AN APPLICATION OF TSCHEBYSCHEFF POLYNOMIALS TO A PROBLEM IN SYMMETRIC FUNCTIONS

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1. Introduction

In this note we consider the following *Problem**: Given that the set of variables $(x_1, x_2, \ldots x_n)$ satisfy the conditions:—

(*a*) $0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$.

(*b*) The first $n-1$ elementary symmetric functions of the x's have fixed values,

i.e.,

$$\left.\begin{aligned} &\Sigma x_1 = k_1 \\ &\Sigma x_1 x_2 = k_2 \\ &\cdots\cdots\cdots\cdots \\ &\Sigma x_1 x_2 \ldots x_{n-1} = k_{n-1} \end{aligned}\right\} \qquad (1)$$

where $k_1, k_2 \ldots k_{n-1}$ are fixed, but *unspecified,* constants.

(*c*) Maximum and minimum values of successive x's occur alternately, *i.e.*, if x_1 has its minimum value, x_2 has its maximum value, x_3 its minimum value and so on. To find, under the above conditions, the limits of variability of the x's, *i.e.*, to find n numbers $(\alpha_1, \alpha_2, \ldots \alpha_n)$ such that

$$0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_{n-1} \leqslant x_n \leqslant \alpha_n \qquad (2)$$

By the imposition of the condition (*c*) we secure the determination not only of the α's but also of the initially unspecified constants $k_1 \ldots k_{n-1}$.

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* This problem was suggested by an example in Goodstein[1] in which, however, there are only two constants k and they are *specified*, viz. $k_1 = 2$, $k_2 = 1$.

According to condition (c) we have the following two sets of particular solutions for $(x_1 \ldots x_n)$:—

	x_1	x_2	x_3	x_4	x_6	. . .
Solution I ..	0	a_2	a_2	a_4	a_4	. . .
Solution II ..	a_1	a_1	a_3	a_3	a_5	. . .

(3)

We shall refer to these two solutions as the two "extreme solutions". The condition (b) requires that the elementary symmetric functions of the quantities in the second row shall be equal to the corresponding functions of the third row. We show that, under the above conditions, the problem has a unique solution and that the a's may be determined as the zeros of certain Tschebyscheff polynomials.

In order to orient ourselves on the problem and gain motivation for the solution, we first consider some particular cases: $n = 3, 4, 5, 6$ and then proceed to the general case of any number of variables.

2. PARTICULAR CASES

(a) $n = 3$, $0 \leqslant x_1 \leqslant a_1 \leqslant x_2 \leqslant a_2 \leqslant x_3 \leqslant a_3$. Here the two extreme solutions for the x's are

$$\left.\begin{array}{l} 0,\ a_2,\ a_2 \\ a_1,\ a_1,\ a_3 \end{array}\right\} \tag{4}$$

The equality of the symmetric functions for the two solutions, imposed by the condition (b) of §1 may be expressed by the following identity ("the polynomial identity"):

$$\left.\begin{aligned} \phi(t) \equiv t(t-a_2)^2 &\equiv (t-a_1)^2(t-a_3) + a_1^2 a_3 \\ &\equiv t^3 - k_1 t^2 + k_2 t \end{aligned}\right\} \tag{P$_3$}$$

Equating coefficients of powers of t in (P$_3$) we get

$$\left.\begin{aligned} 2a_2 &= a_3 + 2a_1 = k_1 \\ a_2^2 &= a_1^2 + 2a_1 a_3 = k_2 \end{aligned}\right\} \tag{5}$$

whence we find readily that

$$a_1 : a_2 : a_3 = 1:3:4 \tag{6}$$

and

$$k_1 : k_2 : a_3 = 6:9:4.$$

If we take $a_3 = 4$, we have $a_1 = 1$, $a_2 = 3$, $a_3 = 4$. Even in this simplest case it is worth noting that since, according to (P_3),

$$\phi(t) = t(t - a_2)^2$$

$$\phi(t) - a_1^2 a_3 = (t - a_1)^2 (t - a_3)$$

$\phi'(t)$ has the roots a_1, a_2, so that we have the identity ("derivative identity")

$$\phi'(t) \equiv 3t^2 - 2k_1 t + k_2 \equiv 3(t - a_1)(t - a_2) \tag{D_3}$$

This gives

$$\left.\begin{aligned} 3(a_1 + a_2) &= 2k_1 \\ 3a_1 a_2 &= k_2 \end{aligned}\right\} \tag{7}$$

Combining the k_1-relation in (5) with the k_1- relation in (7) we get at once

$$a_1 : a_2 : a_3 = 1 : 3 : 4$$

as before. The relations (5) give 4 equations for determining the 4 quantities a_1, a_2, k_1, k_2 in terms of a_3, and nothing new is given by (7). Nevertheless it proves convenient in dealing with higher cases to use the relations corresponding to both (5) and (7). Of course, these relations though redundant are mutually consistent.

