Math. Ann. 226, 1—52 (1977) © by Springer-Verlag 1977

An Algebraic Construction of a Class of Representations of a Semi-Simple Lie Algebra

R. Parthasarathy
Department of Mathematics, Tata Institute of Fundamental Research, Bombay 400005, India

Introduction

It is wellknown that the discrete class representations of a semisimple Lie group form the building blocks for the representation theory of semisimple Lie groups. Several attempts have been made to realize these representations by proving analogoues of the Borel-Weil-Bott theorem for noncompact symmetric spaces. We note that in any such analogue there is no "ab initio" proof that the space of "L^2-harmonic forms" concerned is nonzero. In this paper we give a straightforward algebraic construction of a class of irreducible, infinitesimally unitarizable representations of a semisimple Lie algebra. This class contains a special subseries of the discrete series. Our method is by explicitly constructing (through algebraic results about existence and uniqueness) certain operators on the direct sum of some cohomology spaces (of bundles on a compact flag manifold); the operators so defined will represent the given Lie algebra. Going into the details of this paper one can see that our construction has some applications (among them, for example, is a proof of Blattner's conjecture for the special subseries of the discrete series). In [1] Enright and Varadarajan obtained some modules which include all discrete classes. Recently (i.e. at the time of writing up this paper) Schmid has also obtained some modules which include all discrete series.

We now begin to describe our results in detail.

Let G be a connected noncompact semisimple Lie group with finite center. Let K be a maximal compact subgroup of G. Assume rank of of K = rank of G. Let $T \subset K \subset G$ be a Cartan subgroup of G. Let $t \subset k \subset g$ denote the Lie algebras of T, K, and G respectively. Let $t^C \subset k^C \subset g^C$ denote the complexifications of $t \subset k \subset g$. Let Σ be the set of roots of (t^C, g^C) and $P \subset \Sigma$ a positive system of roots. Let P_k and P_n denote the set of compact and noncompact roots in P respectively so that $P = P_k \cup P_n$. Let $b \subset k^C$ be the Borel subalgebra of k^C, defined by

$$b = t^C + \Sigma_{\alpha \in P_k} g^\alpha$$

where g^α is the root space corresponding to the root α. Let $B \subset K^C$ denote the corresponding Borel subgroup of K^C. Let $g = k + p$ be a Cartan decomposition of g and $g^C = k^C + p^C$ its complexification. Then p^C is stable under the adjoint action of K^C. Let

$$p_+ = \Sigma_{\alpha \in P_n} g^\alpha$$ [1]

so that $p_+ \subset p^C$ and is stable under the adjoint action of B. *Throughout this paper we assume*

$$[[p_+, p_+], p_+] = 0 .$$

For $l \geqq 0$, we denote by $S^l(p_+)$ the l^{th} symmetric power of the B module p_+. We denote by p_- the quotient B module p^C/p_+ and by $\wedge^q p_-$ the q^{th} exterior power of p_-. The characters e^λ on T are in one to one correspondence with elements λ belonging to a lattice F of linear forms on i.t. For $\lambda \in F$ we denote by l_λ the one dimensional representation on B got by extending uniquely the character e^λ on T^C. Let ϱ be half the sum of the positive roots. Consider a $\lambda \in F$ for which

i) $\langle \lambda + \varrho, \alpha \rangle > 0$ for $\alpha \in P_k$ and $\langle \lambda + \varrho, \alpha \rangle \geqq 0$ for $\alpha \in P_n$,

and

ii) $H^i(\wedge^q p_- \otimes l_{\lambda + 2\varrho}) = 0$, for $i < s = \dim K^C/B$, where $H^i(\wedge^q p_- \otimes l_{\lambda + 2\varrho})$

denotes the i^{th} cohomology space of the sheaf of germs of holomorphic sections of the homogeneous holomorphic vector bundle on K^C/B induced by the B module $\wedge^q p_- \otimes l_{\lambda + 2\varrho}$.

For each λ as above, we construct in this paper an irreducible representation ϱ_λ of g on the space $\oplus_{l \geqq 0} H^s(S^l(P_+) \otimes l_{\lambda + 2\varrho})$. It will turn out that for $Y \in k$, $\varrho_\lambda(Y)$ maps the l^{th} summand into itself and defines there the usual action of K^C on the cohomology space $H^s(S^l(p_+) \otimes l_{\lambda + 2\varrho})$. For $X \in p$, $\varrho_\lambda(X)$ is a sum of two operators $\partial(X)$ and $\varepsilon(X)$, where $\partial(X)$ maps the l^{th} summand into the $l-1^{\text{st}}$ summand while $\varepsilon(X)$ maps the l^{th} summand into the $l+1^{\text{st}}$ summand. The map

$$\partial : p^C \otimes H^s(S^l(p_+) \otimes l_{\lambda + 2\varrho}) \to H^s(S^{l-1}(p_+) \otimes l_{\lambda + 2\varrho}) \tag{0.1}$$

($\partial(X \otimes u) = \partial(X)u$) is got as follows: We have the differentiation map

$$p^C \otimes S^l(p_+) \to S^{l-1}(p_+)$$

which is a B module map. Tensoring it with the identity of $l_{\lambda + 2\varrho}$ and inducing we get a vector bundle map of the associated vector bundles. This in turn induces a map in cohomology which can be essentially interpreted as ∂. In fact the operators $\partial(X)$ are our starting point. We then look for operators $\varepsilon(X)$ to build up a representation ϱ_λ with $\varrho_\lambda(X) = \partial(X) + \varepsilon(X)$. Our proof of the existence and uniqueness of the operators $\varepsilon(X) : H^s(S^l(p_+) \otimes l_{\lambda + 2\varrho}) \to H^s(S^{l+1}(p_+) \otimes l_{\lambda + 2\varrho})$ is by induction on l. Since the map $\partial : p^C \otimes H^s(S^l(p_+) \otimes l_{\lambda + 2\varrho}) \to H^s(S^{l-1}(p_+) \otimes l_{\lambda + 2\varrho})$ is by definition got by inducing in cohomology the differentiation map (restricted to polynomials), the exact complex of q-forms ($q = 0, 1, \ldots, \dim p^C$) and exterior differentiation δ^q

[1] Throughout we make the assumption that $[[p_+, p_+], p_+] = 0$. Such positive root systems are called special (or admissible). For any given G, there always exist such positive root systems P. Explicit use of this assumption is made in the proof of Lemma 6.7 and in Section 9 where the unitarity of the limits of discrete classes is proved

must have an analogue by inducing in cohomology, (Lemma 2.2). We have maps

$$\partial: \wedge^q p^C \otimes H^s(S^{l-q}(p_+)\otimes l_{\lambda+2\varrho}) \to \wedge^{q+1} p^C \otimes H^s(S^{l-q-1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{0.2}$$

and exact complexes connecting these. The problem of finding

$$\varepsilon: p^C \otimes H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{0.3}$$

for which $\varepsilon(X)u=\varepsilon(X\otimes u)$ must have the desired properties for building up the representation ϱ_λ is translated into a problem of finding a 1-form ε with some properties for which the exterior derivative should be a given 2-form b. Thus one has to know that the exterior derivative of b is zero Luckily enough this happens to be so. (Lemma 4.2). That b should be the boundary of ε is only a part of the conditions on ε. The other conditions on ε namely the K^C linearity of ε and (3.7 IH) these are the ones which force it to be unique – make the problem complicated. The key lemma for solving the above problem is Lemma 4.3, whose proof is lengthy and by induction.

For our inductive proof of the existence and uniqueness of

$$\varepsilon_l: p^C \otimes H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho}),$$

it is very crucial to know and also desirable to have that

$$\varepsilon_i: p^C \otimes H^s(S^i(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{i+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{0.4}$$

are surjective for $i=0, 1, \ldots, l-1$. The following is roughly the idea of proving then that (0.4) is surjective for $i=l$.

Any $X \in p$ is represented on a certain finite dimensional space of spinors L by an operator $C(X): L \to L$, such that $C(X)C(Y)+C(Y)C(X)=(X, Y)$ where $(\,,\,)$ is a scalar multiple of the restriction of the Killing form to p. One first proves that the operators

$$G: H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L \to H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})\otimes L$$

defined by

$$G(u\otimes s)=\Sigma_i \varepsilon(X_i)u\otimes C(X_i)s$$

(where X_i runs through an orthonormal basis of p) have the properties that one should expect of an "adjoint" – if one exists – of

$$F: H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})\otimes L \to H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L$$

defined by

$$F(u\otimes s)=\Sigma_i \partial(X_i)u\otimes C(X_i)s\,.$$

For example the kernel of the Laplacian $GF+FG$ gives the cohomology groups of the complex connecting the maps F for various k, which incidentally are zero for $k>0$. The maps G were formed from the maps ε. But using properties of the Clifford algebra on p for which the space L of spinors is an irreducible module, properties of ε can be recaptured from G. The technical lemma needed is Lemma 6.7.

As indicated earlier, the representation $\varrho_\lambda(X)$ is defined using the maps $\varepsilon_l(X)$.

When $\lambda+\varrho$ is not singular on any noncompact root α, the question arises whether the representation ϱ_λ that we construct belongs to Harish-Chandra's discrete class $\omega(\lambda+\varrho)$. If $\lambda+\varrho$ is assumed to be sufficiently regular, an affirmative answer can be given using [3, Theorem (II)]. However, the above conclusion is true even without the assumption that $\lambda+\varrho$ is sufficiently regular. The situation is similar to the problem of identification of the Enright-Varadarajan modules [1] with discrete classes. Only for this purpose a result of Schmid, namely [8, (1.3)] is used.

By construction the restriction to k of the representation ϱ_λ decomposes in a manner similar to the k decomposition of discrete class representations. Among the ϱ_λ, the ones corresponding to nonregular $\lambda+\varrho$ are called limits of discrete series. Of course, even for the special system of positive roots P that we work with the ϱ_λ that we construct do not account for all limits of discrete series. This is because of the vanishing condition ii) (However, they account for all discrete classes associated to P, if G is linear). The representation can be shown to be same as the representation given by the module $D_{P,\lambda+2\varrho_n}$ constructed by Enright and Varadarajan [9].

Following is a brief idea of our proof in §9 that the representations ϱ_λ are unitaraziable. Dualizing the vector bundles associated to $S^k(p_+)\otimes l_{\lambda+2\varrho_n}$ and taking suitable direct images, one gets bundles $R(\psi_k)$. $\oplus_k H^0(R(\psi_k))$ is our new representation space and $\bar{\varrho}_\lambda$ the new representation. The proof of the existence of the maps $\varepsilon_k(X)$ throws a hermitian form on $H^0(R(\psi_k))$. The operators $\bar{\varrho}_\lambda(Z)$, $Z\in g$, leave the hermitian form on $\oplus_k H^0(R(\psi_k))$ infinitesimally invariant. The convenience in taking the direct images $R(\psi_k)$ is that the hermitian form on $H^0(R(\psi_k))$ is given by invariant hermitian forms on the fibres of $R(\psi_k)$. Similar to F and G we have $\bar{F}$ and $\bar{G}$. Since $\bar{G}$ is like an adjoint of $\bar{F}$ we conclude, with appropriate induction hypotheses that the hermitian form on $H^0(R(\psi_{k+1}))\otimes L$ must be positive definite on the image of $\bar{G}$. This image contains sections of $R(\psi_{k+1})\otimes a$ line sub-bundle of L. Eventually we conclude that the hermitian form on the fibres of $R(\psi_{k+1})$ must be positive definite.

§1

G denotes, as in the introduction, a connected noncompact semisimple Lie group, not necessarily linear. G is assumed to have a compact Cartan subgroup. We fix one such, denoted T, and let K be a maximal compact subgroup of G containing T. $g\supset k\supset t$ denotes the corresponding Lie algebras. We denote by p the subspace of g in the Cartan decomposition $g=k+p$ and consider p as a K module through the adjoint action. The superscript C will denote complex conjugation. The root system for (g^C, t^C) will be denoted by Σ. For a root $\alpha\in\Sigma$, g^α denotes the one dimensional eigenspace of α in g^C. We fix an arbitrary[1] positive root system P and set,

$$P_k=\{\alpha\in P; g^\alpha\subseteqq k^C\},$$

and

$$P_n=\{\alpha\in P; g^\alpha\subseteqq p^C\}.$$

Roots in P_k are positive compact roots and roots in P_n are positive noncompact roots. We put

$$\varrho=\tfrac{1}{2}\langle P\rangle,\quad \varrho_k=\tfrac{1}{2}\langle P_k\rangle \quad\text{and}\quad \varrho_n=\tfrac{1}{2}\langle P_n\rangle,$$

where for a subset Y, $\langle Y \rangle$ denotes the sun of the elements in Y. We once for all let

$$m = \# P_n \quad \text{and} \quad s = \# P_k . \tag{1.0}$$

$\hat{T}$ denotes the character group of T. We let $t^* = \mathrm{Hom}(t, R)$, the real dual space of t. Elements $e^\mu \in T$ are in one-one correspondence with elements μ in a lattice $F \subseteqq it^* \subseteqq t^{*C}$. As in usual in the finite dimensional representation theory for k, for a dominant integral linear from $v \in it^*$, (dominant w.r.t P_k), τ_v shall denote the finite dimensional irreducible representation of k^C, with highest weight v. (1.1)

Let B be the Borel subgroup of K^C whose Lie algebra is

$$b = t^C + \Sigma_{\alpha \in P_k} g^\alpha .$$

Then $K/T = K^C/B$ becomes a K^C homogeneous compact complex manifold for a linear form $\mu \in \mathrm{Hom}(t^C, C)$, we denote by l_μ the one-dimensional b-module obtained by extending trivially on the nilpotent radical $\Sigma_{\alpha \in P_k} g^\alpha$. If e^μ is a character on T, l_μ also stands for the B-module given by it. By restriction, the K^C module p^C becomes a B module. Then

$$p_+ = \Sigma_{\alpha \in P_n} g^\alpha \tag{1.2}$$

is a B-submodule of p^C. Then

$$p_- = p^C / p_+ \tag{1.3}$$

is also a B module. The Killing forms (,) of g^C restricts to a nondegenerate symmetric bilinear form on p^C, which is positive definite on p. We have

$$(p_+, p_+) = 0 .$$

Therefore, the B module p_- is the dual of the B module p_+. For a holomorphic B-module m, $H^i(m)$ stands for the i^{th} cohomology space with coefficients in the sheaf of germs of holomorphic sections of the homogeneous holomorphic vector bundle over K^C/B associated to m.

We now define

$$\tilde{F} = \{\lambda \in F | \langle \lambda + \varrho, \alpha \rangle > 0, \alpha \in P_k, \text{ and } \langle \lambda + \varrho, \alpha \rangle \geqq 0, \alpha \in P_n\} . \tag{1.4}$$

2ϱ being the sum of all the roots in P, $2\varrho \in F$. Thus,

$$\lambda + 2\varrho \in F . \tag{1.5}$$

As above, $l_{\lambda + 2\varrho}$ denotes the one dimensional B module given by $\lambda + 2\varrho$. If m is a B module, we denote by $S^l(m)$ the l-th symmetric power of $m (l = 0, 1, 2, \ldots)$. We now consider the tensor product $S^l(p_+) \otimes l_{\lambda + 2\varrho}$ and the cohomology spaces $H^i(S^l(p_+) \otimes l_{\lambda + 2\varrho})$ $(0 \leqq i \leqq s, l = 0, 1, 2, \ldots)$ of these modules. As usual, these cohomology spaces naturally become finite dimensional K modules.

For a fixed λ belonging to a suitable subset of $\tilde{F}$, the K module

$$\bigoplus_{l=0}^{s} H^s(S^l(p_+) \otimes l_{\lambda + 2\varrho})$$

shall be the basic object of interest to us. It is on this space that we will construct the representation ϱ_λ of g. To make this construction possible, we need some information about the cohomology spaces $H^i(\wedge^q p_- \otimes l_{\lambda+2\varrho})$. For this we make the following

Definition 1.1. We say that $\lambda \in \tilde{F}$ satisfies the condition (*), when $H^i(\wedge^q p_- \otimes l_{\lambda+2\varrho})=0$ for every $i<s$ and for every q, $0 \leqq q \leqq m$.

Henceforth, we make the assumption $\lambda \in \tilde{F}$ and that the condition (*) holds[2] *for λ.* (1.6)

Remark. If $\lambda \in \tilde{F}$ and if $(\lambda+\varrho_k+\langle Q\rangle, \alpha) \geqq 0$, for every $\alpha \in P_k$ and every $Q \subseteq P_n$, then λ satisfies the condition (*). (See Lemma 4.1, [3]).

Lemma 1.1. *Let $\lambda \in \tilde{F}$ and suppose that condition (*) holds for λ. Then for all $l=0, 1, 2, \ldots$*

$$H^i(S^l(p_+) \otimes l_{\lambda+2\varrho})=0,$$

for all $i<s$.

Proof. The proof of this is contained in the proof of Lemma 5.3, [3]. One has to note that for $\lambda \in \tilde{F}$ the condition (*) of this paper is the same as the condition (#) for $\lambda+2\varrho_n$ of [3] q.e.d.

§2

In this article we will define a map

$$\partial: p^C \otimes_C H^s(S^{l+1}(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^l(P_+) \otimes l_{\lambda+2\varrho})$$

$(l=0, 1, 2, \ldots)$ and study its properties. First we fix the following conventions. Let m be a K^C module. By restriction to B, m becomes a B module. The associated vector bundle over K^C/B is holomorphically trivial and $m \simeq H^0(m)$ as K^C modules. If another B module n is given, then the cup product of cohomology spaces gives a K^C module isomorphism

$$m \otimes H^i(n) \simeq H^0(m) \otimes H^i(n) \xrightarrow{\sim} H^i(m \otimes n) \tag{2.1}$$

The Killing form $(,)$ of g^C is nondegenerate and symmetric on p^C. By this bilinear form the K^C module p^C is isomorphic to the K^C module $\mathrm{Hom}(p^C, C)$. We make use of this isomorphism in the future, whenever convenient to do so, without making further mention. The inclusion $p_+ \to p^C$, gives rise to an inclusion

$$S^l(p_+) \to S^l(p^C) \tag{2.2}$$

which respects the B module structures. Since p^C has been identified to its dual space, $S^l(p^C)$ is the space of polynomials on p^C of homogeneous degree l. Thus one has the differentiation map

$$\delta: p^C \otimes S^{l+1}(p^C) \to S^l(p^C) \tag{2.3}$$

given by $\delta(X \otimes f)=\delta_X f$. δ is easily seen to be K^C linear.

[2] There exists an integer N such that for every odd integer $N'>N$ and every $\lambda \in \tilde{F}$ (cf. (1.4)) $N'(\lambda+\varrho)-\varrho \in \tilde{F}$ and the cohomology vanishing condition is satisfied for $N'(\lambda+\varrho)-\varrho$. In particular the set of limits of discrete classes that we construct is nonempty

Remark 2.1. Let $S(p^C)=\bigoplus_{l=0}^{\infty} S^l(p^C)$ denote the symmetric algebra on p^C. For $X\in p^C$, let

$$\delta_X : p^C \to S(p^C) \tag{2.4}$$

be the linear map given

$$\delta_X(Y)=(X, Y)\in S^0(p^C), \tag{2.5}$$

the bilinear form being the Killing form. There exists a unique extension

$$\delta_X : S(p^C)\to S(p^C) \tag{2.6}$$

of (2.4) as a derivation of the algebra $S(p^C)$. This gives rise to a map

$$\delta : p^C\otimes S(p^C)\to S(p^C)$$

and one has

$$\delta\{p^C\otimes S^{l+1}(p^C)\}\subseteqq S^l(p^C).$$

The map in (2.3) is the same.

Under the map (2.3)

$$\delta(p^C\otimes S^{l+1}(p_+))\subseteqq S^l(p_+). \tag{2.7}$$

Thus one gets a map also denoted by δ,

$$\delta : p^C\otimes S^{l+1}(p_+)\to S^l(p_+), \tag{2.8}$$

which is a B linear map. Tensoring (2.8) with the one dimensional B module $l_{\lambda+2\varrho}$, we get a map also denoted by δ,

$$\delta : p^C\otimes S^{l+1}(p_+)\otimes l_{\lambda+2\varrho}\to S^l(p_+)\otimes l_{\lambda+2\varrho} \tag{2.9}$$

which is B linear. Take the associated vector bundles on K/T; (2.9) induces a bundle map between these vector bundles, which in turn induces K-linear maps between the i^{th} cohomology spaces of the corresponding sheaves of germs of holomorphic sections. Thus one has a K linear map

$$\partial : H^s(p^C\otimes S^{l+1}(p_+)\otimes l_{\lambda+2\varrho})\to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \tag{2.10}$$

By (2.1) the first cohomology space can be identified with $p^C\otimes H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho})$. Thus, we get the desired map

$$\partial : p^C\otimes H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho})\to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}). \tag{2.11}$$

For $X\in p^C$ and $u\in H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho})$, we now define

$$\partial(X)u=\partial(X\otimes u) \tag{2.12}$$

Thus, $\forall\, X\in p^C$, we have the operator

$$\partial(X) : H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})\to H^s(S^{l-1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{2.13}$$

($\partial(X)$ is identically zero, on the 0-th summand.)

Our aim is to define operators

$$\varepsilon(X): \oplus H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to \oplus H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}),$$

for every $X \in p^C$, such that $\varepsilon(X)\{H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})\} \subseteq H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho})$, so as to have the following: For $Y \in k^C$, define

$$\varrho_\lambda(Y): \oplus_l H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to \oplus_l H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$$

so that $\varrho_\lambda(Y)$ maps the l^{th} summand into itself and defines there the representation of k^C on $H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. For $X \in p^C$, define

$$\varrho_\lambda(X) = \partial(X) + \varepsilon(X).$$

For $Z = Y + X \in g^C$, where $Y \in k^C$ and $X \in p^C$, define

$$\varrho_\lambda(Z) = \varrho_\lambda(Y) + \varrho_\lambda(X).$$

Then the operators $\varrho_\lambda(Z)$, $(Z \in g^C)$, shall define on irreducible representation of g.

The proof of the existence and uniqueness of the operators $\varepsilon(X)$ goes through Sections 3, 4, 5, and 6.

