A NOTE ON GENERALISED MEAN VALUE THEOREMS

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1. In this Note we indicate a new approach to some generalisations of the classical mean value theorems of Lagrange and Cauchy in the Differential Calculus. We obtain new forms for these generalised formulæ and also derive some further generalisations suggested by these forms.

2. The usual form of Taylor's theorem is

$$f(x+h) - f(x) - hf'(x) - \ldots - \frac{h^n}{n!} f^{(n)}(x) = R_{n+1}.$$

Writing $a - x$ for h we have

$$f(a) - f(x) - (a-x) f'(x) - \ldots - \frac{(a-x)^n}{n!} f^{(n)}(x) = R_{n+1}.$$

This is more convenient in what follows as the right-hand side is identically replaced by the closed expression

$$\frac{(a-x)^{n+1}}{n!} D^n \left[\frac{f(x) - f(a)}{x-a}\right]$$

where $D \equiv d/dx$.

We now state the following generalisation of the Cauchy mean value theorem:

$$\frac{D^n \left[\dfrac{f(x) - f(a)}{x-a}\right]}{D^p \left[\dfrac{\phi(x) - \phi(a)}{x-a}\right]} = \left(\frac{\xi - a}{x-a}\right)^{n-p} \frac{f^{(n+1)}(\xi)}{\phi^{(p+1)}(\xi)}, \; x < \xi < a, \ldots \quad \ldots \quad \text{(A)}$$

In the next paragraph we reduce this, by a change of variable, to an equivalent form which is shown to follow very simply from Cauchy's mean value

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theorem. We here add a few remarks to show the connection of (A) with various known results. Thus writing (A) in the form

$$\frac{\dfrac{(a-x)^{n+1}}{n!}\,D^n\left[\dfrac{f(x)-f(a)}{x-a}\right]}{\dfrac{(a-x)^{p+1}}{p!}\,D^p\left[\dfrac{\phi(x)-\phi(a)}{x-a}\right]}=(a-\xi)^{n-p}.\frac{p!}{n!}\frac{f^{(n+1)}(\xi)}{\phi^{(p+1)}(\xi)}$$

we get

$$\frac{f(a)-f(x)-(a-x)f'(x)-\ldots\ldots-\dfrac{(a-x)^n}{n!}f^{(n)}(x)}{\phi(a)-\phi(x)-(a-x)\phi'(x)-\ldots\ldots-\dfrac{(a-x)^p}{p!}\phi^{(p)}(x)}$$

$$=\frac{p!}{n!}(a-\xi)^{n-p}\frac{f^{(n+1)}(\xi)}{\phi^{(p+1)}(\xi)} \quad \ldots \quad \ldots \quad (A_1)$$

which is a known generalisation of Cauchy's formula (*cf.* Mahajani, *Elementary Analysis*, 4th edition, p. 119).

If in (A_1) we write $\phi(t)=(t-x)^{q+1}$, where q is any number (not necessarily an integer) such that $q+1>p$, we then have

$$\phi(a)=(a-x)^{q+1},\ \phi(x)=\phi'(x)=\ldots\ldots=\phi^{(p)}(x)=0$$

$$\phi^{(p+1)}(t)=(q+1)q(q-1)\ldots\ldots(q-p+1)(t-x)^{q-p}.$$

Hence the numerator of the left-hand side of (A_1) is

$$R_{n+1}=\frac{p!}{n!}\frac{(a-x)^{q+1}(a-\xi)^{n-p}f^{(n+1)}(\xi)}{(q+1)\,q(q-1)\ldots.(q-p+1)(\xi-x)^{q-p}}$$

If we now put $a-x=h$, $\xi=x+\theta h$, $0<\theta<1$, this gives

$$R_{n+1}=\frac{p!}{n!}\cdot\frac{(1-\theta)^{n-p}}{\theta^{q-p}}\cdot\frac{h^{n+1}f^{(n+1)}(x+\theta h)}{(q+1)\,q(q-1)\ldots.(q-p+1)} \quad \ldots \quad (A_2)$$

which is a known form for R_{n+1} (see Edwards, *Differential Calculus*, p. 511). As a particular case, if in (A_2) we put $q=p-\frac{1}{2}$, we get

$$R_{n+1}=\frac{p!}{n!}\frac{2^{p+1}\theta^{\frac{1}{2}}(1-\theta)^{n-p}h^{n+1}f^{(n+1)}(x+\theta h)}{1\cdot3\cdot5\ldots.(2p+1)} \quad \ldots \quad \ldots \quad (A_3)$$

Any number of such particular cases may be derived by taking $q=p-\beta$, where β is a prescribed proper fraction.