(*b*) $n = 4$, $0 \leqslant x_1 \leqslant a_1 \leqslant x_2 \leqslant a_2 \leqslant x_3 \leqslant a_3 \leqslant x_4 \leqslant a_4$. The two extreme solutions are here

$$\begin{matrix} 0 & a_2 & a_2 & a_4 \\ a_1 & a_1 & a_3 & a_3 \end{matrix}$$

and the polynomial identity is

$$\left.\begin{aligned} \phi(t) &\equiv t(t - a_2)^2 (t - a_4) \\ &\equiv (t - a_1)^2 (t - a_3)^2 - a_1^2 a_3^2 \\ &\equiv t^4 - k_1 t^3 + k_2 t^2 - k_3 t \end{aligned}\right\} \tag{P_4}$$

Now $\phi'(t)$ has the roots a_1, a_2, a_3 so that the derivative identity is

$$\left.\begin{aligned} \phi'(t) &\equiv 4t^3 - 3k_1 t^2 + 2k_2 t - k_3 \\ &\equiv 4(t - a_1)(t - a_2)(t - a_3) \end{aligned}\right\} \tag{D_4}$$

Equating coefficients of powers of t in (P_4) and (D_4) we get the following two sets of relations:—

$$\left.\begin{aligned} 2(\alpha_1+\alpha_3) &= 2\alpha_2+\alpha_4 = k_1 \\ 2\alpha_1\alpha_3(\alpha_1+\alpha_3) &= \alpha_2{}^2\alpha_4 = k_3 \\ \alpha_1{}^2+4\alpha_1\alpha_3+\alpha_3{}^2 &= \alpha_2{}^2+2\alpha_2\alpha_4 = k_2 \end{aligned}\right\} \quad (8)$$

$$\left.\begin{aligned} 4(\alpha_1+\alpha_2+\alpha_3) &= 3k_1 \\ 4(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1) &= 2k_2 \\ 4\alpha_1\alpha_2\alpha_3 &= k_3 \end{aligned}\right\} \quad (9)$$

Combining the k_1-relations in (8) and (9) we get

$$\alpha_1+\alpha_3 : \alpha_2 : \alpha_4 = 2:1:2 \quad (10)$$

Again, combination of the k_3-relations in (8) and (9) gives

$$\alpha_1\alpha_3 = \tfrac{1}{8}\,\alpha_4{}^2 \quad (11)$$

Thus α_1, α_3 are the roots of

$$f(x) = x^2 - \alpha_4 x + \tfrac{1}{8}\alpha_4{}^2 = 0 \quad (12)$$

and α_2 is the root of $\psi(x) = 2x - \alpha_4 = 0$ (13)

Taking $\alpha_4 = 4$ we get: $\alpha_1 = 2-\sqrt{2}$, $\alpha_2 = 2$, $\alpha_3 = 2+\sqrt{2}$, $\alpha_4 = 4$.

(*c*) $n = 5$, $0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \leqslant x_3 \leqslant \alpha_3 \leqslant x_4 \leqslant \alpha_4 \leqslant x_5 \leqslant \alpha_5$.

The two extreme solutions are now

$$0,\ \alpha_2,\ \alpha_2,\ \alpha_4,\ \alpha_4$$

$$\alpha_1,\ \alpha_1,\ \alpha_3,\ \alpha_3,\ \alpha_5$$

and the polynomial identity becomes

$$\left.\begin{aligned} \phi(t) &\equiv t(t-\alpha_2)^2(t-\alpha_4)^2 \\ &\equiv (t-\alpha_1)^2(t-\alpha_3)^2(t-\alpha_5)+\alpha_1{}^2\alpha_3{}^2\alpha_5 \\ &\equiv t^5 - k_1t^4 + k_2t^3 - k_3t^2 + k_4t \end{aligned}\right\} \quad (\mathrm{P}_5)$$

The derivative identity is

$$\left.\begin{aligned} \phi'(t) &\equiv 5t^4 - 4k_1t^3 + 3k_2t^2 - 2k_3t + k_4 \\ &\equiv 5(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)(t-\alpha_4) \end{aligned}\right\} \quad (\mathrm{D}_5)$$

From these we derive the relations:

$$\left.\begin{aligned} 2(\alpha_2+\alpha_4) &= 2(\alpha_1+\alpha_3)+\alpha_5 = k_1 \\ \alpha_2{}^2+4\alpha_2\alpha_4+\alpha_4{}^2 &= \alpha_1{}^2+4\alpha_1\alpha_3+\alpha_3{}^2+2\alpha_5(\alpha_1+\alpha_3) = k_2 \\ 2\alpha_2\alpha_4(\alpha_2+\alpha_4) &= 2\alpha_1\alpha_3(\alpha_1+\alpha_3)+\alpha_5(\alpha_1{}^2+\alpha_3{}^2) = k_3 \\ 2\alpha_2\alpha_4(\alpha_2+\alpha_4) &= 2\alpha_1\alpha_3(\alpha_1+\alpha_3)+\alpha_5(\alpha_1{}^2+\alpha_3{}^2) = k_3 \\ \alpha_2{}^2\alpha_4{}^2 &= \alpha_1{}^2\alpha_3{}^2+2\alpha_1\alpha_3\alpha_5(\alpha_1+\alpha_3) = k_4 \end{aligned}\right\} \quad (14)$$

$$\left.\begin{aligned} 5\Sigma\alpha_1 &= 4k_1 \\ 5\Sigma\alpha_1\alpha_2 &= 3k_2 \\ 5\Sigma\alpha_1\alpha_2\alpha_3 &= 2k_3 \\ 5\alpha_1\alpha_2\alpha_3\alpha_4 &= k \end{aligned}\right\} \tag{15}$$