We begin by proving the exactness of a certain complex connecting the maps ∂, for various l, defined in (2.11).

Since $\dim p_+ = m$, $\dim p^C = 2m$. Let q be an integer such that $0 \leqq q \leqq 2m-1$. Using the maps δ of (2.8) we define maps

$$\delta^q: S^{l+1}(p_+)\otimes \wedge^q p^C \to S^l(p_+)\otimes \wedge^{q+1} p^C \tag{2.14}$$

to be the restriction of exterior differentiation

$$S^{l+1}(p^C)\otimes \wedge^q p^C \to S^l(p^C)\otimes \wedge^{q+1} p^C$$

It is clear that under exterior differentiation $S^{l+1}(p_+)\otimes \wedge^q p^C$ is mapped into $S^l(p_+)\otimes \wedge^{q+1} p^C$. Note that δ^q is a B module homomorphism. The maps in (2.14) fit into the exact sequences in the following lemma.

Lemma 2.1. *Let $m = \dim p_+$ so that $2m = \dim p^C$. Define $S^k(p_+) = 0$, if $k < 0$.*

If $l > m$, one has the following exact complex:

$$0 \to S^l(p_+) \xrightarrow{\delta 0} S^{l-1}(p_+)\otimes \wedge^1 p^C \xrightarrow{\delta 1} S^{l-2}(p_+)\otimes \wedge^2 p^C \ldots \to S^{l-2m}(p_+)\otimes \wedge^{2m} p^C \to 0 \tag{2.15}$$

Let $l \leqq m$. The quotient map $p^C \to p_- = p^C/p_+$ induces maps $\wedge^q p^C \to \wedge^q p_-$. We now have the following exact complex.

$$\begin{aligned}0 \to S^l(p_+) &\xrightarrow{\delta 0} S^{l-1}(p_+)\otimes \wedge^1 p^C \xrightarrow{\delta 1} \ldots \to \ldots \to S^1(p_+)\otimes \wedge^{l-1} p^C \\ &\to S^0(p_+)\otimes \wedge^l p^C \to \wedge^l p_- \to 0\end{aligned} \tag{2.16}$$

Proof. The proof of this lemma can be easily extracted using the method in [2, Proof of 3, Theorem p. 27]. q.e.d.

The maps in (2.15) and (2.16) are all B module homomorphisms. Tensoring with the one dimensional B module $l_{\lambda+2\varrho}$ gives rise to the following exact complexes of B module homomorphisms

For $l>m$

$$0\to S^l(p_+)\otimes l_{\lambda+2\varrho}\xrightarrow{\delta^0}S^{l-1}(p_+)\otimes\wedge^1p^C\otimes l_{\lambda+2\varrho}\xrightarrow{\delta^1}\dots\to S^{l-2m}(p_+)\otimes\wedge^{2m}p^C\otimes l_{\lambda+2\varrho}\to 0 \tag{2.17}$$

is an exact complex of B module homomorphisms.

For $l\leqq m$

$$0\to S^l(p_+)\otimes l_{\lambda+2\varrho}\xrightarrow{\delta^0}S^{l-1}(p_+)\otimes\wedge^1p^C\otimes l_{\lambda+2\varrho}\xrightarrow{\delta^1}\dots\to S^1(p_+)\otimes\wedge^{l-1}p^C\otimes l_{\lambda+2\varrho}\to S^0(p_+)\otimes\wedge^lp^C\otimes l_{\lambda+2\varrho}\to\wedge^lp_-\otimes l_{\lambda+2\varrho}\to 0 \tag{2.18}$$

is an exact complex of B module homomorphisms.

Lemma 2.2. *For $l>m$, (2.17) induces an exact complex of K module homomorphisms*

$$0\to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})\xrightarrow{\partial^0}H^s(S^{l-1}(p_+)\otimes\wedge^1p^C\otimes l_{\lambda+2\varrho})\xrightarrow{\partial^1}\dots\to H^s(S^{l-2m}(p_+)\otimes\wedge^{2m}p^C\otimes l_{\lambda+2\varrho})\to 0 \tag{2.19}$$

For $l\leqq m$, (2.18) induces an exact complex of K module homomorphisms.

$$0\to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})\xrightarrow{\partial^0}H^s(S^{l-1}(p_+)\otimes\wedge^1p^C\otimes l_{\lambda+2\varrho})\xrightarrow{\partial^1}\dots\to H^s(S^0(p_+)\otimes\wedge^lp^C\otimes l_{\lambda+2\varrho})\to H^s(\wedge^lp_-\otimes l_{\lambda+2\varrho})\to 0 \tag{2.20}$$

Proof. Consider (2.18) $W^{l,q}=$ the image of δ^q in $S^{l-q-1}(p_+)\otimes\wedge^{q+1}p\otimes l_{\lambda+2\varrho}$. We then have a short exact sequence

$$0\to W^{l,q-1}\to S^{l-q}(p_+)\otimes\wedge^qp^C\otimes l_{\lambda+2\varrho}\to W^{l,q}\to 0 \tag{2.21}$$

By Lemma 1.1 and Remark 2.1, we know that the i^{th} cohomology spaces of the middle term are zero for $i<s$. Considering the long cohomology exact sequence associated to (2.21), we can deduce the following:

If for a q, $H^i(W^{l,q})=0$, for $i<s$, then $H^i(W^{l,q-1})=0$, for $i<s$ and we have an exact sequence

$$0\to H^s(W^{l,q-1})\to H^s(S^{l-q}(p_+)\otimes\wedge^qp^C\otimes l_{\lambda+2\varrho})\to H^s(W^{l,q})\to 0 \tag{2.22}$$

But from (2.18), we have the short exact complex

$$0\to W^{l,l-1}\to S^0(p_+)\otimes\wedge^lp^C\otimes l_{\lambda+2\varrho}\to\wedge^lp_-\otimes l_{\lambda+2\varrho}\to 0\,.$$

Since $\lambda\in\tilde{F}$ and satisfies condition (∗), $H^i(\cdot)=0$, for $i<s$, for the last two terms. From the long cohomology exact sequence the same vanishing also holds for $W^{l,l-1}$ Thus the conclusion in (2.22) holds for all q. The exactness of the complex in (2.20) follows now. The exactness of the complex in (2.19) is proved similarly. q.e.d.

Remark 2.2. Identify

$$H^s(S^{l-q}(p_+)\otimes\wedge^qp^C\otimes l_{\lambda+2\varrho})\simeq H^s(S^{l-q}(p_+)\otimes l_{\lambda+2\varrho})\otimes\wedge^qp^C\,.$$

Let $X_1, X_2, \ldots, X_{2m}$ be an orthonormal basis of p w.r.t the Killing form. Then the map $\partial^0 : H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{l-1}(p_+)\otimes \wedge^1 p^C \otimes l_{\lambda+2\varrho})$ is given by

$$\partial^0(u) = \sum_{i=1}^{2m} \partial(X_i)u \otimes X_i\,, \tag{2.23}$$

where $\partial(X_i)$ is defined in (2.12). Also, the map

$$\partial^1 : H^s(S^{l-1}(p_+)\otimes \wedge^1 p^C \otimes l_{\lambda+2\varrho}) \to H^s(S^{l-2}(p_+)\otimes \wedge^2 p^C \otimes l_{\lambda+2\varrho})$$

is given by

$$\partial^1(v\otimes X) = \sum_{i=1}^{2m} \partial(X_i)v \otimes X_i \wedge X \tag{2.24}$$

where $v \in H^s(S^{l-1}(p_+)\otimes l_{\lambda+2\varrho})$ and $X \in p^C$. (2.23) and (2.24) can both be easily deduced from the definitions.

§3

In this article we discuss the existence of certain maps $\varepsilon(X) : H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho})$, $(X \in p^C)$. These maps will be later used to build up an irreducible representation of g^C on $\otimes_l H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. The existence and uniqueness is established by an inductive argument on l. For notational convenience, we put

$$H_l = H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \tag{3.1}$$

and denote the representation of K^C (and also k^C) on H_l by τ_l. It will also be convenient to add suffixes to the maps ∂ of (2.11). Thus we rewrite (2.11) as

$$\partial_l : p^C \otimes H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{l-1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{3.2}$$

We also change the notation in (2.12) and rewrite for $X \in p^C$, and $u \in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$

$$\partial_l(X)u = \partial_l(X \otimes u) \tag{3.3}$$

A simple computation using the K linearity of (3.1) shows that for $X \in p^C$, $T \in k^C$ and $u \in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$

$$\partial_l(X)\tau_l(T) = \tau_{l-1}(T)\partial_l(X) + \partial_l[X, T] \tag{3.4}$$

Lemma 3.1. *For $u \in H$, $X, Y \in p^C$, we have*

$$\partial_{l-1}(X)\partial_l(Y)u = \partial_{l-1}(Y)\partial_l(X)u\,.$$

Proof. Of course $l \geqq 2$. By Lemma 2.2, $\partial^1\partial^0(u) = 0$. But by Remark (2.2)

$$\partial^1\partial^0(u) = \sum_{i<j} \{\partial_{l-1}(X_i)\partial_l(X_j)u - \partial_{l-1}(X_j)\partial_l(X_i)u\}\, X_i \wedge X_j\,.$$

The lemma now follows. q.e.d.

We now make the following induction hypothesis: *Induction Hypotheses:* Let $l \geqq 1$. Suppose for $k = 0, \ldots, l-1$ we have defined K^C linear maps

$$\varepsilon_k : p^C \otimes H^s(S^k(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{3.5 IH}$$

with the following properties:

Denoting for $X \in p^C$, $u \in H_k (= H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}))$

$$\varepsilon_k(X)u = \varepsilon_k(X \otimes u), \tag{3.6 IH}$$

they should satisfy

$$\varepsilon_k(X)\varepsilon_{k-1}(Y)v = \varepsilon_k(Y)\varepsilon_{k-1}(X)v, \tag{3.7 IH}$$

for $X, Y \in p^C$ and $v \in H_{k-1} \cdot (0 < k \leqq l-1)$. For $X, Y \in p^C$ and $u \in H_k (0 \leqq k \leqq l-1)$,

$$\begin{aligned} \partial_{k+1}(X)\varepsilon_k(Y)u - \partial_{k+1}(Y)\varepsilon_k(X)u = \tau_k[X, Y]u - \varepsilon_{k-1}(X)\partial_k(Y)u \\ + \varepsilon_{k-1}(Y)\partial_k(X)u \end{aligned} \tag{3.8 IH}$$

[The motivation for desiring (3.8 IH) comes from the fact that we want the operators $\tau(T)$ $(T \in k^C)$ and $\partial(X) + \varepsilon(X)$ $(X \in p^C)$ to define a representation of g^C.] Note that for $k=0$, (3.8 IH) becomes $\partial_1(X)\varepsilon_0(Y)u - \partial_1(Y)\varepsilon_0(X)u = \tau_0(X, Y]u$.

Moreover, our induction hypothesis also includes – this is very crucial – that for $k = 0, 1, \ldots, l-1$ the map ε_k of (3.5 IH) has the following property.

$$\varepsilon_k(p^C \otimes H^s(S^k(p_+) \otimes l_{\lambda+2\varrho})) = H^s(S^{k+1}p_+) \otimes l_{\lambda+2\varrho}) \tag{3.9 IH}$$

Some more induction hypotheses will be stated (p. 13) after some preparation.

In the next section we will prove the existence and uniqueness of a k^C linear map

$$\varepsilon_l : p^C \otimes H^s(S^l(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^{l+1}(p_+) \otimes l_{\lambda+2\varrho})$$

satisfying (3.7 IH) and (3.8 IH) for $k = l$.

In §5, we show that (for $l = 0$)

$$\varepsilon_0 : p^C \otimes H^s(l_{\lambda+2\varrho}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho})$$

exists satisfying (3.8 IH) for $k=0$ and is unique.

In §6, we prove the surjectivity (3.9 IH) for ε_0 and ε_l.

We complete this section after dualizing the exact complexes of §2. This will be used in the next section in proving the existence of the map ε. By Serre's duality Theorem, the dual of

$$H^s(S^l(p_+) \otimes l_{\lambda+2\varrho}) \simeq H^0(S^l(p_-) \otimes l_{-\lambda-2\varrho_n}) \tag{3.10}$$

canonically. Here, ϱ_n equals half the sum of the noncompact positive roots. Here $(S^l(p_-))$ is the dual of $S^l(p_+)$ and $l_{-\lambda-2\varrho}$ is the one dimensional B module given by $-\lambda - 2\varrho_n$. By dualizing the exact complexes (2.17) and (2.18) and taking the induced maps in cohomology is the same as dualizing the complexes (2.19) and (2.20) respectively. We thus have the following

Lemma 3.2. *For $l > m$, the dual complex of* (2.17) *induces an exact complex of K^C module homomorphisms*

$$\begin{aligned} &0 \leftarrow H^0(S^l(p_-) \otimes l_{-\lambda-2\varrho_n}) \xleftarrow{\eta^0} H^0(S^{l-1}(p_-) \otimes \wedge^1 p^C \otimes l_{-\lambda-2\varrho n}) \\ &\xleftarrow{\eta^1} \ldots \leftarrow H^0(S^{l-2m}(p_-) \otimes \wedge^{2m} p^C \otimes l_{-\lambda-2\varrho_n}) \leftarrow 0 \end{aligned} \tag{3.11}$$

(3.11) is precisely the complex dual to (2.19).

For $l \leqq m$, the dual of (2.18) induces an exact complex of K^C module homomorphisms

$$\begin{aligned} 0 \leftarrow H^0(S^l(p_-) \otimes l_{-\lambda-2\varrho_n}) &\xleftarrow{\eta^0} H^0(S^{l-1}(p_-) \otimes \wedge^1 p^C \otimes l_{-\lambda-2\varrho_n}) \\ &\xleftarrow{\eta^1} \ldots \leftarrow H^0(S^0(p_-) \otimes \wedge^l p^C \otimes l_{-\lambda-2\varrho_n}) \leftarrow H^0(\wedge^l p_+ \otimes l_{-\lambda-2\varrho_n}) \leftarrow 0 \end{aligned} \tag{3.12}$$

(3.12) is also the complex dual to (2.20).

Using the maps

$$\eta^0 : p^C \otimes H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \to H^0(S^{k+1}(p_-) \otimes l_{-\lambda-2\varrho_n})$$

of Lemma 3.2, we now define for $X \in p^C$ and $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$

$$\eta_k(X)(\xi) = \eta^0(X \otimes \xi) \quad \text{and note} \quad \eta_k(X)\eta_{k-1}(Y) = \eta_k(Y)\eta_{k-1}(X) \tag{3.13}$$

We should remember that we have fixed the positive root system P, that $\lambda \in \tilde{F}$ with respect to this P and satisfies the condition (*). Of course the choice of P was arbitrary[①]. For our needs in the next section it is convenient to consider also another positive root system which is closely related to P.

Let $\varkappa$ be an element of the Weyl group of (k^C, t^C) such that $\varkappa P_k = -P_k$. The latter is a subgroup of the Weyl group of (g^C, t^C). Thus $-\varkappa P$ is also a positive root system of (g^C, t^C). Symbolically we denote by $F_{-\varkappa}$, the set which is defined analogous to F. The following notation should be regarded carefully. No confusion will arise if one bears in mind that $\varkappa$ acts on t^C but not on g^C.

With respect to $-\varkappa P$, the B module which is the analogue of the B module p_+ of (1.2) is symbolically denoted $\varkappa p_-$. The analogue of p_- is denoted $\varkappa p_+$.

Since $\varkappa$ is an element of the Weyl group of $(k^C, t^C)\varkappa$ acts on $\mathrm{Hom}(t^C, C)$. We now have the following

Lemma 3.3. *Let $\lambda \in \tilde{F}$ and satisfy the condition (*). Then $-\varkappa\lambda \in \tilde{F}_{-\varkappa}$ and satisfies*

$$H^i(\wedge^q \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)}) = 0 \tag{3.14}$$

for $i < s$ and all q.

Proof. Let $\tilde{K}^C$ be a connected simply connected covering of K^C. Let $\tilde{T}^C$ be a cartan subgroup of K^C lying above T^C and B a Borel subgroup of K^C lying above B. Let σ be a Dynkin outer automorphism of K^C. So,

$$\sigma(\tilde{B}) = \tilde{B}\,, \quad \sigma(\tilde{T}) = \tilde{T} \quad \text{and} \quad \sigma/T = -\varkappa \tag{3.15}$$

and also

$$\sigma(g^\alpha) = g^{-\varkappa\alpha}\,, \qquad \forall \alpha \in P_k\,. \tag{3.16}$$

Using σ, we will twist the homogeneous holomorphic vector bundle associated to the B module $\wedge^q p_- \otimes l_{\lambda+2\varrho}$ and obtain a new homogeneous holomorphic vector bundle. Note that $\tilde{K}/\tilde{B} \simeq K/B$.

If $y \in \tilde{K}^C$ and if $\mu(y)$ denotes the old action of y on the vector bundle, we define a new action μ^σ by

$$\mu^\sigma(y) = \mu(\sigma y)\,. \tag{3.17}$$

Note that the holomorphic structure of the vector bundle is unchanged. Only the homogeneous structure is changed. Thus any vanishing of cohomology spaces present in the old bundle is present in the new bundle too. The lemma now follows if one observes that the twisting yields a homogeneous bundle which is identical to the homogeneous bundle induced by the B module $\wedge^q \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)}$. q.e.d.

Remark 3.1. Let $\tilde{K}^C$ and $\sigma: \tilde{K}^C \to \tilde{K}^C$ be as in the proof of Lemma 3.3. Let $\tau: K^C \to \mathrm{Aut}(V)$ be a finite dimensional representation of K^C. Let $\tilde{\tau}$ denote the representation of K^C obtained by composing. The representation $\tau \circ \sigma$ of $\tilde{K}^C$ can be easily seem to be the dual (noncanonically) of τ (by inspecting the weights of the former, which are obtained by applying $-\varkappa$ to the weights of the latter). In particular, $\tau \circ \sigma$ comes down to be a representation of K^C. Again, if E is a homogeneous holomorphic vector bundle on $\tilde{K}^C/\tilde{B} = K^C/B$, let E^σ denote the homogeneous holomorphic vector bundle obtained by twisting by σ as in the proof of Lemma 3.2. If τ^i denotes the representation of K^C on $H^i(E)$, then the representation of K^C on $H^i(E^\sigma)$ is precisely the one obtained by twisting the representation τ^i by σ, as in the beginning of this remark. Thus the representation of K^C on $H^i(E^\sigma)$ is isomorphic (noncanonically) to the dual of τ^i.

Corollary 3.4. *As K^C modules,*

$$H^s(S^l(p_+) \otimes l_{\lambda+2\varrho}) \simeq H^0(S^l(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}), \tag{3.18}$$

for $l = 0, 1, 2, \ldots$.

Proof. By Serre's duality theorem

$$H^0(S^l(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \quad \simeq \text{dual of} \quad H^s(S^l(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)})$$

But by Remark 3.1,

$$H^s(S^l(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \quad \simeq \text{dual of} \quad H^s(S^l(p_+) \otimes l_{\lambda+2\varrho}) \qquad \text{q.e.d.}$$

We now define maps, for $X \in p^C$

$$\eta_k(X): H^0(S^k(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^0(S^{k+1}(\varkappa p_+) l_{\varkappa(\lambda+2\varrho_n)}) \tag{3.18}$$

analogous to (3.13) working with $-\varkappa P$ and $-\varkappa\lambda \in \tilde{F}_{-\varkappa}$.

We are now in a position to state the final part of the induction hypothesis. We assume that for $k = 0, 1, \ldots, l K^C$ linear isomorphisms

$$j^k: H^0(S^k(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \tag{3.19 IH}$$

are given satisfying for $X \in p^C$ and $\xi \in H^0(S^k(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)})$ $(0 \leqq k \leqq l-1)$

$$j^{k+1} \eta_k(X)(\xi) = \varepsilon_k(X) j^k(\xi) \tag{3.20 IH}$$

where $\eta_k(X)$ is the map defined in (3.18) and $\varepsilon_k(X)$ are the maps (3.6 IH).

Proposition 4.1. *Let $l \geqq 1$. Suppose for $0 \leqq k \leqq l-1$, we have defined K^C linear maps* (3.5 *IH*)

$$\varepsilon_k: p^C \otimes H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho}),$$

having all the properties of the induction hypothesis of §3. *Then there exists a unique K^C linear map*

$$\varepsilon_l: p^C \otimes H^s(S^l(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^{l+1}(p_+) \otimes l_{\lambda+2\varrho}) \tag{4.1}$$

satisfying (3.7 IH) *and* (3.8 IH) *for $k = l$.*

The proof of this proposition needs a lot of preparation.
Let $X, Y \in p^C$ and $u \in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. We define

$$a(X, Y): H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \tag{4.2}$$

by $a(X, Y)u =$ the right hand side of (3.8 IH) for $k=l$, i.e.,

$$a(X, Y)u = \tau_l[X, Y]u - \varepsilon_{l-1}(X)\partial_l(Y)u + \varepsilon_{l-1}(Y)\partial_l(X)u \tag{4.3}$$

Note that

$$a(X, Y) \quad \text{is alternating in} \quad X \quad \text{and} \quad Y. \tag{4.4}$$

Also the obvious map

$$a: \wedge^2 p^C \otimes H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$$

is K^C linear.

Definition 4.1. Given vector spaces L and M and $\forall X, Y, Z \in p^C$ given maps

$$\mu(X, Y, Z): L \to M$$

which are multilinear in X, Y and Z, we define

$$\mathrm{alt}_{X,Y,Z}\,\mu(X, Y, Z): L \to M$$

to be the map given by

$$\mathrm{alt}_{X,Y,Z}\,\mu(X, Y, Z) = \sum_\sigma \mathrm{sgn}(\sigma)\mu(\sigma X, \sigma Y, \sigma Z) \tag{4.5}$$

where σ runs through the group of permutations of X, Y, Z and $\mathrm{sgn}(\sigma) = \pm 1$ denotes the signum of σ.

