. *Set*

$$\overline{\pi}(\ell\xi+\nu) = \sum_{\tau} \rho(\nu',\tau) \otimes \widetilde{\pi}(\xi+\nu''+\epsilon_\tau)$$

(in the Grothendieck group of representations of U_λ). *Then* $\{\overline{\pi}(\ell\xi+\nu)\}_{\nu\in\Lambda}$ *is a coherent family of virtual representations of* U_λ. *Furthermore, the virtual module* $\overline{\pi}(\ell\xi+\nu)$ *for dominant* $\ell\xi+\nu$ *(with respect to the positive system in (ii) above) is actually represented in the Grothendieck group by a* U_λ*- module.*

$\mathbf{A_2}$

If Λ_1, Λ_2 are the fundamental weights, let us denote the weight $m\Lambda_1 + n\Lambda_2$ by the pair (m,n). The fundamental parallelopiped $\{(m,n) \mid 1 \le m \le \ell, 1 \le n \le \ell\}$ can be divided into two 'alcoves' described as follows: 'lowest' alcove $= \{(m,n) \mid 1 \le m \le \ell, 1 \le n \le \ell, 1 \le m+n \le \ell\}$, and 'Highest' alcove $= \{(m,n) \mid 1 \le m \le \ell, 1 \le n \le \ell, \ell \le m+n \le 2\ell\}$.

If μ is a dominant integral weight let $Y(\mu)$ denote the Weyl module for U_λ (where λ is a primitive $\ell - th$ root of unity) whose character $\chi(m,n)$ is given by

$$c(e^\mu) \;=\; \frac{\sum_{\tau\in W} (-1)^{\ell(\tau)} e^{\tau\mu}}{\sum_{\tau\in W} (-1)^{\ell(\tau)} e^{\tau\rho}}.$$

and let $\overline{Y}(\mu)$ denote the unique irreducible quotient of $Y(\mu)$ if the latter is nonzero and zero otherwise. The following decomposition of the Weyl module into irreducibles is known (see [APW, section 11]).

(3.17) If (m,n) is in the lowest alcove then

i) $Y(m,n) = \overline{Y}(m,n)$

ii) $Y(\ell - n, \ell - m) = \overline{Y}(\ell - n, \ell - m) + \overline{Y}(m,n)$

If (m,n) is in the lowest alcove then $(n, \ell - m - n)$ and $(\ell - m - n, m)$ are also in the lowest alcove and hence we also have

iii) $Y(n, \ell - m - n) = \overline{Y}(n, \ell - m - n)$

iv) $Y(m+n, \ell - n) = \overline{Y}(m+n, \ell - n) + \overline{Y}(n, \ell - m - n)$

v) $Y(\ell - m - n, m) = \overline{Y}(\ell - m - n, m)$

vi) $Y(\ell - m, m+n) = \overline{Y}(\ell - m, m+n) + \overline{Y}(\ell - m - n, m)$

Let $\nu = (\ell m'' + m, \ell n'' + n)$ where $m'', n'' \in Z^+$ and m, n are as above (i.e., (m,n) is in the lowest alcove). By the theorem in section 2, to write down the member of the coherent family (of virtual representations of U_λ) attached to $(\ell m'' + m, \ell n'' + n)$ first we need an expression for $e^{(m,n)}.St$. Such an expression can be found by carrying out the calculation indicated in the proof of proposition (iii) in section 2. We find that

(3.18)

$$
\begin{aligned}
&e^{(m,n)}.St = \\
&e^{(\ell,\ell)} Y(m,n) \\
&+ e^{(0,\ell)} Y(\ell - m, m+n) \\
&+ e^{(-\ell,\ell)} Y(n, \ell - m - n) \\
&+ e^{(0,0)} Y(\ell - n, \ell - m) \\
&+ e^{(\ell,-\ell)} Y(\ell - m - n, m) \\
&+ e^{(\ell,0)} Y(m+n, \ell - n)
\end{aligned}
$$

(See Fig.6.)