Here the functions on the l.h.s. are the elementary symmetric functions in $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Combining the k_1-relations of (14) and (15) we get

$$\left.\begin{aligned} \alpha_1 + \alpha_3 &= \frac{3}{4}\alpha_5 \\ \alpha_2 + \alpha_4 &= \frac{5}{4}\alpha_5 \end{aligned}\right\} \tag{16}$$

while from the k_4-relations we have

$$\left.\begin{aligned} \alpha_1\alpha_3 &= \frac{1}{16}\alpha_5^2 \\ \alpha_2\alpha_4 &= \frac{5}{16}\alpha_5^2 \end{aligned}\right\} \tag{17}$$

Thus α_1, α_3 are the roots of

$$f(x) \equiv x^2 - \frac{3}{4}\alpha_5 x + \frac{1}{16}\alpha_5^2 = 0 \tag{18}$$

and α_2, α_4 are the roots of

$$\psi(x) \equiv x^2 - \frac{5}{4}\alpha_5 x + \frac{5}{16}\alpha_5^2 = 0 \tag{19}$$

Observe that

$$\frac{5}{2} f(x) = x^{\frac{1}{2}} \frac{d}{dx}\{x^{\frac{1}{2}}\psi(x)\}$$

With $\alpha_5 = 4$, we get:

$$\alpha_1 = \frac{3-\sqrt{5}}{2},\ \alpha_2 = \frac{5-\sqrt{5}}{2},\ \alpha_3 = \frac{3+\sqrt{5}}{2},\ \alpha_4 = \frac{5+\sqrt{5}}{2},\ \alpha_5 = 4$$

(*d*) $n = 6$. With six variables, the polynomial identity is

$$\left.\begin{aligned} \phi(t) &\equiv t(t-\alpha_2)^2(t-\alpha_4)^2(t-\alpha_6) \\ &\equiv (t-\alpha_1)^2(t-\alpha_3)^2(t-\alpha_5)^2 - \alpha_1^2\alpha_3^2\alpha_5^2 \\ &\equiv t^6 - k_1t^5 + k_2t^4 - k_3t^3 + k_4t^2 - k_5t \end{aligned}\right\} \tag{P$_6$}$$

and the derivative identity becomes

$$\left.\begin{aligned}\phi'(t) &\equiv 6t^5 - 5k_1t^4 + 4k_2t^3 - 3k_3t^2 + 2k_4t - k_5\\ &\equiv 6(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)(t-\alpha_4)(t-\alpha_5)\end{aligned}\right\} \qquad (D_6)$$

Equating coefficients of powers of t and combining the k_1-, k_2- and k_5-relations from the two sets so obtained we get

$$\left.\begin{aligned}&\frac{\alpha_1+\alpha_3+\alpha_5}{3} = \frac{\alpha_2+\alpha_4}{2} = \frac{\alpha_6}{2}\\ &\frac{\alpha_1\alpha_3+\alpha_3\alpha_5+\alpha_5\alpha_1}{9} = \frac{\alpha_2\alpha_4}{3} = \frac{\alpha_6^2}{16}\\ &\alpha_1\alpha_3\alpha_5 = \frac{1}{32}\alpha_6^3\end{aligned}\right\} \qquad (20)$$

Hence α_1, α_3, α_5 are the roots of

$$f(x) \equiv x^3 - \frac{3}{2}\alpha_6 x^2 + \frac{9}{16}\alpha_6^2 x - \frac{1}{32}\alpha_6^3 = 0 \qquad (21)$$

while α_2, α_4 are the roots of

$$\psi(x) \equiv x^2 - \alpha_6 x + \frac{3}{16}\alpha_1^2 = 0 \qquad (22)$$

It will be noted that here $\psi(x) = \frac{1}{3}f'(x)$.

With $\alpha_6 = 4$, we thus have

$\alpha_1 = 2-\sqrt{3},\ \alpha_2 = 1,\ \alpha_3 = 2,\ \alpha_4 = 3,\ \alpha_5 = 2+\sqrt{3},\ \alpha_6 = 4.$

To summarise, we have the following solutions in the particular cases considered above (taking $\alpha_n = 4$ always):

$n = 3,\ (\alpha_1, \alpha_2, \alpha_3) = (1, 3, 4)$

$n = 4,\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (2-\sqrt{2}, 2, 2+\sqrt{2}, 4)$

$n = 5,\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \left(\frac{3-\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}, 4\right)$

$n = 6,\ (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (2-\sqrt{3}, 1, 2, 3, 2+\sqrt{3}, 4).$

It will be observed that for $n = 4$ and $n = 6$, one of the α's has the value $2\,(=\frac{1}{2}\alpha_n)$ and that the remaining α's apart from α_n itself, are symmetrically situated with respect to this.