Lemma 4.2. *For every $X, Y, Z \in p^C$,*

$$\mathrm{alt}_{X,Y,Z}\,a(X, Y)\varepsilon_{l-1}(Z) \equiv 0 \tag{4.6}$$

and

$$\mathrm{alt}_{W,X,Y}\,\partial_l(W)a(X, Y) \equiv 0 \tag{4.7}$$

Proof. Using 4.4

$$\begin{aligned}
&1/2\,\mathrm{alt}_{X,Y,Z}\,a(X, Y)\varepsilon_{l-1}(Z)\\
&= a(X, Y)\varepsilon_{l-1}(Z) - a(X, Z)\varepsilon_{l-1}(Y) + a(Y, Z)\varepsilon_{l-1}(X)\\
&= \tau_l[X, Y]\varepsilon_{l-1}(Z) - \varepsilon_{l-1}(X)\partial_l(Y)\varepsilon_{l-1}(Z) + \varepsilon_{l-1}(Y)\partial_l(X)\varepsilon_{l-1}(Z)\\
&\quad - \tau_l[X, Z]\varepsilon_{l-1}(Y) + \varepsilon_{k-1}(X)\partial_l(Z)\varepsilon_{l-1}(Y) - \varepsilon_{l-1}(Z)\partial_l(X)\varepsilon_{l-1}(Y)\\
&\quad + \tau_l[Y, Z]\varepsilon_{l-1}(X) - \varepsilon_{l-1}(Y)\partial_l(Z)\varepsilon_{l-1}(X) + \varepsilon_{l-1}(Z)\partial_l(Y)\varepsilon_{l-1}(X)\\
&= \tau_l[X, Y]\varepsilon_{l-1}(Z) - \varepsilon_{l-1}(X)\,(\partial_l(Y)\varepsilon_{l-1}(Z) - \partial_l(Z)\varepsilon_{l-1}(Y))\\
&\quad + \tau_l[Z, X]\varepsilon_{l-1}(Y) - \varepsilon_{l-1}(Z)\,(\partial_l(X)\varepsilon_{l-1}(Y) - \partial_l(Y)\varepsilon_{l-1}(X))\\
&\quad + \tau_l[Y, Z]\varepsilon_{l-1}(X) - \varepsilon_{l-1}(Y)\,(\partial_l(Z)\varepsilon_{l-1}(X) - \partial_l(X)\varepsilon_{l-1}(Z))
\end{aligned}$$

(Now use (3.8 IH) for $k=l-1$ for the second terms on each line)

$$\begin{aligned}&=\tau_l[X, Y]\varepsilon_{l-1}(Z)-\varepsilon_{l-1}(X)\{\tau_{l-1}[Y, Z]-\varepsilon_{l-2}(Y)\partial_{l-1}(Z)+\varepsilon_{l-2}(Z)\partial_{l-1}(Y)\}\\ &\quad+\tau_l[Z, X]\varepsilon_{l-1}(Y)-\varepsilon_{l-1}(Z)\{\tau_{l-1}[X, Y]-\varepsilon_{l-2}(X)\partial_{l-1}(Y)+\varepsilon_{l-2}(Y)\partial_{l-1}(X)\}\\ &\quad+\tau_l[Y, Z]\varepsilon_{l-1}(X)-\varepsilon_{l-1}(Y)\{\tau_{l-1}[Z, X]-\varepsilon_{l-2}(Z)\partial_{l-1}(X)+\varepsilon_{l-2}(X)\partial_{l-1}(Z)\}\end{aligned}$$

Using (3.7 IH) all the terms in which τ_l or τ_{l-1} does not appear are seen to cancel. On the other hand using the K^C linearity of the maps (3.5 IH), one deduces similar to (3.4), the following. For $X\in p^C$, $T\in k^C$ and $u\in H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$,

$$\varepsilon_k(X)\tau_k(T)=\tau_{k+1}(T)\varepsilon_k(X)+\varepsilon_k[X, T] \tag{4.8}$$

Thus

$$\tau_l[X, Y]\varepsilon_{l-1}(Z)-\varepsilon_{l-1}(Z)\tau_{l-1}[X, Y]=\varepsilon_{l-1}[[X, Y], Z]$$

Now to prove $1/2\ \mathrm{alt}_{X,Y,Z}a(X, Y)\varepsilon_{l-1}(Z)=0$, we use the above identity together with Jacobi's identity. Thus (4.6) is proved. The proof of (4.7) is similar. (q.e.d).

Using the map (3.19 IH) for $k=l$, we define for $X, Y\in p^C$,

$$b(X, Y): H^0(S^k(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$$

to be the composite $a(X, Y)\circ j^l$. Now using (3.20 IH) Lemma 4.2 can be rephrased as follows.

For $X, Y, Z\in p^C$,

$$\mathrm{alt}_{X,Y,Z}b(X, Y)\eta_{l-1}(Z)\equiv 0 \quad\text{and}\quad \mathrm{alt}_{W,X,Y}\partial(W)b(X, Y)\equiv 0\,. \tag{4.9}$$

Proposition (4.1) will now be deduced from the following

Lemma 4.3. *Let $i\geqq 0$, $j\geqq 1$. Let*

$$b: \wedge^2 p^C\otimes H^0(S^i(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^j(p_+)\otimes l_{\lambda+2\varrho}) \tag{4.10}$$

be a K^C linear map. For $X, Y\in p^C$ define a map

$$b(X, Y): H^0(S^i(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^j(p_+)\otimes l_{\lambda+2\varrho})$$

by $b(X, Y)\xi=b(X\wedge Y\otimes\xi)$. Assume

$$\mathrm{alt}_{W,X,Y}\partial_j(W)b(X, Y)\equiv 0 \quad \textit{and} \quad \mathrm{alt}_{X,Y,Z}b(X, Y)\eta_{i-1}(Z)\equiv 0\,.$$

(The second condition becomes vacuous if $i=0$.) Then, i) *$\exists$ a unique K^C linear map*

$$\mu: p^C\otimes H^0(S^i(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^{j+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{4.11}$$

such that with the usual definition of maps

$$\mu(X): H^0(S^i(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^{j+1}(p_+)\otimes l_{\lambda+2\varrho})$$

($X\in p^C$) the following two properties hold.

For $\forall X, Y\in p^C$,

$$\mu(X)\eta_{i-1}(Y)=\mu(Y)\eta_{i-1}(X) \quad \textit{and} \quad \partial_{j+1}(X)\mu(Y)-\partial_{j+1}(Y)\mu(X)=b(X, Y)\,. \tag{4.12}$$

(first part of 4.12 becomes vacuous if $i=0$.)

ii) *These also exists a unique K^C linear map*

$$v: p^C \otimes H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+) \otimes l_{\lambda+2\varrho}) \tag{4.13}$$

such that when

$$v(X): H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$$

($X \in p^C$) *are defined as usual, the following two properties hold*

$$\partial_j(X) v(Y) = \partial_j(Y) v(X) \tag{4.14}$$

$$v(X)\eta_i(Y) - v(Y)\eta_i(X) = b(X, Y). \tag{4.15}$$

The proof of this lemma is by induction on j and to start induction one needs the following lemma which we prove in the next section. The case $i=0$ of Lemma 4.3 is also proved only in the next section.

Lemma 4.4. *Let $i \geqq 0$. Let*

$$b: \wedge^2 p^C \otimes H^0(S^i(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(l_{\lambda+2\varrho}) \tag{4.16}$$

be a K^C linear map. With $b(X, Y)$ defined as usual, assume

$$\mathrm{alt}_{X,Y,Z}\, b(X, Y)\eta_{i-1}(Z) \equiv 0.$$

(This condition is vacuous if $i=0$.) Then

i) $\exists$ *a unique K^C linear map*

$$\mu: p^C \otimes H^0(S^i(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \tag{4.17}$$

such that

$$\partial_1(X)\mu(Y) - \partial_1(Y)\mu(X) = b(X, Y) \tag{4.18}$$

and

$$\mu(X)\eta_{i-1}(Y) = \mu(Y)\eta_{i-1}(X) \tag{4.19}$$

(4.19) *is vacuous if $i=0$.*

ii) *Then also exists a unique K^C linear map*

$$v: p^C \otimes H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(l_{\lambda+2\varrho}) \tag{4.20}$$

having the following property,

$$v(X)\eta_i(Y) - v(Y)\eta_i(X) = b(X, Y). \tag{4.21}$$

Proof of Lemma 4.3 (the case $i>0$). Since for two vector spaces L and M $\mathrm{Hom}(L, M) = L^* \otimes M$, the map b in (4.10) corresponds to an element

$$\bar{b} \in \wedge^2 p^C \otimes H^s(S^i(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M \tag{4.22}$$

where $M = H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$. For any map b as in (4.10), (i.e., without assuming the other data, for a while), the maps

$$\mathrm{alt}_{X,Y,Z}\, b(X, Y)\eta_{i-1}(Z): H^0(S^{i-1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$$

give rise to a map

$$\wedge^3 p^C \otimes H^0(S^{i-1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+) \otimes l_{\lambda+2\varrho}).$$

The last map corresponds to an element

$$\delta\bar{b} \in \wedge^3 p^C \otimes H^s(S^{i-1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M \tag{4.23}$$

Now let us look at the exact complexes of Lemma 2.2 applied to $-\varkappa\lambda \in \tilde{F}_{-\varkappa P}$. We have in particular maps

$$p^C \otimes H^s(S^{i+1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \xrightarrow{\partial^1} \wedge^2 p^C \otimes H^s(S^i(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)})$$
$$\xrightarrow{\partial^2} \wedge^3 p^C \otimes H^s(S^{i-1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \tag{4.24}$$

Tensoring with $\mathrm{id}_M: M \to M$, the identity map, we have also

$$p^C \otimes H^s(S^{i+1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M$$
$$\xrightarrow{\partial^1} \wedge^2 p^C \otimes H^s(S^i(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M$$
$$\xrightarrow{\partial^2} \wedge^3 p^C \otimes H^s(S^{i-1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M \tag{4.25}$$

(We have denoted $\partial^1 \otimes \mathrm{id}_M$ and $\partial^2 \otimes \mathrm{id}_M$ by ∂^1 and ∂^2 respectively.)
It is straightforward to chock that

$$\delta\bar{b} = C \cdot \partial^2 \bar{b} \tag{4.26}$$

where C is a constant independent of $\bar{b}$. For the particular map b given in (4.10) of the lemma, by data, we know that $\delta\bar{b} = 0$ and so $\partial^2 \bar{b} = 0$. By exactness, there exists an element

$$\bar{v} \in p^C \otimes H^s(S^{i+1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M \tag{4.27}$$

such that under the map ∂^1 in (4.25)

$$\partial^1 \bar{v} = \bar{b}\,. \tag{4.28}$$

The maps ∂^1 and ∂^2 of (4.25) are of course K^C linear maps. The map b in (4.10) being K^C linear, the corresponding element $\bar{b}$ of (4.22) is K^C invariant. By Schur's lemma, therefore, the element $\bar{v}$ of 4.27 can be modified to be K^C invariant with (4.28) still satisfied.

Let

$$v: p^C \otimes H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to M = H^s(S^j(p_+) \otimes l_{\lambda+2\varrho}) \tag{4.29}$$

be the K^C linear map corresponding to $\bar{v}$ of (4.27). Much in the same way as remark (4.26), one now reinterprets (4.28) to mean the following: $\forall X, Y \in p^C$,

$$v(X)\eta_i(Y) - v(Y)\eta_i(X) = b(X, Y) \tag{4.30}$$

We want a v which in addition also satisfies (4.14). At any rate, let us define

$$b': \wedge^2 p^C \otimes H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{j-1}(p_+) \otimes l_{\lambda+2\varrho} \tag{4.31}$$

by

$$b'(X, Y) = \partial_j(X) v(Y) - \partial_j(Y) v(X) \tag{4.32}$$

If $j-1 \geqq 1$,

$$\mathrm{alt}_{W,X,Y}\, \partial_{j-1}(W) b'(X, Y) = 0 \tag{4.33}$$

follows easily from Lemma 3.1. Also,

$$\begin{aligned}\mathrm{alt}_{X,Y,Z}b'(X,Y)\eta_i(Z) &= 2\,\mathrm{alt}_{X,Y,Z}\partial_j(X)v(Y)\eta_i(Z)\\ &= \mathrm{alt}_{X,Y,Z}\partial_j(X)\,(v(Y)\eta_i(Z)-v(Z)\eta_i(Y))\\ &= \mathrm{alt}_{X,Y,Z}\partial_j(X)b(Y,Z) \qquad \text{(by (4.30))}\\ &= 0\,, \qquad \text{by data in the lemma}\,.\end{aligned}$$

Now, we make the following induction hypotheses: "Lemma 4.4 is valid for all $i \geqq 0$ and Lemma 4.3 is valid for all $i \geqq 0$ and $k<j$."

Applying the induction hypotheses to the map b' in (4.31), we conclude that there exists a K^C linear map

$$\mu': p^C \otimes H^0(S^{i+1}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+)\otimes l_{\lambda+2\varrho})$$

such that $\forall X, Y \in p^C$,

$$\mu'(X)\eta_i(Y) = \mu'(Y)\eta_i(X) \tag{4.34}$$

and

$$\partial_j(X)\mu'(Y) - \partial_j(Y)\mu'(X) = b'(X, Y) \tag{4.35}$$

Now, we replace v by $v-\mu'$ in (4.29). Then (4.30) and (4.34) imply

$$(v-\mu')(X)\eta_i(Y) - (v-\mu')(Y)\eta_i(X) = b(X, Y)\,.$$

Also (4.32) and (4.35) imply

$$\partial_j(X)(v-\mu')(Y) = \partial_j(Y)(v-\mu')(X)\,.$$

Thus part (ii) of Lemma (4.3) except for the uniqueness is proved under the induction hypothesis.

We now use the map v of part (ii) of Lemma 4.3 to prove part (i). Let $\xi \in H^0(S^{i+1}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})$. Consider the element

$$e(\xi) = \sum_{r=1}^{2m} X_r \otimes v(X_r)\xi \in p^C \otimes H^s(S^j(p_+)\otimes l_{\lambda+2\varrho}) \tag{4.36}$$

Using (4.14) one sees that under the map

$$\partial^1 : p^C \otimes H^s(S^j(p_+)\otimes l_{\lambda+2\varrho}) \to \wedge^2 p^C \otimes H^s(S^{j-1}(p_+)\otimes l_{\lambda+2\varrho})$$

$\partial^1(e(\xi))=0$. Therefore, by the exactness of the complexes of Lemma 2.2, there exists an element $J(\xi) \in H^s(S^{j+1}(p_+)\otimes l_{\lambda+2\varrho})$ such that $e(\xi)=\partial^0(J(\xi))$. Thus we have defined a map

$$J: H^0(S^{i+1}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{j+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{4.37}$$

such that $\forall X \in p^C$ and $\forall \xi \in H^0(S^{i+1}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})$,

$$v(X)\xi = \partial_{j+1}(X)J(\xi)\,. \tag{4.38}$$

We now define a map

$$\mu: p^C \otimes H^0(S^i(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{j+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{4.39}$$

by taking

$$\mu = J \circ \eta_i \tag{4.40}$$

We claim μ satisfies the condition (4.12) and (4.13) of Lemma 4.3. For,

$$\begin{aligned}\mu(X)\eta_{i-1}(Y) &= J \circ \eta_i(X)\eta_{i-1}(Y) \\ &= J \circ \eta_i(Y)\eta_{i-1}(X) = \mu(Y)\eta_{i-1}(X).\end{aligned}$$

Also,

$$\begin{aligned}\partial_{j+1}(X)\mu(Y) - \partial_{j+1}(Y)\mu(X) &= \partial_{j+1}(X)J\eta_i(Y) - \partial_{j+1}(Y)J\eta_i(X) \\ &= v(X)\eta_i(Y) - v(Y)\eta_i(X), \quad \text{by (4.38)} \\ &= b(X, Y), \quad \text{by (4.15)}.\end{aligned}$$

The assertion about uniqueness (for all i and j) in Lemma 4.3 and also in Lemma 4.4 follow from the following Lemma.

Lemma 4.5. *Let* $\max(k, l) \geqq 1$. *Let*

$$\varphi : p^C \otimes H^0(S^l(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \tag{4.41}$$

be a K^C linear map such that

$$\partial_k(X)\varphi(Y) = \partial_k(Y)\varphi(X), \qquad \forall X, Y \in p^C \tag{4.42}$$

and

$$\varphi(X)\eta_{l-1}(Y) = \varphi(Y)\eta_{l-1}(X), \qquad \forall X, Y \in p^C \tag{4.43}$$

(4.42 *is vacuous if* $k=0$, *whereas* 4.43 *is vacuous if* $l=0$.)

Then $\varphi = 0$.

Proof. We first consider the case $l=0$, so that $k \geqq 1$.

For every $\xi \in H^0(S^l(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)})$, consider the element

$$e(\xi) = \sum_{r=1}^{2m} X_r \otimes \varphi(X_r)\xi \in p^C \otimes H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}).$$

Using (4.42) one sees that $\partial^1(e(\xi)) = 0$. Therefore by the exactness of the complexes of Lemma 2.2, there exists an element $J(\xi) \in H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho})$, such that $e(\xi) = \partial^0(J(\xi))$. Thus we have defined a K^C linear map

$$J : H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho}) \tag{4.44}$$

By Corollary 3.4, as K^C modules $H^0(l_{\varkappa(\lambda+2\varrho_n)})H^0(l_{\varkappa(\lambda+2\varrho_n)}) \simeq H^s(l_{\lambda+2\varrho})$. We assert that ' any K^C linear map

$$H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho}), \qquad (\forall k \geqq 0) \tag{4.45}$$

is identically zero'.

This fact is contained in ([3] Lemma 4.2, 4.3).

This proves the map $J=0$ and the case of the lemma for $l=0$ can be now easily deduced.

Next we consider the case $k=0$. For a while let M denote the K^C module $H^s(l_{\lambda+2\varrho})$. The map φ in (4.41) gives rise to an element

$$\bar{\varphi}\in p^C\otimes H^s(S^l(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})\otimes M\,.$$

The map

$$\wedge^2 p^C\otimes H^0(S^{l-1}(\varkappa p_+)\otimes l_{-\varkappa(\lambda+2\varrho_n)})\to H^s(S^0(p_+)\otimes l_{\lambda+2\varrho})$$

given by

$$X\wedge Y\otimes\xi\mapsto\varphi(X)\eta_{l-1}(Y)\xi-\varphi(Y)\eta_{l-1}(X)\xi$$

corresponds to an element

$$\delta\bar{\varphi}\in\wedge^2 p^C\otimes H^s(S^{l-1}(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})\otimes M\,.$$

As remarked in (4.26) one sees that $\delta\bar{\varphi}=C\cdot\partial^1\bar{\varphi}$. But by (4.43), data, $\delta\bar{\varphi}=0$. By exactness, therefore, there exists

$$\bar{\psi}\in H^s(S^{l+1}(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})\otimes M \tag{4.46}$$

such that under the map ∂^0,

$$\partial^0: H^s(S^{l+1}(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})\otimes M\to p^C\otimes H^s(S^l(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})\otimes M$$

$\partial^0(\bar{\psi})=\bar{\varphi}$. φ being K^C linear, $\bar{\varphi}$ is a K^C fixed element and by Schur's lemma, $\bar{\psi}$ can be taken to be K^C fixed. But by Corollary 3.4 and the assertion (4.45) one concludes $\bar{\psi}=0$. Thus $\bar{\varphi}=0$ and also $\varphi=0$.

Now we consider the general case. The proof is by induction on l. We can assume $k\geqq 1$. We make the induction hypothesis that the assertion of the lemma is proved for $l=0,1,\dots,l'-1$. We want to prove for $l=l'$. The proof given earlier for $l=0$ has certain general features. For $\xi\in H^0(S^{l'}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})$, we construct $e(\xi)\in p^C\otimes H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$ as in there. We then get a map, K^C linear,

$$J: H^0(S^{l'}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})$$

such that $e(\xi)=\partial^0 J(\xi)$, i.e., $\forall X\in p^C$,

$$\varphi(X)=\partial_{k+1}(X)J(\xi)\,.$$

Thus to prove $\varphi=0$, it is enough to prove $J=0$. We now define a map φ' by composing,

$$\eta_{l'-1}: p^C\otimes H^0(S^{l'-1}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^0(S^{l'}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})$$

with

$$J: H^0(S^{l'}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})\to H^s(S^{k+1}(p_+)\otimes l_{\lambda+2})$$

Since $\eta_{l'-1}$ is a surjection, in order to prove $J=0$, it suffices to prove $\varphi'=0$. But the induction hypothesis can be applied to φ': For,

$$\begin{aligned}\partial_{k+1}(X)\varphi'(Y)&=\partial_{k+1}(X)J\eta_{l'-1}(Y)=\varphi(X)\eta_{l'-1}(Y)\\&=\varphi(Y)\eta_{l'-1}(X)=\partial_{k+1}(Y)\varphi'(X)\,.\end{aligned}$$

Also, if $l'-1 \neq 0$,

$$\begin{aligned}\varphi'(X)\eta_{l'-2}(Y) &= J\eta_{l'-1}(X)\eta_{l'-2}(Y) = J\eta_{l'-1}(Y)\eta_{l'-2}(X)\\ &= \varphi'(Y)\eta_{l'-2}(X)\,.\end{aligned}$$

The Lemma 4.5 is now completely proved.

§5. Proof of Lemma 4.4 (for all i) and the Proof of Lemma 4.3 ($i=0$)

The proofs depend upon the following lemma.