Again, if in (A) we write $\phi(t)=t^{p+1}$ the denominator of the L.H.S. is

$$D^p\left[\frac{x^{p+1}-a^{p+1}}{x-a}\right]=p!$$

and we obtain for the numerator

$$R_{n+1} = \frac{(1-\theta)^{n-p}}{n!\,(p+1)}\, h^{n+1} f^{(n+1)}(x+\theta h).$$

which is Schlömilch's form for R_{n+1}. Thus (A) includes the Schlömilch form of the remainder in Taylor's theorem and hence also the Lagrange and Cauchy forms which are obtained from it by taking $p = n$ and $p = o$ respectively.

3. We now return to the result (A). It may be written in the symmetrical form

$$\frac{(x-a)^n D^n \left[\dfrac{f(x)-f(a)}{x-a}\right]}{(x-a)^p D^p \left[\dfrac{\phi(x)-\phi(a)}{x-a}\right]} = \frac{(\xi-a)^{n+1} f^{(n+1)}(\xi)}{(\xi-a)^{p+1} \phi^{(p+1)}(\xi)} \quad \ldots \quad \ldots \quad \text{(B)}$$

Now write $x - a = e^t$, $F(t) = \dfrac{f(x)-f(a)}{x-a}$, $\Phi(t) = \dfrac{\phi(x)-\phi(a)}{x-a}$.

Then $\Delta \equiv d/dt = (x-a)D$ and $(x-a)^n D^n = \Delta(\Delta-1)..(\Delta-n+1)$

$$= \lambda_n(\Delta), \text{ say.}$$

With these substitutions the L.H.S. of (B) becomes

$$\frac{\lambda_n(\Delta)\, F(t)}{\lambda_p(\Delta)\, \Phi(t)}.$$

and on the R.H.S. we get

$$\left[\frac{(x-a)^{n+1} D^{n+1}\{f(x)-f(a)\}}{(x-a)^{p+1} D^{p+1}\{\phi(x)-\phi(a)\}}\right]_{x=\xi} = \left[\frac{\lambda_{n+1}(\Delta)\,\{e^t F(t)\}}{\lambda_{p+1}(\Delta)\,\{e^t \Phi(t)\}}\right]_{t=t_1}, \quad (-\infty < t_1 < t)$$

$$= \left[\frac{e^t \lambda_{n+1}(\Delta+1)\, F(t)}{e^t \lambda_{p+1}(\Delta+1)\, \Phi(t)}\right]_{t=t_1}$$

Thus (B) takes the form

$$\frac{\lambda_n(\Delta)\, F(t)}{\lambda_p(\Delta)\, \Phi(t)} = \left[\frac{\lambda_{n+1}(\Delta+1)\, F(t)}{\lambda_{p+1}(\Delta+1)\, \Phi(t)}\right]_{t=t_1} \quad \ldots \quad \ldots \quad \text{(C)}$$

Now the form (C) can be shown to result immediately from Cauchy's mean value theorem. We first write the latter in the form

$$\left[\frac{f(x)-f(a)}{x-a}\right] \Big/ \left[\frac{\phi(x)-\phi(a)}{x-a}\right] = \frac{f'(\xi)}{\phi'(\xi)} = \left[\frac{(x-a)D\{f(x)-f(a)\}}{(x-a)D\{\phi(x)-\phi(a)\}}\right]_{x=\xi}$$

Introducing the variable t and the operator Δ as above this becomes

$$\frac{F(t)}{\Phi(t)} = \left[\frac{\Delta\{e^t F(t)\}}{\Delta\{e^t \Phi(t)\}}\right]_{t=t_1} = \left[\frac{(\Delta+1)\, F(t)}{(\Delta+1)\, \Phi(t)}\right]_{t=t_1}$$