We shall see that this is true generally for all even values of n. For $n = 3$ and $n = 5$, if we omit α_n, we notice that, of the remaining α's the sum of any pair of quantities equidistant from the two ends is always equal to α_n. This will also be found to be true generally for all odd values of n.

3. General Case of n Variables

In dealing with the general case it is convenient to consider the cases of n odd and n even separately.

Case I.—Odd number of variables, $n = 2m + 1$

$$0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \leqslant \ldots\ldots \leqslant x_{2m} \leqslant \alpha_{2m} \leqslant x_{2m+1} \leqslant \alpha_{2m+1} (= \xi).$$

The two extreme solutions for the x's are

$$\left.\begin{array}{l} 0, \alpha_2, \alpha_2, \alpha_4, \alpha_4, \ldots \alpha_{2m}, \alpha_{2m} \\ \alpha_1, \alpha_1, \alpha_3, \alpha_3, \ldots \alpha_{2m-1}, \alpha_{2m-1}, \xi \end{array}\right\}$$

We have therefore, on account of the condition (b) in § 1, the polynomial identity

$$\left.\begin{aligned} \phi(t) &\equiv t(t-\alpha_2)^2(t-\alpha_4)^2 \ldots (t-\alpha_{2m})^2 \\ &\equiv (t-\alpha_1)^2(t-\alpha_3)^2 \ldots (t-\alpha_{2m-1})^2(t-\xi) + \xi\alpha_1^2\alpha_3^2 - \alpha^2_{2m-1} \\ &\equiv t^{2m+1} - k_1t^{2m} + k_2t^{2m-1} - \ldots - k_{2m-1}t^2 + k_{2m}t \end{aligned}\right\} \quad (\mathrm{P}_{2m+1})$$

For the derivative identity we have

$$\left.\begin{aligned} \phi'(t) &\equiv (2m+1)t^{2m} - 2mk_1t^{2m-1} + (2m-1)k_2t^{2m-2} - \ldots \\ &\qquad -2k_{2m-1}t + k_{2m} \\ &\equiv (2m+1)(t-\alpha_1)(t-\alpha_2)(t-\alpha_3)\ldots(t-\alpha_{2m-1})(t-\alpha_{2m}) \end{aligned}\right\} \quad (\mathrm{D}_{2m+1})$$

Let

$$\left.\begin{aligned} f(t) &\equiv (t-\alpha_1)(t-\alpha_3)\ldots(t-\alpha_{2m-1}) \equiv t^m - s_1t^{m-1} + s_2t^{m-2} - \ldots \\ &\qquad + (-1)^m s_m \\ \psi(t) &\equiv (t-\alpha_2)(t-\alpha_4)\ldots(t-\alpha_{2m}) \equiv t^m - \sigma_1t^{m-1} + \sigma_2t^{m-2} - \ldots \\ &\qquad + (-1)^m\sigma_m \end{aligned}\right\} \quad (23)$$

Introducing (23) in (P_{2m+1}) and (D_{2m+1}) we get

$$\left.\begin{aligned} & t\,[t^m - \sigma_1t^{m-1} + \sigma_2t^{m-2} - \ldots + (-1)^{m-1}\sigma_{m-1}t + (-1)^m\sigma_m]^2 \\ &\equiv [t^m - s_1t^{m-1} + s_2t^{m-2} - \ldots + (-1)^{m-1}s_{m-1}t \\ &\qquad + (-1)^m s_m]^2(t-\xi) + \xi s_m^2 \\ &\equiv t^{2m+1} - k_1t^{2m} + k_2t^{2m-1} - \ldots - k_{2m-1}t^2 + k_{2m}t \end{aligned}\right\} \quad (\mathrm{P}'_{2m+1})$$

$$\left.\begin{aligned} & (2m+1)\,[t^m - \sigma_1t^{m-1} + \sigma_2t^{m-2} - \ldots + (-1)^{m-1}\sigma_{m-1}t + (-1)^m\sigma_m] \\ & [t^m - s_1t^{m-1} + s_2t^{m-2} - \ldots + (-1)^{m-1}s_{m-1}t + (-1)^m s_m \\ &\equiv (2m+1)t^{2m} - 2mk_1t^{2m-1} + (2m-1)k_2t^{2m-2} - \ldots \\ &\qquad - k_{2m-1}\,t + k_{2m} \end{aligned}\right\} \quad (\mathrm{D}'_{2m+1})$$