By the theorem, the member of the coherent family is given by

(3.19)

$$\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m^{''} + m, \ell n^{''} + n) = \\
&\widetilde{\pi}(m^{''} + 1, n^{''} + 1) \otimes Y(m, n) \\
&+\widetilde{\pi}(m^{''}, n^{''} + 1) \otimes Y(\ell - m, m + n) \\
&+\widetilde{\pi}(m^{''} - 1, n^{''} + 1) \otimes Y(n, \ell - m - n) \\
&+\widetilde{\pi}(m^{''}, n^{''}) \otimes Y(\ell - n, \ell - m) \\
&+\widetilde{\pi}(m^{''} + 1, n^{''} - 1) \otimes Y(\ell - m - n, m) \\
&+\widetilde{\pi}(m^{''} + 1, n^{''}) \otimes Y(m + n, \ell - n)
\end{aligned}$$

Using the decompositions 3.17 in 3.19, we get

(3.20)

$$\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m^{''} + m, \ell n^{''} + n) = \\
&\widetilde{\pi}(m^{''} + 1, n^{''} + 1) \otimes \overline{Y}(m, n) \\
&+\widetilde{\pi}(m^{''}, n^{''} + 1) \otimes \{\overline{Y}(\ell - m, m + n) + \overline{Y}(\ell - m - n, m)\} \\
&+\widetilde{\pi}(m^{''} - 1, n^{''} + 1) \otimes \overline{Y}(n, \ell - m - n) \\
&+\widetilde{\pi}(m^{''}, n^{''}) \otimes \{\overline{Y}(\ell - n, \ell - m) + \overline{Y}(m, n)\} \\
&+\widetilde{\pi}(m^{''} + 1, n^{''} - 1) \otimes \overline{Y}(\ell - m - n, m) \\
&+\widetilde{\pi}(m^{''} + 1, n^{''}) \otimes \{\overline{Y}(m + n, \ell - n) + \overline{Y}(n, \ell - m - n)\}
\end{aligned}$$

If $1\leq m^{''}$ and $1\leq n^{''}$, all the parameters $(,)$ for which $\widetilde{\pi}(,)$ occur in the right side of 3.20 are in the positive cone. Hence,the corresponding virtual modules are actual modules. If $0\leq m^{''}$ and $0 = n^{''}$, then $pr_{(m^{''}+1,-1)}\{\pi(m^{''} + 1, 0) \otimes F(1, 0)\} = \pi(m^{''} + 1, -1) + \pi(m^{''}, 1)$. Thus in the right side of 3.20, $\overline{\pi}(m^{''} + 1, -1) \otimes \overline{Y}(\ell - m - n, m) + \overline{\pi}(m^{''}, 1) \otimes \overline{Y}(\ell - m - n, m)$ is an actual module. Similarly, if $0 = m^{''}$ and $1\leq n^{''}$, then $pr_{(-1,n^{''}+1)}\{\pi(0, n^{''} + 1) \otimes F(0, 1)\} = \pi(-1, n^{''} + 1) + \pi(1, n^{''})$. Thus in the right side of 3.20, $\overline{\pi}(-1, n^{''} + 1) \otimes \overline{Y}(n, \ell - m - n) + \overline{\pi}(1, n^{''}) \otimes \overline{Y}(n, \ell - m - n)$ is an actual module. Thus, $\overline{\pi}(\ell m^{''} + m, \ell n^{''} + n)$ is seen to be an actual module.

Next, suppose that (m, n) is in the highest alcove. Thus, $0\leq m \leq \ell, 0 \leq n \leq \ell$ and $\ell \leq m + n \leq 2\ell$. Put $m_0 = \ell - n, n_0 = \ell - m$. Then, (m_0, n_0) is in the lowest alcove. We can see that

(3.21)

$$\begin{aligned}
&e^{(m,n)}.St = \\
&e^{(0,0)}Y(m_0,n_0) \\
&+e^{(0,\ell)}Y(\ell-m_0,m_0+n_0) \\
&+e^{(0,2\ell)}Y(n_0,\ell-m_0-n_0) \\
&+e^{(\ell,\ell)}Y(\ell-n_0,\ell-m_0) \\
&+e^{(2\ell,0)}Y(\ell-m_0-n_0,m_0) \\
&+e^{(\ell,0)}Y(m_0+n_0,\ell-n_0)
\end{aligned}$$

(See Fig.7.)

By the theorem, the member of the coherent family is given by

(3.22)

$$\begin{aligned}
&\overline{\pi}(\nu) = \overline{\pi}(\ell m'' + m, \ell n'' + n) = \\
&\widetilde{\pi}(m'',n'') \otimes Y(m_0,n_0) \\
&+\widetilde{\pi}(m'',n''+1) \otimes Y(\ell-m_0,m_0+n_0) \\
&+\widetilde{\pi}(m'',n''+2) \otimes Y(n_0,\ell-m_0-n_0) \\
&+\widetilde{\pi}(m''+1,n''+1) \otimes Y(\ell-n_0,\ell-m_0) \\
&+\widetilde{\pi}(m''+2,n'') \otimes Y(\ell-m_0-n_0,m_0) \\
&+\widetilde{\pi}(m''+1,n'') \otimes Y(m_0+n_0,\ell-n_0)
\end{aligned}$$

One sees that all the parameters (,) for which $\overline{\pi}(,)$ occurs in 3.22 are in the positive cone. Thus, $\overline{\pi}(\ell m'' + m, \ell n'' + n)$ is an actual module.