Replacing $F(t)$ by $\lambda_n(\Delta)\,\psi(t)$ and $\Phi(t)$ by $\lambda_p(\Delta)\,\chi(t)$, we get

$$\frac{\lambda_n(\Delta)\,\psi(t)}{\lambda_p(\Delta)\,\chi(t)} = \left[\frac{(\Delta+1)\,\lambda_n(\Delta)\,\psi(t)}{(\Delta+1)\,\lambda_p(\Delta)\,\chi(t)}\right]_{t=t_1}$$

$$= \left[\frac{\lambda_{n+1}(\Delta+1)\,\psi(t)}{\lambda_{p+1}(\Delta+1)\,\chi(t)}\right]_{t=t_1}$$

which is precisely (C). If we go back now by writing

$$\psi(t) = \frac{f(x)-f(a)}{x-a}$$

$$\chi(t) = \frac{\phi(x)-\phi(a)}{x-a}$$

$$\lambda_n(\Delta) = (x-a)^n\,D^n, \text{ etc.},$$

we recover (B).

4. We proceed to consider some further generalisations of (A).

Let $Q_n(t) = \sum_{r=0}^{n} q_r\,t^r$ be an arbitrary polynomial. Consider the expression

$$\sum_{r=0}^{n} q_r\,D^r\left[\frac{f^{(n-r)}(x)-f^{(n-r)}(a)}{x-a}\right].$$

By carrying out the differentiations and noting that

$$Q_n^{(r)}(0) = r!\;q_r,\quad Q_n^{(r)}(1) = \sum_{m=r}^{n} \frac{m!}{(m-r)!}\,q_m$$

we obtain the identity

$$(-1)^n(x-a)^{n+1}\sum_{r=0}^{n} q_r\,D^r\left[\frac{f^{(n-r)}(x)-f^{(n-r)}(a)}{x-a}\right]$$

$$= \sum_{m=0}^{n} (-1)^m(x-a)^m\left[Q_n^{(n-m)}(1)\,f^{(m)}(x) - Q_n^{(n-m)}(0)\,f^{(m)}(a)\right] \quad \ldots \text{(D)}$$

Since it is easily verified that

$$\sum_{r=0}^{n} q_r\,D^r\left[\frac{f^{(n-r)}(x)-f^{(n-r)}(a)}{x-a}\right] = \int_0^1 f^{(n+1)}\,[a+t(x-a)]\,Q_n(t)\,dt \quad \cdot \text{(E)}$$

the result (D) is equivalent to a formula of Darboux (Whittaker and Watson, *Modern Analysis*, 4th edition, p. 125).

Now write

$$F(t) = \sum_{m=0}^{n} (-1)^m(x-a)^m\,A_n^{(n-m)}(t)\,f^{(m)}\,[a+t(x-a)]$$

$$\Phi(t) = \sum_{m=0}^{p} (-1)^m(x-a)^m\,B_p^{(p-m)}(t)\,\phi^{(m)}\,[a+t(x-a)]$$

where

$$A_n(t) = \sum_{r=0}^{n} a_r t^r, \; B_p(t) = \sum_{r=0}^{p} b_r t^r.$$

Then it is easily verified that

$$F'(t) = (-1)^n (x-a)^{n+1} A_n(t) f^{(n+1)}[a + t(x-a)]$$

$$\Phi'(t) = (-1)^p (x-a)^{p+1} B_p(t) \phi^{(p+1)}[a + t(x-a)]$$

If we now apply Cauchy's formula

$$\frac{F(1) - F(0)}{\Phi(1) - \Phi(0)} = \frac{F'(t_1)}{\Phi'(t_1)}, \quad 0 < t_1 < 1$$

we get

$$\frac{\sum_{m=0}^{n} (-1)^m (x-a)^m \left[A_n^{(n-m)}(1) f^{(m)}(x) - A_n^{(n-m)}(0) f^{(m)}(a)\right]}{\sum_{m=0}^{p} (-1)^m (x-a)^m \left[B_p^{(p-m)}(1) \phi^{(m)}(x) - B_p^{(p-m)}(o) \phi^{(m)}(a)\right]} =$$

$$= \frac{(-1)^n (x-a)^{n+1} A_n(t_1) f^{(n+1)}(\xi)}{(-1)^p (x-a)^{p+1} B_p(t_1) \phi^{(p+1)}(\xi)}$$