Equating coefficients of $t^{2m}, t^{2m-1}, \ldots. t^{m+1}$ in (P'_{2m+1}) and (D'_{2m+1}) we obtain the two sets of relations:

$$\left.\begin{aligned}
&2\sigma_1 = 2s_1 + \xi = k_1\\
&\sigma_1{}^2 + 2\sigma_2 = s_1{}^2 + 2s_2 + 2\xi s_1 = k_2\\
&2\sigma_3 + 2\sigma_1\sigma_2 = 2s_3 + 2s_1s_2 = \xi\,(2s_2 + s_1{}^2) = k_3\\
&2\sigma_4 + 2\sigma_1\sigma_3 + \sigma_2{}^2 = 2s_4 + 2s_1s_3 + s_2{}^2 + 2\xi\,(s_3 + s_1s_2) = k_4\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&2\sigma_m + 2\sigma_1\sigma_{m-1} + \ldots. = 2s_m + 2s_1s_{m-1} + \ldots. + \xi\,(2s_{m-1}\\
&\qquad + 2s_1s_{m-2} + \ldots.) = k_m
\end{aligned}\right\} \qquad (24)$$

$$\left.\begin{aligned}
&(2m+1)\,(\sigma_1 + s_1) = 2mk_1\\
&(2m+1)\,(\sigma_2 + \sigma_1s_1 + s_2) = (2m-1)\,k_2\\
&(2m+1)\,(\sigma_3 + \sigma_1s_2 + \sigma_2s_1 + s_3) = (2m-2)\,k_3\\
&\ldots\ldots\ldots\ldots\ldots\\
&(2m+1)\,(\sigma_m + \sigma_{m-1}s_1 + \sigma_{m-2}s_2 + \ldots. + \sigma_1s_{m-1} + s_m) = (m+1)\,k_m.
\end{aligned}\right\} \qquad (25)$$

These constitute $3m$ relations for determining the $3m$ quantities $(\alpha_1, \alpha_2, \ldots. \alpha_{2m-1}, \alpha_{2m})$, $(k_1, k_2, \ldots. k_m)$ in terms of ξ*. By combining corresponding relations from (24) and (25) (conveniently labelled by the k's on the r.h.s.), we can solve for the σ_i and s_i in terms of ξ. Thus, by combining the k_1-relations from (24) and (25) we get

$$\frac{\sigma_1}{2m+1} = \frac{s_1}{2m-1} = \tfrac{1}{4}\,\xi.$$

Next combining the k_2-relations and using the values of σ_1, s_1 we get

$$\frac{\sigma_2}{2m+1} = \frac{s_2}{2m+3} = (m-1)\left(\frac{\xi}{4}\right)^2.$$

Proceeding like this we get the following solution:

$$\frac{\sigma_1}{2m+1} = \frac{s_1}{2m-1} = \frac{\xi}{4}$$

$$\frac{\sigma_2}{2m+1} = \frac{s_2}{2m-3} = (m-1)\left(\frac{\xi}{4}\right)^2$$

$$\frac{\sigma_3}{2m+1} = \frac{s_3}{2m-5} = \frac{1}{3}\,(m-2)\,(2m-3)\left(\frac{\xi}{4}\right)^6, \text{ etc.}$$

* If we equate the coefficients of *all* powers of t in (P'_{2m+1}) & (D'_{2m+1}) we obtain $3\,m$ further relations, some of which determine the remaining k's, while the others are identically satisfied in virtue of the rest.

These give

$$\left.\begin{aligned} s_1 &= \binom{2m-1}{1}\left(\frac{\xi}{4}\right) \\ s_2 &= \binom{2m-2}{2}\left(\frac{\xi}{4}\right)^2 \\ &\cdots\cdots\cdots\cdots \\ s_m &= \binom{m}{m}\left(\frac{\xi}{4}\right)^m \end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{aligned} \sigma_1 &= \frac{2m+1}{2m-1}\binom{2m-1}{1}\left(\frac{\xi}{4}\right) \\ \sigma_2 &= \frac{2m+1}{2m-3}\binom{2m-2}{2}\left(\frac{\xi}{4}\right)^2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_m &= \frac{2m+1}{1}\binom{m}{m}\left(\frac{\xi}{4}\right)^m \end{aligned}\right\} \tag{27}$$

Hence, by (23), $(\alpha_1, \alpha_3, \ldots \alpha_{2m-1})$ are the roots of

$$f(x) \equiv x^m - \binom{2m-1}{1}\left(\frac{\xi}{4}\right)x^{m-1} + \binom{2m-2}{2}\left(\frac{\xi}{4}\right)^2 x^{m-2} - \ldots + (-1)^m \binom{m}{m}\left(\frac{\xi}{4}\right)^m = 0 \tag{28}$$

and $(\alpha_2, \alpha_4, \ldots \alpha_{2m})$ are the roots of

$$\psi(x) \equiv \frac{x^m}{2m+1} - \binom{2m-1}{1}\left(\frac{\xi}{4}\right)\frac{x^{m-1}}{2m-1} + \binom{2m-2}{2}\left(\frac{\xi}{4}\right)^2 \frac{x^{m-2}}{2m-3} - \ldots + (-1)^m \binom{m}{m}\left(\frac{\xi}{4}\right)^m = 0 \tag{29}$$