Lemma 5.1. *Let $\lambda \in \tilde{F}$ and satisfy condition $(*)$. For $q>0$ and $j \geqq 0$, there does not exist any nonzero K^C linear homomorphism*

$$H^s(S^j(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(\wedge^q p_- \otimes l_{\lambda+2\varrho})\,.$$

Proof. We recall the spin representation

$$\sigma : o(p^C) \to \mathrm{End}(L) \tag{5.1}$$

(see [4, §1]). Here $o(p^C)$ is the Lie algebra of all endomorphisms T of p^C which leave the Killing form infinitesimally invariant, i.e., $(TX, Y)+(X, TY)=0$, $\forall X, Y \in p^C$. Let $\mathrm{ad}: k^C \to o(p^C)$ be the adjoint representation. Now define a representation χ of k^C on L by

$$\chi = \sigma \circ \mathrm{ad}\,. \tag{5.2}$$

The weights of χ are of the form

$$\varrho_n - \gamma_1 - \dots - \gamma_t\,, \tag{5.3}$$

where $\gamma_1, \dots, \gamma_t$ are distinct noncompact positive roots (see [4, Remark 2.1]). By (1.4), $2\langle \lambda+\varrho, \alpha\rangle/\langle \alpha, \alpha\rangle$ is a nonzero positive integer, $\forall \alpha \in P_k$. Thus $2\langle \lambda+\varrho_n, \alpha\rangle/\langle \alpha, \alpha\rangle$ is a nonnegative integer for every $\alpha \in P_k$. We thus have an irreducible representation

$$\tau_{\lambda+\varrho_n} : k^C \to \mathrm{End}(V_{\lambda+\varrho_n}) \tag{5.4}$$

In [3, §3, Lemma 3.4] it has been proved that when the condition $(*)$ is satisfied, we have an isomorphism of k^C modules:

$$\bigoplus_{q=0}^{m} H^s(\wedge^q p_- \otimes l_{\lambda+2\varrho}) \simeq V_{\lambda+\varrho_n} \otimes L \tag{5.5}$$

(Although the statement in [3, §3] is made under an extra assumption that $\langle \lambda, \alpha\rangle \geqq 0$, $\forall \alpha \in P_k$, this last assumption is not used there; what is just needed there is that one has an irreducible representation with $\lambda+\varrho_n$ as the highest weight and that condition $(*)$ is satisfied.)

Now to prove Lemma 5.1, we turn Schmid's proof of [7, (8.34) Proposition] (where he proved "Blattner's conjecture" $\Rightarrow$ "vanishing theorem") to our advantage.

The highest weight (w.r.t. P_k) of any irreducible component of $V_{\lambda+\varrho_n} \otimes L$ equals the highest weight of $V_{\lambda+\varrho_n}$ plus a weight of L; so by (5.3) such a highest weight is of the form

$$\lambda + \varrho_n + \varrho_n - \gamma_1 - \dots - \gamma_t\,, \qquad \gamma_i \in P_n\,. \tag{5.6}$$

In $V_{\lambda+\varrho_n}\otimes L$, the irreducible k^C module with highest weight $\lambda+2\varrho_n$ occurs with multiplicity one. Under the isomorphism (5.5) this irreducible module is precisely the summand for $q=0$, i.e. $H^s(l_{\lambda+2\varrho})$. Thus by (5.6) the highest weight of any irreducible component of $\sum_{q=1}^{m} H^s(\wedge^q p_-\otimes l_{\lambda+2\varrho})$ is of the form

$$\lambda+2\varrho_n-\gamma_1-\ldots-\gamma_t, \quad t\geqq 1, \quad \gamma_i\in P_n \tag{5.7}$$

By [3, Lemma 4.3], the highest weight of any irreducible component of

$$H^s(S^j(p_+)\otimes l_{\lambda+2\varrho})$$

is of the form

$$w(\lambda+\varrho+\varrho_n+\beta_1+\ldots+\beta_j)-\varrho_k, \quad \beta_i\in P_n \tag{5.8}$$

where w is an element of the Weyl group of (k^C, t^C). If the assertion of the lemma is false, then an irreducible component of $\otimes_{j\geqq 0}H^s(S^j(p_+)\otimes l_{\lambda+2\varrho})$ and an irreducible component of $\bigoplus_{q=1}^{m} H^s(\wedge^q p_-\otimes l_{\lambda+2\varrho})$ would have to be isomorphic. Thus a weight of the form (5.7) and a weight of the form (5.8) would have to be equal. i.e.

$$\lambda+\varrho+\varrho_n+\beta_1+\ldots+\beta_j=w^{-1}(\lambda+2\varrho_n-\gamma_1-\ldots-\gamma_t+\varrho_k) \tag{5.9}$$

Elements in (5.7), being highest weights, are dominant with respect to P_k; so also is ϱ_k. Thus $\lambda+2\varrho_n-\gamma_1-\ldots-\gamma_t+\varrho_k$ is dominant with respect to P_k. For every weight v which is dominant with respect to P_k and for every element w of the Weyl group of (k^C, t^C), $v-w^{-1}v$ is a sum of positive compact roots. Thus, there exist $\alpha_i\in P_k$, such that

$$w^{-1}(\lambda+2\varrho_n-\gamma_1-\ldots-\gamma_t+\varrho_k)=\lambda+2\varrho_n-\gamma_1-\ldots\gamma_t+\varrho_k-\alpha_1-\ldots-\alpha_r \tag{5.10}$$

(5.9) and (5.10) imply

$$-\gamma_1-\ldots-\gamma_t-\alpha_1-\ldots-\alpha_r=+\beta_1+\ldots+\beta_j.$$

But this is a contradiction since $t\geqq 1$. Thus the assertion of the lemma is true. q.e.d.

The Proof of Lemma 4.4 (for all i). First we consider $i>0$. Repeating the arguments from (4.22) to (4.30), we conclude the existence of the map v in (4.20) having the property (4.21). Thus part ii) of Lemma 4.4 is proved for all $i>0$. (The uniqueness was already proved using Lemma 4.5). Now let $i=0$. The K^C linear map

$$b\colon \wedge^2 p^C\otimes H^0(l_{\varkappa(\lambda+2\varrho_n)})\to H^s(l_{\lambda+2\varrho})$$

of (4.16) gives rise to a K^C invariant element

$$\bar{b}\in \wedge^2 p^C\otimes H^s(l_{-\varkappa(\lambda+2\varrho)})\otimes M \tag{5.11}$$

where $M=H^s(l_{\lambda+2\varrho})$. From Lemma 2.2 (applied to $-\varkappa\lambda\in\tilde{F}_{-\varkappa P}$)

$$\begin{aligned} p^C\otimes H^s(S^1(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)}) &\xrightarrow{\partial^1} \wedge^2 p^C\otimes H^s(l_{-\varkappa(\lambda+2\varrho)}) \\ &\to H^s(\wedge^2\varkappa p_+\otimes l_{-\varkappa(\lambda+2\varrho)}) \end{aligned} \tag{5.12}$$

is an exact sequence of K^C modules. Tensoring with $\mathrm{id}_M: M \to M$, we have an exact sequence

$$p^C \otimes H^s(S^1(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M \xrightarrow{\partial^1} \wedge^2 p^C \otimes H^s(l_{-\varkappa(\lambda+2\varrho)}) \otimes M$$
$$\to H^s(\wedge^2 \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M \tag{5.13}$$

of K^C modules. Applying Remark 3.1, $H^s(\wedge^2 \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)})$ is isomorphic to the dual of $H^s(\wedge^2 p_- \otimes l_{\lambda+2\varrho})$. Thus by Lemma 5.1. There is no non-zero K^C invariant element of $H^s(\wedge^2 \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)}) \otimes M$, the last term in (5.13). Hence the exactness of the sequence (5.13) gives the following:

There exists an element $\bar{v}$ of $p^C H^s(S^1(\varkappa p_-) \otimes l_{-\varkappa(\lambda+2\varrho)} \otimes M$ such that

$$\bar{b} = \partial^1(\bar{v}). \tag{5.14}$$

$\bar{v}$ corresponds to a map, K^C linear,

$$v: p^C \otimes H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to M = H^s(l_{\lambda+2\varrho}) \tag{5.15}$$

As usual, we interpret (5.14) to mean

$$b(X, Y) = v(X)\eta_0(Y) - v(Y)\eta_0(X).$$

Thus part (ii) of Lemma (4.4) is proved for all $i \geq 0$. Using part (ii) we will prove part (i). For $\xi \in H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)})$, define $e(\xi) \in H^s(l_{\lambda+2\varrho}) \otimes p^C$ by

$$e(\xi) = \sum_{r=1}^{2m} X_r \otimes v(X_r)\xi$$

as in (4.36). We thus have a K^C linear map

$$e: H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to p^C \otimes H^s(l_{\lambda+2\varrho}) \tag{5.16}$$

Applying Lemma 2.2, we have an exact sequence

$$H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \xrightarrow{\partial^0} p^C \otimes H^s(l_{\lambda+2\varrho}) \to H^s(\wedge^1 p_- \otimes l_{\lambda+2\varrho})$$

The composite of the map e with the second map above is zero by Lemma 5.1 and Corollary 3.4. Thus, there exists a K^C linear map

$$J: H^0(S^{i+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)} \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \tag{5.17}$$

such that

$$e = \partial^0 J \tag{5.18}$$

i.e. $v(X)\xi = \partial_1(X)J(\xi)$. We now define

$$\mu: p^C \otimes H^0(S^i(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \tag{5.19}$$

by $\mu = J \cdot \eta_i$. Then μ has the desired properties of Lemma 4.4 part i) Lemma 4.4 is now completely proved. With this the proof of Lemma 4.3, for $i > 0$, is complete.

Proof of Lemma 4.3 (the case $i=0$). We are given a K^C linear map

$$b: \wedge^2 p^C \otimes H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$$

$(j \geq 1)$. $X_1, X_2, \ldots, X_{2m}$ is a basis of p^C as usual. For $\xi \in H^0(l_{\varkappa(\lambda+2\varrho_n)})$, define $f(\xi) \in \wedge^2 p^C \otimes H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$ by

$$f(\xi) = \Sigma_{r<s} X_r \wedge X_s \otimes b(X_r, X_s)\xi \,.$$

We thus have defined a K^C linear map

$$f: H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to \wedge^2 p^C \otimes H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$$

The data $\mathrm{alt}_{W,X,Y} \partial_j(W) b(X,Y) = 0$ in Lemma 4.3 implies that $\partial^2(f(\xi)) = 0$. By the exactness of the complexes of Lemma 2.2, we conclude that there exists a K^C linear map $e: H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to p^C \otimes H^s(S^{j+1}(p_+) \otimes l_{\lambda+2\varrho})$ such that $f = \partial^1 e$. The contraction

$$p^C \otimes p^C \to C$$

which is the restriction of the Killing form now gives us a map μ, by composing

$$p^C \otimes H^0(l_{\varkappa(\lambda+2\varrho_n)}) \xrightarrow{1 \otimes e} p^C \otimes p^C \otimes H^s(S^{j+1}(p_+) \otimes l_{\lambda+2\varrho}) \longrightarrow H^s(S^{j+1}(p_+) \otimes l_{\lambda+2\varrho})$$

one now reinterprets $f = \partial^1 e$ to mean (4.12) of Lemma 4.3. Thus Lemma 4.3 part i) is proved for $i=0$. For part ii) look at the exact sequence given by Lemma 3.2 (for $-\varkappa\lambda \in \tilde{F}_{-\varkappa P}$)

$$H^0(\wedge^1 \varkappa p_- \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^0(p^C \otimes l_{\varkappa(\lambda+2\varrho_n)}) \xrightarrow{\eta^0} H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)})$$

We claim that the composite of the first map with the map μ in (4.11) (for $i=0$) is zero. For $H^0(\wedge^1 \varkappa p_- \otimes l_{\varkappa(\lambda+2\varrho_n)})$ is the dual of $H^s(\wedge^1 \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)})$. By Remark 3.1 the dual of $H^s(\wedge^1 \varkappa p_+ \otimes l_{-\varkappa(\lambda+2\varrho)})$ is isomorphic to $H^s(\wedge^1 p_- \otimes l_{\lambda+2\varrho})$. Our claim now follows from Lemma 5.1.

Thus the map μ is zero on the Kernel of η^0. Since η^0 is a surjection, we now get a K^C linear map

$$J: H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{j+1}(p_+) \otimes l_{\lambda+2\varrho})$$

such that $\mu(X)\xi = J\eta_0(X)\xi$. We now define for $X \in p^C$,

$$v(X): H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+) \otimes l_{\lambda+2\varrho})$$

by $v(X) = \partial_{j+1}(X)J$. The maps $v(X)$ give rise to a map

$$v: p^C \otimes H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^j(p_+) \otimes l_{\lambda+2\varrho}) \,.$$

This map v serves for part ii) of Lemma 4.3 (for $i=0$) as an easy computation shows.

The proof of Lemma 4.3 for $i=0$ is now complete.

Proposition 5.2. *Let τ_0 denote the representation of K^C (and also of K^C) on $H^s(l_{\lambda+2\varrho})$. There exists a unique K^C linear map*

$$\varepsilon_0: p^C \otimes H^s(l_{\lambda+2\varrho}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \tag{5.19}$$

such that $\forall u \in H^s(l_{\lambda+2\varrho})$ *and* $X, Y \in p^C$,

$$\tau_0[X, Y]u = \partial_1(X)\varepsilon_0(Y)u - \partial_1(Y)\varepsilon_0(X)u \tag{5.20}$$

choose and fix a K^C *linear isomorphism*

$$j^0 : H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(l_{\lambda+2\varrho}) . \tag{5.21}$$

There exists a unique K^C *linear homomorphism*

$$j^1 : H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \tag{5.22}$$

such that (3.20 IH) *is satisfied for* $k=0$.

Proof. Define

$$b : \wedge^2 p^C \otimes H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(l_{\lambda+2\varrho})$$

by $b(X \wedge Y \otimes \xi) = \tau_0[X, Y]j^0(\xi)$. By applying Lemma 4.4 (the case $i=0$) to the map b above we conclude the existence of a map

$$\mu : p^C \otimes H^0(l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho})$$

such that

$$\partial_1(X)\mu(Y)\xi - \partial_1(Y)\mu(X)\xi = \tau_0[X, Y]j^0(\xi) .$$

We now define

$$\varepsilon_0(X \otimes u) = \mu(X \otimes j^{0^{-1}}u) . \tag{5.23}$$

The map ε_0 thus defined serves for (5.19). Again, we consider Lemma 4.4 for $i=0$ and for the map b above. In (5.17) we constructed a map

$$J : H^0(S^1(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho})$$

and μ and J are related by

$$\mu(X)\xi = J\eta_0(X)\xi , \quad X \in p^C \quad \text{and} \quad \xi \in H^0(l_{\varkappa(\lambda+2\varrho_n)}) \tag{5.24}$$

We now take the above map J to be the map j^1 in (5.22). Then, for $\xi \in H^0(l_{\varkappa(\lambda+2\varrho_n)})$ and $X \in p^C$,

$$\begin{aligned} \varepsilon_0(X)j^0(\xi) &= \mu(X)(\xi) , \quad \text{by (5.23)} \\ &= j^1\eta_0(X)(\xi) \quad \text{by (5.24)} . \end{aligned}$$

Thus Proposition (5.2) is proved, except for uniqueness. If there are two maps (5.19) both having property (5.20), then (5.23) gives two maps μ both having the property

$$\partial_1(X)\mu(Y) - \partial_1(Y)\mu(X) = b(X, Y) . \tag{5.25}$$

But this will be a contradiction of Lemma 4.4. Thus the uniqueness of ε_0 is proved. Suppose there are two maps j^1 both serving for (5.22). Define

$$\mu = j^1\eta_0 .$$

Since η_0 is a surjection, the above defines two distinct maps μ both satisfying (5.25). This will contradict Lemma 4.4. Proposition (5.2) is completely proved. q.e.d.

Proposition 5.3. *Let $l \geqq 1$. Let ε_l be the map given by Lemma 4.1 under the induction hypotheses of §3. Then there exists a unique K^C linear map*

$$j^{l+1}: H^0(S^{l+1}(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^{l+1}(p_+)\otimes l_{\lambda+2\varrho}) \tag{5.26}$$

such that (3.20 IH) *is satisfied for $k=l$.*

Proof. Define $a(X, Y)$ as in (4.3). Now define

$$b: \wedge^2 p^C \otimes H^0(S^l(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$$

by

$$b(X\wedge Y\otimes\xi) = a(X, Y) j^l(\xi)$$

where j^l is as in (3.19 IH) for $k=l$. We now apply Lemma 4.3 with $i=j=l$ for the above map b. We now take for j^{l+1} in (5.26) the map J defined in (4.37) for this case. We observe that the map μ given by Lemma 4.3 gives rise to the map ε_l in Proposition 4.1 by the relation

$$\varepsilon_l(X)\mu = \mu(X)\,(j^l)^{-1}(\mu) \tag{5.27}$$

On the other hand μ and J are related by (4.40), i.e.,

$$\mu(X) = J\eta_l(X) \tag{5.28}$$

Now, for $\xi \in H^0(S^l(\varkappa p_+)\otimes l_{\varkappa(\lambda+2\varrho_n)})$ and $X \in p^C$,

$$\begin{aligned} \varepsilon_l(X) j^l(\xi) &= \mu(X)\,(\xi), \quad \text{by (5.27)} \\ &= j^{l+1}\eta_l(X)\,(\xi) \quad \text{by (5.28)} \end{aligned}$$

Thus proving (3.20 IH) for $k=l$. The proof of the uniqueness of j^{l+1} is exactly similar to that of j^1 in the proof of Proposition 5.2.

The surjectivity (3.9 IH) for $k=0, l$ and also the fact that the maps j^1 and j^{l+1} of (5.22) and (5.26) are isomorphisms are proved in the next section.

§6

Proposition 6.1. *The map ε_0 in* (5.19) *and the map ε_l in* (4.1) *are both surjections.*

The proof of this proposition is broken up into lot of lemmas.

Let cliff (p^C) be the Clifford algebra on p^C: (see [4, §1]): Let $X_1, \ldots, X_{2m}$ be on orthonormal basis of $p \subseteq p^C$. Then Cliff (p^C) is generated by $p^C \subseteqq C(p^C)$ subject to the relations

$$X_i^2 = -1 \quad \text{and} \quad X_i X_j = -X_j X_i (i \neq j) \tag{6.1}$$

Let p_2^C be the complex subspace of Cliff (p^C) spanned by $\{X_i X_j | i<j\}$. The orthogonal algebra $o(p^C)$ (see (5.1)) can be naturally identified with the subspace p_2^C and then the representation σ in (5.1) can be extended to a representation

$$C: \mathrm{Cliff}(p^C) \to \mathrm{End}(L) \tag{6.2}$$

of the Clifford algebra. Regarding the representation χ of k^C on L (5.2) we have the following: For $T \in k^C$,

$$\chi(T) = \sum_{i,j=1}^{2m} \frac{([T, X_i], X_j)}{4} C(X_i)C(X_j) \tag{6.3}$$

([4, Lemma 2.1]). We now define a map

$$d: S^k(p_+) \otimes L \to S^{k-1}(p_+) \otimes L \tag{6.4}$$

by

$$d(u \otimes s) = \sum_{i=1}^{2m} \delta_{X_i}(u) \otimes C(X_i)s \tag{6.5}$$

where δ_{X_i} is the differentiation map (2.6). At this point, it is to be remarked that the representation $\chi: k^C \to \mathrm{End}(L)$ need not integrate to a representation of K^C. However, it can be always integrated to a certain covering group K^C of K^C. For convenience of notation only we assume that χ actually integrates to a representation of K^C. If prefered, one can assume that the new covering group is denoted by K^C, forgeting the old group K^C. The K^C module L restricts to a B module L and 6.4 is a B module homomorphism. It is easy to check that

$$d^2 = 0 \tag{6.6}$$

Tensoring with $l_{\lambda+2\varrho}$ one gets from (6.4) another map also denotes by d.

$$d: S^k(p_+) \otimes l_{\lambda+2\varrho} \otimes L \to S^{k-1}(p_+) \otimes l_{\lambda+2\varrho} \otimes L \tag{6.7}$$

Then in cohomology we have a map

$$D_k: H^s(S^k(p_+) \otimes l_{\lambda+2\varrho} \otimes L) \to H^s(S^{k-1}(p_+) \otimes l_{\lambda+2\varrho} \otimes L) \tag{6.8}$$

Since L is a K^C module

$$H^s(S^k(p_+) \otimes l_{\lambda+2\varrho} \otimes L) \simeq H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \otimes L$$

Then the map D_k can be checked to be given by

$$D_k(u \otimes s) = \sum_{i=1}^{2m} \partial_k(X_i)u \otimes C(X_i)s \tag{6.9}$$

We define $D_0 = 0$. Because of (6.6),

$$D_{k-1}D_k = 0\,. \tag{6.10}$$

We now define a map for $k = 0, 1, \ldots,$

$$E_k: H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \otimes L \to H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho}) \otimes L \tag{6.11}$$

by

$$E_k(u \otimes s) = \sum_{i=1}^{2m} \varepsilon_k(X_i)u \otimes C(X_i)s \tag{6.12}$$

We define $E_{-1} = 0$.

Claim. $E_k E_{k-1} = 0 (0 < k \leqq l)$

Define

$$E'_{k-1}: H^0(S^{k-1}(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L \to H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L$$

by

$$E'_{k-1}(v \otimes s) = \sum_{i=1}^{2m} \eta_{k-1}(X_i) v \otimes C(X_i) s .$$

The module L (for $o(p^C)$) is self dual. The map E'_{k-1} is then essentially the transpose of D_k. Then (6.10) implies the claim.