On using the identity (D) on the L.H.S. this gives

$$\frac{\sum_{r=0}^{n} a_r D^r \left[\dfrac{f^{(n-r)}(x) - f^{(n-r)}(a)}{x-a}\right]}{\sum_{r=0}^{p} b_r D^r \left[\dfrac{\phi^{(p-r)}(x) - \phi^{(p-r)}(a)}{x-a}\right]} = \frac{A_n\left(\dfrac{\xi-a}{x-a}\right) f^{(n+1)}(\xi)}{B_p\left(\dfrac{\xi-a}{x-a}\right) \phi^{(p+1)}(\xi)} \quad \ldots \text{(F)}$$

This is the desired generalisation of (A). It reduces to the latter if we take $a_0 = a_1 = \ldots. = a_{n-1} = 0$; $b_0 = b_1 = \ldots. = b_{p-1} = 0$.

The above result can be still further generalised. We set

$$F(t) = \sum_{m=0}^{n} (-1)^m (x-a)^m A_n^{(n-m)}(t) f^{(m)}[a + t(x-a)]$$

$$\Phi(t) = \sum_{m=0}^{p} (-1)^m (x-a)^m B_p^{(p-m)}(t) \phi^{(m)}[a + t(x-a)]$$

$$\Psi(t) = \sum_{m=0}^{q} (-1)^m (x-a)^m C_q^{(q-m)}(t) \psi^{(m)}[a + t(x-a)]$$

where

$$A_n(t) = \sum_{r=0}^{n} a_r t^r, \; B_p(t) = \sum_{r=0}^{p} b_r t^r, \; C_q(t) = \sum_{r=0}^{q} c_r t^r$$

If we put

$$H(t) = \begin{vmatrix} F(t), & F(0), & F(1) \\ \Phi(t), & \Phi(0), & \Phi(1) \\ \Psi(t), & \Psi(0), & \Psi(1) \end{vmatrix}$$

then $H(1) = H(0) = 0$, so that $H'(t_1) = 0,\ 0 < t_1 < 1$,

i.e.,
$$\begin{vmatrix} F'(t_1), & F(0), & F(1) \\ \Phi'(t_1), & \Phi(0), & \Phi(1) \\ \Psi'(t_1), & \Psi(0), & \Psi(1) \end{vmatrix} = 0.$$

Using (D) and simplifying, this can be finally reduced to the form:

$$\begin{vmatrix} A_n\left(\dfrac{\xi - a}{x - a}\right) f^{(n+1)}(\xi), & \sum\limits_{r=0}^{n} a_r D^r\left(\dfrac{1}{x-a}\right) f^{(n-r)}(a), & \sum\limits_{r=0}^{n} a_r D^r\left[\dfrac{f^{(n-r)}(x) - f^{(n-r)}(a)}{x-a}\right] \\ B_p\left(\dfrac{\xi - a}{x - a}\right) \phi^{(p+1)}(\xi), & \sum\limits_{r=0}^{p} b_r D^r\left(\dfrac{1}{x-a}\right) \phi^{(p-r)}(a), & \sum\limits_{r=0}^{p} b_r D^r\left[\dfrac{\phi^{(p-r)}(x) - \phi^{(p-r)}(a)}{x-a}\right] \\ C_q\left(\dfrac{\xi - a}{x - a}\right) \psi^{(q+1)}(\xi), & \sum\limits_{r=0}^{q} c_r D^r\left(\dfrac{1}{x-a}\right) \psi^{(q-r)}(a), & \sum\limits_{r=0}^{q} c_r D^r\left[\dfrac{\psi^{(q-r)}(x) - \psi^{(q-r)}(a)}{x-a}\right] \end{vmatrix} = 0. \quad \text{(G)}$$

If we take in particular $A_n(t) = t^n$, $B_p(t) = t^p$ and $C_q(t) = t^q$, this reduces to Rajagopal's result (see Mahajani, *Analysis*, p. 119).