If we multiply these equations by $(4/\xi)^m$ and put

$$u = 4x/\xi \tag{30}$$

they become

$$\mathscr{F}(u) \equiv \left(\frac{4}{\xi}\right)^m f\left(\frac{\xi u}{4}\right) \equiv u^m - \binom{2m-1}{1}u^{m-1} + \binom{2m-2}{2}u^{m-2} - \ldots + (-1)^m \binom{m}{m} = 0 \tag{31}$$

$$\Psi(u) \equiv \left(\frac{4}{\xi}\right)^m \psi\left(\frac{\xi u}{4}\right) \equiv \frac{u^m}{2m+1} - \binom{2m-1}{1}\frac{u^{m-1}}{2m-1} + \binom{2m-2}{2}\frac{u^{m-2}}{2m-3} - \ldots + (-1)^m \binom{m}{m} = 0 \tag{32}$$

Case II.—Even number of variables, $n = 2m$.

$$0 \leqslant x_1 \leqslant \alpha_1 \leqslant x_2 \leqslant \alpha_2 \leqslant \ldots, \leqslant x_{2m-1} \leqslant \alpha_{2m-1} \leqslant x_{2m} \leqslant \alpha_{2m} (= \xi).$$

The polynomial and derivative identities corresponding to (P_{2m+1}) and (D_{2m+1}) are now, respectively,

$$\left.\begin{aligned}\phi(t) &\equiv t(t-\alpha_2)^2(t-\alpha_4)^2 \ldots (t-\alpha_{2m-2})^2(t-\xi)\\ &\equiv (t-\alpha_1)^2(t-\alpha_3)^2 \ldots (t-\alpha_{2m-1})^2 - \alpha_1{}^2\alpha_3{}^2 \ldots \alpha^2{}_{2m-1}\\ &\equiv t^{2m} - k_1 t^{2m-1} + k_2 t^{2m-2} - \ldots + k_{2m-2}t^2 - k_{2m-1}t\end{aligned}\right\} \quad (P_{2m})$$

and

$$\left.\begin{aligned}\phi'(t) &\equiv 2mt^{2m-1} - (2m-1)k_1 t^{2m-2} + (2m-2)k_2 t^{2m-3} - \ldots\\ &\quad + 2k_{2m-2}t - k_{2m-1}\\ &\equiv 2m(t-\alpha_1)(t-\alpha_2) \ldots (t-\alpha_{2m-2})(t-\alpha_{2m-1})\end{aligned}\right\} \quad (D_{2m})$$

Proceeding as in Case I we now obtain the following results:

$$\left.\begin{aligned}\frac{s_1}{m} &= \frac{\sigma_1}{m-1} = \frac{\xi}{2}\\ \frac{s_2}{m} &= \frac{\sigma_2}{m-2} = (2m-3)\left(\frac{\xi}{4}\right)^2\\ \frac{s_3}{m} &= \frac{\sigma_3}{m-3} = \tfrac{1}{3}(2m-4)(2m-5)\left(\frac{\xi}{4}\right)^3\\ &\ldots\ldots\ldots\ldots\\ \frac{s_{m-1}}{m} &= \frac{\sigma_{m-1}}{1} = \binom{m}{m-1}\left(\frac{\xi}{4}\right)^{m-1}\\ s_m &= \frac{2\sigma_{m-1}}{m} \cdot \left(\frac{\xi}{4}\right).\end{aligned}\right\} \quad (33)$$

Thus we find that $(\alpha_2, \alpha_4, \ldots = \alpha_{2m-2})$ are the roots of

$$\psi_1(x) \equiv x^{m-1} - \binom{2m-2}{1}\left(\frac{\xi}{4}\right)x^{m-2} + \binom{2m-3}{2}\left(\frac{\xi}{4}\right)^2 x^{m-3} - \ldots$$
$$+ (-1)^{m-2}\binom{m+1}{m-2}\left(\frac{\xi}{4}\right)^{m-2} x + (-1)^{m-1}\binom{m}{m-1}\left(\frac{\xi}{4}\right)^{m-1} = 0 \quad (34)$$

and $(\alpha_1, \alpha_3, \ldots \alpha_{2m-1})$ are the roots of

$$f_1(x) \equiv \frac{x^m}{m} - \binom{2m-2}{1}\left(\frac{\xi}{4}\right)\frac{x^{m-1}}{m-1} + \binom{2m-3}{2}\left(\frac{\xi}{4}\right)^2\frac{x^{m-2}}{m-2} \quad (35)$$
$$- \ldots + (-1)^{m-1}\binom{m}{m-1}\left(\frac{\xi}{4}\right)^{m-1} x + (-1)^m \frac{2}{m}\left(\frac{\xi}{4}\right)^m = 0$$