The Killing form (,) of g restricted to k is negative definite. Choose a basis $Y_1, \ldots, Y_t$ of k such that

$$\begin{aligned} B(Y_i, Y_j) &= 0 , \quad \text{if} \quad i \neq j \\ &= -1 \quad \text{if} \quad i = j \end{aligned} \tag{6.13}$$

We now have the following

Lemma 6.2. *Let* $k \geqq 0$.

$$\begin{aligned} &E_{k-1} D_k + D_{k+1} E_k \\ &= -\Sigma_q (\tau_k \otimes \chi)(Y_q)^2 - (\varrho, \varrho) + (\varrho_k, \varrho_k) \\ &\quad + \Sigma_q \tau_k(Y_q)^2 \otimes 1 - \Sigma_i \{\varepsilon_{k-1}(X_i) \partial_k(X_i) + \partial_{k+1}(X_i) \varepsilon_k(X_i)\} \otimes 1 \end{aligned} \tag{6.14}$$

Proof.

$$\begin{aligned} E_{k-1} D_k &= \sum_{i,j=1}^{2m} \varepsilon_{k-1}(X_i) \partial_k(X_j) \otimes C(X_i) C(X_j) \\ &= \Sigma_{i<j} \{\varepsilon_{k-1}(X_i) \partial_k(X_j) - \varepsilon_{k-1}(X_j) \partial_k(X_j)\} \otimes C(X_i) C(X_j) \\ &\quad - \Sigma_i \varepsilon_{k-1}(X_i) \partial_k(X_i) \otimes 1 \quad \text{(using (6.1))} \end{aligned}$$

$$\begin{aligned} D_{k+1} E_k &= \sum_{i,j=1}^{2m} \partial_{k+1}(X_i) \varepsilon_k(X_j) \otimes C(X_i) C(X_j) \\ &= \Sigma_{i<j} \{\partial_{k+1}(X_i) \varepsilon_k(X_j) - \partial_{k+1}(X_j) \varepsilon_k(X_i)\} \otimes C(X_i) C(X_j) \\ &\quad - \Sigma_i \partial_{k+1}(X_i) \varepsilon_k(X_i) \otimes 1 \end{aligned}$$

Thus,

$$\begin{aligned} E_{k-1} D_k + D_{k+1} E_k &= \Sigma_{i<j} \tau_k [X_i, X_j] \otimes C(X_i) C(X_j) \\ &\quad - \Sigma_i \{\varepsilon_{k-1}(X_i) \partial_k(X_i) + \partial_{k+1}(X_i) \varepsilon_k(X_i)\} \otimes 1 \quad \text{(using (3.8 IH))} \\ &= 1/2\, \Sigma_{i,j} \tau_k [X_i, X_j] \otimes C(X_i) C(X_j) \\ &\quad - \Sigma_i \{\varepsilon_{k-1}(X_i) \partial_k(X_i) + \partial_{k+1}(X_i) \varepsilon_k(X_i)\} \otimes 1 \end{aligned} \tag{6.15}$$

Consider the first term in (6.15). We have

$$
\begin{aligned}
&1/2\, \Sigma_{i,j} \tau_k[X_i, X_j] \otimes C(X_i)C(X_j) \\
&= -1/2\, \Sigma_{i,j} \sum_{q=1}^{t} ([X_i, X_j], Y_q)\tau_k(Y_q) \otimes C(X_i)C(X_j) \\
&= -1/2\, \Sigma_q \tau_k(Y_q) \otimes \Sigma_{i,j}([Y_q, X_i], X_j)C(X_i)C(X_j) \\
&= -2\, \Sigma_q \tau_k(Y_q) \otimes \chi(Y_q) \quad \text{(by 6.3)} \\
&= -\Sigma_q(\tau_k \otimes \chi)(Y_q)^2 + \Sigma_q \tau_k(Y_q)^2 \otimes 1 + \Sigma_q 1 \otimes \chi(Y_q)^2 .
\end{aligned}
$$

By [4, Lemma 2.2] the third summation above equals $-(\varrho, \varrho) + (\varrho_k, \varrho_k)$. Thus from (6.15)

$$
\begin{aligned}
E_{k-1}D_k + D_{k+1}E_k = &-\Sigma_q(\tau_k \otimes \chi)(Y_q)^2 - (\varrho, \varrho) + (\varrho_k, \varrho_k) + \Sigma_q \tau_k(Y_q)^2 \otimes 1 \\
&- \Sigma_i \{\varepsilon_{k-1}(X_i)\partial_k(X_i) + \partial_{k+1}(X_i)\varepsilon_k(X_i)\} \otimes 1
\end{aligned}
$$

q.e.d.

Lemma 6.3. *The linear map of $H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \otimes L$ into itself ($k \geqq 0$) given by*

$$
-\Sigma_q \tau_k(Y_q)^2 + \Sigma_i \{\varepsilon_{k-1}(X_i) + \partial_{k+1}(X_i)\varepsilon_k(X_i)\}
$$

is independent of k. It equals scalar multiplication by $(\lambda+\varrho, \lambda+\varrho) - (\varrho, \varrho)$.

Proof. The proof is by induction on k. The case $k=0$ will be proved in the next section. Let $k>0$. Assume the lemma to be true for the values $0, 1, \ldots, k-1$. Let A_k denote the operator in the lemma. We will prove that for every $X \in p^C$,

$$
\partial_k(X)A_k = A_{k-1}\partial_k(X) \tag{6.16}
$$

Granting this for a while we will show that $A_k = (\lambda+\varrho, \lambda+\varrho) - (\varrho, \varrho)$. Indeed, for any $u \in H^s(S^k(p_+) \otimes l_{\lambda+2\varrho})$ and for any $X \in p^C$, from (6.16),

$$
\begin{aligned}
\partial_k(X)A_k(u) &= A_{k-1}\partial_k(X)u \\
&= \{(\lambda+\varrho, \lambda+\varrho) - (\varrho, \varrho)\}\partial_k(X)u
\end{aligned}
$$

by induction hypothesis

$$
= \partial_k(X)\, \{(\lambda+\varrho, \lambda+\varrho) - (\varrho, \varrho)\}u .
$$

Thus, the elements $A_k(u)$ and $\{(\lambda+\varrho, \lambda+\varrho) - (\varrho, \varrho)\}u$ both go into the same element in $p^C \otimes H^s(S^{k-1}(p_+) \otimes l_{\lambda+2\varrho})$ under the map ∂^0. Since ∂^0 is an injection, it follows that $A_k(u) = \{(\lambda+\varrho, \lambda+\varrho) - (\varrho, \varrho)\}u$.

Now we prove (6.16).

$$
\begin{aligned}
\partial_k(X)\tau_k(Y_q)^2 &= \tau_{k-1}(Y_q)\partial_k(X)\tau_k(Y_q) + \partial_k[X, Y_q]\tau_k(Y_q) \quad \text{by (3.4))} \\
&= \tau_{k-1}(Y_q)^2\partial_k(X) + \tau_{k-1}(Y_q)\partial_k[X, Y_q] \\
&\quad + \partial_k[X, Y_q]\tau_k(Y_q) \quad \text{(by (3.4) again)}
\end{aligned}
$$

Thus,

$$\Sigma_q \partial_k(X)\tau_k(Y_q)^2 = \Sigma_q \tau_{k-1}(Y_q)^2 \partial_k(X) + \Sigma_q \tau_{k-1}(Y_q)\partial_k[X, Y_q] + \Sigma_q \partial_k[X, Y_q]\tau_k(Y_q) \tag{6.17}$$

$$\begin{aligned}\partial_k(X)\varepsilon_{k-1}(X_i)\partial_k(X_i) = {} & \partial_k(X_i)\varepsilon_{k-1}(X)\partial_k(X_i) + \tau_{k-1}[X, X_i]\partial_k(X_i) \\ & -\varepsilon_{k-2}(X)\partial_{k-1}(X_i)\partial_k(X_i) + \varepsilon_{k-2}(X_i)\partial_{k-1}(X)\partial_k(X_i) \\ & \text{(by 3.8 IH)}\end{aligned} \tag{6.18}$$

$$\begin{aligned}\partial_k(X)\partial_{k+1}(X_i)\varepsilon_k(X_i) &= \partial_k(X_i)\partial_{k+1}(X)\varepsilon_k(X_i) \\ &= \partial_k(X_i)\partial_{k+1}(X_i)\varepsilon_k(X) + \partial_k(X_i)\tau_k[X, X_i] \\ &\quad - \partial_k(X_i)\varepsilon_{k-1}(X)\partial_k(X_i) \\ &\quad + \partial_k(X_i)\varepsilon_{k-1}(X_i)\partial_k(X) \quad \text{(by 3.8 IH)}\end{aligned} \tag{6.19}$$

Claim.

$$\sum_{i=1}^{2m} \partial_{k-1}(X_i)\partial_k(X_i) = 0 \tag{6.20}$$

The K^C linear map

$$\Sigma_i \partial_{k-1}(X_i)\partial_k(X_i): H^s(S^k(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{k-2}(p_-)\otimes l_{\lambda+2\varrho})$$

is by the definition of the maps $\partial(X)$, clearly got by inducing in cohomology the B linear map

$$\Sigma_i \delta_{X_i}\delta_{X_i}: S^k(p_+)\otimes l_{\lambda+2\varrho} \to S^{k-2}(p_+)\otimes l_{\lambda+2\varrho} \tag{6.21}$$

where δ_X are the maps in (2.6) (tensored with the identity map of $l_{\lambda+2\varrho}$). But $\Sigma_i \delta_{X_i}\delta_{X_i}$ is the "laplacian" and "holomorphic" polynomials are "harmonic". Thus (6.21) is zero and the claim is proved.

Summing (6.18) and (6.19) over $i = 1, \dots, 2m$, using 6.20 and noting that $\partial_{k-1}(X)\partial_k(X_i) = \partial_{k-1}(X_i)\partial_k(X)$, we get

$$\begin{aligned}&\partial(X)\Sigma_i \varepsilon_{k-1}(X_i)\partial_k(X_i) + \partial_{k+1}(X_i)\varepsilon_k(X_i) \\ &= \Sigma_i \varepsilon_{k-2}(X_i)\partial_{k-1}(X_i) + \partial_k(X_i)\varepsilon_{k-1}(X_i)\cdot\partial(X) \\ &\quad + \Sigma_i \tau_{k-1}[X, X_i]\partial_k(X_i) + \Sigma_i \partial_k(X_i)\tau_k[X, X_i]\end{aligned} \tag{6.22}$$

Now (6.16) can be concluded from (6.17) and (6.22): For

$$\begin{aligned}\Sigma_q \tau_{k-1}(Y_q)\partial_k[X, Y_q] &= \Sigma_q \tau_{k-1}(Y_q)\Sigma_i([X, Y_q], X_i)\partial_k(X_i) \\ &= -\Sigma_{i,q}(Y_q, [X, X_i])\tau_{k-1}(Y_q)\partial_k(X_i) \\ &= \Sigma_i \tau_{k-1}[X, X_i]\partial_k(X_i)\end{aligned}$$

and similarly

$$\Sigma_q \partial_k[X, Y_q]\tau_k(Y_q) = \Sigma_i \partial_k(X_i)\tau_k[X, X_i] \qquad \text{q.e.d.}$$

The result of Lemma 6.2 can be restated as follows.

Let $k \geqq 0$. Then

$$E_{k-1}D_k + D_{k+1}E_k = -\Sigma_q(\tau_k \otimes \chi)(Y_q)^2 - (\lambda+\varrho, \lambda+\varrho) + (\varrho_k, \varrho_k) \tag{6.23}$$

We now have the following

Lemma 6.4. *The linear map $E_{k-1}D_k + D_{k+1}E_k$ of $H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L$ into itself is an isomorphism for $k>0$. For $k=0$, the map is, of course, D_1E_0 and its image equals the image of D_1.*

Proof. $k \geqq 0$ is arbitrary. Let $V_\xi \subseteq H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L$ be an irreducible K^C submodule on which $E_{k-1}D_k + D_{k+1}E_k$ is zero. ξ is the highest weight of the K^C module V_ξ, with respect to P_k. It is well known that

$$-\Sigma_q(\tau_k \otimes \chi)(Y_q)^2|_{V_\xi} = (\xi+\varrho_k, \xi+\varrho_k) - (\varrho_k, \varrho_k) \tag{6.24}$$

(6.23) and (6.24) now imply

$$(\xi+\varrho_k, \xi+\varrho_k) = (\lambda+\varrho, \lambda+\varrho) \tag{6.25}$$

Since $V_\xi \subseteq H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L$, the K^C module $H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L\otimes V_\xi^*$ has a nonzero K^C invariant element. Since $L \simeq L^*$, we then conclude that an irreducible submodule of $V_\xi \otimes L$ and an irreducible submodule of $H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$ are isomorphic. Let v be the highest weight of such an irreducible component V_v. Since $V_v \subseteq V_\xi \otimes L$, v is the form

$$v = \xi + \varrho_n - r_1 - \ldots - r_t, \qquad r_i \in P_n \tag{6.26}$$

On the other hand, since $V_v \subseteq H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$, v has the form

$$v = w(\lambda+\varrho+\varrho_n+\beta_1+\ldots+\beta_k) - \varrho_k \tag{6.27}$$

for some element w of the Weyl group of (k^C, t^C). Thus

$$\lambda+\varrho+\varrho_n+\beta_1+\ldots+\beta_k = w^{-1}(\xi+\varrho-r_1-\ldots-r_t) \tag{6.28}$$

Repeating the arguments after 5.9 (which is the same as the arguments in the proof of [7, (8.34) Proposition]) we conclude there exist elements $\alpha_1, \ldots, \alpha_r \in P_k$ such that

$$\lambda+\varrho+\varrho_n+\beta_1+\ldots+\beta_k = \xi+\varrho-r_1-\ldots-r_t-\alpha_1-\ldots-\alpha_r \tag{6.29}$$

Thus

$$\xi+\varrho_k = \lambda+\varrho+\beta_1+\ldots+\beta_k+r_1+\ldots+r_t+\alpha_1+\ldots+\alpha_r \tag{6.30}$$

Hence,

$$\begin{aligned}(\xi+\varrho_k, \xi+\varrho_k) = {} & (\lambda+\varrho, \lambda+\varrho) + 2(\lambda+\varrho, \beta_1+\ldots+\beta_k+r_1+\ldots+r_t+\alpha_1+\ldots+\alpha_r) \\ & + (\beta_1+\ldots+\beta_k+r_1+\ldots+r_t+\alpha_1+\ldots+\alpha_r, \beta_1+\ldots+\beta_k \\ & + r_1+\ldots+r_t+\alpha_1+\ldots+\alpha_r)\end{aligned} \tag{6.31}$$

On the right hand side the third term is nonnegative. Also, we have assumed that $\lambda \in \tilde{F}$. By (1.4), then, the second term is also nonnegative. Using (6.25) we then conclude that

$$\beta_1+\ldots+\beta_k+r_1+\ldots+r_t+\alpha_1+\ldots+\alpha_r = 0 \tag{6.32}$$

But β_i, r_i, α_i are all positive roots. Thus we conclude

$$k=0, \quad r=0 \quad \text{and} \quad t=0. \tag{6.33}$$

At this stage, the lemma is proved for $k>0$. For $k=0$, let $V_\xi \subseteqq H^s(l_{\lambda+2\varrho}) \otimes L$ be any irreducible K^C submodule. $D_1 E_0(V_\xi)=0$ if and only if $\xi=\lambda+\varrho_n$. This follows from (6.32) and (6.30).

Claim. The irreducible module with highest weight $\lambda+\varrho_n$ occurs with multiplicity one in

$$H^s(l_{\lambda+2\varrho}) \otimes L. \tag{6.34}$$

By [3, Lemma 4.3], $H^s(l_{\lambda+2\varrho})$ is the irreducible K^C module $V_{\lambda+2\varrho_n}$ with highest weight $\lambda+2\varrho_n$. Now, the multiplicity of $V_{\lambda+\varrho_n}$ in $V_{\lambda+2\varrho_n} \otimes L=$ the dimension of K^C invariants in $V^*_{\lambda+\varrho_n} \otimes L \otimes V_{\lambda+\varrho 2_n}=$ the multiplicity of $V_{\lambda+2\varrho_n}$ in $V_{\lambda+\varrho_n} \otimes L$. The last number is well known to be one.

Thus the image of

$$D_1 E_0 : V_{\lambda+2\varrho_n} \otimes L \to V_{\lambda+2\varrho_n} \otimes L$$

is the unique K^C submodule of $V_{\lambda+2\varrho_n} \otimes L$ which is complimentary to $V_{\lambda+\varrho_n} \subseteqq V_{\lambda+2\varrho_n} \otimes L$. Since the image of $D_1 E_0$ is contained in the image of

$$D_1 : H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \otimes L \to H^s(l_{\lambda+2\varrho}) \otimes L,$$

to prove the lemma for $k=0$, it is enough to prove D_1 is not onto.

As remarked in (6.8), D_1 is the map induced in cohomology by the B module map

$$d : p_+ \otimes l_{\lambda+2\varrho} \otimes L \to l_{\lambda+2\varrho} \otimes L$$

given by

$$d(X \otimes e \otimes s) = \sum_{i=1}^{2m} \delta_{X_i}(X) \otimes e \otimes C(X_i)s.$$

((6.7) and (6.5)). By 2.5, the map d is simply

$$X \otimes e \otimes s \mapsto e \otimes C(X)s.$$

It is well known that

$$L/C(p_+)L \simeq l_{-\varrho_n}$$

the one dimensional B module given by $-\varrho_n$. Thus one has a B module exact sequence

$$p_+ \otimes l_{\lambda+2\varrho} \otimes L \xrightarrow{d} l_{\lambda+2\varrho} \otimes L \to l_{\lambda+\varrho_n+2\varrho_k} \to 0 \tag{6.35}$$

In cohomology (6.35) induces a K^C module exact sequence

$$H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \otimes L \xrightarrow{D_1} H^s(l_{\lambda+2\varrho}) \otimes L \to H^s(l_{\lambda+\varrho_n+2\varrho_k}) \to 0.$$

Thus D_1 is not onto. Now the lemma is proved for all $k \geqq 0$. q.e.d.

Corollary 6.5. *Let $0 \leqq k \leqq l$. We have*

$$D_{k+1}E_k(H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L) = D_{k+1}(H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})\otimes L) \tag{6.36}$$

Proof. The case $k=0$ is already proved in Lemma 6.4. Now assume $k>0$. Let $v \in H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})\otimes L$. By Lemma 6.4, $\exists u \in H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})\otimes L$, such that

$$D_{k+1}(v) = E_{k-1}D_k u + D_{k+1}E_k u\,. \tag{6.37}$$

Applying D_k and using (6.10) we see that

$$D_k E_{k-1} D_k u = 0\,.$$

By the claim following (6.12), one also has

$$E_k E_{k-1} D_k u = 0\,.$$

Thus $(E_{k-1}D_k + D_{k+1}E_k)E_{k-1}D_k u = 0$. Hence, by Lemma 6.4, $E_{k-1}D_k u = 0$. Now, from (6.37)

$$D_{k+1}(v) = D_{k+1}E_k(u)\,.$$

The corollary is proved. q.e.d.

We recall the maps $\varepsilon_k (0 \leqq k \leqq l): p^C \otimes H^s(S^k(p_+)\otimes l_{\lambda+2\varrho}) \to H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})$. We put

$$F_{k+1} = \text{the image of} \quad \varepsilon_k \tag{6.38}$$

By the definition of the maps E_k in (6.12), one sees that

$$\text{image of} \quad E_k \subseteqq F_{k+1} \otimes L \tag{6.39}$$

Corollary 6.5 now implies.

Corollary 6.6. *Let $0 \leqq k \leqq l$. Then*

$$D_{k+1}(H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})\otimes L) = D_{k+1}(F_{k+1}\otimes L) \tag{6.40}$$

Proposition 6.1 will follow from

Lemma 6.7. *Let $k \geqq 0$. Let F_{k+1} be a K^C submodule of $H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})$ and suppose*

$$D_{k+1}(H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})\otimes L) = D_{k+1}(F_{k+1}\otimes L) \tag{6.41}$$

Then, $F_{k+1} = H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})$.

Proof. We make specific use of the assumption that $[p_+, [p_+, p_+]] = 0$. Let $Q \subseteqq K^C$ be the parabolic subgroup defined by

$$Q = \{k \in K^C \mid \mathrm{Ad}(k) \text{ leaves } p_+ \text{ stable}\}\,.$$

Let q be the Lie subalgebra of k^C corresponding to Q. In [3, §9] it has been shown that $[p_+, p_+] = \underline{u}$, where $\underline{u}$ is the nilradical of q. Let $Q = MU$ be the Levi decomposition of Q. Then for $k \in U$, $\mathrm{Ad}(k)X = X$ for all $X \in p_+$. Let $W_{\lambda+2\varrho_n}$ be the irreducible M module with highest weight $\lambda + 2\varrho_n$ (with respect to $B \cap M$). By extending trivially over U, $W_{\lambda+2\varrho_n}$ becomes now a MU module. Hence also $S^k(p_+)\otimes W_{\lambda+2\varrho_n}$ is a Q module. Note that U acts trivially on $S^k(p_+)\otimes W_{\lambda+2\varrho_n}$. When a holomorphic Q module W is given, we denote by $H^i(K^C/Q; W)$ the K^C module given by the i-th cohomology space with coefficients in the sheaf of germs of holomorphic sections of the holomorphic vector bundle over K^C/Q associated to W. Let ϱ_q denote half the sum of those positive compact roots for which the root space lies in u. Let $l_{2\varrho_q}$ denote the one dimensional Q module given by $2\varrho_q$. One can then see that

$$H^{i+r}(S^k(p_+)\otimes l_{\lambda+2\varrho})\simeq H^i(K^C/Q; S^k(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q})$$

for every i and k. Here $r=\dim K^C/B-\dim K^C/Q$. (See [3, p. 172] for necessary references.)

From the Borel-Weil-Bott theorem for the parabolic subgroup Q of K^C, one can easily see the following.

Suppose V is a K^C module and put

$$W=V^u=\{v\in V; xv=0, \text{ for any } x\in u\}$$

so that W is a Q module. Then

$$H^i(K^C/Q; W\otimes l_{2\varrho_q})=0 \tag{6.42}$$

for $0\leqq i<\bar{s}=\dim K^C/Q$ and

$$H^{\bar{s}}(K^C/Q; W\otimes l_{2\varrho_q})\simeq V \tag{6.43}$$

We now come to the proof of the lemma. We label the two half spin representations L' and L'' so as to get Q module surjections $\bar{q}':L'\to l_{-\varrho_n}$ and $\bar{q}'':L''\to l_{-\varrho_n}\otimes p_+$. We have the following commutative diagram the arrows of which are explained below.

$$\begin{array}{ccc}
S^k(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes L'' & \xleftarrow{\ d\ } & S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes L' \\
\downarrow{\scriptstyle 1\otimes\bar{q}''} & & \downarrow{\scriptstyle 1\otimes\bar{q}'} \\
S^k(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes p_+\otimes l_{-\varrho_n} & \xleftarrow{\ \delta\ } & S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes l_{-\varrho_n}
\end{array}$$

The Q module map d is defined similar to (6.7). The map δ is got by tensoring $u\mapsto\Sigma\delta_{X_i}(u)\otimes X_i$ of $S^{k+1}(p_+)$ into $S^k(p_+)\otimes p_+$ with the identity map in the other factors. Inducing in cohomology we have the following commutative diagram.

$$\begin{array}{ccc}
H^{\bar{s}}(H^C/Q, S^k(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes L'') & \xleftarrow{\ d\#\ } & H^{\bar{s}}(K^C/Q, S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}l_{2\varrho_q}\otimes L') \\
\downarrow{\scriptstyle (1\otimes\bar{q}'')_\#} & & \downarrow{\scriptstyle (1\otimes\bar{q}')_\#} \\
H^{\bar{s}}(K^C/Q, S^k(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes p_+\otimes l_{-\varrho_n}) & \xleftarrow{\ \delta\#\ } & H^s(K^C/Q, S^{k+1}(p_+)\otimes \\
 & & W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes l_{-\varrho_n})
\end{array}$$

Here $d_\#$ is induced in cohomology by the Q module map d defined similarly to (6.7). Also the Q module map δ is an injection and its image in

$$S^k(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes p_+\otimes l_{-\varrho_n}$$

is a Q module direct summand (a complementary module being the tensor product of $W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q}\otimes l_{-\varrho_n}$ with the kernel of the symmetrisation map $S^k(p_+)\otimes p_+\to S^{k+1}(p_+)$). Thus, $\delta_\#$ is an injection.