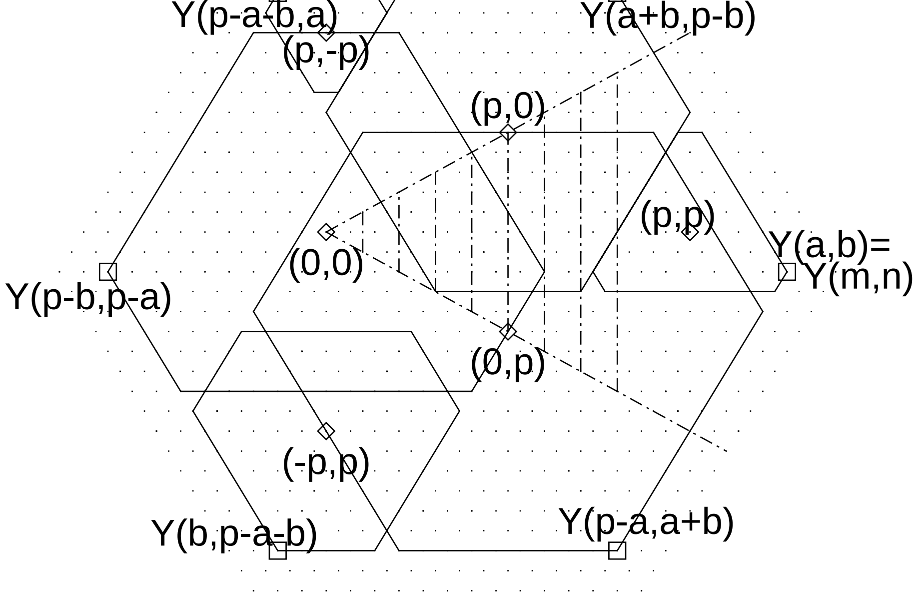
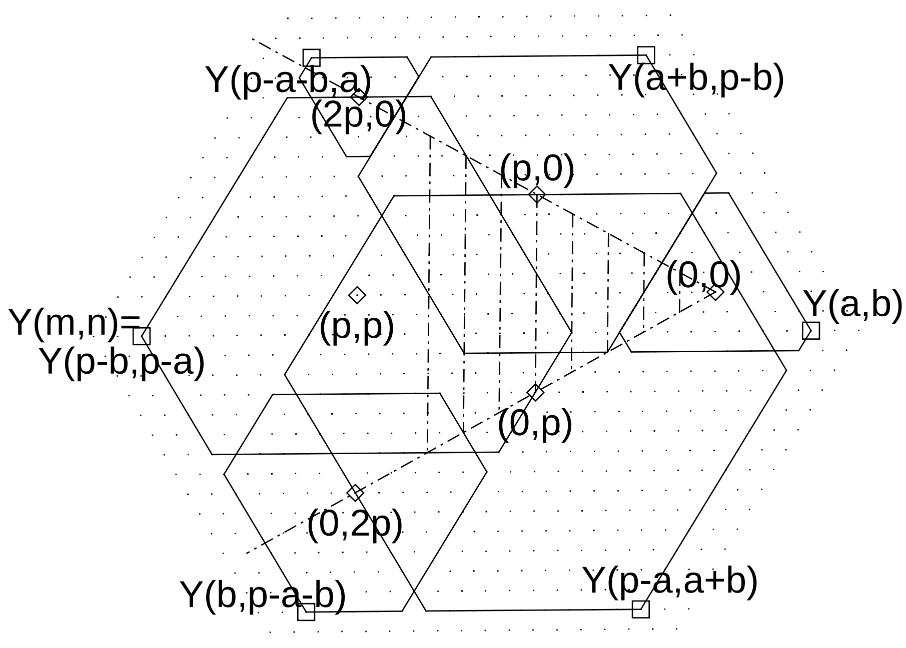


Fig.6. See Eqns 3.18,3.19. p=\ell, (a,b)=(m,n) \in lowest alcove. (m'',n'')=(0,0). Fig.7. See Eqns 3.21,3.22. (a,b)=(m_0,n_0). (m,n) \in highest alcove.



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