If as in (30) we write $u = 4x/\xi$, these equations become

$$\Psi_1(u) \equiv \left(\frac{4}{\xi}\right)^{m-1} \psi_1\left(\frac{\xi u}{4}\right) \equiv u^{m-1} - \binom{2m-2}{1} u^{m-2} + \binom{2m-3}{2} u^{m-3}$$

$$- \ldots . + (-1)^{m-2} \binom{m+1}{m-2} u + (-1)^{m-1} \binom{m}{m-1} = 0$$

$$\mathscr{F}_1(u) \equiv \left(\frac{4}{\xi}\right)^{m} f_1\left(\frac{\xi u}{4}\right) \equiv \frac{u^m}{m} - \binom{2m-2}{1} \frac{u^{m-1}}{m-1} + \binom{2m-3}{2} \frac{u^{m-2}}{m-2} - \ldots .$$

$$+ (-1)^{m-1} \binom{m}{m-1} u + (-1)^m \frac{2}{m} = 0 \tag{37}$$

It is to be observed that in Case I, for example,

$$f(x) = 2x^{\frac{1}{2}} \frac{d}{dx} \{x^{\frac{1}{2}} \psi(x)\}$$

so that the roots of $f(x) = 0$ separate those of $\psi(x) = 0$, as indeed they should. A similar remark applies to Case II also, where

$$\psi_1(x) = \frac{d}{dx} f_1(x).$$

The problem is thus reduced to showing that the equations have real roots and to obtaining these roots explicitly if possible. We shall show that both these objects can be attained by reducing the polynomials on the l.h.s. of these equations to Tschebyscheff polynomials.

4. Completion of the Solution by Reduction to Tschebyscheff Polynomials

Consider $\mathscr{F}(u)$. We have

$$\mathscr{F}(u) = \sum_{r=0}^{m} (-1)^r \binom{2m-r}{r} u^{m-r}$$

$$= (-1)^m \sum_{r=0}^{m} (-1)^r \binom{m+r}{m-r} u^r$$

[reversing the order of terms]

$$= (-1)^m \sum_{r=0}^{m} (-1)^r \binom{m+r}{2r} u^r$$

$$= (-1)^m \,\mathrm{F}\left(m+1, \; -m; \; \tfrac{1}{2}; \; \frac{u}{4}\right)$$

in the notation of hypergeometric functions.

Restoring x in place of u, we get

$$f(x) = \left(\frac{\xi}{4}\right)^m \mathscr{F}\left(\frac{4x}{\xi}\right) = (-1)^m \left(\frac{\xi}{4}\right)^m \mathrm{F}\left(m+1, \; -m; \; \tfrac{1}{2}; \; \frac{x}{\xi}\right) \tag{38}$$

Similarly we find that

$$\psi(x) = \left(\frac{\xi}{4}\right)^m \Psi\left(\frac{4x}{\xi}\right) = (-1)^m \left(\frac{\xi}{4}\right)^m \mathrm{F}\left(m+1, -m; \frac{3}{2}; \frac{x}{\xi}\right) \tag{39}$$

$$\psi_1(x) = \left(\frac{\xi}{4}\right)^{m-1} \Psi_1\left(\frac{4x}{\xi}\right) = (-1)^{m-1} \left(\frac{\xi}{4}\right)^{m-1} \mathrm{F}\left(m+1, -m+1; \frac{3}{2}; \frac{x}{\xi}\right) \tag{40}$$

$$f_1(x) = \left(\frac{\xi}{4}\right)^m \mathscr{F}_1\left(\frac{4x}{\xi}\right) = (-1)^m \left(\frac{\xi}{4}\right)^m \cdot \frac{2}{m}\mathrm{F}\left(m, -m; \tfrac{1}{2}; \frac{x}{\xi}\right) \tag{41}$$

We now make use of the following known results[2]:—

$$\mathrm{F}\left(\frac{1+\mu}{2}, \frac{1-\mu}{2}; \tfrac{1}{2}; x^2\right) = \frac{\cos(\mu \operatorname{arc}\sin x)}{(1-x^2)^{\frac{1}{2}}} \tag{42a}$$

$$\mathrm{F}\left(\frac{1+\mu}{2}, \frac{1-\mu}{2}; \frac{3}{2}; x^2\right) = \frac{\sin(\mu \operatorname{arc}\sin x)}{\mu x} \tag{42b}$$

$$\mathrm{F}\left(\frac{\mu}{2}, -\frac{\mu}{2}; \tfrac{1}{2}; x^2\right) = \cos(\mu \operatorname{arc}\sin x) \tag{42c}$$

$$\mathrm{F}\left(1+\tfrac{1}{2}\mu, 1-\tfrac{1}{2}\mu; \frac{3}{2}; x^2\right) = \frac{\sin(\mu \operatorname{arc}\sin x)}{\mu x(1-x^2)^{\frac{1}{2}}} \tag{42d}$$