Identifying $H^s(S^{k+1}(p_+)\otimes l_{\lambda+2\varrho})$ with $G_{k+1}=H^{\bar{s}}(K^C/Q, S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q})$, the K^C submodule F_{k+1} of G_{k+1} corresponds to a K^C submodule F_{k+1} of $H^{\bar{s}}(K^C/Q)S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q})$.

The K^C submodules $F_{k+1}\otimes L'$ and $G_{k+1}\otimes L'$ of the top right hand side corner of the above diagram have the same image in the bottom left hand side. Since $\delta_\#$ is an injection

$$(1\otimes\bar{q}')_\#(F_{k+1}\otimes L')=(1\otimes\bar{q}')_\#(G_{k+1}\otimes L') \tag{6.66}$$

(In this version the vanishing condition (*) in (1.6) is not used).

Now we note the following: Since the unipotent radical U of Q acts trivially on $S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}$, the K^C submodule F of $H^{\bar{s}}(K^C/Q, S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q})$ are in one-one correspondence with Q submodules Φ of $S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}$, so that

$$\begin{aligned} &F=H^{\bar{s}}(K^C/Q, \Phi\otimes l_{2\varrho_q})\\ &F\leftrightarrow\Phi \end{aligned} \tag{6.44}$$

Explicitly, $\Phi=\{v\in F|x\cdot v=0, \forall x\in u\}$. We denote by Φ_{k+1} the Q submodule of $S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}$, corresponding to F_{k+1}. Clearly,

$$(1\otimes\bar{q}')_\#(F_{k+1}\otimes L')=H^{\bar{s}}(K^C/Q, \Phi_{k+1}\otimes l_{2\varrho_q}\otimes l_{-\varrho_n}) \tag{6.68}$$

Again there exists a one-one correspondence between K^C submodules F' of $H^{\bar{s}}(K^C/Q, S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{-\varrho_n}\otimes l_{2\varrho_q})$ and Q submodules Φ' of $S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{-\varrho_n}$ because the latter is a Q submodule of

$$\{\xi\in S^{k+1}(p^C)\otimes V_{\lambda+\varrho_n}|x\cdot\xi=0, \forall x\in u\}\,.$$

Here, $V_{\lambda+\varrho_n}$ is an irreducible K^C module with highest weight $\lambda+\varrho_n$ and we have identified

$$W_{\lambda+2\varrho_n}\otimes l_{-\varrho_n}=\{u\in V_{\lambda+\varrho_n}|x\cdot u=0, \forall x\in u\}\,.$$

Note the following: Let F be a K^C submodule of $H^{\bar{s}}(K^C/Q, S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{2\varrho_q})$ and Φ the corresponding Q submodule of $S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}$. Let Φ' be the Q submodule of $S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{-\varrho_n}$, defined by $\Phi'=\Phi\otimes l_{-\varrho_n}$ and let F' be the corresponding K^C submodule of $H^{\bar{s}}(K^C/Q, S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{\varrho_n}\otimes l_{2\varrho_q})$. Then, $(1\otimes\bar{q}')_\#(F\otimes L')=F'$. Moreover, if F_1 is properly contained in F_2, then Φ_1, Φ_1', F_1' are properly contained in Φ_2, Φ_2', F_2' respectively.

Now, in view of (6.66) and (6.68)

$$\Phi_{k+1}\otimes l_{-\varrho_n}=S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\otimes l_{-\varrho_n}\,,$$

hence

$$\Phi_{k+1}=S^{k+1}(p_+)\otimes W_{\lambda+2\varrho_n}\,,$$

and hence

$$F_{k+1}=G_{k+1}.$$

This completes the proof of Lemma (6.7) q.e.d.

In the next section we take up the case $k=0$ of Lemma 6.3.

§7. Proof of Lemma 6.3 for $k=0$

Consider the representation $\tau_{\lambda+\varrho_n}\otimes\varkappa$ of H^C on $V_{\lambda+\varrho_n}\otimes L$. The irreducible K^C module $\tau_{\lambda+2\varrho_n}$ on $V_{\lambda+2\varrho_n}$ with highest weight $\lambda+2\varrho_n$ occurs with multiplicity one in $\tau_{\lambda+\varrho_n}\otimes\varkappa$. We thus have an inclusion,

$$V_{\lambda+2\varrho_n}\to V_{\lambda+\varrho_n}\otimes L. \tag{7.1}$$

On the symmetric space $G/K, \tau_{\lambda+2\varrho_n}$ and $\tau_{\lambda+\varrho_n}\otimes\varkappa$ induce G-homogeneous vector bundles E_0 and E respectively. Because of (7.1) we have an injection

$$E_0\to E$$

This induces an injection

$$C_k^\infty(E_0)\to C_K^\infty(E),$$

where C_K^∞ denotes the space of C^∞ sections which are K finite.

We now use the results of [3^C. The Dirac operator

$$D: C_K^\infty(E)\to C_K^\infty(E), \tag{7.2}$$

is defined in [3, §3]. Let

$$H_K(E_0)=\{s\in C_K^\infty(E_0)|Ds=0\}. \tag{7.3}$$

We now define a filtration in $C_K^\infty(E_0)$ as follows: $C_K^\infty(E_0)$ can be naturally identified to the space

$$C_K^\infty(G, V_{\lambda+2\varrho_n})^0=\{f: G\to V_{\lambda+2\varrho_n}|.$$

i) f is infinitely differentiable and $f(gk)=\tau_{\lambda+2\varrho_n}(k^{-1})f(g)$ and

ii) f is left K finite$\}$. (7.4)

Any $f\in C_K^\infty(G, V_{\lambda+2\varrho_n})^0$ is completely determined by the function $\bar{f}: p\to V_{\lambda+2\varrho_n}$, defined by

$$\bar{f}(X)=f(\exp X). \tag{7.5}$$

For, writing $g\in G$ as $\exp X\cdot k$ for $X\in p$ and $k\in K$,

$$f(\exp X\cdot k)=\tau_{\lambda+2\varrho_n}(k^{-1})\bar{f}(X)$$

in view of i) in (7.4). $\bar{f}$ has a formal Taylor expansion as an element of $\pi_{l=0}^\infty(S^l(p^C)\otimes V_{\lambda+2\varrho_n})$. (By our identification $p^C\simeq(p^C)^*$, $S^l(p^C)$ is to be regarded as the space of homogeneous l^{th} degree polynomial functions on p^C). We now define

$$C_{K,i}^\infty=\{f\in C_K^\infty(G, V_{\lambda+2\varrho_n})^0|$$

the formal Taylor expansion of $\tilde{f}$ has component zero in $S^l(p^C)\otimes V_{\lambda+2\varrho_n}$, for

$l=0,1,\dots,i-1\}$.

Since $C_K^\infty(E_0)=C_K^\infty(G,V_{\lambda+2\varrho_n})^0$ by our identification, we now have a filtration,

$$C_K^\infty(E_0)=C_{K,0}^\infty\supset C_{K,1}^\infty\supset C_{K,2}^\infty\dots. \tag{7.6}$$

One has the obvious identification

$$C_{K,l}^\infty/C_{K,l+1}^\infty\simeq S^l(p^C)\otimes V_{\lambda+2\varrho_n}. \tag{7.7}$$

We thus have a projection

$$T^l\colon C_{K,l}^\infty\to S^l(p^C)\otimes V_{\lambda+2\varrho_n}. \tag{7.8}$$

The filtration 7.6 induces a filtration on $H_K(E_0)$ (7.3) by intersection.

$$H_K(E_0)=H_{K,0}(E_0)\supseteqq H_{K,1}(E_0)\supseteqq H_{K,2}(E_0)\supseteqq\dots \tag{7.9}$$

where

$$H_{K,i}(E_0)=H_K(E_0)\cap C_{K,i}^\infty. \tag{7.10}$$

Under the condition (*) of §1 for λ, it has been shown in [3, proof Lemma 5.3] that

$$T^l(H_{K,l}(E_0))\subseteqq i_\#(H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})) \tag{7.11}$$

where $i_\#$ is as defined in (6.49). (Here we have made the identifications

$$H^s(S^l(p^C)\otimes l_{\lambda+2\varrho})\simeq S^l(p^C)\otimes H^s(l_{\lambda+2\varrho})\simeq S^l(p^C)\otimes V_{\lambda+2\varrho_n}).$$

By the methods of Schmid in [5] and [6] one can show that under the condition (*) of §1 on λ

$$H_K(E_0)\neq 0. \tag{7.12}$$

Choose $l\geqq 0$ and $s\in H_{K,l}(E_0)$ such that $T^l(s)\neq 0$. If $l\neq 0$, then $\Sigma X_i\otimes\partial_l(X_i)(T^l(s))\neq 0$. Thus for some $X\in p^C$, $\partial_l(X)(T^l(s))\neq 0$, i.e. $T^{l-1}(\pi(X)s)\neq 0$. Note that $\pi(X)s\in H_{K,l-1}(E_0)$. Descending further if necessary, we can find $s\in H_K(E_0)$ such that $T^0(s)\neq 0$. Hence, $T^0(H_{K,0}(E_0))=i_\#(H^s(l_{\lambda+2\varrho}))$. (We will hereafter drop $i_\#$ in (7.11)). Because of (11) and because the multiplicity of the irreducible K^C module $H^s(l_{\lambda+2\varrho})$ in $\otimes_{l\geqq 0}H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$ is one ([3, Lemma 4.2 and Lemma 4.3]) there exists a unique irreducible K^C submodule $\tilde{V}_{\lambda+2\varrho_n}$ of $H_K(E_0)$ such that

$$H_K(E_0)=\tilde{V}_{\lambda+2\varrho_n}\otimes H_{K,1}(E_0) \tag{7.13}$$

$$T^0|\tilde{V}_{\lambda+2\varrho_n}\colon\tilde{V}_{\lambda+2\varrho_n}\to i_\#(H^s(S^0(p_+)\otimes l_{\lambda+2\varrho})) \tag{7.14}$$

is an isomorphism.

G acts in the usual way on the space of C^∞ sections of the G homogeneous vector bundle E_0. Denote by $\hat{\pi}$ this representation of G. Then for s belonging a suitable subspace of C^∞ sections of E_0, one can define for $X\in g$

$$\hat{\pi}(X)(s)=\left.\frac{d}{dt}\right|_{t=0}\hat{\pi}(\exp tX)s \tag{7.15}$$

At any rate the above is well defined for all X and for all $s \in H_K(E_0)$ and moreover

$$\hat{\pi}(X)(H_K(E_0)) \subseteqq H_K(E_0).$$

Fairly elementary arguments actually show the following. For $X \in p \subseteqq g$,

$$\hat{\pi}(X)(H_{K,l}(E_0)) \subseteqq H_{K,l-1}(E_0). \tag{7.16}$$

Also, if $s \in C^{\infty}_{K,l}$, then for $X \in p$,

$$\hat{\pi}(X)(s) \in C^{\infty}_{K,l-1}. \tag{7.17}$$

In view of (7.7), $\hat{\pi}(X)$ induces a map

$$\hat{\delta}_X : S^l(p^C) \otimes V_{\lambda+2\varrho_n} \to S^{l-1}(p^C) \otimes V_{\lambda+2\varrho_n} \tag{7.18}$$

As can be expected the differentiation (7.15) induces differentiation in (7.18) i.e.

$$\hat{\delta}_X = C \cdot \delta_X \otimes 1 \tag{7.19}$$

where C is a nonzero constant independent of X and δ_X is as in (2.6).

Hence, for $s \in H_{K,l}(E_0)$ and $X \in p$,

$$T^{l-1}(\hat{\pi}(X)s) = C\hat{c}(X)T^l(s) \tag{7.20}$$

(where $\hat{c}(X)$ is as defined in (2.12)) as follows from the commutativity of the diagram

$$\begin{array}{ccc} H^s(S^{l-1}(p^C) \otimes l_{\lambda+2\varrho}) & \xleftarrow{\delta_X \otimes 1} & H^s(S^l(p^C) \otimes l_{\lambda+2\varrho}) \\ {\scriptstyle i_\#}\uparrow & & {\scriptstyle i_\#}\uparrow \\ H^s(S^{l-1}(p_+) \otimes l_{\lambda+2\varrho}) & \xleftarrow{\delta(X)} & H^s(S^l(p_+) \otimes l_{\lambda+2\varrho}). \end{array}$$

On the other hand, consider the map

$$\bar{\varepsilon}_0 : p^C \otimes \tilde{V}_{\lambda+2\varrho_n} \to H_K(E_0)$$

given by

$$\bar{\varepsilon}_0(X \otimes s) = \hat{\pi}(X)s. \tag{7.21}$$

The map $\bar{\varepsilon}_0$ is clearly K^C linear. No irreducible component of $p^C \otimes \tilde{V}_{\lambda+2\varrho_n}$ has highest weight $\lambda+2\varrho_n$.

Thus, no irreducible subspace in the image of $\bar{\varepsilon}_0$ has highest weight $\lambda+2\varrho_n$. Hence

$$\bar{\varepsilon}_0(p^C \otimes \tilde{V}_{\lambda+2\varrho_n}) \subseteqq H_{k,1}(E_0). \tag{7.22}$$

Thus there exists a unique map

$$\varepsilon'_0 : p^C \otimes H^s(l_{\lambda+2\varrho}) \to H^s(S^1(p_+) \otimes l_{\lambda+2\varrho}) \tag{7.23}$$

such that for $X \in p^C$ and $s \in \tilde{V}_{\lambda+2\varrho_n}$,

$$T^1(\bar{\varepsilon}_0(X \otimes s)) = \varepsilon'_0(X \otimes T^0 s), \tag{7.24}$$

We define for $X \in p^C$ and $u \in H^s(l_{\lambda+2\varrho})$,

$$\varepsilon'_0(X)u = \varepsilon'_0(X \otimes u). \tag{7.25}$$

From (7.21), (7.24) and (7.25) we have for $X\in p^C$ and $s\in\tilde{V}_{\lambda+2\varrho_n}$,

$$\varepsilon_0'(X)(T^0 s)=T^1(\hat{\pi}(X)s)\,. \tag{7.26}$$

We have for $s\in\tilde{V}_{\lambda+2\varrho_n}$, $X, Y\in p^C$,

$$T^0(\hat{\pi}(X)\hat{\pi}(Y)s-\hat{\pi}(Y)\hat{\pi}(X)s)=T^0(\hat{\pi}[X,Y]s)\,.$$

Thus, from (7.20)

$$C\cdot\partial(X)(T^1(\hat{\pi}(Y)s))-C\cdot\partial(Y)(T^1(\hat{\pi}(X)s))=\tau_0[X,Y]\,T^0 s\,.$$

Now using (7.26)

$$C\cdot\partial(X)\varepsilon_0'(Y)T^0 s-C\partial(Y)\varepsilon_0'(X)T^0 s=\tau_0[X,Y]\,T^0 s\,. \tag{7.27}$$

Hence, for

$$u\in H^s(l_{\lambda+2\varrho}),\ C\cdot\partial(X)\varepsilon_0'(Y)u-C\cdot\partial(Y)\varepsilon_0'(X)u=\tau_0[X,Y]\,u\,. \tag{7.28}$$

Thus,

$$C\cdot\varepsilon_0'(X)=\varepsilon_0(X) \tag{7.29}$$

where $\varepsilon_0(X)$ is as in Proposition 5.2.

For

$$s\in H_K(E_0)\subseteqq C_K^\infty(E_0)\subseteqq C_K^\infty(E)\,,$$

by (7.3) $D(Ds)=0$. On the other hand, it is proved in [4, Proposition 3.2] that

$$\begin{aligned}D(Ds)=\Sigma_q\hat{\pi}(Y_q)^2 s-\Sigma_i\hat{\pi}(X_i)^2 s\\ +\{(\lambda+\varrho,\lambda+\varrho)-(\varrho,\varrho)\}\,s\,,\end{aligned}$$

where $\{Y_q\}$, $\{X_i\}$ are bases of k and p in Lemma 6.3. Thus for $s\in\tilde{V}_{\lambda+2\varrho_n}$,

$$\begin{aligned}\{(\lambda+\varrho,\lambda+\varrho)-(\varrho,\varrho)\}T^0 s&=-\Sigma_q T^0(\hat{\pi}(Y_q)^2 s)+\Sigma_i T^0(\hat{\pi}(X_i)\hat{\pi}(X_i)s)\\ &=-\Sigma_q\tau_0(Y_q)^2T^0 s+\Sigma_i C\partial(X_i)T^1(\hat{\pi}(X_i)s)\,, &&\text{by (7.20)}\\ &=-\Sigma_q\tau_0(Y_q)^2T^0 s+\Sigma_i C\partial(X_i)\varepsilon_0'(X_i)T^0 s\,, &&\text{by (7.26)}\\ &=-\Sigma_q\tau_0(Y_q)^2T^0 s+\Sigma_i\partial(X_i)\varepsilon_0(X_i)T^0 s\,. &&\text{by (7.29)}\end{aligned}$$

Thus, Lemma 6.3 is proved for $k=0$.

§8

G, $t^C\subseteqq k^C\subseteqq g^C$ are as in the foregoing sections. p_+ is the subspace of g^C spanned by the root vectors corresponding to noncompact roots $\alpha\in P$, where P is a positive system of roots of (t^C, g^C). Let $\varrho=$ half the sum of the roots in P. F is the lattice in $\mathrm{Hom}(\sqrt{-1}\,t,\mathbb{R})$ defined by the characters on T. Let

$$\begin{aligned}\tilde{F}=\{\lambda\in F\,|\,\langle\lambda+\varrho,\alpha\rangle>0,\ \forall\alpha\in P_k \quad\text{and}\\ \langle\lambda+\varrho,\alpha\rangle\geqq 0,\ \forall\alpha\in P_n\}\,.\end{aligned}$$

Finally, for $\lambda \in \tilde{F}$, we say that λ satisfies the condition $(*)$ when $H^i(\wedge^q p_- \otimes l_{\lambda+2\varrho})=0$, $\forall i<s$ and $\forall_q \geqq 0$.

In the statement of all the theorems in this and the next section, we assume that P is chosen such that the condition

$[[p_+, p_+], p_+]=0$

is satisfied and $\lambda \in \tilde{F}$ is chosen such that the condition $(*)$ is satisfied.

We then recall again the following facts for clarification.

1) For any G admitting discrete series there is always a choice of P such that $[[p_+, p_+], p_+]=0$. For $SO(2n, 1)$ and $SU(n, 1)$ any choice of P has this property.

2) For $\lambda \in \tilde{F}$, if $|\langle \lambda+\varrho, \alpha\rangle|$ is sufficiently large for every compact root α then the condition $(*)$ is satisfied [3, Lemma 4.1].

3) There are lots of $\lambda \in \tilde{F}$ such that $\langle \lambda+\varrho, \alpha\rangle$ is zero for a noncompact root α and condition $(*)$ is satisfied. For this combine Remark 2 with the following observation: There exists an integer N_0 such that for every integer $N>N_0$, $\lambda \in \tilde{F} \Rightarrow N(\lambda+\varrho)-\varrho \in \tilde{F}$ and satisfies condition $(*)$.

4) Let $\tilde{F}'=\{\lambda \in \tilde{F} | \langle \lambda+\varrho, \alpha\rangle>0, \forall \alpha \in P\}$. Then the (disjoint) union of the $\tilde{F}'$as P varies over all the positive rootsystems containing a fixed P_k parametrizes the discrete series for G. If G is linear and if P satisfies $[[p_+, p_+], p_+]=0$, then $\lambda \in \tilde{F}' \Rightarrow \lambda$ satisfies the condition $(*)$ [3, §9].

Theorem 8.1. *Suppose P satisfies $[[p_+, p_+], p_+]=0$. Let $\lambda \in \tilde{F}$ and assume that λ satisfies the condition $(*)$. For $k \geqq 0$, there exist unique K^C linear maps*

$$\varepsilon_k : p^C \otimes H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho}) \tag{8.1}$$

with the following properties:

Denoting for $X \in p^C$, $u \in H^s(S^k(p_+) \otimes l_{\lambda+2\varrho})$

$$\varepsilon_k(X)u=\varepsilon_k(X \otimes u) \tag{8.2}$$

they satisfy

$$\varepsilon_{k+1}(X)\varepsilon_k(Y)=\varepsilon_{k+1}(Y)\varepsilon_k(X) \tag{8.3}$$

for all $X, Y \in p^C$. Also, with $\partial_k(X)$ defined as in (3.3) for $k \geqq 1$,

$$\partial_{k+1}(X)\varepsilon_k(Y)-\partial_{k+1}(Y)\varepsilon_k(X)=\tau_k[X, Y]-\varepsilon_{k-1}(X)\partial_k(Y)+\varepsilon_{k-1}(Y)\partial_k(X). \tag{8.4}$$

For $k=0$, we have

$$\partial_1(X)\varepsilon_0(Y)-\partial_1(Y)\varepsilon_0(X)=\tau_0[X, Y]. \tag{8.5}$$

Moreover, for all $k \geqq 0$,

$$\varepsilon_k(p^C \otimes H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}))=H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho}). \tag{8.6}$$

Proof. For $k=0$, the existence and uniqueness of ε_0 with the desired properties follows from Proposition 5.2 and Proposition (6.1).