Using these and recalling the definitions of the Tschebyscheff polynomials[3]:

$$\mathrm{T}_n(x) = \cos(n \operatorname{arc}\cos x)$$

$$\mathrm{U}_n^*(x) = (1-x^2)^{-\frac{1}{2}} \sin\{(n+1)\operatorname{arc}\cos x\}$$

we find that

$$\left.\begin{aligned}
f(x) &= \left(\frac{\xi}{4}\right)^m \mathrm{U}_{2m}^*(\sqrt{x/\xi} = \left(\frac{\xi}{4}\right)^m \frac{\sin\{(2m+1)\operatorname{arc}\cos\sqrt{x/\xi}\}}{(1-x/\xi)^{\frac{1}{2}}}\\
\psi(x) &= \left(\frac{\xi}{4}\right)^m \cdot \frac{\mathrm{T}_{2m+1}(\sqrt{x/\xi})}{(2m+1)\sqrt{x/\xi}} = \left(\frac{\xi}{4}\right)^m \cdot \frac{\cos(2m+1)\operatorname{arc}\cos\sqrt{x/\xi}}{(2m+1)\sqrt{x/\xi}}\\
\psi_1(x) &= \left(\frac{\xi}{4}\right)^{m-1} \frac{\mathrm{U}^*_{2m-1}(\sqrt{x/\xi})}{x^{\frac{1}{2}}} = \left(\frac{\xi}{4}\right)^{m-1} \cdot \frac{\sin\{2m \operatorname{arc}\cos\sqrt{x/\xi}\}.}{\sqrt{x(1-x/\xi)}}\\
f_1(x) &= \left(\frac{\xi}{4}\right)^m \cdot \frac{2}{m}\mathrm{T}_{2m}\sqrt{x/\xi} = \frac{2}{m}\left(\frac{\xi}{4}\right)^m \{\cos 2m \operatorname{arc}\cos\sqrt{x/\xi}\}
\end{aligned}\right\} \tag{43}$$

The above identifications assume that $0 \leqslant x \leqslant \xi$ which, of course, is satisfied since the a's lie in $(0, \xi)$. But it may be mentioned that the reduction of $f(x)$, $\psi(x)$, $f_1(x)$, $\psi_1(x)$ to Tschebyscheff polynomials may be carried out without any restricting condition on x, by starting from the definitions

$$\mathrm{T}_n(x) = \mathrm{F}\left(n, -n; \tfrac{1}{2}; \frac{1-x}{2}\right)$$

$$\mathrm{U}_n^*(x) = (n+1)\,\mathrm{F}\left(n+2, -n; \frac{3}{2}; \frac{1-x}{2}\right)$$

and using some formulæ of transformation of hypergeometric functions, It is wellknown that the roots of the polynomials $T_n(x)$, $U_n^*(x)$ are all real and lie in $(-1, +1)$. Therefore the above expressions show that the equations for the α's have all their roots real and located in $(0, \xi)$. Further, these expressions yield the following explicit solutions for the α's in the general case:

(*a*) $n = 2m + 1$:

$$\left.\begin{aligned} \alpha_{2j-1} &= \xi \cos^2 \frac{j\pi}{2m+1} \\ \alpha_{2j} &= \xi \cos^2 \frac{(2j-1)\pi}{2(2m+1)} \end{aligned}\right\} (j = 1, 2 \ldots . m) \tag{44}$$

$$\alpha_{2m+1} = \xi$$

(*b*) $n = 2m$:

$$\left.\begin{aligned} \alpha_{2j} &= \xi \cos^2 \frac{j\pi}{2m}, \; j = 1, 2, \ldots . (m-1) \\ \alpha_{2j-1} &= \xi \cos^2 \frac{(2j-1)\pi}{4m}, \; j = 1, 2, \ldots . m \end{aligned}\right\} \tag{45}$$

$$\alpha_{2m} = \xi$$

It is clear that for all even $n (= 2m)$, one of the α's has the value ξ. In fact, if $m = 2q$

$$\alpha_{2q} = \tfrac{1}{2}\xi$$

while if $m = 2q - 1$,

$$\alpha_{2q-1} = \tfrac{1}{2}\xi$$

i.e., for all m:

$$\alpha_m = \tfrac{1}{2}\xi.$$

It is also easy to see that, in this case, the remaining α's are symmetrically situated w.r.t. α_m. Further it is easily verified that for odd $n (= 2m + 1)$,

$$\alpha_1 + \alpha_{2m} = \alpha_2 + \alpha_{2m-1} = \alpha_3 + \alpha_{2m-2} = \ldots . = \xi.$$

Added in Proof

It will be observed that, if we take the origin at α_0 instead of at O, then for *any* n (odd or even) we have

$$\xi = \alpha_0 + \alpha_n = \alpha_1 + \alpha_{n+1} = \cdots \text{ etc.,}$$

$$= 2\alpha_{n/2} \text{ if } n \text{ is even}$$

Thus there is symmetry from the two ends in all cases.

These properties are illustrated by the particular solutions enumerated in §2.

References

1. Goodstein .. *Mathematical Analysis*, Oxford, 1948, p. 356, Ex. 12:01.
2. Magnus and Oberhettinger .. *Spezielle Functionen d. math. Phys.*, 2nd ed., p. 11.
3. ——— .. *Ibid.*, pp. 103, 104.