Further by Proposition 5.2, the map ε_0 so constructed has the following property:

Choose and fix a K^C linear isomorphism j^0 as in (5.21). Then there exists a unique K^C linear homomorphism j^1 as in (5.22) such that (3.20 IH) is satisfied for $k=0$. By looking at (3.20 IH), it is clear that image of $j^1 \supseteqq$ image of

$$\varepsilon_0 = H^s(S^1(p_+)\otimes l_{\lambda+2\varrho}), \quad \text{by (8.6)}$$

Thus, for dimension reasons j^1 is an isomorphism.

Now, we prove the theorem by induction: Let $l \geqq 1$. Suppose for $k=0,1,\ldots,l-1$, the existence and uniqueness of ε_k satisfying (8.3), (8.4) and (8.6) has been established; suppose the maps ε_k so constructed have the following property: For $k=0,1,\ldots,l$ there exists unique K^C linear isomorphisms j^k as in (3.19 IH) so that (3.20 IH) is satisfied for $k=0,1,\ldots,l-1$.

Under the above induction hypothesis, the existence and uniqueness of ε_l satisfying (8.3) and (8.4) is proved in Proposition 4.1. The property (8.6) for the map ε_l so constructed is proved in Proposition 6.1. In Proposition 5.3, the existence of a unique map j as in (3.19 IH) for $k=l+1$, satisfying (3.20 IH) for $k=l$ is proved.

The proof of Theorem 8.1 is complete.

Theorem 8.2. *Suppose P satisfies $[[p_+,p_+],p_+]=0$. Let $\lambda\in\tilde{F}$ and assume that λ satisfies the condition $(*)$. Let*

$$H_\lambda = \oplus_{l\geqq 0} H^s(S^l(p_+)\otimes l_{\lambda+2\varrho}).$$

For $X\in p^C$, let $\varrho_\lambda(X): H_\lambda\to H_\lambda$ be the linear map defined by

$$\varrho_\lambda(X)u = (\partial_l(X)+\varepsilon_l(X))u$$

for $u\in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$ $(\partial_0(X)=0)$. Here $\varepsilon_l(X)$ is given by Theorem 8.1. For $Y\in k^C$ let $\varrho_\lambda(Y): H_\lambda\to H_\lambda$ be the linear map given by

$$\varrho_\lambda(Y)u = \tau_l(Y)u$$

for $u\in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. For $Z=Y+X$ where $Y\in k^C$ and $X\in p^C$, define

$$\varrho_\lambda(Z) = \varrho_\lambda(Y)+\varrho_\lambda(X).$$

Then ϱ_λ defines an irreducible representation of g on H_λ.

Proof. If $Y_1, Y_2\in k^C$, it is clear that $\varrho_\lambda[Y_1,Y_2] = \varrho_\lambda(Y_1)\varrho_\lambda(Y_2)-\varrho_\lambda(Y_2)\varrho_\lambda(Y_1)$.

Let $X_1, X_2\in p^C$ and $u\in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. We have to compute $\varrho_\lambda(X_1)\varrho_\lambda(X_2)u - \varrho_\lambda(X_2)\varrho_\lambda(X_1)u$.

$$\begin{aligned}\varrho_\lambda(X_1)\varrho_\lambda(X_2)u &= \varrho_\lambda(X_1)(\partial_l(X_2)u+\varepsilon_l(X_2)u)\\ &= \partial_{l-1}(X_1)\partial_l(X_2)u+\varepsilon_{l-1}(X_1)\partial_l(X_2)u\\ &\quad +\partial_{l+1}(X_1)\varepsilon_l(X_2)u+\varepsilon_{l+1}(X_1)\varepsilon_l(X_2).\end{aligned}$$

Similarly, write down the expression for $\varrho_\lambda(X_2)\varrho_\lambda(X_1)u$ and take the difference. Note that

$$\varepsilon_{l+1}(X_1)\varepsilon_l(X_2) = \varepsilon_{l+1}(X_2)\varepsilon_l(X_1) \quad \text{by (8.3)}.$$

Also,

$$\partial_{l-1}(X_1)\partial_l(X_2) = \partial_{l-1}(X_2)\partial_l(X_1) \quad \text{by Lemma 3.1}.$$

Then (8.4) implies that

$$\varrho_\lambda(X_1)\varrho_\lambda(X_2)-\varrho_\lambda(X_2)\varrho_\lambda(X_1)=\varrho_\lambda[X_1, X_2].$$

Now let $X\in p^C$ and $Y\in k^C$. The K^C linearity of ∂_l and the K^C linearity of ε_l imply that for $u\in H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$

$$\tau_{l-1}(Y)\partial_l(X)u=\partial_l(X)\tau_l(Y)u+\partial_l[Y, X]u$$

and

$$\tau_{l+1}(Y)\varepsilon_l(X)u=\varepsilon_l(X)\tau_l(Y)u+\varepsilon_l[Y, X]u$$

i.e.

$$\varrho_\lambda(Y)\partial_l(X)u=\partial_l(X)\varrho_\lambda(Y)u+\partial_l[Y, X]u$$

and

$$\varrho_\lambda(Y)\varepsilon_l(X)u=\varepsilon_l(X)\varrho_\lambda(Y)u+\varepsilon_l[Y, X]u.$$

Adding up we see that,

$$\varrho_\lambda(Y)\varrho_\lambda(X)u=\varrho_\lambda(X)\varrho_\lambda(Y)u+\varrho_\lambda[Y, X]u.$$

Thus indeed ϱ_λ is a representation of g.

Now suppose H' is a nonzero g invariant subspace of H_λ. Let l be the least nonnegative integer such that

$$H' \subseteqq \oplus_{k\geqq l} H^s(S^k(p_+)\otimes l_{\lambda+2\varrho}).$$

We claim $l=0$. For, suppose not. Choose an element $u\in H'$ which has a nonzero component in $H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. Since the map

$$\partial^0: H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})\to p^C\otimes H^s(S^{l-1}(p_+)\otimes l_{\lambda+2\varrho})$$

defined by

$$\partial^0(v)=\Sigma_i X_i\otimes\partial_l(X_i)v$$

is an injection (Lemma 2.2), if $v\neq 0$, there exists an X_i such that $\partial_l(X_i)v\neq 0$. Thus, one can choose $X\in p^C$ such that $\partial_l(X)u_l\neq 0$, where u_l is the component of u in $H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$. Then it is clear that $\varrho_\lambda(X)u$ has nonzero component $\partial_l(X)u_l$ in $H^s(S^{l-1}(p_+)\otimes l_{\lambda+2\varrho})$. This contradicts the minimality of l since $\varrho_\lambda(X)u\in H'$. Thus $l=0$. Since the multiplicity of $H^s(l_{\lambda+2\varrho})$ in $\otimes_{l>0}H^s(S^l(p_+)\otimes l_{\lambda+2\varrho})$ is zero, one now concludes that $H^s(l_{\lambda+2\varrho})\subseteqq H'$.

Now let $k>0$ and suppose that $H^s(S^i(p_+)\otimes l_{\lambda+2\varrho})\subseteqq H'$, for $i=0, 1, \ldots, k-1$. Since

$$\varepsilon_{k-1}: p^C\otimes H^s(S^{k-1}(p_+)\otimes l_{\lambda+2\varrho})\to H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$$

is surjective, any $u\in H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$ can be expressed as

$$u=\Sigma_i\varepsilon_{k-1}(X_i)v_i,\ X_i\in p^C$$

where $v_i\in H^s(S^{k-1}(p_+)\otimes l_{\lambda+2\varrho})$. Since

$$\varrho_\lambda(X_i)v_i=\partial_{k-1}(X_i)v+\varepsilon_{k-1}(X_i)v_i$$

we have then,

$$u \equiv \Sigma_i \varrho_\lambda(X_i) v_i \bmod (H^s(S^{k-2}(p_+) \otimes l_{\lambda+2\varrho}))$$

i.e. $u \in H' \bmod (\Sigma_{i<k} H^s(S^i(p_+) \otimes l_{\lambda+2\varrho}))$. So, in fact, $u \in H'$. Thus we have concluded

$$H' = H_\lambda .$$

q.e.d.

Theorem 8.3. *Suppose P satisfies $[[p_+, p_+], p_+] = 0$. Let $\lambda \in \tilde{F}$ and assume that λ satisfies the condition $(*)$. Let $Q(P, \lambda+\varrho)$ be the universal g module constructed in [8, § 3,4]. Let $\pi_{P,\lambda+\varrho}$ be the representation of g on $Q(P, \lambda+\varrho)$. Then $\pi_{P,\lambda+\varrho} \simeq \varrho_\lambda$. In particular the g module $Q(P, \lambda+\varrho)$ is irreducible.*

Proof. By using [Lemma 8, (4.2)] one can show that there is a nonzero g module homomorphism f of $Q(P, \lambda+\varrho)$ into H_λ, the representation space of ϱ_λ. But since ϱ_λ is irreducible, f has to be a surjection. On the other hand using [Lemma 8, (3.8)] one can show that for any irreducible representation δ of k, the multiplicity of δ of in $\pi_{P,\lambda+\varrho}$ is less than or equal to the multiplicity of δ in ϱ_λ. This proves the theorem. q.e.d.

Theorem 8.4. *Suppose P satisfies $[[p_+, p_+], p_+] = 0$. Let $\lambda \in \tilde{F}$ and assume that λ satisfies the condition $(*)$. Let ϱ_λ be the irreducible representation of g given by Theorem 8.2. Let τ_μ be an irreducible finite dimensional representation of k with highest weight μ with respect to P_k. Then the multiplicity of τ_μ in ϱ_λ equals*

$$\Sigma_{w \in W_K} \varepsilon(w) Q(w(\mu+\varrho_k) - (\lambda + 2\varrho_n + \varrho_k))$$

where W_K is the Weyl group of K and $Q(w(\mu+\varrho_k) - (\lambda+2\varrho_n+\varrho_k))$ denotes the number of distinct ways in which $w(\mu+\varrho_k) - (\lambda+2\varrho_n+\varrho_k)$ can be expressed as a sum of positive noncompact roots, i.e. roots in P_n.

Proof. We have only to sum up over m the multiplicity of τ_μ in $H^s(S^m(p_+) \otimes l_{\lambda+2\varrho})$. The theorem now follows from [3, Lemma 4.3] q.e.d.

§9. The Unitarity of the Representations ϱ_λ

First to get a hermitian form, what we need is a family of conjugate linear maps

$$\bar{j}_k : H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \to H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \tag{9.1}$$

preserving the K actions and having properties similar to (3.20 IH) rather than the C-linear maps

$$j_k : H^0(S^k(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho n)}) \to H^s(S^k(p_+) \otimes l_{\lambda+2\varrho})$$

in (5.26), which preserve the K^c actions and have the properties (3.20 IH). By Serre's duality, the module on the left in (9.1) is canonically the dual of the module on the right. Thus $\bar{j}_k$ can be used to define a hermitian form. But the important thing is to prove that $\bar{j}_k$ is positive definite. Naturally the positivity conditions in

(1.4) will enter into the proof. And as in the rest of the paper, especially §6, tensoring with the spin module $\varkappa: k \to \mathrm{End}(L)$ seems to help.

For $x \in p^c$ and any $k \geqq 0$, recall the maps

$$\eta_k(X): H^0(S^k(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)}) \to H^0(S^{k+1}(\varkappa p_+) \otimes l_{\varkappa(\lambda+2\varrho_n)})$$

in (3.18). We denote by

$$\bar{\varepsilon}_k(X): H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \to H^0(S^{k+1}(p_-) \otimes l_{-\lambda-2\varrho_n})$$

the negative of the analogously defined maps (3.13).

Lemma 9.1. *Let* $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$, $u \in H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho})$ *and* $X \in p^c$. *Then*

$$\langle \bar{\varepsilon}_k(X)\xi, u \rangle = -\langle \xi, \partial_{k+1}(X) u \rangle .$$

(Here, $\langle , \rangle$ is given by Serre's duality).

Proof. We have, from (3.13)

$$\begin{aligned} \langle \bar{\varepsilon}_k(X)\xi, u \rangle &= -\langle \eta^0(X \otimes \xi), u \rangle \\ &= -\langle X \otimes \xi, \partial^0 u \rangle , \qquad \text{by Lemma 3.2} \\ &= -\langle X \otimes \xi, \Sigma X_i \otimes \partial_{k+1}(X_i) u \rangle \\ &= -\Sigma \langle X, X_i \rangle \langle \xi, \partial_{k+1}(X_i) u \rangle \\ &= -\langle \xi, \partial_{k+1}(X) u \rangle \quad \text{q.e.d.} \end{aligned}$$

We now *define* for $X \in p^c$, $k \geqq 0$

$$\bar{\partial}_{k+1}(X): H^0(S^{k+1}(p_-) \otimes l_{-\lambda-2\varrho_n}) \to H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$$

to be the negative of the transpose of $\varepsilon_k: H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho})$. Also, *define* for $Y \in k^c$,

$$\bar{\tau}_k(Y): H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \to H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$$

to be the negative of the transpose of

$$\tau_k(Y): H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \to H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) .$$

Then $\bar{\tau}_k$ defines a representation of k^c dual to τ_k and is precisely the representation of k^c on $H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$ considered previously.

Lemma 9.2. *The maps* $\bar{\partial}_k(X): H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \to H^0(S^{k-1}(p_-) \otimes l_{-\lambda-2\varrho_n})$ *defined above are the unique maps with the following property:*

i) $\bar{\partial}_{k-1}(X)\bar{\partial}_k(Y) = \bar{\partial}_{k-1}(Y)\bar{\partial}_k(X)$, $\forall X, Y \in p^c$ *and* $k \geqq 2$.

ii) *The map* $\bar{\partial}_k: p^c \otimes H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) H^0(S^{k-1}(p_-) \otimes l_{-\lambda-2\varrho_n})$ *defined by* $\bar{\partial}_k(X \otimes \xi) = \bar{\partial}_k(X)\xi$ *is* K^c *module map.*

iii) $\forall X, Y \in p^c$, $\bar{\tau}_k[X, Y] = \bar{\varepsilon}_{k-1}(X)\bar{\partial}_k(Y) - \bar{\varepsilon}_{k-1}(Y)\bar{\partial}_k(X)$

$$+ \bar{\partial}_{k+1}(X)\bar{\varepsilon}_k(Y) - \bar{\partial}_{k+1}(Y)\bar{\varepsilon}_k(X)$$

Proof. This is got by dualizing Theorem 8.1. q.e.d.

Similarly dualizing Theorem 8.2, we have

Lemma 9.3. *Let* $\bar{H}_\lambda = \otimes_{k \geqq 0} H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n})$. *For* $X \in p^c$. *Let* $\bar{\varrho}_\lambda(X): \bar{H}_\lambda \to \bar{H}_\lambda$ *be the linear map defined by*

$$\bar{\varrho}_\lambda(X)\xi = (\bar{\partial}_k(X) + \bar{\varepsilon}_k(X))\xi$$

for $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n})$. *For* $Y \in k^c$, *let* $\bar{\varrho}_\lambda(Y)$ *be given by*

$$\bar{\varrho}_\lambda(Y)\xi = \bar{\tau}_k(Y)\xi$$

for $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n})$. *Extend* $\bar{\varrho}_\lambda$ *linearly to all of* g^c. *Then* $\bar{\varrho}_\lambda$ *defines an irreducible representation of* g^c *on* $\bar{H}_\lambda$.

We now choose and fix a conjugate linear K module map

$$\bar{j}_0 : H^0(l_{-\lambda - 2\varrho_n}) \to H^s(l_{\lambda + 2\varrho})$$

such that for $\xi, \xi' \in H^0(l_{-\lambda - 2\varrho_n})$, $\langle \xi, \bar{j}_0 \xi' \rangle$ defines a K invariant positive definite hermitian form on $H^0(l_{-\lambda - 2\varrho_n})$. We now have the following

Lemma 9.4. *There exist unique conjugate linear K module maps*

$$\bar{j}_k : H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n}) \to H^s(S^k(p_+) \otimes l_{\lambda + 2\varrho})$$

with $\bar{j}_0$ *as already chosen, having the following property: For* $X \in p$ *and*

$$\xi \in H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n}),$$

$$\bar{j}_{k+1}(\bar{\varepsilon}_k(X)(\xi)) = \varepsilon_k(X)(\bar{j}_k(\xi)). \tag{9.2}$$

The maps $\bar{j}_k$ *also satisfy*

$$\bar{j}_{k-1}\bar{\partial}_k(X) = \partial_k(X)\bar{j}_k. \tag{9.3}$$

Proof. For the existence and uniqueness of $\bar{j}_k$, in view of Proposition 5.3, it suffices to prove that there exist conjugate linear K module maps

$$a_k : H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n}) \to H^0(S^k(\kappa p_+) \otimes l_{\kappa(\lambda + 2\varrho_n)}) \tag{9.4}$$

such that for $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda - 2\varrho_n})$ and $X \in p$

$$a_{k+1}(\bar{\varepsilon}_k(X)\xi) = \eta_k(X)(a_k(\xi)) \tag{9.5}$$

and such that a_k is onto for every $k \geqq 0$. For then the existence and uniqueness of $\bar{j}_k$ would follow from those of $j_k = \bar{j}_k(a_k)^{-1}$ in Proposition 5.3. (We have taken $a_0 = (j_0)^{-1}\bar{j}_0$). Also, to prove the existence of the surjective maps (9.4) with the properties (9.5), it suffices, by dualizing and using Lemma 9.1 to prove the existence of surjective conjugate linear K module maps

$$b_k : H^s(S^k(\varkappa p_-) \otimes l_{-\kappa(\lambda + 2\varrho)}) \to H^s(S^k(p_+) \otimes l_{\lambda + 2\varrho})$$

such that for $u \in H^s(S^{k+1}(\varkappa p_-) \otimes l_{-\varkappa(\lambda + 2\varrho)})$ and $X \in p$

$$b_k(\partial_{k+1}(X)u) = \partial_{k+1}(X)(b_{k+1}(u)) \tag{9.6}$$

(b_0 = the transpose of a_0). For then $a_k = (-1)^k$ transpose of b_k, will do the job. We define b_1 as follows: Take $u \in H^s(S^1(\varkappa p_-) \otimes l_{-\kappa(\lambda + 2\varrho)})$. Choose orthonormal basis $X_1, X_2, \ldots, X_{2m}$ of p and consider the element

$$e(u) \in p^c \otimes H^s(l_{\lambda + 2\varrho})$$

defined by

$$e(u)=\Sigma_i X_i \otimes b_0(\partial_1(X_i)u)$$

Clearly $u \mapsto e(u)$ is a conjugate linear K module map of $H^s(S^1(\kappa p_-)\otimes l_{-\kappa(\lambda+2\varrho)})$ into $p^c \otimes H^s(l_{\lambda+2\varrho})$. By Lemma 5.1, the composite of the above map with $p^c \otimes H^s(l_{\lambda+2\varrho}) \to H^s(l_{\lambda+2\varrho}\otimes p_-)$ is identically zero. Thus, from the exactness of the completes (2.20), for $u \in H^s(S^1(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})$ there exists

$$b_1(u) \in H^s(S^1(p_+)\otimes l_{\lambda+2\varrho})$$

such that $e(u)=\partial^0(b_1(u))$. Since ∂^0 is an injection, if $u \neq 0$, $e(u) \neq 0$ and $b_1(u) \neq 0$. The conjugate linear map b_1 is the desired map. For dimension reasons b_1 is a surjection.

Now let $m>1$ and suppose that b_k has been defined for $k<m$ satisfying (9.6) in the appropriate range. Let $u \in H^s(S^m(\varkappa p_-)\otimes l_{-\varkappa(\lambda+2\varrho)})$. Define an element $e(u) \in p^c \otimes H^s(S^{m-1}(p_+)\otimes l_{\lambda+2\varrho})$ by

$$e(u)=\Sigma X_i \otimes b_{m-1}(\partial_m(X_i)u)$$

Recall the map ∂^1 in (2.19). By 2.24

$$\begin{aligned}\partial^1(e(u)) &= \Sigma_{i,j} X_j \wedge X_i \otimes \partial_{m-1}(X_j)(b_{m-1}\partial_m(X_i)u)\\ &= \Sigma_{i,j} X_j \wedge X_i \otimes b_{m-2}(\partial_{m-1}(X_j)\partial_m(X_i)u)\\ &= 0, \text{ because of Lemma 3.1 .}\end{aligned}$$

Thus by the exactness in Lemma 2.2, there exists a unique element $b_m(u) \in H^s(S^m(p_+)\otimes l_{\lambda+2\varrho})$ such that $e(u)=\partial^0(b_m(u))$. The map $u \mapsto b_m(u)$ is the required map.

We have to verify (9.3), i.e. we have to prove that for $X \in p$,

$$\bar{\partial}_k(X)=(\bar{j}_{k-1})^{-1}\partial_k(X)\bar{j}_k$$

Denote by $\bar{\partial}_k(X)$, the map on the right hand side above. We verify that $\bar{\partial}_k(X)$ has all the properties required for the uniqueness assertion in Lemma 9.2. The properties i) and ii) of Lemma 9.2 obviously hold for $\bar{\partial}_k$. The property iii) for $\bar{\partial}_k$ follows combining (9.2) and (8.4) and noting that $\bar{j}_k$ commutes with the k action. q.e.d.

We now define a hermitian form on $H^0(S^k(p_-)\otimes l_{-\lambda-2\varrho_n})$ by

$$(\xi,\xi')=\langle \xi, \bar{j}_k \xi' \rangle .$$

We have to first verify

Lemma 9.5. *Let $\xi, \xi' \in H^0(S^k(p_-)\otimes l_{-\lambda-2\varrho_n})$ letting conj z the complex conjugate of z, $(\xi,\xi')=$ conj (ξ',ξ)*

Proof. Define $\bar{i}_k: H^0(s^k(p_-)\otimes l_{-\lambda-2\varrho_n}) \to H^s(S^k(p_+)\otimes l_{\lambda+2\varrho})$ by $\langle \xi', \bar{i}_k(\xi)\rangle =$ conj $\langle \xi, \bar{j}_k(\xi')\rangle$ for $\xi, \xi' \in H^0(S^k(p_-)\otimes l_{-\lambda-2\varrho_n})$. (For fixed ξ, conj $\langle \xi, \bar{j}_k(\xi')\rangle$ depends linearly on ξ', etc.), $\bar{i}_k$ is a conjugate linear K module map. We will prove the lemma by showing that $\bar{i}_k=\bar{j}_k$. For this we will show that $\bar{i}_k$ has the property (9.2) required for the uniqueness assertion in Lemma 9.4. We first verify that for $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$ and $X \in p$

$$\bar{i}_{k+1}\bar{\varepsilon}_k(X)\xi = \varepsilon_k(X)\bar{i}_k(\xi) \tag{9.7}$$

The two sides are elements of $H^s(S^{k+1}(p_+) \otimes l_{\lambda+2\varrho})$. We have to show that they have the same scalar product with

$$\psi \in H^0(S^{k+1}(p_-) \otimes l_{-\lambda-2\varrho_n}).$$

We have

$$\langle \psi, \bar{i}_{k+1}\bar{\varepsilon}_k(X)\xi \rangle = \operatorname{conj} \langle \bar{\varepsilon}_k(X)\xi, \bar{j}_{k+1}(\psi) \rangle.$$

On the other hand

$$\begin{aligned} \langle \psi, \varepsilon_k(X)\bar{i}_k(\xi) \rangle &= -\langle \bar{\partial}_{k+1}(X)\psi, \bar{i}_k(\xi) \rangle \\ &\quad \text{(by the def of } \bar{\partial}_{k+1}) \\ &= -\operatorname{conj} \langle \xi, \bar{j}_k \bar{\partial}_{k+1}(X)\psi \rangle \\ &= -\operatorname{conj} \langle \xi, \partial_{k+1}(X)\bar{j}_{k+1}(\psi) \rangle, \quad \text{(by 9.3)} \\ &= -\operatorname{conj} \langle \bar{\varepsilon}_k(X)\xi, \bar{j}_{k+1}(\psi) \rangle \quad \text{by Lemma 9.1.} \end{aligned}$$

Thus (9.7) is proved. Note that by the choice of $\bar{j}_0$ and the definition of $\bar{i}_0$, $\bar{i}_0 = \bar{j}_0$. Now Lemma 9.4 can be applied q.e.d.

The hermitian forms on $H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$ give rise to a hermitian form on $\bar{H}_\lambda = \otimes k \geqq 0 H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$ by making two different summands orthogonal to each other.

Proposition 9.6. *(Infinitesimal Invariance). The operators $\bar{\varrho}_\lambda(z)$, $z \in g$, leave the above defined hermitian form on $\bar{H}_\lambda$ infinitesimally invariant, i.e.,*

$$(\bar{\varrho}_\lambda(z)\xi, \xi') + (\xi, \bar{\varrho}_\lambda(z)\xi') = 0$$

Proof. We have only to prove that for $X \in p$,

$$\xi_i \in H^0(S^i(p_-) \otimes l_{-\lambda-2\varrho_n}),\ i = k-1, k, k+1$$

$$(\bar{\partial}_k(X)\xi_k, \xi_{k-1}) + (\xi_k, \bar{\varepsilon}_{k-1}(X)\xi_{k-1}) = 0 \tag{9.8}$$

and

$$(\bar{\varepsilon}_k(X)\xi_k, \xi_{k+1}) + (\xi_k, \bar{\partial}_{k+1}(X)\xi_{k+1}) = 0 \tag{9.9}$$

The left side of (9.8) equals

$$\begin{aligned} &\langle \bar{\partial}_k(X)\xi_k, \bar{j}_{k-1}\xi_{k-1} \rangle + \langle \xi_k, \bar{j}_k\bar{\varepsilon}_{k-1}(X)\xi_{k-1} \rangle \\ &= \langle \xi_k, -\varepsilon_{k-1}(X)\bar{j}_{k-1}\xi_{k-1} \rangle + \langle \xi_k, \bar{j}_k\bar{\varepsilon}_{k-1}(X)\xi_{k-1} \rangle \\ &= 0, \quad \text{by (9.2).} \end{aligned}$$

(9.9) is proved similarly q.e.d.

Proposition 9.7. *The hermitian form on $H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$ is positive definite, i.e., for $\xi \in H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n})$, $\xi \neq 0$,*

$$\langle \xi, \bar{j}_k(\xi) \rangle \quad \textit{is real and positive}.$$

A brief idea of the proof of Proposition 9.7 has been given in the introduction. We need several lemmas.

Recall the spin representation $\varkappa: k \to \mathrm{End}(L)$ and $c: \mathrm{cliff}(p^c) \to \mathrm{End}(L)$. We choose and fix a positive definite hermitian form on L such that for $s, s' \in L$ and $X \in p$,

$$(c(X)s, s') + (s, c(X)s') = 0 \tag{9.10}$$

(see [4, Lemma 4.1]). We also choose and fix a nondegenerate bilinear form on L such that for $s, s' \in L$ and $X \in p^c$,

$$\langle c(X)s, s' \rangle + \langle s, c(X)s' \rangle = 0 \tag{9.11}$$

Similar to the maps D_k and E_k ((6.8) and (6.11)), we now define

$$\bar{D}_k : H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L \to H^0(S^{k-1}(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L \tag{9.12}$$

by $\bar{D}_k(\xi \otimes s) = \Sigma \bar{\partial}_k(X_i)\xi \otimes c(X_i)s$ and

$$\bar{E}_k : H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L \to H^0(S^{k+1}(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L$$

by $\bar{E}_k(\xi \otimes s) = \Sigma \bar{\varepsilon}_k(X_i)\xi \otimes c(X_i)s$. It is easy to see that for $\varphi_i \in H^0(S^i(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L$, $i = k-1, k, k+1$ and $\varphi_i^* \in H^s(S^i(p_+) \otimes l_{\lambda+2\varrho}) \otimes L$, $i = k-1, k, k+1$,

$$\langle \bar{D}_k \varphi_k, \varphi_{k-1}^* \rangle = \langle \varphi_k, E_{k-1} \varphi_{k-1}^* \rangle \tag{9.13}$$

and

$$\langle \bar{E}_k \varphi_k, \varphi_{k+1}^* \rangle = \langle \varphi_k, D_{k+1} \varphi_{k+1}^* \rangle \tag{9.14}$$

With respect to the hermitian forms, we have

Lemma 9.8. *For $\varphi_i \in H^0(S^i(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L$, $i = k-1, k, k+1$ we have,*

$$(\bar{E}_k \varphi_k, \varphi_{k+1}) = (\varphi_k, \bar{D}_{k+1} \varphi_{k+1})$$

and

$$(\bar{D}_k \varphi_k, \varphi_{k-1}) = (\varphi_k, \bar{E}_{k-1} \varphi_{k-1}) .$$

Proof. Follows from (9.8), (9.9) and (9.10) q.e.d.

Let Ω_K denote the casimir of K. Combining (6.23), (9.13) and (9.14) we have, for $k \geqq 0$,

$$\bar{E}_{k-1} \bar{D}_k + \bar{D}_{k+1} \bar{E}_k = (\tau_k \otimes \chi)(\Omega_K) - (\lambda + \varrho, \lambda + \varrho) + (\varrho_k, \varrho_k) . \tag{9.15}$$

Thus, if $\varphi \in V_{-\mu} \subsetneqq H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L$, where $V_{-\mu}$ is an irreducible k^c submodule with lowest weight $-\mu$, then,

$$(\bar{E}_{k-1} \bar{D}_k + \bar{D}_{k+1} \bar{E}_k)\varphi = \{(\mu + \varrho_k, \mu + \varrho_k) - (\lambda + \varrho, \lambda + \varrho)\} \varphi . \tag{9.16}$$

Moreover, as concluded in (6.30),

$$\mu + \varrho_k = \lambda + \varrho + A ,$$

where A is a sum of positive roots, with $A \neq 0$, if $k>0$. Using (1.4), we conclude that if $k>0$, then

$$(\mu+\varrho_k, \mu+\varrho_k)-(\lambda+\varrho, \lambda+\varrho)>0.$$

We have proved

Lemma 9.9. *Let $k>0$, and let $\varphi \in V_{-\mu} \subsetneqq H^0(S^k(p_-)\otimes l_{-\lambda-2\varrho_n})\otimes L$, where $V_{-\mu}$ is an irreducible k submodule with lowest weight $-\mu$. Then,*

$$((\bar{E}_{k-1}\bar{D}_k+\bar{D}_{k+1}\bar{E}_k)\varphi, \varphi)=c_\mu(\varphi, \varphi),$$

where c_μ is real and positive. In particular if the hermitian form on $H^0(S^k(p_-)\otimes l_{-\lambda-2\varrho_n})$ is positive definite then $((\bar{E}_{k-1}\bar{D}_k+\bar{D}_{k+1}\bar{E}_k)\varphi, \varphi)$ is real and positive if $\varphi \neq 0$.

We now make the following induction hypothesis. Let $m \geqq 1$ and assume that the hermitian form on $H^0(S^k(p_-)\otimes l_{-\lambda-2\varrho_n})$ is positive definite for $k<m$.

Note that by our choice the hermitian form on $H^0(l_{-\lambda-2\varrho_n})$ is positive definite.

Lemma 9.10. *Let $\xi \in H^0(S^{m-1}(p_-)\otimes l_{-\lambda-2\varrho_n})\otimes L$, so that $\bar{E}\xi \in H^0(S^m(p_-)\otimes l_{-\lambda-2\varrho_n})\otimes L$. If $\bar{E}\xi \neq 0$, then, $(\bar{E}\xi, \bar{E}\xi)>0$.*

Proof. Let $S_k=$ the image of the map $\bar{E}_{k-1}$ and $R_k=$ the image of the map $\bar{D}_{k+1}$. For $k>0$, the operator $\bar{E}_{k-1}\bar{D}_k+\bar{D}_{k+1}\bar{E}_k$ has kernel $\{0\}$. Note that for $\varphi \in R_k$, $\bar{D}_k\varphi=0$, while for $\psi \in S_k$, $\bar{E}_k\psi=0$. By these remarks, $S_m=\bar{E}_{m-1}\bar{D}_m(S_m)=\bar{E}_{m-1}(R_{m-1})$. Let $\varphi \in S_m$. Then $\varphi=\bar{E}_{m-1}(\psi)$ for a unique $\psi \in R_{m-1}$. If $\varphi \neq 0$, then $\psi \neq 0$. Now,

$$(\varphi, \varphi)=(\bar{E}_{m-1}(\psi), \bar{E}_{m-1}(\psi))=(\bar{D}_m\bar{E}_{m-1}(\psi), \psi). \tag{9.17}$$

Since $\psi \in$ image of $\bar{D}_m$, we have $\bar{D}_m\bar{E}_{m-1}(\psi)=\bar{D}_m\bar{E}_{m-1}(\psi)+\bar{E}_{m-2}\bar{D}_{m-1}(\psi)$. For nonzero ψ, the last expression is nonzero if $m-1>0$. For $m-1=0$, the same assertion is true for nonzero $\psi \in R_0$. This can be seen as follows. Noting $\bar{D}_0=0$, $\bar{E}_0\bar{D}_1\bar{E}_0(\psi)=(\bar{D}_2\bar{E}_1+\bar{E}_0\bar{D}_1)\bar{E}_0(\psi) \neq 0$, since $E_0(\psi) \neq 0$ for nonzero $\psi \in R_0$. Applying these remarks to (9.17) and using Lemma 9.9 for $k=m-1$, the lemma follows q.e.d.

With our induction hypothesis, according to Lemma 9.10, the hermitian form on $H^0(S^m(p_-)\otimes l_{-\lambda-2\varrho_n})\otimes L$, is positive definite at least on the image of $\bar{E}_{m-1}$.

The one dimensional B module l_{ϱ_n} is a submodule of L. Thus, we have an inclusion as K^c modules,

$$\begin{aligned} H^0(S^m(p_-)\otimes l_{-\lambda-2\varrho_n}\otimes l_{\varrho_n}) &\to H^0(S^m(p_-)\otimes l_{-\lambda-2\varrho_n}\otimes L) \\ &\simeq H^0(S^m(p_-)\otimes l_{-\lambda-2\varrho_n})\otimes L. \end{aligned}$$

Lemma 9.11. *$H^0(S^m(p_-)\otimes l_{-\lambda-2\varrho_n}\otimes l_{\varrho_n})$ is contained in the image of $\bar{E}_{m-1}$.*

Proof. Let $S^{m-1}(p_-)\otimes L \xrightarrow{\bar{e}_{m-1}} S^m(p_-)\otimes L$ be the transpose of $d_m: S^m(p_+)\otimes L \to S^{m-1}(p_+)\otimes L$, where $d_m(u\otimes s)=\Sigma\, \delta_{X_i}u\otimes c(X_i)s$. Let $\bar{e}_{m-1}$ also stand for the map

$$S^{m-1}(p_-)\otimes l_{-\lambda-2\varrho_n}\otimes L \to S^m(p_-)\otimes l_{-\lambda-2\varrho_n}\otimes L,$$

got by tensoring with the identity map on the factor $l_{-\lambda-2\varrho_n}$. Then $\bar{E}_{m-1}$ is clearly the map got by inducing in cohomology from the map $\bar{e}_{m-1}$. We claim that

$$\text{the image of } \bar{E}_{m-1} = \text{the kernel of } \bar{E}_m. \tag{9.18}$$

First observe that we have the following exact sequence 'at the fiber level'.

$$0 \to l_{-\lambda-2\varrho_n} \otimes l_{\varrho_n} \to S^0(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes L \xrightarrow{\bar{e}_0} \dots$$
$$\xrightarrow{\bar{e}_{k-1}} S^k(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes L \xrightarrow{\bar{e}_k} \dots . \tag{9.19}$$

The verification of this fairly easy. Note that for each term of the above exact sequence, for the associated vector bundles, $H^i(\,)=0$, for $i>0$. Inducing from this in cohomology (see proof of Lemma 2.2) we then have an exact sequence

$$0 \to H^0(l_{-\lambda-2\varrho_n} \otimes l_{\varrho_n}) \to H^0(S^0(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes L)$$
$$\xrightarrow{\bar{E}_0} \dots \xrightarrow{\bar{E}_{k-1}} H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes L) \xrightarrow{\bar{E}_k} \dots .$$

Thus the claim (9.18) is proved. To prove the lemma we have only to observe that $S^m(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes l_{\varrho_n} \subsetneqq$ kernel of $\bar{e}_m$ q.e.d.

Corollary 9.12. *Under the inclusion*

$$H^0(S^m(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes l_{\varrho_n}) H^0(S^m(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes L) \simeq H^0(S^m(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L\,,$$

the hermitian form on $H^0(S^m(p_-) \otimes l_{-\lambda-2\varrho_n}) \otimes L$ *restricts to a positive definite hermitian form on* $H^0(S^m(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes l_{\varrho_n})$.

For the rest of the proof of Proposition (9.7), we use the assumption that $[[p_+, p_+], p_+]=0$. We use the notation in the proof of Lemma 6.7. Let $W_{-\lambda-2\varrho_n}$ be the irreducible M^c module with lowest weight $-\lambda-2\varrho_n$ (relative to $M^c \cap B$). Regard $W_{-\lambda-2\varrho_n}$ as a Q module in the usual way. For the same reason as in [3, page 172], we have the identifications

$$H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n}) \simeq H^0(K^c/_Q, S^k(p_-) \otimes W_{-\lambda-2\varrho_n})$$
$$H^s(S^k(p_+) \otimes l_{\lambda+2\varrho}) \simeq H^{\bar{s}}(K^c/_Q, S^k(p_+) \otimes W_{\lambda+2\varrho_n} \otimes l_{2\varrho_q}) \tag{9.20}$$

(See page 66)

$$H^0(S^k(p_-) \otimes l_{-\lambda-2\varrho_n} \otimes l_{\varrho_n}) \simeq H^0(K^c/_Q, S^k(p_-) \otimes W_{-\lambda-2\varrho_n} \otimes l_{\varrho_n})\,,$$

via these identifications, the conjugate linear K module maps $\bar{j}_k$ give rise to conjugate linear K module maps

$$\bar{j}_k : H^0(K^c/_Q, S^k(p_-) \otimes W_{-\lambda-2\varrho_n}) \to H^{\bar{s}}(K^c/_Q, S^k(p_+) \otimes W_{\lambda+2\varrho_n} \otimes l_{2\varrho_q})\,, \tag{9.21}$$

and hermitian forms give rise to hermitian forms. As is well known, $K^c/_Q = K/_M$ where $M = Q \cap K$. An M invariant hermitian form on $S^k(p_-) \otimes W_{-\lambda-2\varrho_n}$ induces a K invariant hermitian form on $H^0(K^c/_Q, S^k(p_-) \otimes W_{-\lambda-2\varrho_n})$ by integration on $K/_M$ (with respect to a K invariant volume element in $K/_M$ of total measure 1). We now observe the following: Let V be a finite dimensional (not necessarily irreducible) K^c module. Let

$$W = V^U = \{v \in V \mid yv = v, \forall y \in U\}\,,$$

where U is the unipotent radical of Q. W is a Q module. Let $W^* =$ dual of W. By the Borel-Weil-Bott theorem,

$$H^i(K^c/_Q; W^*) = 0 \; (i>0)$$

and

$$H^0(K^c/_Q; W^*) \simeq V^*, \qquad \text{canonically}.$$

In this context, it should be noted that $\mathrm{End}_M(W^*)$ is canonically isomorphic to $\mathrm{End}_K(H^0(K^c/_Q, W^*)$. As already pointed out an M invariant hermitian form on W^*, induces a K invariant hermitian form on V^*. But, what is more important is the following assertion:

Lemma 9.13. *The above correspondence is a bijection from the space $h_M(W^*)$ of M invariant hermitian forms on W^* onto the space $h_K(V^*)$ of K invariant hermitian forms on V^*. Under this correspondence an M invariant hermitian form on W^* is positive definite if and only if the corresponding element of $h_K(V^*)$ is positive definite.*

Proof. Since $h_M(W^*)$ and $h_K(V^*)$ are spanned over C by the set of positive definite elements $h_M^p(W^*)$ and $h_K^p(V^*)$ respectively, it suffices to prove that $h_M^p(W^*)$ is bijectively mapped $h_K^p(V^*)$. The assertion is obvious when V^* is irreducible. In general given an element of $h_K^p(V^*)$, choose an orthogonal decomposition $V^* = \oplus V_i^*$ into irreducibles. Let $W = \oplus W_i^*$ denote the corresponding decomposition of the M^c module W^*. It is now clear what element of $h_M^p(W^*)$ we should take to get the given element of $h_K^p(V^*)$ as the image. q.e.d.

We are now in a position to complete the proof of Proposition 9.7. $S^k(p_-)\otimes W_{-\lambda-2\varrho_n}$ and $S^k(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes l_{\varrho_n}$ are isomorphic to an appropriate W^* of Lemma (9.13) by choosing V to be an appropriate K^c submodule of $S^k(p^c)\otimes V_{\lambda+2\varrho_n}$ in the first case and $S^k(p^c)\otimes V_{\lambda+\varrho_n}$ in the second case. Thus, we conclude that our hermitian forms on $H^0(S^m(p_-)\otimes W_{-\lambda-2\varrho_n})$ and $H^0(S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes l_{\varrho_n})$ are induced by *unique* M invariant hermitian forms, say h_1 on $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}$ and h_2 on $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes l_{\varrho_n}$. But h_1 and h_2 are related as follows:

The hermitian form on $H^0(S^m(p_-)\otimes W_{-\lambda-2\varrho_n})\otimes L$ what we have been considering is the "product" of the one on $H^0(S^m(p_-)\otimes W_{-\lambda-2\varrho_n})$ induced by h_1 and the K invariant hermitian form (9.10) on L. It is clearly induced by the M invariant hermitian form on $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes L$ which is the product of h_1 on the factor $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}$ and our hermitian metric (9.10) on L. This hermitian form on $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes L$ restricts to a M invariant hermitian form on $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes l_{\varrho_n}$ which then induces a hermitian form on $H^0(S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes l_{\varrho_n})$. The latter is clearly the restriction of our hermitian form on $H^0(S^m(p_-)\otimes W_{-\lambda-2\varrho_n})\otimes L$. But this restriction is induced by a unique M invariant hermitian form on $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}\otimes l_{\varrho_n}$, namely h_2. Thus we have related h_1 and h_2.

By Corollary 9.12 and Lemma 9.13, h_2 is positive definite. According to the above relationship h_2 is the "product" of h_1 on the factor $S^m(p_-)\otimes W_{-\lambda-2\varrho_n}$ and the positive definite M invariant hermitian form on l_{ϱ_n} got by restricting the one on L. Hence h_1 is positive definite and Proposition 9.7 is proved.

In conclusion, by Proposition 9.6 and Proposition 9.7 $\bar{\varrho}_\lambda$ is an irreducible unitary representation. Hence ϱ_λ, the dual of $\bar{\varrho}_\lambda$ is also an irreducible unitary representation.

References

1. Enright, T.J., Vardarajan, V.S.: On an infinitesimal characterisation of the discrete series. To appear in Ann. of Math.
2. Gunning, R.C., Rossi, H.: Analytic functions of several complex variables. Englewood Cliffs: Prentice-Hall 1965
3. Hotta, R., Parthasarathy, R.: Multiplicity formulae for Discrete Series. Inventiones Math. **26**, 133—178 (1974)
4. Parthasarathy, R.: Dirac operator and the discrete series. Ann. of Math. **96**, 1—30 (1972)
5. Schmid, W.: Homogeneous complex manifolds and representations of semisimple Lie groups. Thesis at Univ. of California-Berkeley, 1967; Proc. Nat. Acad. Sci. USA **59**, 56—59 (1968)
6. Schmid, W.: On the realization of the discrete series of a semisimple Lie group. Rice Univ. Studies **56**, 99—108 (1970)
7. Schmid, W.: On the characters of the discrete series (the Hermitian-symmetric case). To appear in Inventiones Math.
8. Schmid, W.: Some properties of square-integrable representations of semisimple Lie groups. Preprint

Received September 29, 1975